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Two Papers on non well-founded Sets

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Expressing Infinity without Foundation

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Abstract

The Axiom of Infinity can be expressed by stating the existence of sets satisfying a formula which involves restricted universal quantifiers only, even if the Axiom of Foundation is not assumed.

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The problem of expressing the existence of infinite sets in the first order set theoretic language by means of formulae of low logical complexity has been addressed in [PP88] and [PP90b]. While the usual formulations of the Infinity Axiom (*Inf*) make use of formulae involving (at least) alternations of universal and existential restricted quantifiers, [PP88] provided the first example of a formula involving only restricted universal quantifiers, whose satisfiability entails the existence of infinite sets, provided the Foundation Axiom (*FA*) is assumed together with the usual axioms of Zermelo-Fraenkel except, of course, the Infinity Axiom. It was then observed in [PP90b] that an even shorter formula had the same property. As explained in [PP88] the above problem is related to the so-called decision problem for fragments of set theory (see [CFO90]).

Set theories not assuming *FA* but rather contradicting it in various forms have come to attract considerable interest (see [Acz88]) and the corresponding decision problem has begun to be investigated (see [PP90a]). It is therefore of particular interest to ask whether there are restricted purely universal formulae which are satisfiable but not finitely satisfiable, even when *FA* is dropped.

In this note we show that a positive answer can be obtained through an appropriate merging of the two formulae in [PP88] and [PP90b], although none of them suffices alone.

Let $\mathcal{L}_\in$ be the first order set theoretic language with identity, based on the membership relation $\in$. A formula of $\mathcal{L}_\in$ is restricted if it does not contain quantifiers except for the restricted quantifiers $(\forall x \in y)$ and $(\exists x \in y)$.

Let ZF^- denote $ZF - Inf$ and ZF^{--} denote $ZF^- - FA$. In ZF^{--} one can define the ordinals as transitive sets well ordered by $\in$ and the non-zero natural numbers as successor ordinals with zero and successor ordinals only, as elements. Finiteness is taken to stand for equinumerousity with a natural number and Inf can be stated as the existence of a set containing all the natural numbers.

In ZF^-, but not in ZF^{--}, Inf is equivalent to any of the other formulations of the Infinity Axiom in use. Note that Inf states the satisfiability of a formula that, besides restricted quantifiers of both types, involves also an unrestricted universal quantifier needed to express well foundedness.

Let φ_1 and φ_2 be the following formulae from [PP88] and [PP90b]:

$$\begin{gathered} a \neq b \ \wedge \ a \notin b \ \wedge \ b \notin a \ \wedge \\ (\forall x \in a)(\forall u \in x)(u \in b) \ \wedge \ (\forall x \in b)(\forall u \in x)(u \in a) \ \wedge \ (\forall x \in a)(x \notin b) \ \wedge \\ (\forall x, y \in a)(\forall z, w \in b)(z \in x \ \wedge \ x \in w \ \wedge \ w \in y \ \rightarrow \ z \in y) \ \wedge \\ (\forall x, y \in b)(\forall z, w \in a)(z \in x \ \wedge \ x \in w \ \wedge \ w \in y \ \rightarrow \ z \in y), \end{gathered}$$

and

$$\begin{gathered} a \neq b \ \wedge \ a \notin b \ \wedge \ b \notin a \ \wedge \\ (\forall x \in a)(\forall u \in x)(u \in b) \ \wedge \ (\forall x \in b)(\forall u \in x)(u \in a) \ \wedge \\ (\forall x \in a)(\forall y \in b)(x \in y \ \vee \ y \in x). \end{gathered}$$

From [PP88] and [PP90b] we have the following property:

PROPOSITION 1

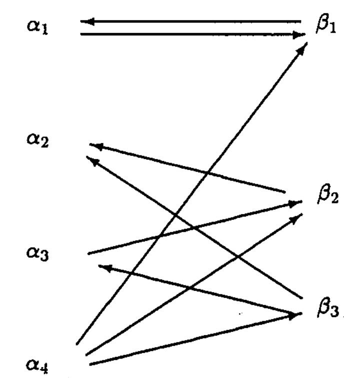
1) $ZF^- \vdash (\exists a, b)\varphi_1(a, b) \ \rightarrow \ Inf$;

2) $ZF^- \vdash (\exists a, b)\varphi_2(a, b) \ \rightarrow \ Inf$.

Actually 2) follows immediately from 1) since $ZF^- \vdash \varphi_2(a, b) \rightarrow \varphi_1(a, b)$. [PP90b] provides specific examples which show that $ZF^- \nvdash \varphi_1(a, b) \rightarrow \varphi_2(a, b)$.

PROPOSITION 2 $ZF^{--} \nvdash (\exists a, b)\varphi_1(a, b) \rightarrow Inf$.

Proof. Consider the following graph G_1 :



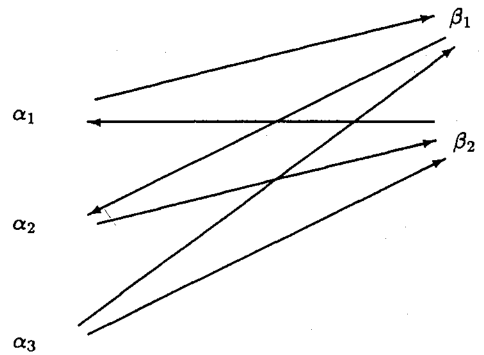
By making use of a suitable permutation of the universe it is easy to define a model $\mathcal{M}$ of $ZF^{--} + \neg Inf$ in which there are elements a, a_1, a_2, a_3, a_4 and b, b_1, b_2, b_3 such that

$$\begin{aligned} a &= \{a_1, a_2, a_3, a_4\}^{\mathcal{M}}; \\ b &= \{b_1, b_2, b_3\}^{\mathcal{M}}; \\ a_i \in^{\mathcal{M}} b_j &\Leftrightarrow \alpha_i \rightarrow \beta_j \quad \text{for } 1 \leq i \leq 4, \ \ 1 \leq j \leq 3; \\ b_i \in^{\mathcal{M}} a_j &\Leftrightarrow \beta_i \rightarrow \alpha_j \quad \text{for } 1 \leq i \leq 3, \ \ 1 \leq j \leq 4. \end{aligned}$$

It is then straightforward to check that a and b satisfy φ_1 in $\mathcal{M}$. ■

PROPOSITION 3 $ZF^{--} \nvdash (\exists a,b)\varphi_2(a,b) \to Inf$.

Proof. Use the same kind of argument as for the previous property starting with the following graph G_2 :



■

Remark. Since G_1 and G_2 are extensional graphs, i.e. two different nodes have different sets of predecessors, the consistency results following from Proposition 2 and Proposition 3 can be improved to claim that the existence of finite sets satisfying φ_1 and φ_2 is actually a theorem of $ZF^{--} + BA_1$ where BA_1 stands for the weak form of Boffa's antifoundation axiom discussed in [Acz88].

The transitive closures of the finite sets a and b described by G_1 and G_2 contain a loop of the form $a_1 \in a_2 \in a_1$ and $a_1 \in b_1 \in a_2 \in b_2 \in a_1$ respectively. That is no exception. Let us begin by noticing that because the two conjuncts

$$(\forall x \in a)(\forall u \in x)(u \in b)$$

and

$$(\forall x \in b)(\forall u \in x)(u \in a)$$

are in both φ_1 and φ_2, if a and b satisfy either φ_1 or φ_2 then $a \cup b$ contains both the transitive closure of a and the transitive closure of b. Thus if a and b satisfy either φ_1 or φ_2 and are finite, then they are actually *hereditarily finite*, by which, in absence of FA, it is meant that they have finite pictures (like G_1 and G_2 in the above examples),(see [Acz88]).

PROPOSITION 4 *(In ZF^{--}) If a and b are finite and $\varphi_1(a,b)$ holds, then there is a_1 in a and b_1 in b such that $a_1 \in b_1$ and $b_1 \in a_1$.*

Proof. From the proof in [PP88] it follows that if a and b are finite and $\varphi_1(a,b)$ holds, then $a \cup b$ cannot be well founded. Hence $a \cup b$ must contain a cycle with respect to membership, say $c_1 \in c_2 \in \ldots \in c_n \in c_1$.

