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AILA PREPRINT
n. 6 gennaio 1991

Direttore Responsabile: Ruggero Ferro
Iscrizione al Registro Stampa del Tribunale di Padova n. 1235 del 26.9.1990

Pubblicato con il contributo di:
Cassa di Risparmio di Padova e Rovigo
Stampa: Rotografica Padova

NON MONOTONIC CONSEQUENCE BASED ON INTUITIONISTIC LOGIC

GISELE FISCHER SERVI

1. - ABSTRACT AND INTRODUCTION.

Research in AI has recently begun to address the problems of *non deductive reasoning*, i. e. the problems that arise when, on the basis of approximate or incomplete evidence, we form well reasoned but possibly false judgments. Attempts to simulate such reasoning fall in two main categories: the numerical approach based on probabilities and the non numerical one which tries to reconstruct non deductive reasoning as a special type of deductive process. In this paper, we are concerned with the latter usually known as *non monotonic deduction*, because the set of theorems does not increase monotonically with the set of axioms.

It is generally acknowledged that non monotonic (n. m.) formalisms (e. g. [C], [MC1], [MC2] [MD], [MD-D], [R1], [R2], [S]) are plagued by a number of difficulties. A key issue concerns the fact that most systems do not produce an axiomatizable set of validities. Thus, the chief objective of this paper is to develop an alternative approach in which the set of n. m. inferences, that somehow qualify as being deductively sound, is r. e.

The basic idea here is to reproduce the situation in first Order Logic where the metalogical concept of deduction translates into the logical notion of material implication. Since n. m. deductions are no longer truth preserving, our way to deal with a change in the metaconcept is to extend the standard logical apparatus so that it can reflect the new metaconcept. In other words, the intent is to study a concept of *non monotonic implication* that goes hand in hand with a notion of n. m. deduction. And in our case, it is convenient that the former be characterized within the more tractable context of monotonic logic.

To be precise, let us introduce the logical constant «A non monotonically implies B» ($A>B$) and define n. m. inference ($A \mid\!\sim B$) as

(#) $$A \mid\!\sim B \quad \text{iff} \quad \vdash A>B.$$

The problem now amounts to giving an adequate characterization of the right hand side in (#). The first question that must be answered concerns the interpretation of the standard logical constants. Generally speaking, n. m. logic refers to the following situation: it is required that we go beyond the available data A by making a reasonable but defeasible assumption B. Our claim is that this is a situation in which Intuitionistic rather than Classical Logic is appropriate. For, a basic idea in Intuitionistic Logic is that the reasoning subject may not have all the information needed to decide in each moment whether a statement is true. Similarly, we are interested here in a logic that allows for the statement «$A>B$» to be true when B is uncertain, i. e. when B has no definite truth value. And this cannot be coped with, in a classical setting.

Our next concern is how to formalize the notion of non monotonic implication (>). This concept is not a logical item of our vocabulary, so the obvious question is just what is it supposed to mean that «A non monotonically implies B». We propose to fix the intuitive meaning of «$A>B$» with any one of the following: «if A, then usually B», «if A, then typically (mostly) B», «if A, then it is reasonable to presume B». As far as its formal meaning is concerned, it could be suggested that the conditional of Counterfactual Logics ([Lw], [St]) presents enough analogies with n. m. inference to be useful in this context. But it has been noted ([N], [D]), that these logics are problematic from the point of view of n. m. reasoning since they allow from A and $A>B$ the *monotonic* inference B. This proves to be fatal because the very idea of n. m. implication is that A and $A>B$ could be true, even when B turns out to be false.

More promising ideas can be found in [G2], where the author presents two modal extensions of Intuitionistic first Order Logic. One of them, named γ, codifies the behaviour of the binary sentential operator «on the basis of... expect that...». The explicit motivation, in that paper, is to solve some problems connected with Mc Dermott's analysis of reasoning on propositions that can *coherently* be assumed (see [MD-D]). So the presentation of γ does not seem to be related to the rationale for (#). Nonetheless, we think that γ provides an attractive starting point for an investigation into the concept of n. m. implication.

In [G2], the discussion of the modal axioms and rules of γ is purely informal. The present paper proceeds in the analysis of this system by showing that it also has a suitable formal semantics. Using the latter, we can see that γ is too weak to capture some logical properties of n. m. inference that have been discussed in the literature. As a contribution to this debate, we introduce a proper extension γ_1 of γ, that does satisfy these properties. In §4, we establish completeness for both γ and γ_1, and see that the set of n. m. *logical* consequences based on γ (γ_1 resp.) is r. e. Moreover in §5, we show that the propositional subset of the n. m. consequences based on γ is decidable. In §6, a discussion of n. m. deduction *in a theory* reveals that both γ and γ_1 are insufficient, so we extend them to Γ and Γ_1 respectively and provide each of these new calculi with successful semantics. Last, we briefly discuss how the present approach relates to other known proposals.

2. - NON MONOTONIC INFERENCE BASED ON γ.

In order to make this paper more readable, we recall the axioms and rules of the monotonic calculus γ:

(0) Intuitionistic Predicate Calculus (see for example [F])

(1) $A>B \wedge A>C \longleftrightarrow A>(B\wedge C)$;

(2) $\dfrac{A \longrightarrow B}{A > B}$; (3) $\dfrac{A \longleftrightarrow A'}{A>B \longleftrightarrow A'>B}$; (4) $\dfrac{A>\bot}{\neg A}$;

(5) $A>B \wedge (A\wedge B)>C \longrightarrow A>C$;

(6) $A>B \vee \neg(A>B)$.

where A, B, C belong to the set $\mathcal{F}$ of wff.'s in the language $\mathcal{L}$ of γ.

For a brief discussion of the above, see the original paper [G2].

Notation. - α) $\vdash A$ stands for «A is a theorem of γ»;
β) $\Sigma \vdash B$ stands for «there are $A_1,\dots,A_n \in \Sigma$ such that $A_1\wedge\dots\wedge A_n \longrightarrow B$ is a theorem of γ».

We introduce

Definition 2.1. - When $\vdash A>B$, we will say that B *is non monotonically inferred from* A (write: $A \mid\!\sim B$)

and gain some insight into the logical properties of the n. m. operator $\mid\!\sim$:

Theorem 2.2. - *For any* $A,B,C \in \mathcal{F}$

(7) $A \mid\!\sim A$.

(8) $A \mid\!\sim B$ *and* $A \mid\!\sim C$ *iff* $A \mid\!\sim B\wedge C$.

(9) *if* $A \mid\!\sim B$ *and* $A\wedge B \mid\!\sim C$ *then* $A \mid\!\sim C$.

(10) *if* $A \vdash B$ *then* $A \mid\!\sim B$.

(11) $A \mid\!\sim \bot$ *iff* $\vdash \neg A$.

(12) *if* $\not\vdash \neg A$ *and* $A \mid\!\sim B$ *then* $A \mid\!\not\sim \neg B$.

(13) *if* $\not\vdash \neg A$ *and* $A \mid\!\sim B$ *then* $A \not\vdash \neg B$.

(14) *From* $\vdash A \longrightarrow B$ *and* $C \mid\!\sim A$ *infer* $C \mid\!\sim B$.

(15) *if* $A \mid\!\sim B$ *or* $A \mid\!\sim C$ *then* $A \mid\!\sim (B\vee C)$.

Proof. Because of Def. 2.1, (7), (8), (9) and (10) are obvious. In order to prove (11), take first $A \mid\!\sim \bot$; then by Def. 2.1 and (4) result obtains. Conversely use (2) and Def. 2.1. Now suppose to the contrary in (12); then by (8) and (11) we have $\vdash \neg A$, a contradiction. Moreover, (13) follows from (12) according to (10). As for (14), infer from hypotheses $\vdash C>A$ and $\vdash (A\wedge C)>B$ (use (2) and Def. 2.1) so by (5) and Def. 2.1, $C \mid\!\sim B$. To prove (15), use (14).

Properties (7), (8), (10) are expected; (9) holds for many concepts of non monotonic inference discussed in the literature. The next three properties are reasonable: (11) says that A is n. m. inconsistent iff we are demonstrably sure of $\neg A$; according to (12), incompatible statements cannot be n. m. inferred from the same consistent formula so that any non monotonic consequence of a coherent formula is consistent with it (see (13)). Moreover, by (14) n. m. inference respects standard deduction. Note finally that the converse of (15) should not hold.

At this point, we need to discover whether γ has enough features related to non monotonicity. This will require some semantical arguments, so we postpone a discussion of this topic to the end of §3, after we develop a suitable formal interpretation for γ.

3. - γ-MODELS AND VALIDITY.

The following semantical formal system will be shown to be adequate for γ.

Definition 3.1. - Consider a structure $\mathcal{S}=(S,R_0,R,\sigma,[\ ])$ such that $S\neq\emptyset$, $R_0 \subseteq S\times S$, $R \subseteq S\times\wp(S)\times S$; σ is a function which assigns to each $s\in S$, a set $\sigma(s)$ of individual constants and $[\ ]$ is a function that assigns to each wff. A a set $[A]$ of elements of S. Then $\mathcal{S}$ is a *γ-model* if it satisfies the following conditions. First, some familiar intuitionistic properties:

(16) R_0 is reflexive and transitive and if $s,t \in S$ and $(s,t) \in R_0$ then $\sigma(s) \subseteq \sigma(t)$.

Second, some restraints on the ternary relation R: for $A,B \in \mathcal{F}$,

(17) if $(s,[A],t) \in R$ then $t \in [A]$ $(s,t \in S)$

(18) if $[A] \neq \varnothing$ then $(t,[A],u) \in R$, for some $t,u \in S$.

