Direttore Responsabile: Ruggero Ferro
Iscrizione al Registro Stampa del Tribunale di Padova n. 1235 del 26.9.1990

Pubblicato con il contributo di:
Cassa di Risparmio di Padova e Rovigo

Stampa Veronese (Padova)
Laser Fotocomposizioni (Padova)

COUNTING THE MAXIMAL INTERMEDIATE CONSTRUCTIVE LOGICS

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Abstract.

A proof is given that the set of maximal intermediate propositional logics with the disjunction property and the set of maximal intermediate predicate logics with the disjuntion property and the explicit definability property have the power of continuum. To prove our results, we introduce various notions which might be interesting by themselves. In particular, we illustrate a method to generate wide sets of pairwise "constructively incompatible constructive logics". We use a notion of "semiconstructive" logic, and define wide sets of "constructive" logics by representing the latter as "limits" of decreasing sequences of "semiconstructive" ones. Also, we introduce some generalizations of the usual filtration techniques for propositional logics. For instance, "filtrations over rank formulas" are used to show that any two different logics belonging to a suitable uncountable set of "constructive" logics are "constructively incompatible".

0: Introduction.

Despite their interest and relevance in such areas as constructivism, questions concerning the characterization of maximal intermediate propositional logics with the disjunction property (or of maximal intermediate predicate logics with the disjunction property and the explicit definability property) have been scarcely investigated in the literature. Likewise, little interest has been devoted to the study of the strongly related problem of the "constructive incompatibility of constructive logics", where:

a) An intermediate propositional logic is constructive if it satisfies the disjunction property; an intermediate predicate logic is constructive if it satisfies the disjunction property and the explicit definability property. We do not pretend, however, that these properties qualify any notion of "constructivism" from a philosophical point of view.

b) Two intermediate propositional (respectively, predicate) constructive logics are constructively incompatible if there is no intermediate propositional (respectively, predicate) constructive logic containing both.

c) The existence of more than one maximal intermediate propositional (respectively, predicate) constructive logic entails the existence of constructively incompatible intermediate propositional (respectively, predicate) constructive logics.

d) Since Zorn's lemma entails that any intermediate constructive logic is contained in a maximal one, from the existence of constructively incompatible intermediate constructive logics one deduces the existence of more than one maximal intermediate constructive logic.

Only in the last ten years special attention has been devoted to such questions. In 1982 Kirk showed in [9] the constructive incompatibility of two intermediate propositional constructive logics, namely, the logic KP of Kreisel and Putnam [10] and the logic D_1 of Gabbay and De Jongh [6]. As discussed in [13], although Kirk's result is true, its proof in [9] is not fully correct. Anyway, this result was implicit in a paper that had appeared ten years before [1]. In that paper the author proved the existence of infinitely many intermediate propositional constructive logics, each of them being characterized by a superintuitionistic axiom-schema in one variable. Among these, let us mention the following logics: Scott's logic ST [10], which is characterized by the axiom-schema denoted F_9 in [1] and intuitionistically equivalent to $((\neg\neg A \rightarrow A) \rightarrow A \vee \neg A) \rightarrow \neg A \vee \neg\neg A$; and the logic AST, which is characterized by the axiom-schema denoted F_{11} in [1] and intuitionistically equivalent to $(((\neg\neg A \rightarrow A) \rightarrow A \vee \neg A) \rightarrow \neg A \vee \neg\neg A) \rightarrow \neg\neg A \vee (\neg\neg A \rightarrow A)$. AST stands for "anti-ST". Now, ST and AST are constructively incompatible. But the author, who probably was not interested in this kind of questions, did not mention this fact.

After Kirk's paper, questions concerning constructive incompatibility, or the number of the maximal constructive logics, have been considered with increasing interest. For instance, in [11] Maksimova raises the problem of determining the number of maximal intermediate propositional constructive logics. In [13] and in W. Poloni, *Strumenti metamatematici per lo studio di logiche costruttive, analisi della massimalità*, Tesi di Laurea, Dipartimento di Scienze della Informazione, Università degli Studi di Milano, 1988, it is shown that there are infinitely many pairwise

constructively incompatible intermediate propositional constructive logics. It follows that there are infinitely many maximal intermediate propositional constructive logics. Also, in [17] the paper [12] of Maksimova is quoted where the existence of infinitely many intermediate propositional constructive logics is proved. We do not know whether the proof technique in [12] is the one used in [13] and in Poloni's thesis, which indirectly establishes the existence of maximal constructive logics by first proving the existence of a suitable set of pairwise costructively incompatible construcive logics, and then applying the axiom of choice. A different method is used in two unpublished papers by the authors, namely, P. Miglioli, *Nota su logiche proposizionali costruttive massimali* (manuscript), 1989, and M.Ferrari, *Logiche intermedie costruttive massimali*, Tesi di Laurea, Dipartimento di Scienze della Informazione, Università degli Studi di Milano, 1990. In these works a direct semantical characterization, in terms of the usual Kripke semantics, is provided for a countably infinite family of maximal intermediate propositional constructive logics including the well known Medvedev's logic. This proves the existence of infinitely many maximal intermediate propositional constructive logics *without using the axiom of choice*.

In the present paper we evaluate the cardinality of the set of maximal intermediate propositional, and also of predicate constructive logics. As we shall see, this cardinality is $2^{\aleph_0}$ for both sets. We are not able to directly characterize all maximal logics, but only to single out a set of $2^{\aleph_0}$ intermediate constructive logics which are pairwise constructively incompatible. As in [13] and in Poloni's thesis, our proof will turn out to be highly non effective.

Our paper is organized as follows:

After the preliminary definitions, we introduce the notion of an intermediate propositional logic L being *semiconstructive* in another logic L', in the sense that a disjunction $A \vee B$ is in L only if either A or B is in L'. A similar notion is given in §9 for predicate logics. One of the key ideas of the paper will be to define uncountable sets of intermediate constructive logics by characterizing the latter as "limits of decreasing sequences of semiconstructive logics" (see §2). To obtain uncountably many sequences of semiconstructive logics, we define a binary tree T of logics, where the two logics L_1 and L_2 immediately following a logic L are semiconstructive in L. In this way, the $2^{\aleph_0}$ paths of the tree give rise to $2^{\aleph_0}$ constructive logics. The definition of the tree T is given in §4, while §3 introduces some axiom-schemes (together with their semantical characterizations in terms of Kripke frames) to be combined in §4 in order to build up the logics of T. These logics will be characterized both syntactically (in terms of superintuitionistic axiom-schemes) and semantically (as the logics generated by suitable classes of posets). Soundeness and completeness theorems will be proved, in §3 and in §4, without using filtration techniques. The semantical characterization of the logics of T will give rise to a semantical characterization of the constructive logics associated with the paths of the tree (see §4). At this point, the problem is to show that any two constructive logics $\overline{L}^T_1$ and $\overline{L}^T_2$ associated with two different paths of T are constructively incompatible. Taking into account the structure of the two classes of posets $\mathfrak{F}_1$ and $\mathfrak{F}_2$ generating $\overline{L}^T_1$ and $\overline{L}^T_2$, it is indeed possible to prove that the logic generated by $\mathfrak{F}_1 \cap \mathfrak{F}_2$ (i.e., the "intersection of the semantics" of $\overline{L}^T_1$ and $\overline{L}^T_2$) cannot be contained in any intermediate propositional constructive logic. So, to state the constructive incompatibility of $\overline{L}^T_1$ and $\overline{L}^T_2$ it would be sufficient to show that the smallest propositional intermediate logic containing both $\overline{L}^T_1$ and $\overline{L}^T_2$ is *exactly* the one generated by $\mathfrak{F}_1 \cap \mathfrak{F}_2$.

Unfortunately, this cannot be proved using the methods of §1-4. However, we will prove a related fact, which is sufficient for our purposes. With every path of the tree T, we will associate a *second* logic $\overline{\overline{L}}^T$ such that $\overline{\overline{L}}^T \subseteq \overline{L}^T$, $\overline{L}^T$ being called the first logic. We will show that, for any two paths of T, if $\overline{\overline{L}}^T_1$ and $\overline{\overline{L}}^T_2$ are the second logics associated with these paths, then the smallest logic containing $\overline{\overline{L}}^T_1$ and $\overline{\overline{L}}^T_2$ cannot be included in any intermediate propositional constructive logic.

The second logics associated with the paths of T are defined in §7, both in syntactical terms (by means of axiom-schemes related to the logics in the paths) and in semantical terms (by classes of posets related to the semantical characterizations of the logics in the paths). From our proofs of the completeness theorems we deduce the impossibility of extending any two such logics into an intermediate constructive logic (Corollary 3, §8). Thus, from the proof given in §8 that the second logics associated with the paths of T are included in the corresponding first logics, we immediately deduce that the first logics associated with any two different paths of T are constructively incompatible; hence, by Zorn's lemma, there are $2^{\aleph_0}$ distinct maximal intermediate propositional constructive logics (Theorem 7, §8).

To prove the completeness theorems for the second logics associated with the paths of T, we develop a filtration technique which involves two aspects:
a) the definition of the models obtained by filtration from the canonical models;
b) the definition of particular formulas generating the filtrations.
For a) we describe in §5 a variant of the filtration method used by Gabbay and De Jongh in [6]. For b), in order to prove the completeness of a given logic L with respect to some class $\mathfrak{F}$ of posets using our filtration technique, for every $A \notin L$ we build up a model $\underline{K}_A$ in such a way that A turns out to be invalid in $\underline{K}_A$, and $\underline{K}_A$ is built on a poset of $\mathfrak{F}$. Now, the first requirement is automatically satisfied if A itself is taken as the filtration formula. But, to satisfy the second requirement, it may be appropriate to choose a filtration formula Θ with some desirable properties and *containing* A *as a subformula*. To define a formula Θ suitable to our completeness proofs, in §6 we will introduce the notions of v-rank and of formula extensively completed up to the v-rank $\bar{r}$.

Finally, in §9 we prove that the cardinality of the set of maximal intermediate predicate constructive logics equals $2^{\aleph_0}$. The proof heavily depends on the previous proof for the propositional case. A tree T^* of predicate logics is put in one to one correspondence with the propositional logics of T. The elements of T^* are semantically presented as the logics generated by classes of posets generating the corresponding propositional logics of T. It turns out that the two logics L^*_1 and L^*_2 of T^* immediately following a logic L^* are "weakly" semiconstructive in L^*. As in the propositional case, we then obtain that the predicate logic associated with any path of T^* is constructive. On the other hand, the logic associated with any path of T^* contains a propositional part coinciding with the first logic associated with the corresponding path of T. Therefore, the predicate logics associated with two different paths of T^* are constructively incompatible. Hence the cardinality result follows for the maximal intermediate predicate constructive logics.

The cardinality results presented in this paper have been previously discussed by the authors in the above quoted papers by P. Miglioli, *Nota su logiche proposizionali costruttive massimali* and by M. Ferrari, *Logiche intermedie costruttive massimali*. In the first work we give a sketch of the proofs both for the propositional and for the predicate case, while in the second paper a detailed proof of the propositional result is provided. The latter result has been independently proved also by Chagrov and by Galanter, as the authors have learned from [3], where Chagrov's paper [2] and Galanter's paper [7] are quoted.