From $\varphi_1(a,b)$ it follows that an element of an element of a cannot be itself an element of a, since otherwise $a \cap b \neq \emptyset$. Similarly an element of an element of b cannot be itself an element of b . From that it follows that $c_1, \ldots, c_n$ contains alternatively an element of a and an element of b, and furthermore that n is even. Then by induction on n, using the condition

$$(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_1 \in y_1 \wedge y_1 \in x_2 \wedge x_2 \in y_2 \;\to\; x_1 \in y_2),$$

when $c_1 \in a$, or the condition

$$(\forall x_1, x_2 \in b)(\forall y_1, y_2 \in a)(x_1 \in y_1 \wedge y_1 \in x_2 \wedge x_2 \in y_2 \;\to\; x_1 \in y_2)$$

when $c_1 \in b$, it follows immediately that in any such cycle we must have $c_1 \in c_n$, and our claim is proved by taking c_1 for a_1 and c_n for b_1 . ■

As a straightforward consequence of the previous Proposition we have that if we add the conjunct

$$(\forall x \in a)(\forall y \in b)(x \in y \to y \notin x)$$

then we obtain a formula φ_1' involving only restricted universal quantifiers whose satisfiability entails the existence of infinite sets even if FA is dropped.

COROLLARY 1 $ZF^{--} \vdash (\exists a,b)(\varphi_1(a,b) \wedge (\forall x \in a)(\forall y \in b)(x \in y \to y \notin x) \to Inf$.

Proof. By Proposition 4, the existence of a and b satisfying φ_1' entails the existence of non finite sets. By a well known argument (see [Lev79]), using the Power Set Axiom, the existence of a non finite set implies Inf in ZF^{--} . ■

A proof entirely similar to the one given for Proposition 4 shows that:

PROPOSITION 5 *(In ZF^{--}) If a and b are finite and $\varphi_2(a,b)$ holds and $a \cap b = \emptyset$, then there are a_1, a_2 in a and b_1, b_2 in b such that $a_1 \in b_1 \in a_2 \in b_2 \in a_1$ or $b_1 \in a_1 \in b_2 \in a_2 \in b_1$.*

Therefore the addition to φ_2 of $(\forall x \in a)(x \notin b)$ and the two further conditions

$$(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_1 \in y_1 \wedge y_1 \in x_2 \wedge x_2 \in y_2 \;\to\; y_2 \notin x_1)$$

and

$$(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_1 \in y_1 \wedge y_1 \in x_2 \wedge x_2 \in y_2 \;\to\; y_2 \notin x_1)$$

also yields a formula φ_2' which satisfies Corollary 1.

It follows immediately from [PP88] that φ_1' and φ_2' are irredundant, in the sense that if one drops one of the conjuncts than this property fails. We can however provide an even simpler formula whose satisfiability entails Inf in ZF^{--}. Such a formula is of particular interest also in connection with the decision problem since, unlike φ_1' and φ_2', it does not introduce conjuncts which are implications whose consequent is a negative literal.

Let $\varphi(a, b)$ be the following formula:

$$a \neq b \wedge a \notin b \wedge b \notin a \wedge$$
$$(\forall x \in a)(\forall y \in x)(y \in b) \wedge (\forall x \in a)(\forall y \in x)(y \in b) \wedge$$
$$(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_1 \in y_1 \wedge y_1 \in x_2 \wedge x_2 \in y_2 \rightarrow x_1 \in y_2) \wedge (\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$$

PROPOSITION 6 $ZF^{--} \vdash (\exists a, b)\varphi(a, b) \rightarrow Inf$.

Proof. Working in ZF^{--} we show that if a and b satisfy φ then $a \cup b$ cannot be finite. The conclusion follows as in Proposition 5.

Assuming that a and b satisfy φ we show that if X is a finite non empty subset of $a \cup b$ then there is an element $c_X \in X$ such that either $c_X \in a$ and $X \cap b \subseteq c_X$ or $c_X \in b$ and $X \cap a \subseteq c_X$. That is proved by induction on the cardinality of X.

If X is a singleton the claim is clear since $a \cap b = \emptyset$. So assume X has more than one element. If $X \cap a = \emptyset$ then every element in X can be taken as c_X; otherwise pick $a_i \in X \cap a$ and let $X' = X \setminus \{a_i\}$.

By induction hypothesis there is $c_{X'}$ in X' satisfying our claim. If $c_{X'} \in a$ and $X' \cap b \subseteq c_{X'}$ then, since $a \cap b = \emptyset$, $X \cap b = X' \cap b$, hence $X \cap b \subseteq c_X'$ and we can take c_X to be $c_{X'}$ itself. On the other hand, if $c_{X'} \in b$ and $X' \cap a \subseteq c_{X'}$ we have two cases:

case 1. $a_i \in c_{X'}$. Then it still suffices to let $c_X = c_{X'}$;

case 2. $a_i \notin c_{X'}$. Then, since $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$, $c_{X'} \in a_i$. If $b \cap X \subseteq a_i$, then of course it suffices to let $c_X = a_i$. Otherwise there must be a $b_i \in b \cap X$ such that $b_i \notin a_i$. Then, as above, $a_i \in b_i$. In this case it suffices to let $c_X = b_i$. In fact since $(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_1 \in y_1 \wedge y_1 \in x_2 \wedge x_2 \in y_2 \rightarrow x_1 \in y_2)$ from $X' \cap a \subseteq c_{X'}$, $c_{X'} \in a_i$ and $a_i \in b_i$, it follows that $X' \cap a \subseteq b_i$ and therefore that $X \cap a \subseteq b_i$.

We can now prove that $a \cup b$ cannot be finite.

First of all $a \cup b \neq \emptyset$, since otherwise $a = b = \emptyset$ and $\varphi(a, b)$ would fail.

If $a \cup b$ were finite and non-empty, then we could take $a \cup b$ for X in the above claim and conclude that there is $c \in a \cup b$ such that either $c \in a$ and $(a \cup b) \cap b = b \subseteq c$ or $c \in b$ and $(a \cup b) \cap a = a \subseteq c$. In the former case it would follow that $c = b$ since from $c \in a$ it follows that $c \subseteq b$, because of the condition $(\forall x \in a)(\forall y \in x)(y \in b)$. But then $b \in a$ contradicting $b \notin a$ as required by φ. Similarly in the latter case we would get $a \in b$ contrary to $a \notin b$. ∎

Obviously Proposition 6 still holds if we exchange a with b.

As for φ_1 in [PP88] it is easy to see that if any of the conjunct in φ is dropped then the resulting formula turns out to be satisfiable by finite (although not necessarily well founded) sets.

Remark. It would be interesting to prove Proposition 6 without using the Power Set Axiom.

Note that the implication in Proposition 1, Corollary 1, and Proposition 6 can be reversed since $\varphi, \varphi_1, \varphi_2, \varphi_1'$ as well as φ_2' are all satisfied by $\omega' = \{f_0, f_1, \ldots\}$ and $\omega'' = \{g_0, g_1, \ldots\}$, where $f_0 = \emptyset, g_n = \{f_0, \ldots, f_n\}, f_{n+1} = \{g_0, \ldots, g_n\}$, and the existence of ω' and ω'' is ensured in ZF^{--} provided Inf is assumed. Thus the existential closures of $\varphi, \varphi_1, \varphi_2, \varphi_1'$ as well as φ_2' can all be used to express the Axiom of Infinity in ZF^{--}. To that end the presence of nesting of quantified variables, as in $(\forall x \in a)(\forall y \in x)(y \in b)$ is in general unavoidable, in fact it is unavoidable in any restricted universal formula which is satisfiable but not finitely satisfiable (see [PP88]). Furthermore the presence of at least two free variables is necessary, in fact if a restricted universal formula with just one free variable is satisfied by a set a and there is a finite descending chain of memberships starting with a and ending with the empty set, its Mostowski's collapse provides an hereditarily finite set satisfying the same formula, on the other hand if no such chain exists then under Aczel's antifoundation axiom AFA, a is just the finite set Ω.

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The Decision Problem for Restricted Universal Quantification in Set Theory and the Axiom of Foundation

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Abstract

The still unsettled decision problem for the restricted purely universal formulae,($(\forall)_0$ -formulae), of the first order set-theoretic language based over $=, \in$, is discussed in relation with the adoption or rejection of the axiom of foundation. Assuming the axiom of foundation, the related finite set-satisfiability problem, for the very significant subclass of the $(\forall)_0$ -formulae consisting of the formulae involving only nested variables of level 2, is proved to be semidecidable on the ground of a reflection property over the hereditarily finite sets, and various extensions of this result are obtained.