(19) If for every $t \in S$ $((s,[A],t) \in R$ implies $t \in [B])$, then for every $t \in S$ $((s,[A],t) \in R$ implies $(s,[A] \cap [B],t) \in R)$. $(s \in S)$

(20) $(s,t) \in R_0$ implies that, for every $u \in S$, $(s,[A],u) \in R$ is equivalent to $(t,[A],u) \in R$ $(s,t \in S)$.

Third, [ ] satisfies the usual intuitionistic constraints, i. e.

(21) if $\Phi(s)$ is the set of all formulas in $\mathcal{F}$ which may be constructed using only constants in $\sigma(s)$, then for A atomic, $s \in [A]$ implies $A \in \Phi(s)$.

(22) if A is atomic then $[A]$ is R_0-closed.

(23) $[A \wedge B] = [A] \cap [B]$;

$[\neg A] = \{s \in S\text{: } A \in \Phi(s) \ \&\ (s,t) \in R_0 \text{ implies } t \notin [A]\}$;

and so on.

Last, a condition for evaluating formulas containing >,

(24) $[A>B] = \{s \in S\text{: } A>B \in \Phi(s) \text{ and if } (s,[A],t) \in R, \text{ then } t \in [B]\}$.

A formula A is said to be *true in* a γ-model $\mathcal{S}$ if $\{s \in S\text{: } A \in \Phi(s)\} \subseteq [A]$. Also A is said to be γ-*valid* if A is true in all γ-models.

Notation. - As it is customary, we shall often write $s \Vdash A$ for

$s \in [A]$. Moreover, we replace $(s,t) \in R_0$ by sR_0t.

Given this definition we have,

Theorem 3.2. - *Let $\mathcal{S}$ be a γ-model. Let $A \in \mathcal{F}$ and $s,t \in S$. Then,*

(α) *$s \Vdash A$ implies $A \in \Phi(s)$;*

(β) *if $s \Vdash A$ and sR_0t, then $t \Vdash A$.*

Proof. (α) Immediate by induction on the complexity of A.
(β) We only show the case $A=B>C$. So let $s \Vdash B>C$ and sR_0t. Then for all $u \in S$,

(i) $(s,[B],u) \in R$ implies $u \Vdash C$.

Now take u such that $(t,[B],u) \in R$; by (20), $(s,[B],u) \in R$ and from (i) get $u \Vdash C$; so $t \Vdash A$.

Let us see that Def. 3.1 converts into an informal account of reasoning under presumptions. The main idea in this intuitive reading is that presumptions are based on what is «usual» in a situation or «typical» for a member of a class. These typical aspects are not understood to exhaust «full meaning», but rather are taken to fix a «stereotyped meaning» for the concept or the statement involved. So, given a γ-model $\mathcal{S}$, interpret S to be a collection of states of knowledge and sR_0t as indicating that the state t contains (possibly) more knowledge that the state s. Furthermore, consider the set $[A]$ to represent the *full meaning* assigned to A in the model and consider the set $\{t\text{: } (s,[A],t) \in R\}$ to fix the *stereotyped meaning of A in s*. Given this interpretation, conditions (17) to (20) are sensible: for instance, (17) states that stereotyped meaning is a part of full meaning and (18) says that if A has a (non empty) full meaning, then it also has, at some stage, a (non empty) stereotyped one. According to (19), if the stereotyped meaning of A is included in the full meaning of B, then it is also included in the stereotyped meaning of $A \wedge B$ (see (23)). Condition (20) indicates that knowledge of more or less facts does not affect stereotyped meaning. Also, the truth condition (24) reveals that, in s, all presumptions relative to A are independent of the facts established in s, if s does not happen to belong to the stereotyped meaning of A. Consequently, «if A, then it is reasonable to presume B» may be true in s even if it turns out that «A and non B» is a *fact of* s. Briefly, presumptions are based on stereotyped meaning only, so they can be contradicted by facts.

We set to prove

Theorem 3.3. - *Every theorem of* γ *is* γ*-valid.*

Proof. By induction on the length of proofs in γ. Throughout this proof, let $\mathcal{S}$ be a γ-model and $s,t,u,v \in S$. For all axioms and rules in (0) the proof is standard. To show that (1) is valid note that, by (24), $s \Vdash A>B$ and $s \Vdash A>C$ iff

(i) $(A>B),(A>C) \in \Phi(s)$ and $t \Vdash B \wedge C$ for $(s,[A],t) \in R$.

But (i) holds iff $s \Vdash A>(B \wedge C)$.

Suppose now that $s \Vdash A \rightarrow B$, for all s. Let $(s,[A],t) \in R$; by (17), $t \Vdash A$ which by hypothesis yields $t \Vdash B$. From Thm. 3.2(α), $(A>B) \in \Phi(s)$, so $s \Vdash A>B$ and (2) is valid.

As for (3), it is sufficient to say that $s \Vdash A>B$ iff $s \Vdash A'>B$ when $[A]=[A']$.

To prove the validity of (4) assume that $s \Vdash A>\perp$ for every s, but $t \Vdash A$, for some t. Then $[A] \neq \varnothing$ and by (18) there are u,v such that $(u,[A],v) \in R$; impossible because of hypothesis and (24).

Now for (5): assume both $t \Vdash A>B$ and $t \Vdash (A \wedge B)>C$. The first assumption implies that for every u, if

(ii) $(t,[A],u) \in R$,

then $u \in [B]$. So using (19) we infer that for every u satisfying (ii),

(iii) $(t,[A] \cap [B],u) \in R$.

Clearly $[A] \cap [B]=[A \wedge B]$ (see (23)). Now let u satisfy (ii); then by (iii) and second assumption, $u \Vdash C$; since $A>C \in \Phi(t)$, we have $t \Vdash A>C$.

Last suppose that $s \nVdash A>B$; so for some t, $(s,[A],t) \in R$ and $t \nVdash B$. Assume now that $sR_0 v$. Because of (20), $(v,[A],t) \in R$, thus $v \nVdash A>B$ and hence $s \Vdash \neg(A>B)$.

The following describes an important class of γ-models:

Definition 3.4. - Let (17')-(20') be obtained from (17)-(20) by substituting every occurence of $[A]$ and $[B]$ with generic subsets $Q,Q' \subseteq S$. Then, the triple (S,R_0,R) is said to be a *full* γ*-frame* if $S \neq \varnothing$, R_0 is a preorder relation on S and R, R_0 satisfy conditions (17')-(20'). Moreover, a *full* γ*-model* is a structure $(S,R_0,R,\sigma,[\ ])$ where (S,R_0,R) is a full γ-frame and σ, $[\ ]$ are functions satisfying conditions (16), (21)-(24).

Clearly, a full γ-model, is a γ-model; such a γ-model is said to be *based on* a full γ-frame. *In a full* γ*-model the function* $[\ ]$ *is completely and freely determined by its values on atomic wff's.* In fact:

Theorem 3.5. - *Let* $S \neq \varnothing$, $R_0 \subseteq S \times S$, *and* $R \subseteq S \times \wp(S) \times S$. *Let* $[\ ]_0$ *be a function which assigns to each atomic formula* P *a set* $[P]_0 \subseteq S$. *Then* $[\ ]_0$ *can be uniquely estended to a function* $[\ ]$ *defined on* **all** *formulas satisfying* (21)-(24).

So, if (S,R_0,R) is a full γ-frame, the function $[\ ]_0$ is arbitrary, and $[\ ]$ is obtained according to theorem 3.5, then $(S,R_0,R,\sigma,[\ ])$ is a γ-model (in fact a full one).

Let us see whether γ is an adequate base for n. m. inference.

Theorem 3.6. - *The following wff.'s are* not *theorems of* γ:

(25) $A>B \rightarrow (A \wedge A')>B$.

(26) $A>B \wedge A>C \rightarrow (A \wedge B)>C$.

(27) $A>B \wedge B>C \rightarrow A>C$.

(28) $A>(B \vee C) \rightarrow A>B \vee A>C$.

Proof. Let $S=\{s,t,u\}$; name the non empty proper subsets of S:

$Q_1=\{s\}$; $Q_2=\{t\}$; $Q_3=\{u\}$; $Q_4=\{s,t\}$; $Q_5=\{s,u\}$;$Q_6=\{t,u\}$; $Q_7=S=\{s,t,u\}$.

Put $R_0=\emptyset$ and let R be determined by the following triples:

(s,Q_1,s); (s,Q_2,t); (s,Q_3,u); (s,Q_4,t); (s,Q_5,s); (s,Q_6,t); (s,Q_6,u); (s,Q_7,t).

It is easy to see that (S,R_0,R) is a full γ-frame. Let σ be the constant function on S with value $\emptyset$ and $[\ ]$ a function on $\mathcal{F}$ satisfying both (21)-(24) and

(i) $[P]=Q_7$; $[P']=Q_3$; $[P'']=Q_4$

for distinct 0-ary predicates P, P' and P''. Thm. 3.5 ensures that (i) determines a γ-model $\mathcal{S}$ based on that frame. Now, $[P]=Q_7$ and only $(s,Q_7,t)\in R$; according to (i), $t \Vdash P''$, therefore $s \Vdash P>P''$. On the other hand, $[P \wedge P']=Q_3$ and $(s,Q_3,u)\in R$; since $u \nVdash P''$ (see (i)), $s \nVdash (P \wedge P')>P''$. So (25) is invalid in $\mathcal{S}$.

To falsify (26), replace (i) with

(ii) $[P]=Q_7$; $[P']=Q_6$; $[P'']=Q_2$.

Since $[P]=Q_7$ and $[P \wedge P']=Q_6$, clearly $s \Vdash (P>P') \wedge (P>P'')$ but $s \nVdash (P \wedge P')>P''$.

To falsify (27) in $\mathcal{S}$, replace (i) with

(iii) $[P]=Q_3$; $[P']=Q_7$; $[P'']=Q_4$.