1: Preliminary definitions.

The set of the propositional well formed formulas (wff) is defined as usual, using the connectives $\neg, \vee, \wedge, \rightarrow$. If A is a wff, $\mathcal{V}_A$ will be the set of propositional variables of A. We say that a wff A is *negatively saturated* iff all its variables are within the scope of $\neg$.

INT (respectively, CL) will denote both an arbitrary calculus for intuitionistic propositional logic (respectively, for classical propositional logic) and the set of intuitionistically valid wff's (respectively, the set of classically valid wff's). An *intermediate propositional logic* will be any consistent set L of wff's containing INT and closed under detachment and substitution. *Throughout this paper, the term "logic" will mean an intermediate propositional logic.* As is well known, for every logic L, we have $INT \subseteq L \subseteq CL$. Following tradition, we will define logics as sets of theorems of deductive systems, following tradition. If S is a set of axiom-schemes, then the deductive system (logic) obtained by adding to INT the axiom-schemes of S will be denoted INT+S. If L_1 and L_2 are logics, then L_1+L_2 will be the smallest logic containing both L_1 and L_2.

A logic L will be said to satisfy the *disjunction property* iff $A \vee B \in L$ implies $A \in L$ or $B \in L$ (for every A and B); a logic satisfying the disjunction property will be referred to as a *constructive logic*. The set of all logics, as well as the set of all constructive logics has the power of continuum [8,20]. A constructive logic L will be said to be *maximal* iff there is no constructive logic L' such that $L \subseteq L'$ and $L \neq L'$.

We assume the reader to be familiar with the notion of *Kripke model* $K = \langle P, \leq, \Vdash \rangle$, where $P = \langle P, \leq \rangle$ is a *poset* and $\Vdash$ is the *forcing relation*, defined between elements of P and atomic formulas (propositional variables) and extended in the usual way to arbitrary wff's; we say that $\underline{K}$ is *built on the poset* $\underline{P}$, or that $\underline{P}$ is the *underlying poset of* $\underline{K}$. A poset $\underline{P}$ is said to be *principal* iff $\underline{P}$ has a least element r (called the *root* of the poset). We will only consider principal posets and Kripke models built on them. To avoid unnecessary repetitions, we will consider "poset" as synonimous with "principal poset". Sometimes we will

explicitly indicate the least element r of the poset $\underline{P}$ or of the Kripke model $\underline{K}$, by writing respectively $\underline{P} = \langle P, \leq, r \rangle$ and $\underline{K} = \langle P, \leq, r, \Vdash \rangle$. For any element α of a poset $\underline{P} = \langle P, \leq \rangle$, P_α, or, if necessary, also $\langle P_\alpha, \leq \rangle$ denotes the principal subordering generated by α in $\underline{P}$, i.e., the restriction of $\langle P, \leq \rangle$ to the set $P_\alpha = \{\beta / \beta \in P \text{ and } \alpha \leq \beta\}$. The poset $\underline{P}_\alpha$ will be called the *cone of* α *in* P.
An element f of $\underline{P} = \langle P, \leq \rangle$ will be called *final* (in $\underline{P}$) iff, for every $f' \in P$, $f \leq f'$ implies $f = f'$. For any $\underline{P} = \langle P, \leq \rangle$ and any $\alpha \in P$, $Fin(\alpha)_P$ will denote the set $\{f / \alpha \leq f$ and f is a final element in $\underline{P}\}$. When ambiguities cannot arise, we will write $Fin(\alpha)$ instead of $Fin(\alpha)_{\underline{P}}$. We remark that $Fin(\alpha)$ is nonempty if $\underline{P}$ is finite. A nonfinal element α of $\underline{P} = \langle P, \leq \rangle$ is *prefinal* (in $\underline{P}$) iff all immediate successors of α in $\underline{P}$ are final.

If $\mathcal{F}$ is a nonempty class of posets, $\mathcal{K}(\mathcal{F})$ will denote the set $\{\underline{K} = \langle P, \leq, r, \Vdash \rangle / \langle P, \leq, r \rangle \in \mathcal{F}\}$, and $\mathcal{L}(\mathcal{F})$ will denote the set $\{A /$ for every $\underline{K} = \langle P, \leq, r, \Vdash \rangle \in \mathcal{K}(\mathcal{F})$, $r \Vdash A\}$, i.e., the set of all wff's which are valid in every Kripke model of $\mathcal{K}(\mathcal{F})$. As is well known, for every nonempty $\mathcal{F}$, $\mathcal{L}(\mathcal{F})$ is a logic [14].

If Γ is any set of wff's and L is any logic, by $\Gamma \vdash_L A$ we will mean that there are $B_1, \ldots, B_n$ such that $\{B_1, \ldots, B_n\} \subseteq \Gamma$ and $B_1 \wedge \ldots \wedge B_n \rightarrow A \in L$. By $\Gamma \nvdash_L A$ we will mean that $\Gamma \vdash_L A$ does not hold. A *saturated set* Γ will be any consistent set Γ of formulas closed under INT-provability (i.e., $\Gamma \vdash_{INT} A$ implies $A \in \Gamma$) and under the disjunction property (i.e., $A \vee B \in \Gamma$ implies $A \in \Gamma$ or $B \in \Gamma$). If L is a logic, $L \subseteq \Gamma$ and Γ is saturated, then Γ is closed under L-provability: in this case, we will say that Γ is *L-saturated*. If Γ is a saturated set, by the *canonical model generated by* Γ, in symbols, $\mathcal{C}(\Gamma)$, we mean the Kripke model $\underline{K} = \langle P, \leq, r, \Vdash \rangle$ satisfying the following properties:

1) $P = \{\Gamma' / \Gamma \subseteq \Gamma'$ and Γ' is saturated$\}$;
2) for any two $\Gamma', \Gamma'' \in P$, $\Gamma' \leq \Gamma''$ iff $\Gamma' \subseteq \Gamma''$;
3) $r = \Gamma$;
4) for any $\Gamma' \in P$ and any propositional variable p, $\Gamma' \Vdash p$ iff $p \in \Gamma'$.

If Γ is L-saturated, then all elements of $\mathcal{C}(\Gamma)$ include L; in this case, we say that $\mathcal{C}(\Gamma)$ is the *L-canonical model generated by* Γ, and we write $\mathcal{C}_L(\Gamma)$ instead of $\mathcal{C}(\Gamma)$. The following facts are well known [19]: if $\Gamma^* \nvdash_L A$, then there is Γ such that $\Gamma^* \subseteq \Gamma$, Γ is L-saturated and $A \notin \Gamma$; if Γ is L-saturated, Γ' is an element of $\mathcal{C}_L(\Gamma)$ and B is a wff, then $\Gamma' \Vdash B$ in $\mathcal{C}_L(\Gamma)$ iff $B \in \Gamma'$.

For each $n \geq 2$, let (FIN_n) be the following axiom-schema:

$$(FIN_2): \quad \neg A_1 \vee (\neg A_1 \to \neg A_2) \vee (\neg A_1 \to \neg\neg A_2);$$

$$(FIN_n): \quad \neg A_1 \vee (\neg A_1 \to \neg A_2) \vee (\neg A_1 \wedge \neg A_2 \to \neg A_3) \vee \ldots$$
$$\ldots \vee (\neg A_1 \wedge \ldots \wedge \neg A_{n-1} \to \neg A_n) \vee (\neg A_1 \wedge \ldots \wedge \neg A_{n-1} \to \neg\neg A_n).$$

Let $FIN_n = INT + \{(FIN_n)\}$, and, for each $n \geq 2$, let $\mathcal{F}_{FIN_n}$ be the class of posets $\langle P, \leq \rangle$ such that, for every $\alpha \in P$, α is followed by at least one final element and $|Fin(\alpha)| \leq n$. Here, as usual, $|X|$ denotes the cardinality of X. It is easy to prove that, for every $n \geq 2$, $FIN_n = \mathfrak{L}(\mathcal{F}_{FIN_n})$. Moreover, for every logic $L' \supseteq FIN_n$, and every L'-saturated set Γ, $\mathfrak{C}_{L'}(\Gamma)$ is built on a poset belonging to $\mathcal{F}_{FIN_n}$.
For an alternative axiomatization of the logics FIN_n, see, e.g., [5].

For each $i \geq 1$, let DE_i be the axiom-schema defined inductively as follows:

$$(DE_1): \quad A_1 \vee \neg A_1;$$

$$(DE_{i+1}): \quad A_{i+1} \vee (A_{i+1} \to (DE_i)).$$

Also, let $DE_i = INT + \{(DE_i)\}$ and, for every $i \geq 1$, $\mathcal{F}_{DE_i}$ be the class of posets $\langle P, \leq \rangle$ such that, for every $\alpha \in P$, α has depth $\leq i$ in $\underline{P}$ (the final states have depth $= 1$). It is easy to prove that, for every $i \geq 1$, $DE_i = \mathfrak{L}(\mathcal{F}_{DE_i})$.
Moreover, for every logic $L' \supseteq DE_i$, and every L'-saturated set Γ, $\mathfrak{C}_{L'}(\Gamma)$ is built on a poset belonging to $\mathcal{F}_{DE_i}$.
An alternative axiomatization of the logics DE_i was given by Nagata in [15]. See also [5].

2: Semiconstructive logics.

Definition 1: Let L and L' be two logics; L *is semiconstructive in* L' iff whenever $A \vee B \in L$ then either $A \in L'$ or $B \in L'$.

If L'=CL, we say that L is a semiconstructive logic. The following fact is immediate:

Proposition 1: Let L and L' be two logics such that L is semiconstructive in L': then $L' \supseteq L$.□

Proposition 2: Let $\{L_i\}_{i \geq 1}$ be a sequence of logics such that, for every $i \geq 1$, L_{i+1} is semiconstructive in L_i; then $\overline{L} = \bigcap_{i \geq 1} L_i$ is a constructive logic.
Proof: It is easy to prove that $\overline{L}$ is a logic. Let $A \vee B \in \overline{L}$, while $A \notin \overline{L}$ and $B \notin \overline{L}$. Then there are $j, k \geq 1$ such that $A \notin L_j$ and $B \notin L_k$. Let $h = \max(j,k)$. Since, by Proposition 1, $L_i \supseteq L_{i+1}$, we have $A \notin L_h$ and $B \notin L_h$. From $A \vee B \in \overline{L}$, we get $A \vee B \in L_{h+1}$. But this contradicts the hypothesis that L_{h+1} is semiconstructive in L_h.□

Definition 2: Let $\mathcal{F}$ and $\mathcal{F}'$ be two classes of posets. We say that $\mathcal{F}$ *links* $\mathcal{F}'$ iff the following condition is satisfied: for any two posets $\underline{P}_1$ and $\underline{P}_2$ of $\mathcal{F}'$, there is a poset $\underline{P} \in \mathcal{F}$ together with elements α and β of $\underline{P}$ such that $\underline{P}_\alpha$ is isomorphic to $\underline{P}_1$, $\underline{P}_\beta$ is isomorphic to $\underline{P}_2$, and $\underline{P}_\alpha$ and $\underline{P}_\beta$ are disjoint.