When variables are restricted to range only over sets, in universes with infinitely many urelements, the set-satisfiability problem is shown to be solvable provided the axiom of foundation is assumed; if it is not, then the decidability of a related derivability problem still holds. That, in turn, suggests the alternative adoption of an antifoundation axiom under which the set-satisfiability problem is also solvable (of course with different answers).

Turning to set theory without urelements, assuming a form of Boffa's antifoundation axiom, the complement of the set-satisfiability problem for the full class of Δ_0 -formulae is shown to be semidecidable; a result that is known not to hold, for the set-satisfiability problem itself, even for a very restricted subclass of the Δ_0 -formulae.

Key phrases: Set-satisfiability problem, antifoundation axioms

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1 Introduction

Due to the incompleteness of any consistent axiomatization of a sufficiently strong mathematical theory, the well known decision problem for first order logic has two possible set theoretic versions. Namely:

1) the *derivability problem*: to establish, given a sentence φ, whether φ is derivable or not in a given formal system for set theory like, say, ZF,

2) *the validity problem*: to establish, given a sentence φ, whether φ holds in the *intended* set theoretic universe.

Although apparently more vaguely defined, it is the latter form of the problem that has been mostly tackled in the research work that has been done in this area, see ([CFO90]). The form the problem has generally taken has been that of determining, for any given formula in a syntactically defined class, whether there are sets satisfying it or not. For this reason the problem studied has been often referred to as a satisfiability problem. Since, however, it is Problem 1 that is equivalent to the ordinary logical satisfiability problem, for the sake of clarity we will refer to the above version of Problem 2 as the set-satisfiability problem. When the sets satisfying a formula are required to be finite, we have the finite set-satisfiability problem that has also been considered in the literature.

Problem 1 and Problem 2 are rather naturally related. In general, as is shown in [PP88a], a positive solution to Problem 2 entails a positive solution to Problem 1, since it yields the completeness of an appropriate theory with respect to the class of formulae under consideration. On the other hand positive solutions to Problem 1 have been obtained that do not entail a positive solution to the corresponding Problem 2, although they indicate possible strengthening of our set theoretic assumptions under which Problem 2 has also a positive solution.

In both forms the decision problem is quite sensitive to the axiomatic of set theory, even for very restricted and simply defined classes of formulas. A difference is that, unlike Problem 1, Problem 2 may well turn out to be not even semidecidable ([PP90d]).

In this work, the above issues will be discussed with reference to the use of the restricted universal quantification in the $\{\in, =\}$-language and to the axiom of foundation.

2 Reductions and reflection

Let us recall that the Δ_0 formulae are the formulae in the first order language consisting of the equality and membership relation and a constant for the empty-set, in which only restricted quantifiers are involved. The Δ_0 formulae that can be transformed in a purely universal prenex form are called $(\forall)_0$ formulae. They can be characterized as follows (see [PP90a]):

PROPOSITION 2.1 *$\varphi \in (\forall)_0$ iff it is equivalent to a conjunction*

1. *$\varphi_1 \wedge \ldots \wedge \varphi_m$ such that:*
2. *$\forall i, 1 \leq i \leq m$*

$$\varphi_i = (\forall x^i_1 \in y^i_1) \ldots (\forall x^i_{j_i} \in y^i_{j_i})(\ell^i_1 \vee \ldots \vee \ell^i_{h_i}),$$

where ℓ^i_k, for $1 \leq k \leq h_i$, is a literal of the form $a \in b, a \notin b, a = b$ or $a \neq b$ with a and b variables.

Note. If we allow in the previous definition only literals of the form $a \in b$ and $a \neq b$, the expressive power of the class of formulae defined does not change.

As a matter of fact, given a formula $\varphi \in (\forall)_0$ it is always possible to write down a formula $\varphi' \in (\forall)_0$ such that the literals in the matrices of the conjuncts in φ' are all of the form $a \in b$ or $a \neq b$, and such that φ and φ' are equisatisfiable. To see this it is enough to observe that $a \notin b$ can be replaced by $(\forall z_a \in b)(z_a \neq a)$ and that, thanks to extensionality, $a = b$ can be replaced by $(\forall z_b \in b)(z_b \in a) \wedge (\forall z_a \in a)(z_a \in b)$. After these changes (and possibly after a renaming of bound variables) φ' is obtained by simply shifting the universal quantifiers in their correct position according to the previous definition.

Analogously we could allow, in the definition of $(\forall)_0$-formulae, the use of the constant $\emptyset$ for the empty-set, without changing the expressive power for the class. In this case every occurrence of $\emptyset$ should be eliminated by substituting it with a new variable $z_\emptyset$, and adding the conjunct $(\forall z \in z_\emptyset)(z \neq z)$.

For further reference let us note that the above observations are in no way related to the assumption of the axiom of foundation.

DEFINITION 2.1 *given $\varphi \in (\forall)_0$ we will call <u>level of nesting of φ</u> the natural K in the longest chain of the form:*

$$(\forall x_0 \in x)(\forall x_1 \in x_0) \ldots (\forall x_K \in x_{K-1}),$$

in one of the prefixes of one of the conjuncts of φ.

We will indicate by $(\forall)_{0,K}$ the subclass of the $(\forall)_0$-formulae containing all formulae having level of nesting $\leq K$.

Although partial results have been obtained, the decision problem, in both forms, for the class of the $(\forall)_0$-formulae, remains still open. On one hand, various subclasses of the $(\forall)_0$-formulae have been proved to be decidable (see [BFOS81] [PP90a] [CCP88]) and among these let us recall the class $(\forall)_{0,0}$ that, beside being decidable, has the property that if $\varphi \in (\forall)_{0,0}$ is satisfiable, then it is satisfied by hereditarily finite sets (when this situation holds we will say that the given class reflects over $\mathcal{HF}$). On the other hand, [PP90d] shows that the addition to the language of a unary predicate $P(x)$ intended to mean that *x is a pair* is already sufficient to make it unsolvable.

Two reductions of the general set-satisfiability problem are possible. By the first of these two reductions the satisfiability of a given $\varphi \in (\forall)_0$ is equivalent to the satisfiability of an easy to determine $\bar{\varphi} \in (\forall)_{0,2}$. This implies that the decision problem for $(\forall)_{0,2}$ is no easier than the decision problem for the whole class $(\forall)_0$.

PROPOSITION 2.2 *Given $\varphi(x_1,\ldots,x_n) \in (\forall)_0$ we can determine $\bar{\varphi}(x_1,\ldots,x_n) \in (\forall)_{0,2}$ such that* [1]

$$ZF^- \vdash \exists \vec{x}\varphi \leftrightarrow \exists \vec{x}\bar{\varphi}.$$

Proof. For each quantified conjunct φ_i in φ of the form:

$$\varphi_i = (\forall x_1^i \in y_1^i)\ldots(\forall x_{j_i}^i \in y_{j_i}^i)\varphi_i^M.$$

In $\bar{\varphi}$ we will have the following conjunct $\bar{\varphi}_i$ associated with φ_i:

$$\bar{\varphi}_i = (\forall x_1^i \in \bar{x})\ldots(\forall x_{j_i}^i \in \bar{x})(\varphi_i^M \vee x_1^i \notin y_1^i \vee x_{j_i}^i \notin y_{j_i}^i).$$

Then the set of conjuncts of $\bar{\varphi}$ will be completed by the following formulae:

1) $(\forall y_1 \in \bar{x})(\forall y_2 \in y_1)(y_2 \in \bar{x})$;

2) $x_1 \in \bar{x}$;

⋮

n+1) $x_n \in \bar{x}$.

The idea is simply that $\bar{x}$ must play the role of a sufficiently large transitive set (a sort of *universe*) that can be the bound for any of the restricted quantifiers in φ.

Let us prove that φ and $\bar{\varphi}$ are equisatisfiable:

α) if $Mx_1,\ldots,Mx_n$ are n sets satisfying φ then, denoting by $trcl(a)$ the transitive closure of a set a, we have that

$$Mx_1,\ldots,Mx_n, trcl(\{Mx_1,\ldots,Mx_n\})$$

are $n+1$ sets that satisfy $\bar{\varphi}$ when substituted for the variables $x_1,\ldots,x_n,\bar{x}$ respectively.