As for (28), replace (i) with

(iv) $[P]=Q_4$; $[P']=Q_5$.

Take $A = \neg(P \wedge P')$, $B=P$, $C=P'$. Note that $Q_6=[A]$.

It has been argued in ([G3]) that restricted versions of the Tarski-Scott conditions for monotonic provability relations, viz. reflexivity (7), restricted transitivity (9) and restricted monotonicity (26), are the minimal logical properties that befit a large class of n. m. inferences (see also [S]). From this point of view and given Def. 2.1, we can interpret Thm. 3.6 as also saying that γ is too weak to capture a genuine non monotonic inference relation. So consider the logic γ_1 obtained by adding (26) to γ, and put

Definition 3.7. - $A \mid\!\sim_1 B$ iff $A>B$ is a theorem of γ_1 ($\vdash_1 A>B$).

Corollary 3.8. - *Replace* $\mid\!\sim$ *with* $\mid\!\sim_1$ *in* (7)-(15). *The resulting properties obtain. Moreover for any* A, B, $C \in \mathcal{F}$,

(29) *if* $A \mid\!\sim_1 B$ *and* $A \mid\!\sim_1 C$, *then* $A \wedge B \mid\!\sim_1 C$.

(30) *if* $A \mid\!\sim_1 B$ *and* $B \mid\!\sim_1 A$, *then* $A \mid\!\sim_1 C$ *iff* $B \mid\!\sim_1 C$.

(31) *if* $A \mid\!\sim_1 B \wedge C$ *then* $A \wedge B \mid\!\sim_1 C$.

Proof. Def. 3.7 yields (29). We prove (30): from (5) we have

$$A>B \rightarrow ((A \wedge B)>C \rightarrow A>C) \quad \text{and} \quad A>B \rightarrow ((A \wedge B)>C \rightarrow B>C)$$

and from (26) we have

$$A>B \wedge A>C \rightarrow (A \wedge B)>C \quad \text{and} \quad B>A \wedge B>C \rightarrow (A \wedge B)>C$$

Intuitionistic propositional calculus yields $\vdash_1((A>B)\wedge(B>A) \longrightarrow$ $\longrightarrow (A>C \longleftrightarrow B>C))$. As for (31), use (8) and (29).

Remark. - Formula (30) says that the concept of n. m. inference based on γ_1 allows for a proper definition of n. m. deductive equivalence ($A \vdash_1 B$ and $B \vdash_1 A$).

To define γ_1-*validity*, extend Def. 3.1 with the converse of the consequent in (19), i. e.

Definition 3.9. - $\mathcal{S}=(S,R,R,\sigma,[\ ])$ is a γ_1-*model* if $\mathcal{S}$ is a γ-model and

(32) if for every t, $(s,[A],t)\in R$ implies $t\in[B]$), then for every t, $(s,[A]\cap[B],t)\in R$ implies $(s,[A],t)\in R$.

A wff. A is γ_1-*valid* if A is true in every γ_1-model.

Again, if we interpret n. m. reasoning as reasoning on the basis of stereotypes, (32) and (19) say that, when the stereotyped meaning of A is included in the full meaning of B, the stereotyped meanings of A and $A\wedge B$ coincide.

Theorem 3.10. - *Every theorem of* γ_1 *is* γ_1-*valid.*

Proof. To show that (26) is valid, consider a γ_1-model $\mathcal{S}$ and let $t\in S$ be such that $t \Vdash A>B \wedge A>C$. Because of the first conjunct, we have that for all $u\in S$, if $(t,[A],u)\in R$ then $u\in[B]$. So the antecedent of (32) is satisfied, therefore if $u\in S$

(i) $(t,[A\wedge B],u)\in R$ implies $(t,[A],u)\in R$.

Consider now, any $v\in S$ such that $(t,[A\wedge B],v)\in R$. By (i) and the second conjunct in hypothesis, $v \Vdash C$ and thus $t \Vdash (A\wedge B)>C$.

Remark. - The γ-models constructed in the proof of Thm. 3.6 can be transformed into γ_1-models, by adding $(s,Q_7,u)\in R$. Using suitable

models it can be seen that neither γ nor γ_1 are *prime*. Last, *full* γ_1-models are defined by extending Def. 3.4 with

(32') as (32) except that every occurence of $[A]$ and $[B]$ is replaced by Q,Q' respectively $(Q,Q'\subseteq S)$.

4. - COMPLETENESS THEOREMS.

To prove completeness for both γ and γ_1, we use the same basic semantic methods that have been successfully applied to Intuitionistic and Modal Logic, i. e. we construct a canonical model which falsifies all non theorems. First, recall (see [F]) that a set of formulas Σ is *nice* in γ (γ_1 resp.) with respect to a set of constants Ω if

(i) $\{A$: A is a theorem of γ (of γ_1 resp.)$\} \subseteq \Sigma$,

(ii) Σ is closed with respect to Modus Ponens,

(iii) every formula in Σ uses only parameters in Ω,

(iv) Σ is consistent, prime and rich.

Throughout this paragraph, considerable use will be made of

Main Lemma. - *Let* $\Sigma\subseteq\mathcal{F}$ *and* $A\in\mathcal{F}$. *Let* Ω *be the set of constants in* Σ *or* A. *Let* $\{a_1,...\}$ *be a countable collection of distinct constants not in* Ω *and let* $\Omega'=\Omega\cup\{a_1,...\}$. *If* $\Sigma \not\vdash A$, *then* Σ *can be extended to a set* s *which is nice in* γ (γ_1 *resp.*), *with respect to* Ω' *and such that* $A \notin s$.

Proof. The proof in [F, p. 68] applies to γ (γ_1 resp.) also.

Second, arrange, as in [F], the collection of all constants in the language of γ as follows:

$\delta^1 = a^1_1, a^1_2, ..., a^1_n, ...$
$\delta^2 = a^2_1, ..., a^2_n, ...$
....
$\delta^n = a^n_1, ...$
....

and let $\Omega^n = \delta^1 \cup \delta^2 \cup ... \cup \delta^n$.

Consider now,

Definition 4.1. - $\mathcal{S}^c = (S^c, R^c_0, R^c, \sigma^c, [\ ])$ is a *canonical model* for γ (γ_1 respectively) if

(i) S^c is the collection of all sets s that are nice in γ (γ_1 resp.) with respect to some Ω_i ($i<\omega$);

(ii) σ^c is a function that assigns Ω_j to $s \in S^c$ if s is nice with respect to Ω_j;

(iii) for any $s,t \in S^c$, $sR_0{}^c t$ iff $\sigma^c(s) \subseteq \sigma^c(t)$ and $s \subseteq t$;

(iv) for any $A \in \mathcal{F}$, $[A] = \{s \in S^c: A \in s\}$;

(v) $(s,[A],t) \in R^c$ iff for every $B \in \mathcal{F}$, if $A>B \in s$ then $B \in t$ $\quad (A \in \mathcal{F})$.

We say that a canonical model $\mathcal{S}^c$ is the *least canonical model* if, given any $Q \subseteq S^c$, $(s,Q,t) \notin R^c$, whenever there is no $A \in \mathcal{F}$ such that $Q=[A]$.

In order to show that the canonical models are γ-models (γ_1-models resp.), we establish first,

Lemma 4.2. - *The relation R^c, as given by* Def. 4.1, *is well defined.*

Proof. Note that $[A]=[A']$ implies $\vdash A \longleftrightarrow A'$. So by (3) $(s,[A],t) \in R^c$ iff $(s,[A'],t) \in R^c$.

Lemma 4.3. - *If* $(s,[A],t) \in R^c$ *then* $t \in [A]$.

Proof. Suppose that $(s,[A],t) \in R^c$. Since $\vdash A>A$, (see (7)), $A>A \in s$

and hence $t \in [A]$.

Lemma 4.4. - *For any $s \in S^c$, if $(A>B) \notin s$, then there is $t \in S^c$ such that $(s,[A],t) \in R^c$ and $t \notin [B]$.*

Proof. Let $X = \{C \in \mathcal{F}: A>C \in s\}$. Clearly, $X \not\vdash B$, for if not, there would be $C_1, ..., C_n \in X$ such that $\vdash C_1 \wedge ... \wedge C_n \rightarrow B$. Using (14) and (1), we have $\vdash A>C_1 \wedge ... \wedge A>C_n \rightarrow A>B$; but for each $i \leq n$, $A>C_i \in s$, hence $(A>B) \in s$, contradiction. Now apply main Lemma to X and B.

Lemma 4.5. - *If $s \in S^c$ and $(A>\bot) \notin s$ then there is $t \in S^c$ such that $(s,[A],t) \in R^c$.*

Proof. Apply Lemma 4.4 with $B=\bot$.

Lemma 4.6. - *If $[A] \neq \varnothing$ then there are $s,t \in S^c$ such that $(s,[A],t) \in R^c$.*

Proof. Deny the consequent, i. e. for all $s \in S^c$, there are no $t \in S^c$ such that $(s,[A],t) \in R^c$. By Lemma 4.5, for all $s \in S^c$, $A>\bot \in s$ and thus $\vdash A>\bot$. By (3) we have $\vdash \neg A$, hence $[A]=\varnothing$.

Lemma 4.7. - *Condition* (19) *holds for* R^c.

Proof. - Suppose to the contrary, i. e.

(i) if $(s,[A],t) \in R^c$ then $t \in [B]$ $\quad (t \in S^c)$,

but for some $t_0 \in S^c$ such that

(ii) $(s,[A],t_0) \in R^c$,

there is $C \in \mathcal{F}$ such that

(iii) $(A \wedge B)>C \in s$ and $C \notin t_0$.