The next proposition gives a sufficient condition for a logic L to be semiconstructive in a logic L'. Its proof is routine and is left to the reader.

Proposition 3: Let L and L' be two logics, and let $\mathcal{F}_L$ and $\mathcal{F}_{L'}$ be two classes of posets such that $L = \mathfrak{L}(\mathcal{F}_L)$ and $L' = \mathfrak{L}(\mathcal{F}_{L'})$. If $\mathcal{F}_L$ links $\mathcal{F}_{L'}$, then L is semiconstructive in L'.□

Let $GS = INT + \{(FIN_2), (DE_2)\}$, and let $\mathcal{F}_{GS}$ be the class of posets $\langle P, \leq \rangle$ with at most two final elements and with depth ≤ 2.
From the properties of the logics DE_2 and FIN_2 described in §1, we immediately obtain:

Theorem 1: GS coincides with $\mathfrak{L}(\mathcal{F}_{GS})$. Moreover, for every logic $L' \supseteq GS$ and every L'-saturated set Γ, $\mathfrak{C}_{L'}(\Gamma)$ is built on a poset belonging to $\mathcal{F}_{GS}$.□

' As proved in Poloni's thesis quoted in the Introduction, GS is the largest semiconstructive logic.

3: The axiom-schemes $(CONE(L)_m)$ and $(DIFFIN_{n,k})$.

Let us assume that $L=INT+\{(A_L)\}$ is a logic satisfying the following condition:

(c) There is a class of posets $\mathfrak{F}_L$ such that $L=\mathfrak{L}(\mathfrak{F}_L)$, and:

(c1) for every $\underline{P}\in\mathfrak{F}_L$, the root of $\underline{P}$ has a finite depth and $\underline{P}$ has at most m final elements (where m is a fixed natural number not depending on $\underline{P}$);

(c2) for every logic $L^*\supseteq L$ and for every L^*-saturated set Γ, the underlying poset of the canonical model $\mathfrak{C}_{L^*}(\Gamma)$ belongs to $\mathfrak{F}_L$.

For any arbitrary but fixed L satisfying condition (c), let $(CONE(L)_m)$ be the following axiom-schema:

$$((A_L)'\to\neg B_1\vee\ldots\vee\neg B_m)\to\neg B_1\vee\ldots\vee\neg B_m,$$

where $(A_L)'$ is any instance of (A_L). In order to provide the semantics for this axiom-schema, for each $m\geq 2$ we define the class $\mathfrak{F}_{CONE(L)m}$ of all posets $\underline{P}$ with finite depth such that: if $\alpha\in P$, and $f_1,\ldots,f_n$ ($2\leq n\leq m$) are final elements of $\underline{P}$ with $Fin(\alpha)\supseteq\{f_1,\ldots,f_n\}$, then there exists $\beta\in P$ such that $\alpha\leq\beta$, $Fin(\beta)\supseteq\{f_1,\ldots,f_n\}$, and $\underline{P}_\beta\in\mathfrak{F}_L$. Thus, e.g., if $|Fin(\alpha)|\geq m$, then the paths connecting α in $\underline{P}$ with the final states, must all reach states whose *cones* in $\underline{P}$ belong to $\mathfrak{F}_L$ and have m final states.

The class $\mathfrak{F}_{CONE(L)m}$ depends on the choice of the class $\mathfrak{F}_L$ satisfying Condition (c); such a class need not be uniquely determined. In the following, there will be no possibility of confusion about the class $\mathfrak{F}_L$ on which $\mathfrak{F}_{CONE(L)m}$ depends.

Proposition 4: Let $m\geq 2$, and let L be a logic satisfying Condition (c). Then every instance of the schema $(CONE(L)_m)$ belongs to $\mathfrak{L}(\mathfrak{F}_{CONE(L)m})$.

Proof: Assume the contrary. Then for some $m\geq 2$, and some logic L satisfying Condition (c), there is an instance $(CONE(L)_m)'$ of the schema $(CONE(L)_m)$ and

a Kripke mode $\underline{K}=\langle P,\leq,r,\Vdash\rangle\in\mathfrak{K}(\mathfrak{F}_{CONE(L)m})$ such that $r\nVdash(CONE(L)_m)'$. Hence, there is an element $\alpha\in P$ such that:

(i) $\alpha\Vdash(A_L)'\to\neg B_1\vee\ldots\vee\neg B_m$;

(ii) $\alpha\nVdash\neg B_1\vee\ldots\vee\neg B_m$, where $(A_L)',\neg B_1,\ldots,\neg B_m$, are the wff's used in $(CONE(L)_m)'$.

From (ii) we obtain m not necessarily distinct final elements $f_1,\ldots,f_m$ of $\underline{P}=\langle P,\leq\rangle$ such that: $\{f_1,\ldots,f_m\}\subseteq Fin(\alpha)$ and $f_i\Vdash B_i$ for every $1\leq i\leq m$. By the definition of $\mathfrak{F}_{CONE(L)m}$, it follows that there exists $\beta\in P$ such that $\alpha\leq\beta$, $Fin(\beta)\supseteq\{f_1,\ldots,f_m\}$, and $\underline{P}_\beta\in\mathfrak{F}_L$. From (c) we have that $\beta\Vdash(A_L)'$. Hence, $\beta\Vdash\neg B_1\vee\ldots\vee\neg B_m$ in $\underline{K}$. This contradicts the fact that $Fin(\beta)\supseteq\{f_1,\ldots,f_m\}$.□

Proposition 5: Let $m\geq 2$ and let L be a logic satisfying Condition (c). Let L' be a logic such that $(CONE(L)_m)\in L'$ and, for every L'-saturated set Γ, the canonical model $\mathfrak{C}_{L'}(\Gamma)$ has a finite depth and a finite number of final elements (e.g., consider a L' containing, for some n and i, both FIN_n and DE_i). Then the underlying poset $\underline{P}$ of $\mathfrak{C}_{L'}(\Gamma)$ belongs to $\mathfrak{F}_{CONE(L)m}$.

Proof: Otherwise, there are distinct elements $\Gamma',\Gamma^f_1,\ldots,\Gamma^f_n$ of $\mathfrak{C}_{L'}(\Gamma)$, with $2\leq n\leq m$, such that $Fin(\Gamma')\supseteq\{\Gamma^f_1,\ldots,\Gamma^f_n\}$, but, for every Γ'' of $\mathfrak{C}_{L'}(\Gamma)$ such that $\Gamma'\leq\Gamma''$ and $Fin(\Gamma'')\supseteq\{\Gamma^f_1,\ldots,\Gamma^f_n\}$, we have $\underline{P}_{\Gamma''}\notin\mathfrak{F}_L$. Since, by hypothesis, $\mathfrak{C}_{L'}(\Gamma)$ has finite depth, we may assume that the following additional condition holds: for every element Γ'' of $\mathfrak{C}_{L'}(\Gamma)$ such that $\Gamma'\leq\Gamma''$ and $\Gamma'\neq\Gamma''$, $Fin(\Gamma'')\not\supseteq\{\Gamma^f_1,\ldots,\Gamma^f_n\}$. Since the elements of $\mathfrak{C}_{L'}(\Gamma)$ are L'-saturated sets, and the number of final elements of $\mathfrak{C}_{L'}(\Gamma)$ is finite, we can find wff's $A_1,\ldots,A_n$ such that, for every i with $1\leq i\leq n$, the following conditions hold:

$\Gamma^f_i\Vdash A_i$ and, for every j with $1\leq j\leq n$ and $j\neq i$, $\Gamma^f_j\Vdash\neg A_i$;

for every $\Gamma^f\in Fin(\Gamma')$ such that $\Gamma^f\notin\{\Gamma^f_1,\ldots,\Gamma^f_n\}$, $\Gamma^f\Vdash\neg A_i$.

Let $(A_L)'$ be an instance of (A_L) such that $\Gamma'\nVdash(A_L)$; such an instance must exist by Condition (c2) of (c), since $\underline{P}_{\Gamma'}\notin\mathfrak{F}_L$. One easily proves that $\Gamma'\Vdash(A_L)'\to\neg A_1\vee\ldots\vee\neg A_n$ in $\mathfrak{C}_{L'}(\Gamma)$. As a matter of fact, $\Gamma'\nVdash(A_L)'$ in $\mathfrak{C}_{L'}(\Gamma)$ and, for every Γ'' such that $\Gamma'\leq\Gamma''$ and $\Gamma'\neq\Gamma''$, there is a final element Γ^f_i, with $1\leq i\leq n$, such that $\Gamma^f_i\notin Fin(\Gamma'')$. Hence $\Gamma''\Vdash\neg A_i$, by our

choice of $A_1,\dots,A_n$. Since $\mathfrak{C}_{L'}(\Gamma)$ is a canonical model, every instance of the axiom-schema $(CONE(L)_m)$ is forced in Γ'.
Since $n\leq m$, $\Gamma'\Vdash\neg A_1\vee\dots\vee\neg A_n$. But this is impossible, since, for every $1\leq i\leq n$, $\Gamma^f_i\Vdash A_i$ and $\Gamma'\leq\Gamma^f_i$.□

For every $h\geq 2$, we define the following axiom-schema:

$$(DIFFIN_h):\ (\neg A\rightarrow\neg B_1\vee\dots\vee\neg B_h)\rightarrow\neg(\neg A\wedge B_1\wedge\neg B_2\wedge\dots\wedge\neg B_h)\vee$$
$$\vee\neg(\neg A\wedge\neg B_1\wedge B_2\wedge\neg B_3\wedge\dots\wedge\neg B_h)\vee\dots$$
$$\dots\vee\neg(\neg A\wedge\neg B_1\wedge\dots\wedge\neg B_{h-1}\wedge B_h).$$

For each $h\geq 2$, let $\mathcal{F}_{DIFFIN_h}$ be the class of posets $\underline{P}=\langle P,\leq,r\rangle$ such that every element of $\underline{P}$ is followed by at least one final element and, for every $\alpha\in P$ and set $\{f_1,\dots,f_h\}$ of *different* final elements of $\underline{P}$ such that $Fin(\alpha)\supseteq\{f_1,\dots,f_h\}$, there exists a $\beta\in P$ such that $\alpha\leq\beta$ and $Fin(\beta)=\{f_1,\dots,f_h\}$.