[1] by $\exists\vec{x}\varphi$ and $\forall\vec{x}\varphi$ we denote, respectively, the existential and universal closures of the formula φ

β) vice versa, if $\bar{\varphi}$ is satisfied by $Mx_1,\ldots,Mx_n,M\bar{x}$, then it is immediate to check that $Mx_1,\ldots,Mx_n$ satisfy φ.

Finally notice that $\bar{\varphi}$ meets all the requirements to be a $(\forall)_0$-formula and that its only nested conjunct is the formula at point 1) above, which has level of nesting two. ■

Thus every $(\forall)_0$ formula is equivalent (in ZF^-) to a $(\forall)_{0,0}$ formula containing a unique positive occurrence of the predicate $Trans$. On the other hand every $(\forall)_{0,0}$ formula containing any number of occurrences of $Trans$, no matter whether positive or negative, is equisatisfiable with a $(\forall)_{0,2}$ formula since occurrences of $\neg Trans(x)$ can be eliminated, by adding two additional variables z_1 and z_2, in favor of $z_1 \in x \wedge z_2 \in z_1 \wedge z_2 \notin x$. Therefore we have:

PROPOSITION 2.3 *The decision problems for the class $(\forall)_0$ and $(\forall)_{0,0} + Trans$ are equivalent.*

Let us note that the proof of proposition 2.2 depends on the existence of the transitive closure of any given set.

Under the assumption of the well foundeness of the universe this is always true, whereas for non well founded universes the transitive closure of a set a can be defined as

$$\bigcup_{i\in\omega} \bigcup^i a,$$

but this requires the axiom of infinity.

Without the axiom of infinity, the existence of the transitive closure of a set cannot be a theorem, since in models of $ZF^- - \infty + (\forall x)(Finite(x))$ having an infinity of elements forming a descending chains with respect to the membership relation, the transitive closure of a set in such a chain would provide an infinite element in the model.

The decision problem for the $(\forall)_0$-formulae can also be reduced to the decision problem for a subclass of the $(\forall\exists)_0$-formulae which is defined as follows:

DEFINITION 2.2 *$\varphi \in (\forall\exists)_0$ if and only if $\varphi \in \Delta_0$ and there exists one of its prenex normal forms in which the quantified prefix is of the form*

$$\forall y_1,\ldots,y_s\exists x_1,\ldots,x_t.$$

A complete and motivated definition of the Δ_0-hierarchy is given in [PP88a].

PROPOSITION 2.4 *given* $\varphi(x_1,\ldots,x_n) \in (\forall)_0$ *we can determine* $\tilde{\varphi}(x_1,\ldots,x_n,\bar{x}) \in (\forall\exists)_0$ *such that no nesting of variables occur in* $\tilde{\varphi}$ *,it has only one existential quantifier and*

$$ZF \vdash \exists\bar{x}\varphi \leftrightarrow \exists\bar{x}\tilde{\varphi}.$$

Proof. $\tilde{\varphi}$ will be obtained in a way analogous to $\bar{\varphi}$ above.

For each quantified conjunct φ_i in φ of the form

$$\varphi_i \quad = \quad (\forall x_1^i \in y_1^i)\ldots(\forall x_{j_i}^i \in y_{j_i}^i)\varphi_i^M.$$

we put the following conjunct $\bar{\varphi}_i$ in φ :

$$\tilde{\varphi}_i \quad = \quad (\forall x_1^i \in \bar{x})\ldots(\forall x_{j_i}^i \in \bar{x})(\varphi_i^M \ \vee \ x_1^i \notin y_1^i \ \vee \ x_{j_i}^i \notin y_{j_i}^i).$$

Then we complete the set of conjuncts in $\tilde{\varphi}$ by adding the following formulas:

1) $(\forall y_1 \in \bar{x})(\forall y_2 \in \bar{x})(y_1 \neq y_2 \rightarrow (\exists y_3 \in \bar{x})(y_3 \in y_1 \ \wedge \ y_3 \notin y_2) \ \vee \ (y_3 \in y_2 \ \wedge \ y_3 \notin y_1))$ to be denoted by $Ext(\bar{x})$,

2) $x_1 \in \bar{x}$,

⋮

n+1) $x_n \in \bar{x}$.

Again the idea is to introduce a new set that will act as a universe and to ask, using a $(\forall\exists)_0$-formula, that it is rich enough to have witnesses for all the disequalities involving pairs of sets in it.

Notice that the transitive closure used in the previous case can be seen to be a particular case of such a universe. In fact it is always the case that an $(n+1)$-tuple of sets satisfying $\bar{\varphi}$ will also satisfy $\tilde{\varphi}$.

The vice versa is not true.

Let us prove that φ and $\tilde{\varphi}$ are equisatisfiable:

α) As in case α) of proposition 3.1.

β) Vice versa, if $\tilde{M}x_1,\ldots,\tilde{M}x_n,\tilde{M}\bar{x}$ is an $(n+1)$-tuple satisfying $\tilde{\varphi}$, define inductively the function f as

$$f(x) = \begin{cases} \{f(y) \mid y \in x \cap \tilde{M}\bar{x}\} & \text{if } x \cap \tilde{M}\bar{x} \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

then let $Mx_i = f(\tilde{M}x_i)$ for $1 \leq i \leq n$.

It is easy to check that since $\tilde{M}\bar{x}$ satisfies the formula at point 1 in $\tilde{\varphi}$, we have that for any $x, y \in \tilde{M}\bar{x}$

$$x = y \quad \leftrightarrow \quad f(x) = f(y),$$

from which it follows that

$$x \in y \quad \leftrightarrow \quad f(x) \in f(y).$$

As we already noticed, the previous two properties are sufficient to guarantee that $Mx_1,\ldots,Mx_n$ satisfy φ , since if this were not true then the inverse images with respect to f of sets satisfying the negation of a conjunct in φ would satisfy the negation of the corresponding conjunct in $\tilde{\varphi}$. ■

Since as far as satisfiability is concerned, negative occurrences of $Ext(x)$ can be eliminated, by adding two additional variables z_1 and z_2 , in favor of $z_1 \in x \ \wedge \ z_2 \in x \ \wedge \ z_1 \neq z_2 \ \wedge \ (\forall y \in x)(y \in z_1 \ \rightarrow \ y \in z_2) \ \wedge \ (\forall y \in x)(y \in z_2 \ \rightarrow \ y \in z_1)$., the same argument leading from proposition 3.1 to proposition 3.2 applies to yield:

PROPOSITION 2.5 *The decision problems for the class* $(\forall)_0$ *and* $(\forall)_{0,0} + Ext$ *are equivalent.*

Notice that like $(\forall x)(\exists y)(x \subseteq y \ \wedge \ Trans(y))$). also $(\forall x)(\exists y)(x \subseteq y \ \wedge \ Ext(y))$ is non derivable in $ZF^- - \infty$, since in non-standard models of $ZF^- - \infty + (\forall x)(Finite(x))$, for any x belonging to an infinite descending chain, $\{x, \emptyset\}$ cannot be extended into any extensional set.

The next result is the reflection property over $\mathcal{HF}$ for the finite s-satisfiability of the class $(\forall)_{0,2}$ relative to a well founded universe. We point out that this result does not entail that this form of reflection holds for the whole $(\forall)_0$ class, since φ and $\bar{\varphi}$ of Prop.3.1 are not satisfied by the same tuple of sets. In fact we can give an example of a $(\forall)_{0,3}$-formula for which such a property does not hold. Thus finite s-satisfiability reflection over $\mathcal{HF}$ for the class $(\forall)_{0,2}$ is the best we can hope for.