Then by (ii) and second part of (iii), $(A>C) \notin s$; now, (5) and first part of (iii) yields $(A>B) \notin s$. At this point, we apply Lemma 4.4 and find that there is $u \in S^c$ such that $(s,[A],u) \in R^c$ and $u \notin [B]$. But this contradicts (i).

Lemma 4.8. - *If* $s,t \in S^c$ *and* $sR^c_0 t$, *then for every* $u \in S^c$ $(s,[A],u) \in R^c$ *iff* $(t,[A],u) \in R^c$.

Proof. Let $(A>B) \in t$ and $(s,[A],u) \in R^c$. If $(A>B) \in s$ then $B \in u$. Moreover, it cannot be $(A>B) \notin s$; for if it were, by (6) $\neg(A>B) \in s$, and since $s \sqsubseteq t$, $\neg(A>B) \in t$ - contradicting our original supposition. Hence, $(t,[A],u) \in R^c$. Conversely, suppose $(t,[A],u) \in R^c$. Since $s \sqsubseteq t$, if $(A>B) \in s$, then $(A>B) \in t$; thus $B \in u$ and $(s,[A],u) \in R^c$.

Theorem 4.9. - *If* $\mathcal{S}^c$ *is a canonical model for* γ (γ_1 *resp.*), *then it is a* γ*-model* (γ_1*-model resp.*).

Proof. Let $\mathcal{S}^c$ be a canonical model for γ. It is obvious that R^c_0 and σ^c satisfy (16). Lemmas 4.3, 4.6, 4.7, 4.8 ensure that (17)-(20) are satisfied. Conditions (21)-(23) are also easily shown. Let us show that

$$(A>B) \in s \textit{ iff for all } t \in S^c,\ (s,[A],t) \in R^c \textit{ implies } t \in [B].$$

So suppose $(A>B) \in s$ and $(s,[A],t) \in R^c$, for some $t \in S^c$. Then, use Def. 4.1(v). Conversely, suppose $(A>B) \notin s$. Then Lemma 4.4 yields the desired result. So $\mathcal{S}^c$ is a γ-model.

Suppose now that $\mathcal{S}^c$ is a canonical model for γ_1. We show that S^c is a γ_1-model. So deny the consequent of (32), i. e. suppose that for some $t \in S^c_1$

(i) $\quad (s,[A \wedge B],t) \in R^c,$

but for some $C \in \mathcal{F}$

(ii) $\quad (A>C) \in s$ and $C \notin t.$

Then because of (i) and (ii), $(A \wedge B)>C \notin s$; thus by (26) we conclude $(A>B) \notin s$. Now apply Lemma 4.4 to find $u \in S^c$ such that $(s,[A],u) \in R^c$ and $u \notin [B]$, thus contradicting the antecedent of (32).

Theorem 4.10. - *Let* $\mathcal{S}^c$ *be the least canonical model for* γ. *Then* $\mathcal{S}^c$ *can be extended to a full* γ*-model.*

Proof. Add the following triples to R^c: if $Q \subseteq S^c$ and there is no $A \in \mathcal{F}$ such that $Q=[A]$, then

$$(s,Q,t) \in R^c \text{ iff } t \in Q.$$

It is easy to check that R^c so extended satisfies conditions (17')-(20').

Theorem 4.11. - γ (γ_1 *resp.*) *is complete with respect to* γ*-models* (γ_1*-models resp.*).

Proof. Thms. 3.3, 3.10, 4.9.

Furthermore:

Theorem 4.12. - γ *is complete with respect to* **full** γ*-models.*

Proof. Thms. 3.3, 4.10.

We were not able to prove that γ_1 is not complete with respect to *full* γ_1-models. Full models are important because they are manageable (see Thm. 3.5 and surrounding comments). In fact, we shall show in the next section that one method used to prove the finite model property uses the fact that the logic is characterized by full models.

5. - DECIDABILITY OF γ_0.

Using the filtration technique (see [G1], [L]), we prove that γ_0, the propositional fragment of γ, is decidable. First, let us call $\mathcal{L}_0$ the propositional language of γ_0 obtained from $\mathcal{L}$ by forgetting all n-ary predicates with n>0. Then,

Definition 5.1. - A structure $\mathcal{S}=(S,R_0,R,[\ ])$ is a *full* γ_0*-model* if $\mathcal{S}$ is a reduct to $\mathcal{L}_0$ of a full γ-model $(S,R_0,R,\sigma,[\ ])$ (see Def. 3.4). Then, $\mathcal{S}=(S,R_0,R)$ is said to be a *full* γ_0*-frame*. Clearly, the full γ_0-frames are exactly the full γ-frames.

The following is an obvious consequence of Thm. 4.11.

Corollary 5.2. - *γ_0 is complete with respect to full γ_0-models.*

We proceed as usual: given a full γ_0-model $\mathcal{S}=(S,R_0,R,[\ ])$ and any set Ψ of wff's in $\mathcal{L}_0$, if $s,t\in S$ put

$$|s|=|t| \quad \text{iff} \quad \text{for every } A\in\Psi,\ s\in[A] \text{ iff } t\in[A].$$

Now for every $Q\subseteq S$, set

$$|Q| = \{|s|: \text{there is } s'\in Q \text{ such that } |s'|=|s|\}.$$

As a special case, note that

$$|S| = \{|s|: s\in S\}. \tag{33}$$

Definition 5.3. - Let $Q\subseteq S$. The set Q is *saturated* if Q is equal to a union of elements in $|S|$.

Let us recall that saturated sets have useful properties, namely:

Lemma 5.4. - *Let $Q,Q'\subseteq S$;*

(i) *if Q is saturated then for any $s\in S$, $|s|\in|Q|$ iff $s\in Q$,*

(ii) *for any Q, there is one and only one saturated set Q' such that $|Q|=|Q'|$,*

(iii) *if Q and Q' are saturated sets so is $Q\cap Q'$.*

From now on, let Ψ be a set of wff's closed under subformulas and negation (in particular, $A,B\in\Psi$ whenever $(A>B)\in\Psi$).

Definition 5.5 - Let $\mathcal{S}=(S,R_0,R,[\ ])$ be a full γ_0-model. Then $\mathcal{S}^+=(|S|,R_0^+,R^+,[\ ]^+)$ is a *filtration through* Ψ *on* $\mathcal{S}$ if

(i) $|S|$ is as in (33);

(ii) $(|s|,|t|)\in R_0^+$ iff there are $s'\in|s|$ and $t'\in|t|$ such that $s'R_0t'$;

(iii) $(|s|,|Q|,|t|)\in R^+$ iff there are $s'\in|s|$, $t'\in|t|$ such that $(s',Q',t')\in R$, where Q' is the only saturated set such that $|Q|=|Q'|$;

(iv) for each 0-ary predicate P, $[P]^+\subseteq|S|$; furthermore $[P]^+=|[P]|$ when $P\in\Psi$.

Note that (iii) is correct because of Lemma 5.4(ii). We show that Def. 5.5 is suitable.

Lemma 5.6. - *Let $\mathcal{S}$ and $\mathcal{S}^+$ be as in* Def. 5.5. *Then*

(i) *if $A\in\Psi$ then $[A]$ is a saturated subset of S;*

(ii) *for every 0-ary predicate $P\in\Psi$, $[P]^+$ is R_0^+ closed;*

(iii) *if $\neg A\in\Psi$, $(|s|,|t|)\in R_0^+$ and $s\in[\neg A]$ then $t\notin[A]$;*

(iv) *if $(A\rightarrow B)\in\Psi$, $(|s|,|t|)\in R_0^+$, $s\in[A\rightarrow B]$ and $s\in[A]$ then $t\in[B]$;*

(v) *if $(A>B)\in\Psi$, $(|s|,|[A]|,|t|)\in R^+$ and $s\in[A>B]$ then $t\in[B]$.*

Proof. (i). Since $A\in\Psi$, $[A]=\bigcup_{s\in[A]}|s|$.

As for (ii), suppose that for some 0-ary predicate $P\in\Psi$, $|s|\in[P]^+$, i. e. $|s|\in|[P]|$. Let $|s|R_0^+|t|$; so there are $s'\in|s|$, $t'\in|t|$ with $s'R_0t'$. But $|s'|\in|[P]|$ and since $P\in\Psi$, Lemmas 5.6(i), 5.4(i) ensure that $s'\in[P]$ and thus $t'\in[P]$. We conclude $|t|=|t'|\in|[P]|=[P]^+$.

Parts (iii), (iv), (v) are easy.

Let $\mathscr{S}$ and $\mathscr{S}^+$ be as in Def. 5.5. Extend $[\ ]^+$ to arbitrary formulas (see Thm. 3.5, applied to R^+ and R_0^+).

Theorem 5.7. - *If $A\in\Psi$, then for all $s\in S$,*

$$|s|\in[A]^+ \ \textit{iff}\ \ s\in[A].$$

Proof. By induction on the complexity of A. We single out case $A=(B>C)$. Let us first prove that

(i) $$|[B]|=[B]^+.$$

Since $B\in\Psi$, $[B]$ is saturated, so

$$s\in[B] \ \text{ iff } \ |s|\in|[B]|.$$

Using the inductive hypothesis,

$$|s|\in[B]^+ \ \text{ iff } \ |s|\in|[B]|.$$

Now let

(ii) $$|s|\in[A]^+$$

and $(s,[B],t)\in R$. Again $[B]$ is saturated so $(|s|,|[B]|,|t|)\in R^+$. From (i), we have $(|s|,[B]^+,|t|)\in R^+$ and by (ii), $|t|\in[C]^+$, i. e. $t\in[C]$ (by the inductive hypothesis). So $s\in[A]$.

Conversely, let $s\in[A]$ and $(|s|,[B]^+,|t|)\in R^+$. By Lemma 5.6(v), $t\in[C]$. The inductive hypothesis yields $|t|\in[C]^+$, so $|s|\in[A]^+$.