Proposition 6: For each $h\geq 2$, every instance of the axiom schema $(DIFFIN_h)$ belongs to $\mathcal{L}(\mathcal{F}_{DIFFIN_h})$.
Proof: Let us assume that there is an istance $(DIFFIN_h)'$ of $(DIFFIN_h)$, with the wff's $A,B_1,\dots,B_h$, and a Kripke model $\underline{K}=\langle P,\leq,r,\Vdash\rangle\in\mathcal{K}(\mathcal{F}_{DIFFIN_h})$ such that $r\nVdash(DIFFIN_h)'$. Hence, there is $\alpha\in P$ such that $r\leq\alpha$, $\alpha\Vdash\neg A\rightarrow\neg B_1\vee\dots\vee\neg B_h$,
$\alpha\nVdash\neg(\neg A\wedge B_1\wedge\neg B_2\wedge\dots\wedge\neg B_h),\dots,\alpha\nVdash\neg(\neg A\wedge\neg B_1\wedge\neg B_2\wedge\dots\wedge\neg B_{h1}\wedge B_h)$.
Therefore, there are h final elements $f_1,\dots,f_h$ of $\underline{K}$ such that $Fin(\alpha)\supseteq\{f_1,\dots,f_h\}$ and, for each $i=1,\dots,h$, $f_i\Vdash\neg A\wedge\neg B_1\wedge\dots\wedge B_i\wedge\neg B_{i+1}\wedge\dots\wedge\neg B_h$. Of course, $f_1,\dots,f_h$ are distinct elements of $\underline{K}$. By the definition of $\mathcal{F}_{DIFFIN_h}$, it follows that there is $\beta\in P$ such that $\alpha\leq\beta$ and $Fin(\beta)=\{f_1,\dots,f_h\}$. Obviously, $\beta\Vdash\neg A$ in $\underline{K}$, and hence $\beta\Vdash\neg B_1\vee\dots\vee\neg B_h$. This is a contradiction, since $Fin(\beta)=\{f_1,\dots,f_h\}$ and for every $1\leq i\leq h$ we have $f_i\Vdash B_i$.□

Proposition 7: Let L be a logic such that $(DIFFIN_h)\in L$ $(h\geq 2)$. Assume that (owing to the presence in L of some other axiom-schemes), for every L-saturated set Γ, the canonical model $\mathfrak{C}_L(\Gamma)$ has a finite depth and a finite number of final elements (e.g., consider a L' containing, for some n and i, both FIN_n and DE_i). Then the underlying poset of $\mathfrak{C}_L(\Gamma)$ belongs to $\mathcal{F}_{DIFFIN_h}$.
Proof: Assume the contrary. Then there are distinct elements $\Gamma',\Gamma^f_1,\dots,\Gamma^f_h$ of $\mathfrak{C}_L(\Gamma)$ such that $Fin(\Gamma')\supseteq\{\Gamma^f_1,\dots,\Gamma^f_h\}$ and, for every element Γ'' of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma'\leq\Gamma''$, $Fin(\Gamma'')\neq\{\Gamma^f_1,\dots,\Gamma^f_h\}$. Since, by hypothesis, $\mathfrak{C}_L(\Gamma)$ has finite depth, we may assume that the following additional condition holds: for every element Γ'' of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma'\leq\Gamma''$ and $\Gamma'\neq\Gamma''$, $Fin(\Gamma'')\not\supseteq\{\Gamma^f_1,\dots,\Gamma^f_h\}$. Since the elements of $\mathfrak{C}_L(\Gamma)$ are L-saturated sets and, by hypothesis, $\mathfrak{C}_L(\Gamma)$ has a finite number of final elements, we can find wff's $A, B_1,\dots, B_h$ such that the following facts hold:
for every i with $1\leq i\leq h$, $\Gamma^f_i\Vdash\neg A$ and $\Gamma^f_i\Vdash B_i$, but, for every j with $1\leq j\leq h$ and $j\neq i$, $\Gamma^f_j\Vdash\neg B_i$;
for every $\Gamma^f\in Fin(\Gamma')$ such that $\Gamma^f\notin\{\Gamma^f_1,\dots,\Gamma^f_h\}$, $\Gamma^f\Vdash A$.
Now, $\Gamma'\nVdash\neg A$. As a matter of fact, by hypothesis, there is at least a final element Γ^f of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma^f\in Fin(\Gamma')$ and $\Gamma^f\notin\{\Gamma^f_1,\dots,\Gamma^f_h\}$. Then, for every Γ'' such that $\Gamma'\leq\Gamma''$ and $\Gamma''\Vdash\neg A$, we have that $\Gamma'\neq\Gamma''$, and hence there is a final element $\Gamma^f_i\in\{\Gamma^f_1,\dots,\Gamma^f_h\}$ such that $\Gamma^f_i\notin Fin(\Gamma'')\subseteq\{\Gamma^f_1,\dots,\Gamma^f_h\}$; thus $\Gamma''\Vdash\neg B_i$. This implies that $\Gamma'\Vdash\neg A\rightarrow\neg B_1\vee\dots\vee\neg B_h$ in $\mathfrak{C}_L(\Gamma)$. Since Γ' is an L-saturated set, Γ' forces every instance of $(DIFFIN_h)$, and we obtain
$\Gamma'\Vdash\neg(\neg A\wedge B_1\wedge\neg B_2\wedge\dots\wedge\neg B_h)\vee\neg(\neg A\wedge\neg B_1\wedge B_2\wedge\neg B_3\wedge\dots\wedge\neg B_h)\vee\dots$
$\dots\vee\neg(\neg A\wedge\neg B_1\wedge\dots\wedge\neg B_{h-1}\wedge B_h)$ in $\mathfrak{C}_L(\Gamma)$.
This contradicts the assumption that $Fin(\Gamma')\supseteq\{\Gamma^f_1,\dots,\Gamma^f_h\}$.□

For each $m\geq 3$ and $2\leq n<m$, we set:

$$(DIFFIN_{n,m}):\quad (DIFFIN_n)\wedge(DIFFIN_{n+1})\wedge\dots\wedge(DIFFIN_m).$$

For each $2\leq n<m$, let $\mathcal{F}_{DIFFIN_{n,m}}$ be the class of posets $\underline{P}=\langle P,\leq,r\rangle$ satisfying the following properties:
(i) every element of $\underline{P}$ is followed by at least one final element;

(ii) for every $\alpha \in P$, h with $n \leq h \leq m$ and set $\{f_1,...,f_h\}$ of *different* final elements of $\underline{P}$ such that $Fin(\alpha) \supseteq \{f_1,...,f_h\}$, there is $\beta \in P$ such that $\alpha \leq \beta$ and $Fin(\beta) = \{f_1,...,f_h\}$.

From Proposition 6 it follows that:

Proposition 8: Let $m \geq 3$ and $2 \leq n < m$. Then every instance of the axiom schema $(DIFFIN_{n,m})$ belongs to $\mathfrak{L}(\mathfrak{F}_{DIFFIN_{n,m}})$.□

From Proposition 7 we get:

Proposition 9: Let $m \geq 3$ and $2 \leq n < m$; let L be a logic such that $(DIFFIN_{n,m}) \in L$ and (owing to the presence in L of some other axiom-schemes), for every L-saturated set Γ, the canonical model $\mathfrak{C}_L(\Gamma)$ has a finite depth and a finite number of final elements (e.g., consider a L' containing, for some n and i, both FIN_n and DE_i). Then the underlying poset of $\mathfrak{C}_L(\Gamma)$ belongs to $\mathfrak{F}_{DIFFIN_{n,m}}$.□

4: The logics $RI(L_h)$ and $LE(L_h)$, the tree T and the logics $\overline{L}^T$.

Definition 3: Let L be a logic characterized by a single axiom-schema (which may be the conjunction of a finite number of superintuitionistic axiom schemas). Let $\mathfrak{F}_L$ be a class of posets such that $L = \mathfrak{L}(\mathfrak{F}_L)$. We say that L *is* an (n,k) *logic* (for $\mathfrak{F}_L$) iff the following conditions hold:

(i) for every poset $\underline{P} = \langle P, \leq, r \rangle \in \mathfrak{F}_L$, $|Fin(r)| \leq n$, and the depth of r in $\underline{P}$ is $\leq k$;

(ii) for every logic L^* such that $L^* \supseteq L$ and for every L^*-saturated set Γ, the underlying poset of $\mathfrak{C}_{L^*}(\Gamma)$ belongs to $\mathfrak{F}_L$.

For instance, GS is a $(2,2)$ logic; in general, if Condition (c) of §3 holds for a logic L, and $L \supseteq FIN_n \cup DE_k$, then L is an (n,k) logic.

Definition 4: Let $n \geq 2$, $k \geq 2$, and let L be an $(n,k)'$ logic for $\mathfrak{F}_L$; we define the two logics $RI(L)$ and $LE(L)$ as follows:

- $RI(L) = INT + \{(FIN_{2n}), (DE_{n+k}), (CONE(L)_n), (DIFFIN_{n,2n-1})\}$;
- $LE(L) = INT + \{(FIN_{2n}), (DE_{n+1}), (CONE(L)_n)\}$.

Given an (n,k) logic L for $\mathfrak{F}_L$ ($n \geq 2$, $k \geq 2$), we introduce the following classes of posets:

$$\mathfrak{F}_{RI(L)} = \mathfrak{F}_{FIN_{2n}} \cap \mathfrak{F}_{DE_{n+k}} \cap \mathfrak{F}_{CONE(L)_n} \cap \mathfrak{F}_{DIFFIN_{n,2n-1}};$$
$$\mathfrak{F}_{LE(L)} = \mathfrak{F}_{FIN_{2n}} \cap \mathfrak{F}_{DE_{k+1}} \cap \mathfrak{F}_{CONE(L)_n}.$$

Theorem 2: Let $h,k \geq 2$. For every logic L such that L is an (n,k) logic for $\mathfrak{F}_L$, we have that:

(1) $RI(L) = \mathfrak{L}(\mathfrak{F}_{RI(L)})$;

(2) $LE(L) = \mathfrak{L}(\mathfrak{F}_{LE(L)})$.

Proof: (1) Since L is an (n,k) logic for $\mathfrak{F}_L$, Condition (c) of §3 holds. Therefore, by the properties of FIN_{2n} and DE_{n+k} explained in §1, and by Propositions 4 and 8, we get $RI(L) \subseteq \mathfrak{L}(\mathfrak{F}_{RI(L)})$. Furthermore, since the axiom schemes (DE_{n+k}) and (FIN_{2n}) belong to $RI(L)$, it follows that, for every $RI(L)$-saturated set Γ, the poset of $\mathfrak{C}_{RI(L)}(\Gamma)$ has a finite depth and a finite number of final elements. Therefore, by Propositions 5 and 9, we get $\mathfrak{L}(\mathfrak{F}_{RI(L)}) \subseteq RI(L)$. Property (2) can be proved in the same way.□

Proposition 10: Let $n,k \geq 2$. Let L be an (n,k) logic for $\mathfrak{F}_L$. Then:

(1) $\mathfrak{F}_L \subseteq \mathfrak{F}_{RI(L)}$;

(2) $\mathfrak{F}_L \subseteq \mathfrak{F}_{LE(L)}$.

Proof: Immediate from the definition of $\mathfrak{F}_{RI(L)}$ and $\mathfrak{F}_{LE(L)}$.□

Using LE(.) and RI(.), we can build a binary tree T whose nodes are logics as follows:

Definition 5: Let LE(.) and RI(.) be the operators defined above. We inductively define the binary tree T as follows:

(1) The root of T is the semiconstructive logic GS.

(2) Let L be the logic corresponding to an arbitrary node of T: then the logics corresponding to the two nodes immediately following L in T are LE(L) and

RI(L) (respectively called the *left logic immediately above* L and the *right logic immediately above* L).