The example makes use of the following formula $\varphi_{\omega',\omega''}(a,b)$, which is in $(\forall)_{0,2}$, presented in [PP88b]:

$$\begin{gathered} a \neq b \wedge a \notin b \wedge b \notin a \wedge \\ (\forall x \in a)(\forall u \in x)(u \in b) \ \wedge \ (\forall x \in b)(\forall u \in x)(u \in a) \ \wedge \ (\forall x \in a)(x \notin b) \wedge \\ (\forall x, y \in a)(\forall z, w \in b)(z \in x \ \wedge \ x \in w \ \wedge \ w \in y \ \rightarrow \ z \in y) \wedge \\ (\forall x, y \in b)(\forall z, w \in a)(z \in x \ \wedge \ x \in w \ \wedge \ w \in y \ \rightarrow \ z \in y) \end{gathered}$$

Consider the following formula $\varphi_3(z,w)$:

$$z \neq \emptyset \neq w \wedge w \subseteq z \wedge w \neq z \wedge (\forall x_1, x_2, x_3 \in z)(x_1 \neq x_2 \rightarrow x_3 = x_1 \vee x_3 = x_2) \wedge$$
$$(\forall x_1, x_2 \in z)(\varphi_{\omega',\omega''}(x_1,x_2) \vee \varphi_{\omega',\omega''}(x_2,x_1) \vee x_1 = x_2)$$

It is straightforward to check that the only possible pairs of elements satisfying $\varphi_3(z,w)$ are such that $Mz = \{A,B\}$, $Mw = \{A\}$ or $Mz = \{A,B\}$, $Mw = \{B\}$ with A,B satisfying $\varphi_{\omega',\omega''}(a,b)$ (and therefore infinite).

PROPOSITION 2.6 *Given* $\varphi(x_1,\dots x_n) \in (\forall)_{0,2}$ *, if* φ *is satisfied by finite sets, then it is satisfied by hereditarily finite sets as well.*

Proof. Let $a_1,\dots,a_n$ be n finite sets satisfying φ .

Let us consider the following procedure that, from now on, we will call *simplifying procedure*:

let $A_0 = \{a_1,\dots,a_n\}$.

Let $A_0^{max} = \{a \mid a \in A_0 \wedge (\forall b \in A_0)(rkb \leq rka)\}$, in general, given a set B we will indicate by B^{max} the set of elements of B having maximal rank.

The simplifying procedure will define a sequence of finite sets $A_0, A_1, \dots, A_p$ in the following way: let ρ_i be the rank of the elements in A_i^{max} and consider the following two cases:

a) ρ_i is a successor ordinal;

b) ρ_i is a limit ordinal.

In case a) if $A_i^{max} = \{a_1^i,\dots,a_{k_i}^i\}$, for any pair of the form $< a_\ell^i, a_m^i >$ such that $1 \leq \ell \neq m \leq k_i$, we consider an element $x_{\ell,m} \in a_\ell^i \setminus a_m^i$ if such an element exists (that is if $a_\ell^i \setminus a_m^i \neq \emptyset$).

Then we define

$$D_i^1 = \{x_{\ell,m} \mid 1 \leq \ell \neq m \leq k_i \wedge a_\ell^i \setminus a_m^i \neq \emptyset\}.$$

Moreover we put

$$F_i = \bigcup\{a \mid a \in A_i^{max} \wedge Finite(x)\} \cup \{c_i(a) \mid a \in A_i^{max}\},$$

where c_i is a *choice function* for A_i^{max} having as images elements of maximal rank on sets having rank which is a successor ordinal (such as the sets in A_i^{max}).

Finally we put

$$A_{i+1} = (A_i \setminus A_i^{max}) \cup D_i^1 \cup F_i.$$

Case b). Let D_i^1 be defined as in case a) , then let y_ℓ^i be an element in a_ℓ^i for $1 \leq \ell \leq k_i$, satisfying the following property:

i) the rank of y_ℓ^i is greater than the rank of any of the elements in $(A_i \setminus A_i^{max}) \cup D_i^1$.

Note that y_ℓ^i 's having the previous property clearly exist, since $(A_i \setminus A_i^{max}) \cup D_i^1$ is finite and that ρ_i is limit.

Moreover it is also the case that no y_ℓ^i belongs to any of the elements of A_0 , since none of them, being finite, can be one of the a_e^i 's (since ρ_i is a limit ordinal).

Let us define

$$D_i^2 = \{y_\ell^i \mid 1 \leq \ell \leq k_i\},$$

and

$$A_{i+1} = (A_i \setminus A_i^{max}) \cup D_i^1.$$

We claim that after a finite number of steps the simplifying procedure will produce an empty A_i : this simply follows from the fact that $\rho_i > \rho_{i+1}$ and any descending chain of ordinals must be finite.

Let p be the first natural number such that $A_p = \emptyset$.

We will use the A_i 's and the D_i^2 's to build a graph G such that one of its set-theoretic realizations will be the model we are seeking.

The nodes in the graph will be the following:

$$A \cup D^2 \cup L^2$$

where $A = \bigcup_{0 \leq i \leq p} A_i$, $D^2 = \bigcup\{D_i^2 \mid limit(\rho_i) \wedge 1 \leq i \leq p\}$ and L^2 is a set such that $L^2 \cap (A \cup D^2) = \emptyset$ and $|L^2| = |D^2|$.

It is useful to think of the set of nodes called L^2 as a set of *labels* to be attached to the nodes in D^2 .

According to this view we will indicate the elements in L^2 as e_ℓ^i assuming that e_ℓ^i is the node associated (is the label attached) to y_ℓ^i .

The edges in G will be divided into two groups:

1. for $a,b \in A \cup D^2$

$$a \rightarrow b \text{ if and only if } a \in b;$$

2. for $1 \leq i \leq p-1$ and $limit(\rho_i)$

$$e_\ell^i \rightarrow y_\ell^i.$$

At this point let us define the following set-theoretic realization of G in $\mathcal{HF}$: let us suppose that R is a natural number sufficiently large to satisfy the following two conditions:

A) R is greater than the number of nodes in G plus one;

B) there are at least $|D^2|$ distinct singletons (that we will indicate by s_ℓ^i) having rank R.

The set associated to a node a will be indicated by $a^\star$ and it is inductively defined as follow:

$$a^\star = \begin{cases} s_\ell^i & \text{if } a \text{ is } e_\ell^i \\ \{b^\star \mid b \rightarrow a\} & \text{otherwise} \end{cases}$$

For any $a, b \in A \cup D^2$ the following two properties hold:

α)

$$a = b \quad \leftrightarrow \quad a^\star = b^\star,$$

β)

$$a \in b \quad \leftrightarrow \quad a^\star \in b^\star.$$

This can be proved by induction on the maximum rank of a and b as follows:

Notice that the only thing to prove for α) is $a \neq b \rightarrow a^\star \neq b^\star$, and for β) is $a^\star \in b^\star \rightarrow a \in b$.

Case 1. a and b are of the form e_ℓ^i.

In this case $a^\star$ and $b^\star$ are of the form s_ℓ^i and therefore α) follows from the fact that distinct e_ℓ^i's have associated distinct s_ℓ^i's.

β) follows from the fact that all the s_ℓ^i's have the same rank.

Case 2. a is of the form e_ℓ^i and b is not.

α): in this case it is enough to observe that $b^\star$ may only have rank strictly smaller than $R+1$ (in case it does not contain an e_ℓ^i in its transitive closure) or strictly greater than $R+1$ (otherwise).

β): the claim follows from the inductive hypothesis since if $a \notin b$ and $a^\star \in b^\star$, then it would follow that $a^\star = c^\star \in b^\star$ with $a \neq c$ contradicting α) on the pair a, c.

Case 3. b is of the form e_ℓ^i and a is not.

α): entirely analogous to case 2 above.

β): follows from the fact that the e_ℓ^i's are singletons and their only element has rank equal to $rk(e_\ell^i) - 1$, whereas $a^\star$ must have rank strictly greater or strictly less than $rk(e_\ell^i) - 1$.

Case 4. a and b are not of the form e_ℓ^i and have the same rank.

α): since $a \neq b$, we have that in A there must be a c such that $c \in a$ and $c \notin b$ (or vice versa), hence the claim follows by induction hypothesis that allows us to conclude that $c^\star \in a^\star$ and $c^\star \notin b^\star$.

β): by inductive hypothesis.

Case 5. a and b are not of the form e_ℓ^i and they have different ranks.

α): let us suppose, without loss of generality, that $rank(a) < rank(b)$. It is straightforward to check that in any of the possible cases (depending upon whether or not $rank(a)$ and $rank(b)$ are limit) we have that there exists a $c \in A$ such that $c \in b$ and $rank(c) \geq rank(a)$.

Therefore $c \notin a$ and hence, by inductive hypothesis, $c^\star \notin a^\star$, which allows us to conclude that $b^\star \neq a^\star$.

β): by inductive hypothesis.

It is immediate to verify that $a_1^\star, \dots, a_n^\star$ (as well as $a^\star$ for any $a \in A$) are in $\mathcal{HF}$. Let us verify that $a_1^\star, \dots, a_n^\star$ satisfy φ:

by way of contradiction, let us suppose that $a_1^\star, \dots, a_n^\star$ satisfy $\neg\varphi$; in this case one of the disjuncts in $\neg\varphi$ must be satisfied by $a_1^\star, \dots, a_n^\star$.