Unfortunately, $\mathscr{S}^+$ is not a γ_0-model because R_0^+ is not necessarily transitive and R^+,R_0^+ are not connected in the way requested by condition (20'). We shall show, however, that it is possible to construct a γ_0-model from $\mathscr{S}^+$. First note:

Lemma 5.8. - *Let $A\in\Psi$ and, for some $s,t\in S$, suppose that $|s|R_0^+|t|$. Then $|s|\in[A]^+$ implies $|t|\in[A]^+$.*

Proof. By hypothesis, there are $s'\in|s|$ and $t'\in|t|$ such that $s'R_0t'$. By Thm. 5.7, $s\in[A]$ and so $s'\in[A]$. But then $t'\in[A]$, which by Thm. 5.7 implies $|t|=|t'|\in[A]^+$.

In the sequel, let R_0^* be the transitive closure of R_0^+.

Lemma 5.9. - *Let $A\in\Psi$ and suppose $|s|R_0^*|t|$. Then $|s|\in[A]^+$ implies $|t|\in[A]^+$.*

Proof. Apply repeatedly Lemma 5.8.

Lemma 5.10. - *Let $(A>B)\in\Psi$. For all $s,t\in S$, if $|s|\in[A>B]^+$ and $|t|R_0^*|s|$, then $|t|\in[A>B]^+$.*

Proof. Suppose that $|t|R_0^*|s|$ but $|t|\notin[A>B]^+$. By Thm. 5.7 and (6), we conclude $t\in[\neg(A>B)]$. But $\neg(A>B)\in\Psi$, so $|t|\in[\neg(A>B)]^+$: impossible, because of initial hypothesis and Lemma 5.9.

Now call $\overline{R}_0$ the equivalence relation generated by R_0^* and let R^* be defined as follows:

Definition 5.11. - For all $s,t\in S$ and $Q\subseteq S$,

$$(|s|,|Q|,|t|)\in R^*$$

iff there is $u\in S$ such that $|s|\overline{R}_0|u|$ and $(|u|,|Q|,|t|)\in R^+$.

Note that $R^+\subseteq R^*$.

Lemma 5.12. - *Let $(A>B)\in\Psi$. For all $s,t\in S$, if $(|s|,[A]^+,|t|)\in R^*$ and $|s|\in[A>B]^+$, then $|t|\in[B]^+$.*

Proof. By definition of R^*, there is $u\in S$ such that

(i) $$|s|\overline{R}_0|u|$$

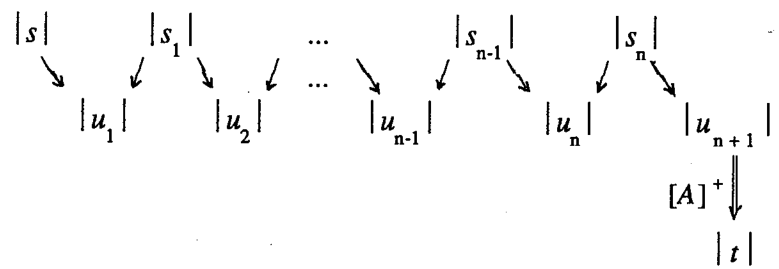
(ii) $$(|u|,[A]^+,|t|)\in R^+.$$

Because of (i) (using reflexivity and transitivity of R_0^*), we can find $s_1,\ldots,s_n,u_1,\ldots u_{n+1}\in S$ such that

$$|u_{n+1}|=|u| \text{ and } |s|R_0^*|u_1|;$$

$$|s_j|R_0^*|u_j| \text{ and } |s_j|R_0^*|u_{j+1}| \qquad (0<j\leq n).$$

Thus (i) and (ii) lead to the following situation



Using hypothesis and Lemmas 5.9 and 5.10 repeatedly, we have $|u|\in[A>B]^+$. Now (ii) yields $|t|\in[B]^+$.

For every 0-ary predicate P, put $[P]^*=[P]^+$ and extend $[\ ]^*$ to arbitrary formulas (see Thm. 3.5 applied to R^* and R_0^*).

Theorem 5.13. - *Let $A\in\Psi$. Then $[A]^+=[A]^*$.*

Proof. By induction on the complexity of A. We illustrate two cases. Let $A=\neg B$ and suppose $|s|\in[A]^+$. Let $|s|R_0^*|t|$; then, according to Lemma 5.9, $|t|\in[A]^+$. Thus $|t|\notin[B]^+$. By the inductive hypothesis $|t|\notin[B]^*$ and $|s|\in[A]^*$. The converse is immediate.

Now let $A=(B>C)$; take $|s|\in[A]^+$ and suppose $(|s|,[B]^*,|t|)\in R^*$. By the inductive hypothesis, $(|s|,[B]^+,|t|)\in R^*$, so we can use Lemma 5.12 and obtain $|t|\in[C]^+$. Again by inductive hypothesis, $|t|\in[C]^*$, so $|s|\in[A]^*$. The converse is immediate.

Theorem 5.14. - *The relation R^+ satisfies* (17')-(19').

Proof. In the course of this proof let $s,s',t,t'\in S$ and $Q,Q'\subseteq S$. Assume $(|s|,|Q|,|t|)\in R^+$, i. e. there are s', t' and there is a saturated set Q' such that $s'\in|s|$, $t'\in|t|$, $|Q|=|Q'|$ and $(s',Q',t')\in R$. But $\mathscr{S}$ is a full γ_0-model so $t'\in Q'$. Since Q' is saturated, $|t|=|t'|\in$ $\in|Q'|=|Q|$ and (17') is proven.

Now suppose that $|Q|\neq\emptyset$, i. e. $|s|\in|Q|$, for some s. Use Lemma 5.4(ii) and find the only saturated set Q' such that $|Q|=|Q'|$. Then

$|s|\in|Q'|$; Q' is saturated, so $s\in Q'$. $\mathscr{S}$ being a full γ_0-model, we can find s',t' such that $(s',Q',t')\in R$. So $(|s'|,|Q|,|t'|)\in R^+$ and (18') holds.

As for (19'), suppose

(i) $\qquad (|s|,|Q|,|t|)\in R^+$ implies $|t|\in|Q'|$, $\qquad (t\in S)$.

Because of Lemma 5.4(ii) we may assume that both Q and Q' are saturated sets, so bearing in mind (i) we have, for every t and every $s'\in|s|$,

(ii) $\qquad (s',Q,t)\in R$ implies $t\in Q'$.

Consider now any t_0 such that $(|s|,|Q|,|t_0|)\in R^+$. So there are $s'\in|s|$ and $t'\in|t_0|$ such that $(s',Q,t')\in R$. Since $\mathscr{S}$ is a full γ_0-model, from (ii) we get that $(s',Q\cap Q',t')\in R$. By Lemma 5.4(iii), $Q\cap Q'$ is saturated, and since $|Q\cap Q'|=|Q|\cap|Q'|$, we have $(|s|,|Q|\cap|Q'|,|t_0|)\in R^+$; so (19') is proved.

Before we show that $(|S|,R_0^*,R^*,[\ ]^*)$ is a γ_0-model we must note

Lemma 5.15. - *Let $s,t\in S$ be such that $|s|R_0^*|t|$. Then for all $u\in S$, $|s|\overline{R}_0|u|$ iff $|t|\overline{R}_0|u|$.*

Proof. Just remember that $\overline{R}_0$ is the equivalence relation generated by R_0^*.

Lemma 5.16. - *Let s,u,Q,Q' be such that*

(i) $\qquad (|s|,|Q|,|t|)\in R^*$ *implies* $|t|\in|Q'|$ $\qquad (t\in S)$

(ii) $\qquad |s|\overline{R}_0|u|$.

Then

$$(|u|,|Q|,|t|)\in R^+ \text{ implies } (|u|,|Q|\cap|Q'|,|t|)\in R^+.$$

Proof. The conclusion of this lemma being the same as that of formula (19') (relatively to R^+), we only need to prove the hypothesis of (19'). Suppose that $(|u|,|Q|,|t|)\in R^+$; from (ii) and (i) we obtain $|t|\in|Q'|$. Apply Thm. 5.14 to get the desired result.

Finally,

Theorem 5.17. - *If* $\mathcal{S}^+$ *is a filtration through* Ψ *on a full* γ_0*-model* $\mathcal{S}$, *then the structure* $\mathcal{S}^*=(|S|,R_0^*,R^*,[\ ]^*)$ *is a full* γ_0*-model.*

Proof. We prove that $(|S|,R_0^*,R^*)$ is a full γ_0-frame. Here s, t, t', u, $v \in S$ and $Q,Q'\subseteq S$.

Let $(|s|,|Q|,|t|)\in R^*$. Then there is u with both $|s|\overline{R}_0|u|$ and $(|u|,|Q|,|t|)\in R^+$. By Thm. 5.14, $|t|\in|Q|$, so (17') is proved.

Now suppose that $|Q|\neq\varnothing$. By Thm. 5.14 there are s,t such that $(|s|,|Q|,|t|)\in R^+\subseteq R^*$.

As for (19'), suppose that

(i) $(|s|,|Q|,|t|)\in R^*$ implies $|t|\in|Q'|$ $(t\in S)$

and let t' be such that $(|s|,|Q|,|t'|)\in R^*$; then we show that $(|s|,|Q|\cap|Q'|,|t'|)\in R^*$. First, by hypothesis there is u such that

(ii) $|s|\overline{R}_0|u|$ and

(iii) $(|u|,|Q|,|t'|)\in R^+$.