By a *path of the tree* T we mean any infinite sequence $L_1, L_2, \ldots, L_n, \ldots$ of nodes of T such that L_1 is the root of T and, for every $n\geq 2$, $L_n=RI(L_{n-1})$ or $L_n=LE(L_{n-1})$. We write $\{L_i\}^T_{i\geq 1}$ to indicate a generic path of the tree T.

Definition 6: Let $\{L_i\}^T_{i\geq 1}$ be a path of the tree T; let $\overline{L}^T=\bigcap_{i\geq 1}L_i$. We will call $\overline{L}^T$ the *(first) logic associated with* $\{L_i\}^T_{i\geq 1}$.

For the logic $\overline{L}^T=\bigcap_{i\geq 1}L_i$ associated with the path $\{L_i\}^T_{i\geq 1}$ of the tree T we let $\mathfrak{F}_{\overline{L}^T}=\bigcup_{i\geq 1}\mathfrak{F}_{L_i}$

In the following we will prove some results about the paths of T and the logics associated with them.

Proposition 11: For every path $\{L_i\}^T_{i\geq 1}$ of T and $j\geq 1$, there is a class of posets $\mathfrak{F}_{L_j}$ together with two integers $n\geq 2$ and $k\geq 2$ such that L_j is a (n,k) logic for $\mathfrak{F}_{L_j}$.

Proof: By easy induction on j, starting from the above propositions. □

Proposition 12: Let a path $\{L_i\}^T_{i\geq 1}$ of T be given, and let $j\geq 1$. Then:
(1) $RI(L_j)$ is semiconstructive in L_j;
(2) $LE(L_j)$ is semiconstructive in L_j.

Proof: We use Proposition 3. Let us assume that (n,k) and $\mathfrak{F}_{L_j}$ characterize the logic L_j as specified in Proposition 11.
(1) Let $\mathfrak{F}_{RI(L_j)}$ be the class of posets defined in terms of the class $\mathfrak{F}_{L_j}$ and characterizing the logic $RI(L_j)$. It is easy to prove that, for every s such that $n\leq s\leq 2n$, and for every set $\{f^*_1,\ldots,f^*_s\}$, there is a poset $\underline{P}^*=\langle P^*,\leq^*,r^*\rangle$ such that $\underline{P}^*\in\mathfrak{F}_{RI(L_j)}$ and $Fin(r^*)=\{f^*_1,\ldots,f^*_s\}$. The proof is by double induction on j and s. Let $\underline{P}_1=\langle P_1,\leq_1,r_1\rangle$ and $\underline{P}_2=\langle P_2,\leq_2,r_2\rangle$ be two posets belonging to

$\mathfrak{F}_{L_j}$. Without loss of generality, we can assume that $P_1\cap P_2=\emptyset$. Let $f_1,\ldots,f_m$ and $g_1,\ldots,g_h$ be the distinct final elements such that $Fin(r_1)=\{f_1,\ldots,f_m\}$ and $Fin(r_2)=\{g_1,\ldots,g_h\}$ (obviously, $m\leq n$ and $h\leq n$; we consider only the nontrivial case where $n<m+h$). Let $\underline{P}=\langle P,\leq,r\rangle$ be a poset of $\mathfrak{F}_{RI(L_j)}$ such that $Fin(r)=\{f_1,\ldots,f_m,g_1,\ldots,g_h\}$, $P\cap P_1=\{f_1,\ldots,f_m\}$ and $P\cap P_2=\{g_1,\ldots,g_h\}$: such a poset exists by the above discussion. Now, let $\underline{P}^{**}$ be the poset obtained from $\underline{P}$ by eliminating the root r. Starting from $\underline{P}^{**}$, we build the poset $\underline{P}'=\langle P',\leq',r'\rangle$ in such a way that
(i) $P'=P^{**}\cup P_1\cup P_2\cup\{r'\}$, with $r'\notin P^{**}\cup P_1\cup P_2$, and
(ii) for $\alpha,\beta\in P'$, $\alpha\leq'\beta$ iff

$\alpha\in P^{**}$, $\beta\in P^{**}$ and $\alpha\leq\beta$
or
$\alpha\in P_1$, $\beta\in P_1$ and $\alpha\leq_1\beta$
or
$\alpha\in P_2$, $\beta\in P_2$ and $\alpha\leq_2\beta$
or
$\alpha=r'$ and $\beta\in P_1$
or
$\alpha=r'$ and $\beta\in P_2$
or
$\alpha=r'$ and $\beta\in P^{**}$
or
$\alpha=\beta=r'$.

It follows that r_1, r_2 and the immediate successors of r in $\underline{P}$ are the immediate successors of r' in $\underline{P}'$. We prove that $\underline{P}'$ belongs to $\mathfrak{F}_{RI(L_j)}$.

It is obvious that $\underline{P}'\in\mathfrak{F}_{FIN_{2n}}$ and $\underline{P}'\in\mathfrak{F}_{DE_{n+k}}$. Moreover, we can write $\underline{P}'\in\mathfrak{F}_{CONE(L_j)_n}\cap\mathfrak{F}_{DIFFIN_{n,2n-1}}$. Indeed, for each $\alpha\in P'$, if $\alpha\in P_1$, then $\underline{P}'_\alpha$ (in $\underline{P}'$) coincides with $\underline{P}_{1\alpha}$ (in $\underline{P}_1$), hence $\underline{P}'_\alpha\in\mathfrak{F}_{L_j}$, whence $\underline{P}'_\alpha\in\mathfrak{F}_{CONE(L_j)_n}\cap\mathfrak{F}_{DIFFIN_{n,2n-1}}$ immediately follows. If $\alpha\in P_2$, then $\underline{P}'_\alpha\in\mathfrak{F}_{CONE(L_j)_n}\cap\mathfrak{F}_{DIFFIN_{n,2n-1}}$. If $\alpha\in P^{**}$, then $\underline{P}'_\alpha$ coincides with $\underline{P}_\alpha$, thus $\underline{P}'_\alpha\in\mathfrak{F}_{CONE(L_j)_n}\cap\mathfrak{F}_{DIFFIN_{n,2n-1}}$. If $\alpha=r'$, $2\leq k\leq n$, $k<m+h$, and $f'_1,\ldots,f'_k$ are distinct final elements such that $\{f'_1,\ldots,f'_k\}\subseteq\{f_1,\ldots,f_m,g_1,\ldots,g_h\}$, then there is $\beta\in P^{**}$ such that $Fin(\beta)\supseteq\{f'_1,\ldots,f'_k\}$. Here, $\underline{P}'_\beta$ coincides with $\underline{P}^{**}_\beta$ and

with $\underline{P}_\beta$. It follows that $\underline{P}'=\underline{P}'_{r'}$ is an element of $\mathcal{F}_{CONE(L_j)_n}$. Finally, if $\alpha=r'$, $n\leq j<m+h$, and $f''_1,\dots,f''_j$ are distinct final elements such that $\{f''_1,\dots,f''_j\}\subseteq\{f_1,\dots,f_m,g_1,\dots,g_h\}$, then there is $\gamma\in P^{**}$ such that $Fin(\gamma)=\{f''_1,\dots,f''_j\}$ (with $\underline{P}'_\gamma=\underline{P}^{**}_\gamma=\underline{P}_\gamma$), whence $\underline{P}'=\underline{P}'_{r'}$ is an element of $\mathcal{F}_{DIFFIN_{n,2n-1}}$.

In a similar way we can prove (2).□

Theorem 3: Let $\overline{L}^T$ be the logic associated with the path $\{L_i\}^T_{i\geq1}$ of T. Then $\overline{L}^T=\mathcal{L}(\mathcal{F}_{\overline{L}^T})=\mathcal{L}(\bigcup_{i\geq1}\mathcal{F}_{L_i})$.

Proof: Let $A\in\overline{L}^T$. Then, by definition, $A\in\bigcap_{i\geq1}L_i$ and so, by Theorem 2, for every $i\geq1$, $A\in\mathcal{L}(\mathcal{F}_{L_i})$. Therefore, for every $i\geq1$, the root of every model in $\mathcal{K}(\mathcal{F}_{L_i})$ forces A. Hence, A is forced on the root of every model in $\mathcal{K}(\bigcup_{i\geq1}\mathcal{F}_{L_i})$. Thus, $A\in\mathcal{L}(\bigcup_{i\geq1}\mathcal{F}_{L_i})$. Conversely, let $A\in\mathcal{L}(\bigcup_{i\geq1}\mathcal{F}_{L_i})$. Then, for every $i\geq1$, $A\in\mathcal{L}(\mathcal{F}_{L_i})$. Hence, by Theorems 1 and 2, for every $i\geq1$, we get $A\in L_i$. Therefore, $A\in\overline{L}^T$. □

Theorem 4: For every path $\{L_i\}^T_{i\geq1}$ of the tree T, the logic $\overline{L}^T$ associated with the path $\{L_i\}^T_{i\geq1}$ is a constructive logic.

Proof: By definition of $\{L_i\}^T_{i\geq1}$ and by Proposition 12, the hypotheses of Proposition 2 are satisfied. Thus, $\overline{L}^T$ is a constructive logic.□

At this point, it is not difficult to prove that, if $\overline{L}^T_1$ and $\overline{L}^T_2$ are the logics associated with two different paths of T, and $\mathcal{F}_{\overline{L}^T_1}$ and $\mathcal{F}_{\overline{L}^T_2}$ are the corresponding classes of posets, then there is no constructive logic L such that $\mathcal{L}(\mathcal{F}_{\overline{L}^T_1}\cap\mathcal{F}_{\overline{L}^T_2})\subseteq L$. From this we could deduce that, if $\overline{L}^T_1+\overline{L}^T_2=\mathcal{L}(\mathcal{F}_{\overline{L}^T_1}\cap\mathcal{F}_{\overline{L}^T_2})$, then $\overline{L}^T_1$ and $\overline{L}^T_2$ are constructively incompatible. The circumstance that there are $2^{\aleph_0}$ paths of the tree T, together with an application of Zorn's lemma, then would yield that there are $2^{\aleph_0}$ maximal

constructive logics. However, from the previous treatment, neither we are able to deduce that $\overline{L}^T_1+\overline{L}^T_2=\mathcal{L}(\mathcal{F}_{\overline{L}^T_1}\cap\mathcal{F}_{\overline{L}^T_2})$, nor we can derive that $\overline{L}^T_1$ and $\overline{L}^T_2$ are constructively incompatible. To establish the constructive incompatibility of the logics associated with different paths of the tree T we need further machinery, to be developed in the next four sections.

5: Selective filtration.