From α) and β) it follows that the disjunct satisfied in $\neg\varphi$ cannot be unquantified, otherwise also $a_1, \dots, a_n$ would satisfy it, contradicting the hypothesis that $a_1, \dots, a_n$ satisfy φ.

Therefore the disjunct satisfied in $\neg\varphi$ must be of the form

(*) $$(\exists x_1^i \in y_1^i) \dots (\exists x_{j_i}^i \in y_{j_i}^i) \neg\varphi_i^M,$$

with $\neg\varphi_i^M$ conjunction of literals of the form $a = b, a \neq b, a \in b, a \notin b$ and a and b variables.

In this case let us notice that since the level of nesting of φ is 2, and since no y_ℓ^i is an element of an a_i (remember that this was a consequence of property i) above), a tuple of elements satisfying (*) will be entirely formed of sets of type $b^\star$ for some $b \in A$ (i.e. no s_ℓ^i will be involved).

Therefore it is immediate to verify using properties α) and β), that the inverse images of the elements in this tuple would still satisfy (*), contradicting the fact that $a_1, \dots, a_n$ satisfy φ.

■

Note that the above proof makes use of the axiom of choice.However its use can be eliminated by building, rather than a sequence $A_0, \dots, A_p$, a tree which takes into account all the possible choices, since there is only a set of them at any given stage.All the paths in the resulting tree will be finite and each one will produce hereditarily finite sets satisfying the original formula.

Even though the simplifying procedure presented in the previous proof is guarantee to stop in a finite number of steps, we have no way to say in how many steps this will happen.

The possibility to give a bound to the number of steps necessary to the simplifying procedure to stop, would permit us to put a bound to the size of the set A and therefore to the size of the transitive closures of $a_1^*, \ldots, a_n^*$. That in turn would give us a decidability result.

In the following we present various generalizations of the previous result, even involving classes for which the finite s-satisfiability problem is known to be undecidable, and hence no bound on the number of steps used by the simplifying procedure can be placed.

Let us consider the following functional operators:

- Pow: that to a set a associates the set of its subsets denoted by $Pow(a)$.
- $\bigcup$: that to a set a associates the set of the elements of elements of a, denoted by $\bigcup(a)$.
- $\times$: that to a pair of sets a, b, associates its cartesian product $a \times b$, i. e. the set of ordered pairs with first component in a and second component in b.

Consider the binary relational operator $\mathcal{R}$ defined by

$$\mathcal{R}(a,b) \quad \leftrightarrow \quad rank(a) \leq rank(b).$$

Let us denote by $(\forall)_{0,2} + Pow + \bigcup + \times + \mathcal{R}$ the class of formulae which are in the form

$$\bigwedge_i \varphi_i,$$

where φ_i is either a $(\forall)_0$-formula or a literal of the form

$$\mathcal{R}(x,y)\ ,\ Pow(x) = y\ ,\ \bigcup x = y\ ,\ x \times y = z\ ,$$

for x, y, z variables.

PROPOSITION 2.7 *If* $\varphi \in (\forall)_{0,2} + Pow + \bigcup + \times + \mathcal{R}$ *is satisfied by finite sets then it is satisfied by hereditarily finite sets.*

Proof. It is easy to see that the property necessary to generalize proposition 3.5 to the class $(\forall)_{0,2} + Pow$ is the following:

$$(Pow(a_i))^* \quad = \quad Pow(a_i^*) \qquad (1)$$

for $1 \leq i \leq n$.

Let us suppose that $a_j = Pow(a_i)$.

First notice that

$$a_j^* = (Pow(a_i))^* \subseteq Pow(a_i^*),$$

in fact if $z^* \in a_j^*$ then $z \in a_j$ from which $z \subseteq a_i$ and therefore $z^* \subseteq a_i^*$.

Vice versa consider $y^* \subseteq a_i^*$ and notice that

$$y \ = \ \{x \mid x^* \in y^*\} \ \subseteq \ a_i.$$

Therefore $y \in a_j$ and since $a_j \subseteq A$ because a_j is finite, we have $y^* \in a_j^*$.

That is

$$Pow(a_i^*) \subseteq a_j^*.$$

A completely analogous proof will apply to the case of the functional operators $\bigcup$ and $\times$.

To deal with the case of the relational operator $\mathcal{R}$ it is enough to be more careful in choosing the elements of the form s_ℓ^i.

Instead of choosing all the s_ℓ^i at the same rank, we will associate s_ℓ^i having the same rank only to y_ℓ^i having the same i.

Clearly this will force us to chose several (but a finite number of) levels playing the role of R in the proof of proposition 3.5, and we shall need a gap of at least n between two levels to guarantee the injectivity of the realization. ∎

Let us observe that the previous result extends the one presented in [CFO88] and proved with a different technique, and that the finite s-satisfiability problem for a subclass of the class involved in the previous proposition has been proved undecidable in [CCP89].

Another strengthening is still possible.

Let D be the class of Δ_0-formulae having level of nesting 2 on the universal quantifiers and level of nesting 1 on the existential quantifiers.

PROPOSITION 2.8 *If* $\varphi \in D + Pow + \bigcup + \times + \mathcal{R}$ *is satisfied by finite sets then it is satisfied by hereditarily finite sets.*

Proof. It is enough to check that the simplifying procedure produces sets which are sufficiently *rich*, and this is the case since the existential quantifiers have level of nesting 1 and since all the elements of the a_i's are in A (because they are finite).

The previous observation can be easily formalized in a proof by contradiction generalizing the one given at the end of proposition 3.5. ∎

3 The role of urelements

Let ZFU be ZF formalized in the language including an additional unary predicate symbol $U(x)$ intended to mean that x is an urelement (or atom), with the appropriate formulation of the axiom of extensionality and foundation and an additional axiom stating that there are infinitely many urelements. A system equivalent to ZF can then be obtained from ZFU by replacing the last axiom with the assumption that there are no urelements at all. ZFU^- is ZFU without foundation.

The class $(\forall)_0^{\bar{U}}$ is obtained from $(\forall)_0$ by relativizing free variables and quantifiers to the complement $\bar{U}$ of U.

The decidability of the set-satisfiability problem for $(\forall)_0^{\bar{U}}$ with respect to well founded universes with infinitely many urelements is ensured by the following proposition.

PROPOSITION 3.1 *ZFU is complete with respect to the existential closure of formulae in $(\forall)_0^{\bar{U}}$.*

Proof. Suppose that given $\varphi(x_1,\dots,x_n) \in (\forall)_0^{\bar{U}}$ there exists a model $\mathcal{M}$ having domain $D_\mathcal{M}$ and interpreting the membership relation $\in$ by $\in_\mathcal{M}$, such that there exist $a_1^\mathcal{M},\dots,a_n^\mathcal{M} \in D_\mathcal{M}$ for which

$$\mathcal{M} \models \varphi(a_1^\mathcal{M},\dots,a_n^\mathcal{M}).$$

For any pair of different elements $a_i^\mathcal{M}, a_j^\mathcal{M}$ $1 \leq i,j \leq n$ by extensionality we can chose another element $a_{i,j}^\mathcal{M}$ such that $(a_{i,j}^\mathcal{M} \in_\mathcal{M} a_i^\mathcal{M} \wedge a_{i,j}^\mathcal{M} \notin_\mathcal{M} a_j^\mathcal{M})$ or $(a_{i,j}^\mathcal{M} \notin_\mathcal{M} a_i^\mathcal{M} \wedge a_{i,j}^\mathcal{M} \in_\mathcal{M} a_j^\mathcal{M})$.

Consider now the following graph $G_1 = (N_1, E_1)$ where:

$$N_1 = \{a_1^\mathcal{M},\dots,a_n^\mathcal{M}\} \cup \{a_{i,j}^\mathcal{M} \mid 1 \leq i,j \leq n\};$$

$$E_1 = \{< a,b > \mid a,b \in N_1 \wedge a \in_\mathcal{M} b\}.$$

Let A be a set of nodes having the same cardinality of the set $\{a_{i,j}^\mathcal{M} \mid 1 \leq i,j \leq n\}$ and such that $A \cap N_1 = \emptyset$.