Because of (i) and (ii) we can apply Lemma 5.16 and obtain

(iv) $(|u|,|Q|,|t|)\in R^+$ implies $(|u|,|Q|\cap|Q'|,|t|)\in R^+$

So (iii) and (iv) imply $(|u|,|Q|\cap|Q'|,|t'|)\in R^+$ which together with (ii) yields $(|s|,|Q|\cap|Q'|,|t'|)\in R^*$.

Last we prove (20'); suppose that $|s|R_0^*|t|$ and $(|s|,|Q|,|v|)\in R^*$. So there is u such that $|s|\overline{R}_0|u|$ and

(v) $(|u|,|Q|,|v|)\in R^+$.

By Lemma 5.15, $|t|\overline{R}_0|u|$ which with (v) yields $(|t|,|Q|,|v|)\in R^*$. Analogously for the converse.

Note that if $\mathcal{S}$ were not a **full** γ_0-model, then the proof of the fact that $\mathcal{S}^*$ is a γ_0-model would not carry through (see Thms. 5.14 and 5.17).

The next result gives a positive answer to a question raised in [G2].

Corollary 5.18. - *The system* γ_0 *has the finite model property and hence it is decidable.*

Proof. Suppose $\not\vdash_{\gamma_0} A$. Let $\mathcal{S}$ be a falsifying full γ_0-model for A (Corollary 5.2). Take Ψ to be a set of formulas containing all the subformulas of A and closed under negations. Use the filtration through Ψ on $\mathcal{S}$ to construct $\mathcal{S}^*$. Thms. 5.7, 5.13 and 5.17 ensure that $\mathcal{S}^*$ is a full γ_0-model that falsifies A. Note that $|S|$ is finite, because such is the set $\{[A]: A\in\Psi\}$.

6. - NON MONOTONIC THEORIES BASED ON γ OR γ_1.

An important fact about γ (γ_1 resp.) is that it is a «domain independent» formalism for n. m. reasoning. By this we mean that the pairs (A,B) such that $A \mid\!\sim B$ form the set of *logically valid* n. m. arguments. By Def. 2.1 (3.7 resp.), this set is determined by a decidable subset of a monotonic logic having the usual recursive properties; so it is r. e.

Now the pure logic of n. m. reasoning is an essential but manifestly insufficient tool in the development of a satisfactory approach. For, as we attempt to extend our incomplete knowledge, we take into account rational epistemic policies, observed regularities or established conventions that are constituent parts of a *theory* we have about a specific domain. It follows that any useful formalization of n. m. reasoning must also include a proper analysis of *n. m. deduction in theories.*

The first natural move in response to this problem, is to consider finite theories only, so that the following makes sense:

First proposal. - Given a finite theory T, let T also stand for the conjunction of its axioms. Then, B is *n. m. deducible in* T iff $\vdash T>B$.

In this section, $\vdash A$ stands for «A is a theorem of either γ or γ_1».

The first proposal is objectionable on both intuitive and technical grounds. To begin with, non monotonic proof theory here has a global characteristic: every axiom of the theory must be taken into account before an attempted proof can succeed. But this is surely counterintuitive: take the canonical example about Tweety flying. Our theory T contains statements like

A_1: $\forall x(\text{Bird}(x) > \text{Flies}(x))$,

A_2: $\forall x(\text{Penguin}(x) \rightarrow \text{Bird}(x) \wedge \neg\text{Flies}(x))$,

A_3: Bird(Tweety),

and possibly some other statements about mice or elephants. Intuitively, you don't presume that Tweety flies because you know all of T but rather because you bear in mind the relevant portion of T.

Worst still, the first proposal cannot handle the Tweety example, since $\not\vdash$ T>Flies(Tweety); yet there is no question that the Tweety story captures some important aspects of n. m. reasoning.

Another objection to the first proposal is that general statements like A_1 and A_2 and particular facts like A_3 have the same standing with respect to n. m. deduction. To be sure, what we consider to be knowledge includes both specific facts and general principles that we are fairly confident about, but each behave differently with respect to n. m. deduction. For, according to our view, non monotonicity is caused here by gaining access to additional facts rather than by changing the theory used to supplement the factual knowledge base (theory revision must be treated separately!). To be more accurate, we think that the present context of knowledge extension calls for the theory to be fixed, the factual base to grow so that *its consequences in the theory* may diminish. In order to express this distinction, consider

Second proposal. - Let T be a theory. Then B is *n. m. deducibile from A in T* iff $T \vdash A{>}B$.

The second proposal deals adequately with the Tweety example (where $T=A_1 \wedge A_2$; $A=A_3$; B=Flies(Tweety)). But it poses another serious problem: not all plausible consequences are obtainable. For instance if T is now

A_4 - $\forall$x(Canary(x) > Yellow(x))

A_5 - $\forall$x(Yellow(x) $\rightarrow \neg$ Green(x)),

from the fact «Canary(Tweety)», we should be able to obtain the n. m. consequence «$\neg$ Green(Tweety)». This turns out to be impossibile, as a simple counterexample shows (note that there is no contrast with rule (14), which requires $A \rightarrow B$ to be a logical theorem).

Another different and interesting idea can be found in [G2]; it is

Gabbay's proposal. - Let T be as in first proposal. Then B is *n. m. deducibile in T* (write: $T \mapsto\!\!\!\!\!\!\rightarrow B$) iff there is $X \in \mathcal{F}$ such that $\vdash T{>}X$ and $T \wedge X \vdash B$.

The idea here, is to describe n. m. deduction (from a finite number of formulas) as the result of a process occurring in two steps: first there is a monotonic evaluation of the correct presumptions, then from these a n. m. conclusion is drawn by standard deduction. But Gabbay's proposal turns out to be equivalent to the first proposal.

Theorem 6.1 - *Let T be as in first proposal. Then $T \mapsto\!\!\!\!\!\!\rightarrow B$ iff $\vdash T{>}B$.*

Proof. Suppose that there is a wff. X such that $\vdash T{>}X$ and $T \wedge X \vdash B$. Then $\vdash T \wedge X \rightarrow B$ which by rule (2) yields $\vdash T \wedge X{>}B$. So by (5), $\vdash T{>}B$. Conversely, let $\vdash T{>}B$; since $T \wedge B \vdash B$ we conclude $T \mapsto\!\!\!\!\!\!\rightarrow B$.

Therefore, all objections to the first proposal also apply to Gabbay's definition. Nonetheless, *this* formulation of the first proposal is attractive for it makes an important distinction - that between drawing reasoned conclusions from an incomplete knowledge base and the deduction mechanism of logic which is applicable to the extended knowledge base. This kind of analysis reflects closely the way we think (see for instance the Default Logic in [R2]): we first look for plausible extensions of our knowledge base, then we reason monotonically on these as if they were part of newly acquired knowledge and what we obtain is a n. m. consequence of our initial knowledge base.

So let us only modify Gabbay's proposal so as to make it immune from the objections discussed above. Consider

Definition 6.2 - Let T be a theory. Then B is *n. m. deducibile from A in T* ($A \mid\!\sim_T B$) if there is $X \in \mathcal{F}$ such that $T \vdash A{>}X$ and $T \vdash A \wedge X \rightarrow B$.

Note that Def. 6.2 is not equivalent to the preceding proposals. It makes a distinction between theory and facts; it deals adequately with exceptions and - until shown to the contrary - it allows for reasoning about typicality. For instance, reconsider the theory $T=\{A_4,A_5\}$. We have,

(i) $T \vdash$ Canary(Tweety) > Yellow(Tweety)

(ii) $T \vdash$ Canary(Tweety)$\wedge$Yellow(Tweety)$\rightarrow \neg$ Green(Tweety).

According to Def. 6.2, (i) and (ii) yield

(Canary(Tweety) $\mid\!\sim_T \neg$ Green(Tweety).

Using Def. 6.2 we have:

Theorem 6.3 - *Let T be a theory and let A,B,C$\in\mathcal{F}$.*

(34) $A \mid\!\sim_T A$.

(35) $A \mid\!\sim_\varnothing B$ *iff* $A \mid\!\sim B$.

(36) *if* $T \wedge A \vdash B$ *then* $A \mid\!\sim_T B$.

(37) *if* $T \vdash A{>}B$ *then* $A \mid\!\sim_T B$.

(38) $A \mid\!\sim_T \bot$ *iff for every B,* $A \mid\!\sim_T B$.

(39) *if* $T \vdash \neg A$ *then* $A \mid\!\sim_T \bot$.

(40) *if* $A \not\mid\!\sim_T \bot$ *and* $A \mid\!\sim_T B$ *then* $A \not\mid\!\sim_T \neg B$.

(41) *if* $A \not\mid\!\sim_T \bot$ *and* $A \mid\!\sim_T B$ *then* $T \wedge A \not\vdash \neg B$.

(42) *if* $T \vdash B \rightarrow C$ *and* $A \mid\!\sim_T B$ *then* $A \mid\!\sim_T C$.

(43) $A \mid\!\sim_T B$ *and* $A \mid\!\sim_T C$ *iff* $A \mid\!\sim_T B \wedge C$.

Proof. Easy.

Remark. - Formula (35) shows that Def. 2.1 captures the *logical* properties of n. m. deduction when its meaning is fixed by Def. 6.2. Property (37) gives a n. m. version of Modus Ponens (compare with [D], where M. P. does not hold for conditional statements). Moreover, if

$$T=\{A{>}X;\ A \rightarrow \neg X\},$$

it is easy to see that $A \mid\!\sim_T \bot$ but $T\cup\{A\}$ is consistent. This proves that both the converse of (36) and that of (39) do not hold.

It turns out, however, that the n. m. deduction operator given in Def. 6.2 is *not restrictedly transitive.* To see this, let T be the theory determined by the following axioms: $P{>}\neg M$; $\neg M \rightarrow P'$; $P \wedge P'{>}M'$; $M' \rightarrow M$; $(P{>}Z) \rightarrow Z$, for every $Z\in\mathcal{F}$, where P,P',M,M' are 0-ary predicates. Using these notations, we first show:

Lemma 6.4. - *If* $P \mid\!\sim_T \bot$, *then* $T \vdash \bot$.