Given any wff H, we let Sf(H) be the set of subformulas of H, while $Sf_{\wedge,\rightarrow,\neg}(H)$ denotes the infinite set of wff's which can be built starting from the elements of Sf(H) only using the connectives $\wedge,\rightarrow,\neg$. Following [4,5], given a Kripke model $\underline{K}=\langle P,\leq,\Vdash\rangle$ and $\alpha,\beta\in P$, we set $\alpha\subseteq_H\beta$ iff, for every $H'\in Sf_{\wedge,\rightarrow,\neg}(H)$, if $\alpha\Vdash H'$ then $\beta\Vdash H'$. We also set $\alpha\equiv_H\beta$ iff $\alpha\subseteq_H\beta$ and $\beta\subseteq_H\alpha$. The relation $\equiv_H$ is an equivalence relation. By a result of Diego and Mc Kay quoted in [4,5], there exists only a finite number of intuitionistically non equivalent wff's built up starting from a finite set of propositional variables and using only the connectives $\wedge,\rightarrow,\neg$. Hence, as in [4,5], one deduces:

Proposition 13: The set of equivalence classes of $\equiv_H$ on the set of elements of $\underline{K}$ is finite.□

As in [4,5], given $\underline{K}=\langle P,\leq,\Vdash\rangle$, we define the model $K/\equiv_H$ to be the Kripke model $\langle P_1,\leq_1,\Vdash_1\rangle$ with the following properties:

1) P_1 is the set of equivalence classes generated by $\equiv_H$ on the set of elemlents of P;

2) if $[\alpha]$ and $[\beta]$ are two elements of P_1 (where $[\gamma]$ is the class of γ), then $[\alpha]\leq_1[\beta]$ iff $\alpha\subseteq_H\beta$;

3) for every variable p such that $p\in Sf_{\wedge,\rightarrow,\neg}(H)$, and for every element $[\alpha]\in P_1$, $[\alpha]\Vdash_1 p$ iff $\alpha\Vdash p$ in $\underline{K}$; for every variable q such that $q\notin Sf_{\wedge,\rightarrow,\neg}(H)$, and for every element $[\alpha]\in P_1$, $[\alpha]\nVdash_1 q$ in $\underline{K}/\equiv_H$.

The main property of $\underline{K}/\equiv_H$ is stated in the following proposition, and can be proved by induction on the wff B as in [4,5]:

Proposition 14: If $B \in Sf_{\wedge,\rightarrow,\neg}(H)$ then, for every element α of $\underline{K}$, $\alpha \Vdash B$ (in $\underline{K}$) iff $[\alpha] \Vdash_1 B$ (in $\underline{K}/\equiv_H$).□

The introduction of the models $\underline{K}/\equiv_H$ is due to Gabbay [4,5] and is a refinement of a previous filtration metod of Segerberg [18]. We will define a different kind of models, obtained by "selective filtrations": our filtration technique can be seen as a variant of the one introduced by Gabbay and De Jongh [6].

Definition 7: Let $\underline{K}=\langle P,\leq,\Vdash\rangle$, let α and β be two elements of P and let H be a wff; we set $\alpha \leftarrow_H \beta$ iff the following two conditions are satisfied:

1) $\alpha \equiv_H \beta$;

2) for every γ such that $\alpha \leq \gamma$ in $\underline{K}$ and $\alpha \not\equiv_H \gamma$, there is a δ in $\underline{K}$ such that $\beta \leq \delta$ and $\gamma \equiv_H \delta$.

An element $\alpha \in P$ is said to be $\leftarrow_H$-*terminal* iff, for every $\beta \in P$ such that $\alpha \leq \beta$ and $\alpha \equiv_H \beta$, we have $\alpha \leftarrow_H \beta$.

The following is an immediate consequence of the finiteness of the number of equivalence classes of $\equiv_H$:

Proposition 15: Let $\underline{K}=\langle P,\leq,\Vdash\rangle$. For every $\alpha \in P$ there is a $\alpha^* \in P$ such that $\alpha \leq \alpha^*$, $\alpha \equiv_H \alpha^*$ and α^* is $\leftarrow_H$-terminal.□

Definition 8: Let $\underline{K}=\langle P,\leq,\Vdash\rangle$ and let α^* be any $\leftarrow_H$-terminal element of $\underline{K}$. An element $\beta^* \in P$ is called a $\leftarrow_H$-*immediate successor of* α^* *in* $\underline{K}$ iff:

1) $\alpha^* \not\equiv_H \beta^*$;

2) $\alpha^* \leq \beta^*$;

3) there is no γ in P such that $\alpha^* \leq \gamma \leq \beta^*$, $\alpha^* \not\equiv_H \gamma$ and $\beta^* \not\equiv_H \gamma$;

4) β^* is $\leftarrow_H$-terminal.

Definition 9: Let α^* be any $\leftarrow_H$-terminal element of $\underline{K}$; the set $\{\alpha^{*s}_1,\ldots,\alpha^{*s}_k\}$ is called a *complete set of* $\leftarrow_H$-*immediate successors of* α^* *in* $\underline{K}$ iff the following conditions are atisfied:

1) $\alpha^{*s}_1,\ldots,\alpha^{*s}_k$ are $\leftarrow_H$-immediate successors of α^* in $\underline{K}$;

2) for every i,j such that $1 \leq i,j \leq k$ and $i \neq j$, $\alpha^{*s}_i \leq_H \alpha^{*s}_j$ does not hold;

3) for every element β^* of $\underline{K}$ such that β^* is a $\leftarrow_H$-immediate successor of α^* in $\underline{K}$, there is an i, $1 \leq i \leq k$, such that $\alpha^{*s}_i \leq_H \beta^*$.

The following proposition is a consequence of Propositions 13 and 15.

Proposition 16: For every $\leftarrow_H$-terminal element α^* of $\underline{K}$ such that $[\alpha^*]$ is not a final element of $\underline{K}/\equiv_H$, there exists $k \geq 1$ together with $\leftarrow_H$-terminal elements $\alpha^{*s}_1,\ldots,\alpha^{*s}_k$ of $\underline{K}$ such that $\{\alpha^{*s}_1,\ldots,\alpha^{*s}_k\}$ is a complete set of $\leftarrow_H$-immediate successors of α^* in $\underline{K}$.□

Let $\underline{K}=\langle P,\leq,r,\Vdash\rangle$. By $\underline{K}^{sel}/H$ we will mean any Kripke model $\langle P_2,\leq_2,r^*,\Vdash_2\rangle$ defined as follows:

1) The least element of the poset $\langle P_2,\leq_2\rangle$ is a $\leftarrow_H$-terminal element r^* of $\underline{K}$ belonging to the equivalence class [r] of the root r of $\underline{K}$.

2) Let α^* be any nonfinal elment of $\langle P_2,\leq_2,r^*\rangle$; then the immediate successors $\alpha^{*s}_1,\ldots,\alpha^{*s}_k$ of α^* in $\langle P_2,\leq_2,r^*\rangle$ are such that $\{\alpha^{*s}_1,\ldots,\alpha^{*s}_k\}$ is a complete set of $\leftarrow_H$-immediate successors of α^* in $\underline{K}$.

3) Let α^* be an element of P_2; then if p is a variable of H then $\alpha^* \Vdash_2 p$ iff $\alpha^* \Vdash p$ in $\underline{K}$; if p is a variable which does not occur in H, then $\alpha^* \nVdash_2 p$.

Using Proposition 13 we immediately obtain:

Proposition 17: For every $\underline{K}$, the model $\underline{K}^{sel}/H$ is finite.□

The model we really need is $\underline{K}^{fin}/H$, and is obtained by taking the quotient of the final states of $\underline{K}^{sel}/H$, as follows:.

$\underline{K}^{fin}/H$ is the Kripke model $\langle P_3,\leq_3,r^*,\Vdash_3\rangle$ obtained from a model $\underline{K}^{sel}/H=\langle P_2,\leq_2,r^*,\Vdash_2\rangle$ by the following procedure:

1) The set of nonfinal elements of $\langle P_3,\leq_3,r^*\rangle$ coincides with the set of nonfinal elements of $\langle P_2,\leq_2,r^*\rangle$ and $\leq_3$ on this set coincides with $\leq_2$.

2) The set of final elements of $\langle P_3,\leq_3,r^*\rangle$ coincides with the set of $\equiv_H$-equivalence classes $[\alpha^{*f}]$ (in $\underline{K}$) such that α^{*f} is a final element of $\langle P_2,\leq_2,r^*\rangle$.

3) If α^* and $[\beta^{*f}]$ are respectively a nonfinal element and a final element of $\langle P_3, \leq_3, r^*\rangle$, then $\alpha^* \leq_3 [\beta^{*f}]$ iff there is an element γ^* of $\underline{K}^{sel}/H$ such that $\alpha^* \leq_2 \gamma^*$ and $\gamma^* \equiv_H \beta^{*f}$.
4) If α^* is a nonfinal element of $\langle P_3, \leq_3, r^*\rangle$ and p is any variable, then $\alpha^* \Vdash_3 p$ iff $\alpha^* \Vdash_2 p$.
5) If $[\beta^{*f}]$ is a final element of $\langle P_3, \leq_3, r^*\rangle$ and p is any variable, then $[\beta^{*f}] \Vdash_3 p$ iff, for every final element γ^{*f} of $\underline{K}^{sel}/H$ such that $\gamma^{*f} \equiv_H \beta^{*f}$, we have that $\gamma^{*f} \Vdash_2 p$.

The finiteness of $\underline{K}^{sel}/H$ implies, a fortiori, the finiteness of $\underline{K}^{fin}/H$. The following propositions [13] give the main properties of $\underline{K}^{fin}/H$:

Proposition 18: For every $A \in Sf_{\wedge,\rightarrow,\neg}(H)$ and for every element α^* of $\underline{K}^{fin}/H$, one has:
(i) If $\alpha^* = [\beta]$ is a final element of $\underline{K}^{fin}/H$, then $\alpha^* \Vdash_3 A$ (in $\underline{K}^{fin}/H$) iff $\beta \Vdash A$ in $\underline{K}$.
(ii) If α^* is not a final element of $\underline{K}^{fin}/H$, then $\alpha^* \Vdash_3 A$ (in $\underline{K}^{fin}/H$) iff $\alpha^* \Vdash A$ in $\underline{K}$.
Proof: By induction on the maximal length of the paths connecting α^* with the final elements of $\underline{K}^{fin}/H$.
Property (i) is the basis of the induction, and can be easily proved by induction on the complexity of the formula A.
To prove (ii), which is the step of the (main) induction, one uses again an auxiliary induction on the complexity of A. The cases $A = B \wedge C$ and $A = B \vee C$ are easily handled using the induction hypothesis.
Let $\alpha^* \Vdash B \rightarrow C$. Then $\alpha^* \Vdash_3 B \rightarrow C$. Otherwise, in $\underline{K}^{fin}/H$ there is a β^* such that $\alpha^* \leq_3 \beta^*$, $\beta^* \Vdash_3 B$, and $\beta^* \nVdash_3 C$. Hence, by the auxiliary induction hypothesis, we get $\beta^* \Vdash B$ and $\beta^* \nVdash C$. This contradicts our assumption that $\alpha^* \Vdash B \rightarrow C$.
Now, let $\alpha^* \Vdash_3 B \rightarrow C$, but $\alpha^* \nVdash B \rightarrow C$. Then there is a β of K (not necessarily being a $\leftarrow_H$-terminal element) such that $\alpha^* \leq \beta$, $\beta \Vdash B$ and $\beta \nVdash C$. Let us assume that $\beta \equiv_H \alpha^*$ is not satisfied. Then in $\underline{K}$ there exists β^* such that β^* is a $\leftarrow_H$-immediate successor of α^* in $\underline{K}$, and $\beta^* \leq \beta$. By definition of $\underline{K}^{fin}/H$, it follows that in $\underline{K}^{fin}/H$ there exists γ^* such that $\gamma^* \subseteq_H \beta^*$ and γ^* is an immediate successor of α^* with respect to the ordering