Then let $G = (N, E)$ be such that:

$$N = N_1 \cup A;$$

$$E = E_1 \cup \{< a, f(a) > \mid a \in A\},$$

where f is a fixed bijection between A and $\{a_{i,j}^\mathcal{M} \mid 1 \leq i,j \leq n\}$.

Because of the definition of the graph G we have that for any pair of nodes $a,b \in N_1$ such that $a \neq b$, there exists a node $x \in N$ such that

$$(< x,a > \in E \wedge < x,b > \notin E) \vee (< x,a > \notin E \wedge < x,b > \in E).$$

In other words the graph G is *extensional* on the elements of N_1.

Now given any model $\mathcal{U}_1$ of ZFU^- since $\mathcal{U}_1$ has sufficiently many atoms, it is very easy to see that in $\mathcal{U}_1$ there is a set $N_{\mathcal{U}_1}$ satisfying the following properties:

i) there is a function $\star : N \longrightarrow N_{\mathcal{U}_1}$ which is a bijection and,

ii) for any pair a,b of elements in N we have

a) $a = b \leftrightarrow a^\star = b^\star$ [2]

b) $a \in_\mathcal{M} b \leftrightarrow a^\star \in_{\mathcal{U}_1} b^\star$.

The set $N_{\mathcal{U}_1}$ can be found, for example, in the following way: suppose $\star$ associates atoms to the elements of A in a one-to-one fashion. Then define inductively

$$x^\star = \{y^\star \mid < y,x > \in E\}$$

for any $x \in N \setminus A$.

Finally take $N_{\mathcal{U}_1} = N^\star = \{x^\star \mid x \in N\}$.

$(a_1^\mathcal{M})^\star,\dots,(a_n^\mathcal{M})^\star$ are n sets in $\mathcal{U}_1$ satisfying φ. In fact, let us suppose, by way of contradiction, that $(a_1^\mathcal{M})^\star,\dots,(a_n^\mathcal{M})^\star$ satisfy $\neg\varphi$. If this is the case they must satisfy the negation of one of the conjuncts in φ, say φ_i. Hence $(a_1^\mathcal{M})^\star,\dots,(a_n^\mathcal{M})^\star$ satisfy the formula $\neg\varphi_i$ which is of the form

$$(\exists x_1^i \in y_1^i)\dots(\exists x_{j_i}^i \in y_{j_i}^i)(\neg\ell_1^i \wedge \dots \wedge \neg\ell_{h_i}^i).$$

Therefore there exist elements $d_1^\star,\dots,d_r^\star$ in the transitive closure of $(a_1^\mathcal{M})^\star,\dots,(a_n^\mathcal{M})^\star$ that satisfy the matrix of $\neg\varphi_i$ (i.e. a conjunction of literals) when substituted for the existentially quantified variables.

At this point it is straightforward to check using a) and b) above, that the elements $d_1,\dots,d_r$ together with $(a_1^\mathcal{M})^\star,\dots,(a_n^\mathcal{M})^\star$ would satisfy the matrix of $\neg\varphi_i$, contradicting the hypothesis that $(a_1^\mathcal{M})^\star,\dots,(a_n^\mathcal{M})^\star$ satisfy φ. ∎

Notice that the previous (inductive) definition makes sense just because E is well founded.

From the proof it follows that a method to check if a formula in $(\forall)_0^{\bar{U}}$ is set-satisfiable in any (equivalently every) well founded universe with infinitely many atoms, is provided by an exhaustive search for a suitable graph, among a finite number of possibilities, quadratically bounded in the number of free variables in φ.

[2] We will denote by $a^\star$ the image of a with respect to $\star$.

When the axiom of foundation is dropped we still have a positive solution to Problem 1.

PROPOSITION 3.2 *For any $\varphi \in (\forall)_0^G$ it is decidable whether or not there exists an $\mathcal{M}$ such that $\mathcal{M} \models ZFU^-$ and $\mathcal{M} \models \varphi$*[3].

Proof. We proceed as in the proof of the previous proposition. However now the graph G may contain cycles and we cannot define $\star$ by induction as before. This case is treated by using the so-called *permutations of the universe* (see [Kri69].

Consider the case in which only one cycle is present. If we delete one edge in the cycle we are back in the previous case and we can define $N_{\mathcal{U}_1}$ as before.

Let us suppose that $< x, y >$ is the edge that has been deleted. Consider the model $\mathcal{U}_2$ defined starting from $\mathcal{U}_1$ and using the permutation of the universe that *swaps* $x^\star$ with $x^\star \cup \{y^\star\}$.

In other words the model $\mathcal{U}_2$ is obtained by taking $D_{\mathcal{U}_2}$ as universe, defining the bijection p from $D_{\mathcal{U}_1}$ to $D_{\mathcal{U}_1}$ that is the identity on all the elements different from $x^\star$ and $x^\star \cup \{y^\star\}$, whereas

$$\mathrm{p}(x^\star) = x^\star \cup \{y^\star\},$$

and

$$\mathrm{p}(x^\star \cup \{y^\star\}) = x^\star.$$

and then letting the membership relation $\in_{\mathcal{U}_2}$ be defined by the following equation:

$$a \in_{\mathcal{U}_2} b \;\; \leftrightarrow \;\; a \in_{\mathcal{U}_1} \mathrm{p}(b).$$

exit exit The two models $\mathcal{U}_1$ and $\mathcal{U}_2$ have the same domain and therefore $N^\star \subseteq D_{\mathcal{U}_2}$; moreover the n sets in $N^\star$ will satisfy properties a) and b) above with respect to $\in_{\mathcal{U}_2}$, therefore

$$\mathcal{U}_2 \models \varphi((a_1^{\mathcal{M}})^\star, \ldots, (a_n^{\mathcal{M}})^\star).$$

The general case with, say, m cycles is treated by defining a sequence of m models $\mathcal{U}_1, \ldots, \mathcal{U}_m$ as before. ■

The previous property immediately yields that the derivability problem, for the universal closures of negations of formulae in $\varphi \in (\forall)_0^G$, with respect to ZFU^-, is decidable. Furthermore the proof suggests an alternative way to strength our set theoretic assumption, so that the set-satisfiability problem for $\varphi \in (\forall)_0^G$ becomes decidable as well. In fact if we assume that for any graph G it is always possible to find elements set-realizing G, mantaining properties a) and b) in the proof of Prop.3.1, the set-satisfiability problem for $\varphi \in (\forall)_0^G$ becomes decidable and the theory becomes complete with respect to to the existential closures of formulae in this class. Furthermore the proof shows that a reflection property over the hereditarily finite sets holds in such a rich universe for formulae in $\varphi \in (\forall)_0^G$, provided the notion of hereditarily finite set in conveniently formulated. That can be done by saying that a set x is hereditarily finite if it is one of $x_1, \ldots, x_n$ satisfying a conjunction of the form:

$$x_1 = \{x_{11}, \ldots, x_{1m_1}\} \;\wedge\; \ldots \;\wedge\; x_n = \{x_{1n}, \ldots, x_{nm_n}\}$$

where $x_1, \ldots, x_n$ are distinct variables and each x_{ij} is drawn from $x_1, \ldots, x_n$. Thus, for example a set x such that $x = \{x\}$ is counted as hereditarily finite.

That brings to evidence the crucial role that is played by the assumption that variables range only over sets in the above result. In fact if we discard that assumption we can produce a formula in $(\forall)_0$ which is satisfiable in the above universe, but it is not finitely satisfiable there.

PROPOSITION 3.3 *If the following formula $\varphi_1(y, w)$:*

$$(\forall y_1 \in y)(\forall y_2 \in y_1)(y_2 \in y) \wedge y \neq \emptyset \wedge \emptyset \notin y \wedge \psi(y) \wedge (\forall w_1 \in w)(w_1 = w) \wedge$$

$$(\forall y_1 \in y)(w \in y_1) \wedge (\forall y_1 \in y)(y_1 \in y_1 \leftrightarrow y_1 = w) \wedge y \notin y.$$

where $\psi(y)$ stands for:

$$(\forall y_1, y_2, y_3 \in y)(y_1 \in y_2 \wedge y_2 \in y_3 \rightarrow y_1 \in y_3).$$

is satisfied with y interpreted as S, then S must contain an infinite descending chain.