Proof. Since $T \vdash (P{>}P) \rightarrow P$, we have $T \vdash P$ (use (2). On the other hand, the hypothesis of this Lemma means that there is $X\in\mathcal{F}$ such that

(i) $T \vdash P{>}X$ and $T \vdash P \wedge X \rightarrow \bot$, i. e.

(ii) $T \vdash P \rightarrow \neg X$.

From (i) and the last axiom in T, we infer $T \vdash X$. But this together with (ii) yields $T \vdash \neg P$; hence $T \vdash \bot$.

Lemma 6.5. - *T is consistent.*

Proof. Consider the structure $\mathcal{S}=(S,R_0,R)$ such that $S=\{s,t,u\}$ and $R_0=\varnothing$. Name the subsets of S as follows: $Q_1=\{s\}$; $Q_2=\{t\}$; $Q_3=\{u\}$; $Q_4=\{s,t\}$; $Q_5=\{s,u\}$; $Q_6=\{t,u\}$; $Q_7=\{s,t,u\}$. Let R be the following set of triples: (s,Q_1,s); (t,Q_2,t); (s,Q_3,u); (s,Q_4,s); (s,Q_4,t); (s,Q_5,u); (t,Q_6,t); (s,Q_7,s); (s,Q_7,t). Now let σ be the constant function on S with value $\varnothing$ and $[]$ the usual truth function such that for 0-ary predicates P,P',M,M',

$$[P] = Q_7 = \{s,t,u\} \qquad [M] = Q_3$$

$$[P'] = Q_5 = \{s,u\} \qquad [M'] = Q_6$$

Now it is easy to check that $\mathcal{S}$ is a full γ_1-frame and so by Thm. 3.5, $\mathcal{S}=(S,R_0,R,\sigma,[])$ is a full γ_1-model. Moreover, $s \Vdash T$ (note that $[P\wedge P']=Q_5$), so T is consistent.

Theorem 6.6 - *For some theory T, the operator $\vdash_T$ is not restrictedly transitive.*

Proof. Take T as above. Note that $P \vdash_T P'$ and $P\wedge P' \vdash_T M$: for, $T \vdash P>\neg M$ and $T \vdash P\wedge\neg M \rightarrow P'$, so we have $P \vdash_T P'$; also, $T \vdash (P\wedge P')>M'$ and $T \vdash P\wedge P'\wedge M' \rightarrow M$ yield $P\wedge P' \vdash_T M$. Now if restricted transitivity held, we could infer

(i) $\qquad P \vdash_T M.$

On the other hand, $T \vdash P>\neg M$ and $T \vdash P\wedge\neg M \rightarrow \neg M$, so $P \vdash_T \neg M$. Using this last result, (i) and (43), we obtain $P \vdash_T \bot$. By Lemma 6.4, $T \vdash \bot$, contradicting Lemma 6.5.

We believe that restricted transitivity is a desirable property; a formalization of n. m. reasoning should account for some «chaining» of non monotonic conclusions. Unrestricted transitivity leads to taking special care to avoid unwanted consequences, but restricted transitivity solves that problem. From this point of view, Thm. 6.6 points out that either there is a serious flaw in our concept of n. m. deduction or that γ_1 (and *a fortiori* γ) is not strong enough to permit chaining in theories. We chose to investigate this second option and propose another calculus, Γ, obtained from γ by substituting (5) with

(44) $$(A\wedge B)>C \rightarrow A>(B \rightarrow C).$$

To evaluate the plausibility of (44) take the instances: A=Bird(Tweety), B=Alive(Tweety), C=Flies(Tweety).

As in the previous case of γ, the calculus Γ determines a n. m. Logic, i. e.

$$A \vdash B \quad \text{iff} \quad \vdash_\Gamma A>B,$$

in which the following holds

(45) $$\text{if } A\wedge B \vdash C \text{ then } A \vdash (B \rightarrow C).$$

Note that (45) extends to the n. m. inference operator half of the Deduction Theorem. The other half cannot hold for otherwise the logic would be monotonic. For a discussion of (44) see also [S].

In the rest of this section, let $A \vdash_T B$ indicate n. m. deduction based on Γ. We show that $\vdash_T$ is restrictedly transitive.

Theorem 6.7. - *If $A \vdash_T B$ and $A\wedge B \vdash_T C$ then $A \vdash_T C$.*

Proof. By hypothesis, there are $X,Y\in\mathcal{F}$ such that

(i) $T \vdash_\Gamma A>X;$ $\qquad$ (ii) $T \vdash_\Gamma A\wedge X \rightarrow B;$

(iii) $T \vdash_\Gamma (A\wedge B)>Y;$ $\qquad$ (iv) $T \vdash_\Gamma A\wedge B\wedge Y \rightarrow C.$

Using (44) and (iii), we have $T \vdash_\Gamma A>(B \rightarrow Y)$, which together with (i) and (2) yields

(v) $$T\vdash_\Gamma A>(X\wedge(B \rightarrow Y)).$$

On the other hand, $T \vdash_{\Gamma} A \wedge X \wedge (B \rightarrow Y) \rightarrow A \wedge B \wedge (B \rightarrow Y)$ (use (ii)), so by (iv) it follows that $T \vdash_{\Gamma} A \wedge X \wedge (B \rightarrow Y) \rightarrow C$. This together with (v) yields the desired result.

Now for semantics: define $\mathcal{S}=(S,R_0,R,\sigma,[])$ to be a Γ-*model* if it satisfies all properties given in Def. 3.1 except that (19) is replaced by

(46) *Let* $s,t,u \in S$ *be such that* $(s,[A],t) \in R$, tR_0u *and* $u \in [B]$. *Then* $(s,[A] \cap [B],u) \in R$.

Intuitively, (46) implies that if the stereotyped meaning of $A \wedge B$ is empty, then the stereotyped meaning of A is included in the full meaning of $\neg B$.

Theorem 6.8. - Γ *is valid in the class of* Γ-*models.*

Proof. Obviously, it is enough to prove that (44) is valid in the class of Γ-models. Take a Γ-model $\mathcal{S}$ and $s,t,u \in S$ such that $(s,[A],t) \in R$. We show that $t \Vdash B \rightarrow C$ when $s \Vdash (A \wedge B) > C$. So let tR_0u and $u \Vdash B$. By (46), $(s,[A] \cap [B],u) \in R$ and thus $u \Vdash C$. Consequently, $s \Vdash A > (B \rightarrow C)$.

Theorem 6.9 - Γ *is complete in the class of* Γ-*models.*

Proof. Consider the canonical model $\mathcal{S}^c$ as given by Def. 5.1, except that now S^c is the set of nice Γ-theories with respect to some Ω_i. We set to prove that $\mathcal{S}^c$ satisfies (46). Let $(s,[A],t) \in R^c$, $t \subseteq u$ and $u \in [B]$ with $s,t,u \in S^c$. Furthermore, suppose that $(s,[A] \cap [B],u) \notin R^c$. So there is $C \in \mathcal{F}$ such that

(i) $(A \wedge B) > C \in s$

but $C \notin u$. Hence $(B \rightarrow C) \notin t$. On the other hand, (i) and (44) imply $A > (B \rightarrow C) \in s$. Therefore, $(s,[A],t) \notin R^c$, a contradiction. At this point use the arguments in the proof of Thm. 4.9 to show that $\mathcal{S}^c$ is a canonical model for Γ.

Remark. - A *full* Γ-*model* $\mathcal{S}=(S,R_0,R,\sigma,[])$ is a full γ-model except that (19') is replaced by

(46') as (46) except that every occurence of $[A]$ and $[B]$ is replaced by arbitrary $Q,Q' \subseteq S$.

It can be shown that no canonical model for Γ is ever full. For, suppose to the contrary, i. e. that a canonical model $\mathcal{S}^c = (S^c R^c R_0^c [],\sigma)$ for Γ is a full Γ-model. Take a non maximal nice Γ-theory, t; so $t \in S^c$ and there is $u \in S^c$ such that $t \subseteq u$ but $t \neq u$. Now let $Q=\{t\}$; then by (18') there are $s,r \in S^c$ such that $(s,Q,r) \in R^c$ and, by (17'), $r=t$. Thus

(i) $(s,Q,t) \in R^c$.

On the other hand, if $B = P \rightarrow P$, then $u \in [B]$ and since $t \subseteq u$, by (i) and (46'), we have that $(s,Q \cap [B],u) \in R^c$. So, by (17'), $u \in Q \cap [B]$, i. e. $u \in Q$, impossible.

At this point we can easily prove

Theorem 6.10 - $\gamma \subseteq \Gamma$.

Proof. To show that (5) is derivable in Γ, we take a canonical model $\mathcal{S}^c=(S^c R^c R_0^c [],\sigma)$ for Γ and, given any $s \in S^c$, we will prove that $s \Vdash$ (5). Suppose that

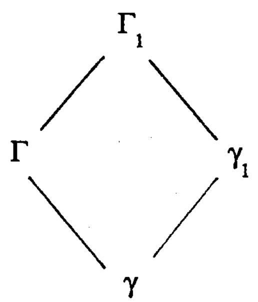
(i) $(A > B) \in s$

and $(A \wedge B) > C \in s$, but $A > C \notin s$; thus, by (46),

(ii) $A > (B \rightarrow C) \in s$,

and by (6) $\neg(A > C) \in s$. So there is $t \in S^c$ such that $(s,[A],t) \in R^c$ but $t \notin [C]$ (Lemma 4.4). By (i), $t \in [B]$ so that $t \notin [B \rightarrow C]$, contradicting (ii).