$\leq_3$ of $\underline{K}^{fin}/H$. Since $\alpha^* \leq_3 \gamma^*$, and $\alpha^* \Vdash_3 B \rightarrow C$, one has that $\gamma^* \Vdash_3 B \rightarrow C$. Also, since the maximal length of the paths in $\underline{K}^{fin}/H$ connecting γ^* with the final elements is less than for α^*, one can apply to γ^* the (main) induction hypothesis, thus deducing that $\gamma^* \Vdash B \rightarrow C$. Since $\gamma^* \subseteq_H \beta^*$, $\beta^* \leq \beta$ and $B \rightarrow C \in Sf_{\wedge,\rightarrow,\neg}(H)$, it follows that $\beta \Vdash B \rightarrow C$; this contradicts the hypothesis that $\beta \Vdash B$ and $\beta \nVdash C$. Therefore, we must have $\beta \equiv_H \alpha^*$, which implies $\alpha^* \Vdash B$ and $\alpha^* \nVdash C$. An application of the auxiliary induction hypothesis shows that $\alpha^* \Vdash_3 B$ and $\alpha^* \nVdash_3 C$, a contradiction.
The case $A = \neg B$ can be treated as the case $A = B \rightarrow C$.□

Propostion 19: Let $\underline{P}_1 = \langle P_1, \leq_1\rangle$ and $\underline{P}_3 = \langle P_3, \leq_3\rangle$ respectively be the underlying poset of $\underline{K}/\equiv_H$ and of $\underline{K}^{fin}/H$. Then, for every nonfinal element $\alpha^* \in P_3$, $Fin([\alpha^*])_{\underline{P}_1} = Fin(\alpha^*)_{\underline{P}_3}$.
Proof: Otherwise, for some $\alpha^* \in P_3$, there is $[\beta^f] \in Fin([\alpha^*])_{\underline{P}_1}$ such that $[\beta^f] \notin Fin(\alpha^*)_{\underline{P}_3}$. Now, for every $[\gamma^f] \in Fin([\alpha^*])_{\underline{P}_1}$ such that $[\beta^f] \neq [\gamma^f]$, one can find $A \in Sf_{\wedge,\rightarrow,\neg}(H)$ such that $\beta^f \Vdash A$ and $\gamma^f \nVdash A$ (in $\underline{K}$). As a matter of fact, according to the definition of $\underline{P}_1$, $\beta^f \subseteq_H \gamma^f$ cannot hold. Hence, being $Fin(\alpha^*)_{\underline{P}_3}$ a finite set, one can find $B \in Sf_{\wedge,\rightarrow,\neg}(H)$ such that $\beta^f \Vdash B$, while, for every $[\gamma^f] \in Fin(\alpha^*)_{\underline{P}_3}$, $\gamma^f \nVdash B$. To see this, take as B the conjunction of the formulas A corresponding to the various states of $Fin(\alpha^*)_{\underline{P}_3}$. By Proposition 18, for every $[\gamma^f] \in Fin(\alpha^*)_{\underline{P}_3}$, $[\gamma^f] \Vdash_3 \neg B$ (in $\underline{K}^{fin}/H$). Hence, $\alpha^* \Vdash_3 \neg B$. By a further application of Proposition 18, one gets $\alpha^* \Vdash \neg B$. But this is impossible, since in $\underline{K}$ there is a β^f such that $\alpha^* \subseteq_H \beta^f$ and $\beta^f \Vdash B$.□

We will apply our selective filtration technique in Section 7. Here we will be interested in the models $\mathfrak{C}_L(\Gamma)^{fin}/H$, obtained from the canonical models $\mathfrak{C}_L(\Gamma)$.

6: Selective filtration over rank formulas.

Definition 10: Let v be a finite set of propositional variables, and let α be an element of the Kripke model $\underline{K}$. We call *atomic v-forcing of* α the set

$v(\alpha)=\{p/\ p\in v \text{ and } \alpha \Vdash p\}\cup\{\neg p/\ p\in v \text{ and } \alpha\Vdash \neg p\}$.

We say that an element α of $\underline{K}$ is v-*final* iff $v(\alpha)$ and v have the same number of elements.

Definition 11: Let α and β be elements of two Kripke models $\underline{K}_1$ and $\underline{K}_2$ respectively (possibly with $\underline{K}_1=\underline{K}_2$), and let v be a finite set of propositional variables. We define the v-*rank* r_α of α (the v-rank r_β of β) and the relations $\alpha \overset{v}{\underset{\le r}{\equiv}} \beta$ and $\alpha \overset{v}{\underset{r}{\equiv}} \beta$ (for any natural number r) by simultaneous induction as follows:

Basis. $r=0$:

α has v-rank 0 iff α is v-final; $\alpha \overset{v}{\underset{\le 0}{\equiv}} \beta$ and $\alpha \overset{v}{\underset{0}{\equiv}} \beta$ iff $r_\alpha=0$, $r_\beta=0$ and $v(\alpha)=v(\beta)$.

Induction step. $r>0$:

- α has v-rank r iff:
  - (i) α has not v-rank $\le r-1$;
  - (ii) there exists an element α_1 of $\underline{K}_1$ such that $\alpha\le\alpha_1$ and $r_{\alpha_1}=r-1$;
  - (iii) for every element α_1 of $\underline{K}_1$ such that $\alpha\le\alpha_1$, $\alpha\neq\alpha_1$ and the v-rank of α_1 is not $\le r-1$, the following holds: $v(\alpha_1)=v(\alpha)$ and, for every α_2 of $\underline{K}_1$ such that $\alpha\le\alpha_2$ and $r_{\alpha_2}\le r-1$, there is an element α_3 of $\underline{K}_1$ such that $\alpha_1\le\alpha_3$ and $\alpha_2 \overset{v}{\underset{\le r-1}{\equiv}} \alpha_3$.
- $\alpha \overset{v}{\underset{r}{\equiv}} \beta$ holds iff:
  - (i) $r_\alpha=r_\beta=r$;
  - (ii) $v(\alpha)=v(\beta)$;
  - (iii) for every element α_1 of $\underline{K}_1$ such that $\alpha\le\alpha_1$ and $r_{\alpha_1}\le r-1$, there exists an element β_1 of $\underline{K}_2$ such that $\beta\le\beta_1$ and $\alpha_1 \overset{v}{\underset{\le r-1}{\equiv}} \beta_1$;
  - (iiii) for every element β_1 of $\underline{K}_2$ such that $\beta\le\beta_1$ and $r_{\beta_1}\le r-1$, there exists an element α_1 of $\underline{K}_1$ such that $\alpha\le\alpha_1$ and $\beta_1 \overset{v}{\underset{\le r-1}{\equiv}} \alpha_1$.
- $\alpha \overset{v}{\underset{\le r}{\equiv}} \beta$ holds iff $\alpha \overset{v}{\underset{\le r-1}{\equiv}} \beta$ or $\alpha \overset{v}{\underset{r}{\equiv}} \beta$.

We remark that at most one v-rank can be assigned to every single element of a Kripke model. Also, a v-rank can be assigned to every element of every finite Kripke model. On the other hand, since only natural numbers are admitted as v-ranks by Definition 11, there may be elements of infinite Kripke models to which no v-rank can be assigned. To associate v-ranks with states of arbitrary Kripke models, one should extend Definition 11 to transfinite ordinals.

As an illustration, we give two examples.

Example 1. Let $F=\{f_1, f_2,\dots,f_n,\dots\}$, $G=\{g_1,g_2,\dots,g_n,\dots\}$ and $H=\{h_1,h_2,\dots,h_n,\dots\}$ be three countable disjoint sets, and let $P=\wp(F\cup G\cup H)-\{\emptyset\}$ be the set of nonempty subsets of $F\cup G\cup H$. Define the relation $\le$ on P in the following way: for every $\alpha,\beta\in P$, $\alpha\le\beta$ iff $\alpha\supseteq\beta$. Then, $\underline{P}=\langle P,\le\rangle$ is an uncountable partial ordering whose final states are singletons of the form $\{f_i\}$ (with $f_i\in F$), $\{g_i\}$ (with $g_i\in G$) and $\{h_i\}$ (with $h_i\in H$) respectively, and whose root is the set $F\cup G\cup H$. Note that, for every $\alpha\in P$ such that $Fin(\alpha)$ is finite, the principal subordering $\underline{P}_\alpha$ generated by α in $\underline{P}$ is a Medvedev model in the sense, e.g., of [5,13]. For every $s\in F\cup G\cup H$, we identify the singleton $\{s\}$ with s itself; as usual, r denotes the root $F\cup G\cup H$ of $\underline{P}$. With these conventions, any nonfinal element α of $\underline{P}$ coincides with the set of final elements following α, and r becomes the set of final elements. Now, let $v=\{p,q\}$ and let $\underline{K}=\langle P,\le,\Vdash\rangle$ be the Kripke model built on the poset $\underline{P}=\langle P,\le\rangle$ whose forcing $\Vdash$ is defined as follows:

1) for every $\alpha\in P$ and variable $p'\notin v$, $\alpha\nVdash p'$;

2) for every $\alpha\in P$ such that either $\alpha\subseteq F$ or $\alpha\subseteq G$ or $\alpha\subseteq H$, $\alpha\Vdash p$ and $\alpha\Vdash q$ (in particular, for every final state s, $s\Vdash p$ and $s\Vdash q$);

3) for every $\alpha,\beta,\gamma\in P$ such that $\alpha\subseteq F$, $\beta\subseteq G$ and $\gamma=\alpha\cup\beta$, $\gamma\Vdash p$ and $\gamma\nVdash q$; 4) for every $\alpha,\beta,\gamma\in P$ such that $\alpha\subseteq F$, $\beta\subseteq H$ and $\gamma=\alpha\cup\beta$, $\gamma\nVdash p$ and $\gamma\Vdash q$;

5) for every $\alpha,\beta,\gamma\in P$ such that $\alpha\subseteq G$, $\beta\subseteq H$ and $\gamma=\alpha\cup\beta$, $\gamma\nVdash p$ and $\gamma\nVdash q$. With these conditions, the model $\underline{K}=\langle P,\leq,\Vdash\rangle$ is uniquely determined and the following properties hold:

a) for every $\alpha\in P$ such that either $\alpha\subseteq F$ or $\alpha\subseteq G$ or $\alpha\subseteq H$, the v-rank of α is 0 ;

b) for every $\alpha,\beta,\gamma\in P$ such that $\alpha\subseteq F$, $\beta\subseteq G$ and $\gamma=\alpha\cup\beta$, the v-rank of γ is 1;

c) for every $\alpha,\beta,\gamma\in P$ such that $\alpha\subseteq F$, $\beta\subseteq H$ and $\gamma=\alpha\cup\beta$, the v-rank of γ is 1;

d) for every $\alpha,\beta,\gamma\in P$ such that $\alpha\subseteq G$, $\beta\subseteq H$ and $\gamma=\alpha\cup\beta$, the v-rank of γ is 1;

e) the v-rank of the root r equals 2 and also coincides with the v-rank of $\{f,g,h\}$, whenever $f\in F$, $g\in G$ and $h\in H$.