Moreover the sets Y and W such that:

1. $Y = \{Y_1, Y_2, Y_3, \ldots\} \cup \{W\}$;
2. $Y_i = \{Y_j \mid \omega > j > i\} \cup \{W\} \qquad \omega > i > 0$;
3. $Y_i \neq Y_j \qquad \omega > i > j > 0$;
4. $W = \{W\}$,

satisfy φ_1.

Proof. Let us rewrite the formula by using some self-explanatory abbreviation:

$$Trans(y) \wedge y \neq \emptyset \wedge \emptyset \notin y \wedge \in\text{-is-transitive-on-y} \wedge w = \{w\} \wedge$$

[3] by convention, $\mathcal{M} \models \varphi$ stands for $\mathcal{M}$ *is a model of the existential closure of* φ.

$(\forall y_1 \in y)(w \in y_1) \wedge$ w-is-the-only-element-in-y-such-that-w$\in$w $\wedge\ y \notin y$.

It is straightforward to check that Y and W satisfy $\varphi_1(y,w)$.

Vice versa let us consider Y', W' satisfying $\varphi_1(y,w)$.

First of all notice that Y' must be different from W' because W' satisfies $w = \{w\}$ whereas Y' satisfies $y \notin y$.

We will prove by induction that there exist sets

$$Y_1', Y_2', Y_3', \ldots$$

such that, if $\omega > j > i > 0$:

1. $Y_i' \in Y'$;
2. $Y_j' \in Y_i'$, $Y_j' \not\ni Y_i'$;
3. $Y_j' \neq Y_i'$;
4. $W' \neq Y_i'$, $W' \in Y_i'$.

Since $Y' \neq W' = \{W'\}$, there must be an element $Z \in Y'$ such that $Z \neq W'$.

Let $Y_1' = Z$. Notice that $W' \in Y_1'$ since the conjunct $(\forall y_1 \in y)(w \in y_1)$ in φ_1 is satisfied by Y'.

Let us suppose that we already established the existence of

$$Y_1', \ldots, Y_p'$$

having properties 1 2 3 4 above.

Since $Y_p' \neq W' = \{W'\}$, there must be an element $Z \in Y_p'$ such that $Z \neq W'$. Since Y' is transitive, $Z \in Y'$, moreover $Z \neq Y_k'$ for $p \geq k > 0$, since otherwise we would have a cycle of the form:

$$Z \in Y_p' \in \ldots \in Y_k' \in Z$$

from which we could conclude that (for example) $Y_p' \in Y_p'$ using the transitivity of $\in$ over Y', and this would contradict the fact that W' is the only element in Y' that belongs to itself.

At this point we can put $Y_{p+1}' = Z$ and proceed in this way to determine the infinite descending chain in Y'. ∎

Let us conclude this section by summarizing the situation about possible formulation of the axiom of infinity in the class $(\forall)_0$.

The formula $\varphi_{\omega',\omega''}(a,b)$, see Sec.2, provides a $(\forall)_0$ formulation of the infinity axiom with respect to a set theory which assumes the existence of no atoms. Another formulation of the infinity axiom by a $(\forall)_0$-formula is given in [PP90c], and the conjunction of these two formulations works also in absence of the axiom of foundation [PP90b].

In the case in which the axiom of foundation in not assumed the formula φ_1 can be seen to express the existence of infinite sets even if we assume the existence of atoms, provided that quantifiers range on atoms as well as on sets, however the satisfiability of φ_1 violates the axiom of foundation.

In case the quantifiers are assumed to range only over sets, by Prop. 3.2, the infinity axiom is not expressible in $(\forall)_0$ (see the proof of proposition 2.1).

4 A semidecidability result assuming Boffa's axiom

We return now to set theory without urelements to show that if a suitable antifoundation axiom is assumed then, although the set-satisfiability problem for a very restricted subclass of the Δ_0 formulae is not semidecidable ([PP90d]), its complement is semidecidable even with respect to the full class of the Δ_0 formulae.

The form of antifoundation axiom we consider is the following:

Every extensional graph is isomorphic to a transitive set.

The axiom is a weaker form of an axiom introduced by Boffa; it is denoted by BA_1 and it is discussed in [Acz88].

PROPOSITION 4.1 *If the universe of sets satisfies BA_1, then the complement of the set-satisfiability problem for the class of Δ_0-formulae is semidecidable.*

Proof. Let $\varphi(x_1,\ldots,x_n) \in \Delta_0$ and let us assume, without loss of generality, that there are no existential quantifiers that are not in the scope of a universal quantifier (this amounts only to possibly increasing the number of free variables in φ).

Let us suppose that there are sets $a_1,\ldots,a_n$ satisfying φ and let us start eliminating all the existential quantifiers by introducing appropriate Skolem functions $f_1,\ldots f_s$.

Notice that since the formula is Δ_0, it is not restrictive to consider $f_1,\ldots,f_s$ as having domain and range included in $trcl(\{a_1,\ldots,a_n\})$.

First of all let us give an idea of what we are seeking:

start with a graph G_0 having n nodes associated to $a_1,\ldots,a_n$ and (as usual) having an edge going from node a to node b if and only if the set associated to node a belongs to the set associated to b.

G_0 might be not extensional, moreover there might be sets in G_0 whose images with respect to some of the Skolem functions $f_1, \dots, f_s$ lies outside G_0. Therefore we build G_1 by adding enough nodes to guarantee extensionality among nodes of G_0 and by adding the images with respect to $f_1, \dots, f_n$ of nodes in G_0.

Now G_1 might be non extensional and some of the images with respect to $f_1, \dots, f_s$ might be missing, for example on nodes in $G_1 \setminus G_0$.

This forces us to continue this way and to build a, possibly infinite, sequence of graphs

$$G_0, G_1, \dots, G_m, \dots$$

Let $G = \bigcup_{i \in \omega} G_i$.

A compactness argument shows that G is extensional and that the images of any node a with respect to $f_1, \dots, f_s$ is in G.

Now the image of $a_1, \dots, a_n$ in a transitive set isomorphic to G, which exists because of BA_1, is an n-tuple of sets satisfying φ.

Now we will give an argument to show the semidecidability of the existence of G, from which we can conclude the semidecidability of the class of Δ_0-formulae.

Notice that this very same argument for the construction of G would apply also in the well-founded case. In that case however, we should look for a graph not containing infinite descending chains and we would not need any axiom to conclude the set-realizability of such a graph since Mostowski's theorem would guarantee it.

Consider a tree T, that we will call *try-tree*, in which all the possibilities for the construction of the sequence

$$G_0, G_1, \dots, G_m, \dots$$

are taken into account.

T can be built as follows: we will always consider graphs G_i with n distinguished nodes $b_1, \dots, b_n$. We will also assume that s functions $F_1^i, \dots, F_s^i$ having a set of tuples in G_i as domain and range included in G_i are associated to G_i.

Then we will say that G_i satisfies $\varphi(x_1, \dots, x_n)$ with respect to N if and only if:

1. N is a set of nodes of G_i, $b_1, \dots, b_n \in N$, different nodes in N have different sets of predecessors in G_i (i.e. G_i is extensional on N);

2. considering any set realization r_i of G_i such that:

(a) r_i is one-to-one;

(b) $r_i(x) = \{r_i(y) \mid x \text{ and } y \text{ are connected in } G_i\}$;

the sets $r_i(b_1), \dots, r_i(b_n)$ together with the functions induced by the $F_1^i, \dots, F_s^i$ on the realizations of nodes in G_i, satisfy the formula

$$\varphi^{<r_i(N)} \quad = \quad (\forall y_1, \dots, y_t \in r_i[N])(\varphi^M)$$

where $(\forall y_1, \dots, y_t)(\varphi^M)$ is the Skolem prenex normal form of φ.

Let k be the maximum among the arities of the Skolem functions $f_1, \dots, f_s$.

We will start defining the root of T as the empty graph. Then all the children of the root will be the graphs G_i with at most $n + s \cdot n^k + \binom{n}{2}$ nodes satisfying φ with respect to $\{b_1, \dots, b_n\}$. Notice that the bound on the number of nodes guarantees that all the possible variants of such graphs are taken into account.

In general the children of a graph $G_i = (N_i, E_i)$ will be all possible graphs G_j with less than $|N_i| + s \cdot |N|^k + \binom{|N_i|}{2}$ nodes, satisfying φ with respect to N_i and such that the functions associated to G_j are extensions of the functions associated to G_i.

At this point it is straightforward to see that T is finite if and only if φ is unsatisfiable. ∎

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