Now define a new calculus Γ_1 which bears the same relation to γ_1 as Γ does to γ. From Thm. 6.8 onward substitute every reference to γ and Γ with γ_1 and Γ_1 respectively. The resulting arguments still hold. Hence we have the following *G-lattice*:



We shall call *G-framework* the set of n. m. operators $\mid\!\sim_T$ based on the G-lattice.

We must check:

Theorem 6.11. - *Deduction in the G-framework can be non monotonic.*

Proof. We will find a theory T, such that $\mid\!\sim_T$ is n. m. Let T be determined by the formulas $(P>P')$ and $(P\wedge\neg P')>Z \longrightarrow Z$, for every $Z\in\mathcal{F}$. Let $\mid\!\sim_T$ belong to the G-framework. Using the same arguments as in the proof of Lemma 6.4, it is easy to see that

(i) if $(P\wedge\neg P')\mid\!\sim_T \bot$, then $T\vdash\bot$

Note that $P \mid\!\sim_T P'$ (see (37)), so if $\mid\!\sim_T$ was monotonic we should have

(ii) $P\wedge\neg P'\mid\!\sim_T P'$

But (ii) means that there is $X\in\mathcal{F}$ such that

(iii) $T\vdash(P\wedge\neg P')>X$

and $T\vdash(P\wedge\neg P'\wedge X) \longrightarrow P'$. Now, we also have $T\vdash(P\wedge\neg P'\wedge X) \longrightarrow \neg P'$, so by (iii)

(iv) $P\wedge\neg P'\mid\!\sim_T \neg P'$.

Therefore (ii) and (iv) imply $P\wedge\neg P'\mid\!\sim_T \bot$ and, using (i), we conclude $T\vdash\bot$. Hence, given any $\mid\!\sim_T$ in the G-framework, if $\mid\!\sim_T$ were monotonic

then T would be (monotonically) inconsistent.

On the other hand, we can construct a Γ_1-model which satisfies T, so $\mid\!\sim_T$ cannot be monotonic. To see this, take $S=\{s,t\}$ and name the nonempty subsets of S as follows: $Q_3=S=\{s,t\}$, $Q_2=\{t\}$, $Q_1=\{s\}$. Now let $R_0=\varnothing$ and let R be determined by the following triples: (s,Q_1,s), (s,Q_2,t) and (s,Q_3,t). It is easy to check that (S,R_0,R) is a full Γ_1-frame. Let us now define a Γ_1-model based on that frame (use Thm. 3.5) by putting: $[P]=\{s,t\}=Q_3$ and $[P']=\{t\}=Q_2$. It is straightforward to verify that in this model $s\Vdash T$.

7 - OTHER SYSTEMS.

The preceding sections served to show that justifiable conceptions of deductive validity will not always rule out non monotonicity.

Although a final verdict on the merits of the G-framework depends on further research, we can already see that it offers a number of advantages. It provides for manageable proof procedures in contrast with the non constructive fixed points used in many n. m. deductive systems. It supports reasoning which cannot be supported by some n. m. systems (see [R3] in connection with the example $T=\{A_4,A_5\}$ in §6). Unlike many existing approaches, it possesses an intuitively clean semantics, whose resulting concept of n. m. consequence is r. e. Admittedly, the present framework will offer a true computational advantage only if a satisfactory implementation of its logic can be devised. But this is one of the many questions about non monotonicity that this paper is not meant to address.

Another interesting question concerns the relationship between the G-framework and other n. m. systems. A comparative study of the respective formal properties might require the construction of a unifying formalism. The G-framework by itself cannot provide for one, because it is connected with a particular province of n. m. reasoning. This is one in which we postulate the existence of a «cognitive» or «causal» model that simulates the way things usually work out in a chosen part of reality. But there are other types of intuitions grounding formalizations of n. m. reasoning. For instance in systems that *reason about closure*, the underlying intuition is that the description of a setting is, in a certain sense, complete. Many such formalisms propose some version of the following scheme:

(47) if B does not follow in **S** from A then $\neg B$ follows from A,

where **S** is a logical system. The addition of scheme (47) to **S**, yields a new formalism Σ which is obviously n. m.. So if we maneuver ourselves into the position described by the antecedent of (47), the scheme sanctions a conclusion which, using our notation, we express as $A \vdash_{\Sigma} \neg B$.

The properties of the n.m. deduction operator $\vdash_{\Sigma}$ will depend upon many things: on the choice of **S**, on the kind of formulas A and B to which (47) applies and on the interpretion given to the concept

(i) «does not follow in **S**».

In fact, (47) comes in many varieties: see for instance, [C], [G4], [MD], [P], [R1], [R2]. But the point is that (47) captures a common pattern of reasoning that can perhaps be explored in some way comparable to the G-framework.

So let us sketch a possibile approach for analysing $\vdash_{\Sigma}$. Take **S** to be a system containing First Order Intuitionistic Logic and let **L** be the language of **S** enriched with two binary modal operators >, <; read

«A<B» as «there is evidence that A does not yield B»;
«A>B» as «on the basis of A it is relatively safe to assume B».

The idea here is to produce an analog of Def. (#) (see §1) and translate into the logic the metalogical concepts that appear in (47). In other words, we interpret > and < as being the logical referents of the metalogical concepts $\vdash_{\Sigma}$ and (i) respectively, and take as a rule of our logic

(a) $$\frac{A < B}{A > \neg B}$$ (47 revisited)

The problem concerns which axioms and rules should be put on these operators.

The meaning of «A<B» was given in very broad terms. For instance, nothing was said about the nature of the required evidence. Actually, we assert «A<B» simply when it is possible to know A without knowing B.

By the same token, the meaning of «A>B» is generic. In fact we take that "on the basis of A it is relatively safe to assume B" if, as far as we know, whenever A obtains, $\neg B$ doesn't.

Bearing in mind these ideas, consider extending any calculus in the G-lattice with (a) and

(b) $\neg(A<B) \longrightarrow (A \longrightarrow B)$

(c) $A \longrightarrow (A<\bot)$

(d) $$\frac{A \longrightarrow B}{\neg(A<B)}$$

(e) $$\frac{A \longrightarrow A'}{(A<B) \longrightarrow (A'<B)}$$

(f) $$\frac{B \longrightarrow B'}{(A<B') \longrightarrow (A<B)}$$

(g) $A>(B\vee\neg B)$

(h) $(A>\neg B) \longrightarrow A>(A<B)$.

Axiom (g) expresses the fact that > is an *implication by closure*, i.e. it is relatively safe to assume that if a statement doesn't hold, its negation does (of course, (g) is meaningful because we are using Intuitionistic Logic).

It is likely that conditions (a)-(h) are insufficient for our purpose. But whether there is a calculus φ in **L** such that the definition

$$A \vdash_{\Sigma} B \quad \text{iff} \quad \vdash_{\varphi} A>B$$

captures reasoning under (47), is a problem that serves to close this paper.

BIBLIOGRAPHY

[C] K. L. Clark, *Negation as failure*, **Logic and Data Bases** (H. Gallaire and J. Minker, eds.) Plenum N. Y., 1978.

[D] J. P. Delgrande, *A first order conditional logic for reasoning about prototypical properties*, Simon Fraser University, Dept. of Computing Science, Tech. Report, Burnaby, 1986.

[FS] G. Fischer Servi, *Logiche non monotone su base intuizionista*, **Atti del III convegno nazionale sulla programmazione logica**, Roma 1988.

[F] M. Fitting, *Intuitionistic Logic, Model theory and Forcing*, North Holland (1969).

[G1] D. Gabbay, *A general filtration method for modal logics*, **Journal of Philosophical Logic**, 1 (1972).

[G2] D. Gabbay, *Intuitionistic basis for Non Monotonic Logic*, **Proc. Conf. on automated deduction**, Springer Lecture Notes in Computer Science, 6, 1982.

[G3] D. Gabbay, *Theoretical foundations for non monotonic reasoning*, **Logics and Models of concurrent systems**, (K. R. Apt ed.) Springer, 1985.

[G4] D. Gabbay, *Modal provability foundations for negation by failure* 1, Doc. Report n. 86/4. Imperial College, London (1989).

[L] E. J. Lemmon, *An introduction to modal logic*, monograph n. 11, Oxford 1977.

[Lw] D. Lewis, *Counterfactuals*, Harvard Univ. Press 1973.

[MC1] J. McCarty, *Circumscription - a form of non monotonic reasoning*, Artificial Intelligence 13 (1986).

[MC2] J. McCarty, *Applications of circumscription to formalizing common sense knowledge*, Artificial Intelligence 28 (1986).

[MD] D. McDermott, *Non Monotonic Logic 2: Non Monotonic Theories*, JACM, 29, 1, 1982.

[MD-D] D. McDermott and J. Doyle, *Non Monotonic Logic* 1, in Artificial Intelligence, 13, 1980.

[M] D. Nute, *Non monotonic reasoning and conditionals*, University of Georgia, Advanced Computational Methods Center, research report 010002, Athens Ga.

[P] R. Parikh, *Logic of knowledge, games and Dynamic logic*, FST-TCS Springer L. N. C. S. 181 (1984).

[R1] R. Reiter, *On closed world data bases*, **Logic and Data Bases** (H. Gallaire and J. Minker eds.) Plenum N. Y. 1978.

[R2] R. Reiter, *A logic for default reasoning*, Artificial Intelligence, 13, (1980).

[R3] R. Reiter, *Non monotonic reasoning*, Annual Reviews of Computer Science (1987).

[S] Y. Shoham, *Reasoning about change*, M. I. T. Press 1988.

[St] R. Stalnaker, *A theory of conditionals*, **Studies in Logical Theories** (N. Rescher ed.) Blackwell Oxford 1968.