<u>Example 2.</u> Let $v=\{p,q\}$; for every natural number $n\geq 0$, we inductively define three Kripke models $\underline{K}_n^1$, $\underline{K}_n^2$ and $\underline{K}_n^3$ as follows:

1) For every k such that $1\leq k\leq 3$, the model $\underline{K}_0^k$ contains only the final state f_k; moreover, for every j with $k\neq j$ and $1\leq j\leq 3$, $f_k\neq f_j$.

2) For every k with $1\leq k\leq 3$, if $p'\notin v$ and $1\leq j\leq 3$, then $f_j\nVdash p'$ in $\underline{K}_0^k$.

3) The following conditions are satisfied:

1_3) $f_1\nVdash p$ and $f_1\nVdash q$ in $\underline{K}_0^1$;

2_3) $f_2\Vdash p$ and $f_2\nVdash q$ in $\underline{K}_0^2$;

3_3) $f_3\nVdash p$ and $f_3\Vdash q$ in $\underline{K}_0^3$.

4) Let $\underline{K}_i^1=\langle P_i^1,\leq_i^1,\Vdash_i^1\rangle$, $\underline{K}_i^2=\langle P_i^2,\leq_i^2,\Vdash_i^2\rangle$ and $\underline{K}_i^3=\langle P_i^3,\leq_i^3,\Vdash_i^3\rangle$. For j,j' with $1\leq j\leq 3$ and $1\leq j'\leq 3$, let r_j be different from every state of $\underline{K}_i^{j'}$ and let $r_j\neq r_{j'}$ for $j\neq j'$. Then, for every h with $1\leq h\leq 3$, we define $\underline{K}_{i+1}^h=\langle P_{i+1}^h,\leq_{i+1}^h,\Vdash_{i+1}^h\rangle$ as follows:

1_4) $P_{i+1}^h=P_i^k\cup P_i^m\cup\{r_h\}$, where $1\leq k,m\leq 3$, $h\neq k$, $h\neq m$ and $k\neq m$;

2_4) r_h is the root of $\underline{P}_{i+1}^h=\langle P_{i+1}^h,\leq_{i+1}^h\rangle$;

3_4) if $\alpha,\beta\in P_{i+1}^h$, then $\alpha\leq_{i+1}^h\beta$ iff

$\alpha=r_h$

or

$\alpha,\beta\in P_i^k$ and $\alpha\leq_i^k\beta$

or

$\alpha,\beta\in P_i^m$ and $\alpha\leq_i^m\beta$;

4_4) $\Vdash_{i+1}^h$ coincides with $\Vdash_i^k$ on the states of $\underline{K}_i^k$ and with $\Vdash_i^m$ on the states of $\underline{K}_i^m$.

Note that the forcing of $\underline{K}_{i+1}^h$ on its root is uniquely determined, for, no variable can be forced on r_h.

Having defined the models $\underline{K}_n^h$ with $n\geq 0$, $1\leq h\leq 3$, we define the infinite Kripke model $\underline{K}'=\langle P',\leq',\Vdash'\rangle$ as follows:

5) $P'=\{r'\}\cup(\bigcup_{n\geq 0,1\leq h\leq 3}P_n^h)$, where $\underline{P}_n^h=\langle P_n^h,\leq_n^h\rangle$ is the poset on which $\underline{K}_n^h$ is built and where $r'\notin\bigcup_{n\geq 0,1\leq h\leq 3}P_n^h$;

6) for every $\alpha,\beta\in P'$, $\alpha\leq'\beta$ iff either $\alpha=r'$, or there are $n\geq 0$ and h with $1\leq h\leq 3$ such that $\alpha,\beta\in P_n^h$ and $\alpha\leq_n^h\beta$;

7) if $\alpha\in P'$ and $\alpha\neq r'$, then there is $n\geq 0$ and h with $1\leq h\leq 3$ such that the principal subordering generated by α in $\underline{P}'=\langle P',\leq'\rangle$ coincides with $\underline{P}_n^h$; for every variable p' (whether or not $p'\in v$), we set $\alpha\Vdash' p'$ iff $\alpha\Vdash_n^h p'$.

Note that the forcing on r' is automatically determined, i.e., r' cannot force any variable.

Now, using Definition 11, we can give the following evaluation of the v-ranks of the states of $\underline{K}'$:

a) Let α be any element of P' different from the root r'. Then, for some $n\geq 0$ and some h with $1\leq h\leq 3$, the principal subordering generated by α in $\langle P',\leq'\rangle$ coincides with $\underline{P}_n^h$ and the v-rank of α is n.

b) The root r' of the model $\underline{K}'$ has no v-rank. This is so because, according to a), for every $\rho\geq 0$ there is α such that $r'\leq\alpha$ and ρ is the v-rank of α.

We omit the easy proof of the following:

Proposition 20: For every finite set of propositional variables, and every $r\geq 0$, we have:

(1) $\overset{v}{\underset{r}{\equiv}}$ is an equivalence relation over the class $\mathfrak{K}_v^r$ of elements with v-rank r.

(2) $\overset{v}{\underset{\leq r}{\equiv}}$ is an equivalence relation over the class $\bigcup_{0\leq r'\leq r}\mathfrak{K}_v^{r'}$ of elements with v-rank $r'\leq r$.□

Proposition 21: For every finite set v of propositional variables and natural number $r\geq 0$, the number of the equivalence classes generated by $\overset{v}{\equiv}_{r}$ over $\mathfrak{R}^{r}_{v}$ is finite, and so is the number of equivalence classes generated by $\overset{v}{\equiv}_{\leq r}$ over $\bigcup_{0\leq r'\leq r}\mathfrak{R}^{r'}_{v}$.□

The proof of the above proposition can be obtained by an easy induction on r (see G. Faglia, *Relazioni fra modelli di Kripke per lo studio della decidibilità di logiche intermedie costruttive,* Tesi di Laurea, Dipartimento di Scienze dell'Informazione, Università degli Studi di Milano, 1987). Using Proposition 21, we can prove:

Proposition 22: For every $r\geq 0$, $\underline{K}=\langle P,\leq,\Vdash\rangle$, $\alpha\in P$, and every finite v, if α has no v-rank, or α has a v-rank $>r$, then there is $\beta\in P$ such that $\alpha\leq\beta$ and the v-rank of β coincides with r.

Proof: We consider only the nontrivial case, proving, by induction on ρ, that, for every ρ, the following holds: for every α, if α has no v-rank, then there is β_ρ such that $\alpha\leq\beta_\rho$ and ρ is the v-rank of β_ρ.

First of all, since v is finite, one easily shows that for every α, there is β_0 such that $\alpha\leq\beta_0$ and for every $p\in v$, either $\beta_0\Vdash p$ or $\beta_0\Vdash\neg p$; thus, $|v(\beta_0)|=|v|$, i.e., 0 is the v-rank of β_0, and we have the basis of our induction argument.

To prove the induction step, let $r\geq 0$, and let us assume that for every α such that α has no v-rank, there is β_r such that $\alpha\leq\beta_r$ and r is the v-rank of β_r. Let α be an element of P having no v-rank. According to our induction hypothesis, it follows that the set of α-states over r is nonempty, where we say that $\alpha'\in P$ is an *α-state over* r iff $\alpha\leq\alpha'$, α' has no v-rank $\leq r$ (but α' may have v-rank $>r$) and there is $\beta'_r\in P$ such that $\alpha'\leq\beta'_r$ and r is the v-rank of β_r. Now, since v is finite and since, by Proposition 21, the set of equivalence classes generated by $\overset{v}{\equiv}_{\leq r}$ over $\bigcup_{0\leq r'\leq r}\mathfrak{R}^{r'}_{v}$ is finite, there must exist $\beta_{r+1}\in P$ such that β_{r+1} is a *minimal* α-state over r. Here we mean that β_{r+1} is an α-state over r satisfying the following condition:

1) For every $\beta'\in P$ such that $\beta_{r+1}\leq\beta'$ and β' is an α-state over r, the following properties are satisfied:

1_1) $v(\beta_{r+1})=v(\beta')$;

2_1) for every $\beta''\in P$ such that $\beta_{r+1}\leq\beta''$ and β'' has a v-rank $\leq r$, there is $\beta'''\in P$ such that $\beta'\leq\beta'''$ and $\beta''\overset{v}{\equiv}_{\leq r}\beta'''$.

We claim that $r+1$ is the v-rank of β_{r+1}. Indeed, Conditions (i) and (ii), ensuring that $r+1$ be the v-rank of β_{r+1}, are immediately satisfied. As for Condition (iii), let $\beta'\in P$ be such that $\beta_{r+1}\leq\beta'$, $\beta_{r+1}\neq\beta'$ and β' has not v-rank $\leq r$. Then, β' has a v-rank $>r$, unless β' has no v-rank at all. In the first case, one immediately gets a $\beta'_r\in P$ such that $\beta'\leq\beta'_r$, and r is the v-rank of β'_r, in the second case, the same conclusion holds, because of our induction hypothesis. Thus, in both cases, β' is an α-state over r; from the minimality of β_{r+1}, one now obtains that $v(\beta_{r+1})=v(\beta')$, and for every $\beta''\in P$ such that $\beta_{r+1}\leq\beta''$ and β'' has a v-rank $\leq r$, there is a $\beta'''\in P$ such that $\beta'\leq\beta'''$ and $\beta''\overset{v}{\equiv}_{\leq r}\beta'''$. Thus, also Condition (iii) holds. This concludes the proof of the proposition.□

We also have (see the above quoted thesis of Faglia):

Proposition 23: Let $r\geq 0$ and let v be a finite set of propositional variables. Let α and β be elements of the Kripke models $\underline{K}_1$ and $\underline{K}_2$ respectively, possibly with $\underline{K}_1=\underline{K}_2$. Further, let $\alpha\overset{v}{\equiv}_{\leq r}\beta$. Then, for every wff A containing only the variables of v, $\alpha\Vdash A$ in $\underline{K}_1$ iff $\beta\Vdash A$ in $\underline{K}_2$.

Proof: The proof is by induction on the complexity of A; we will consider only the case $A=B\rightarrow C$.

First of all, as an easy consequence of Definition 11, we have that if $\alpha\overset{v}{\equiv}_{\leq r}\beta$ and $\alpha\leq\alpha'$ in $\underline{K}_1$ then there exists β' such that $\beta\leq\beta'$ in $\underline{K}_2$ and $\alpha'\overset{v}{\equiv}_{\leq r}\beta'$. Note that the v-rank of α' need not be smaller than the v-rank of α. Now, let $\alpha\overset{v}{\equiv}_{\leq r}\beta$, $\alpha\Vdash B\rightarrow C$ in $\underline{K}_1$, and suppose that $\beta\nVdash B\rightarrow C$ in $\underline{K}_2$. Then there is β' such that $\beta'\Vdash B$ and $\beta'\nVdash C$ in $\underline{K}_2$. By the above discussion, there is α' such that $\alpha\leq\alpha'$ in $\underline{K}_1$ and $\beta'\overset{v}{\equiv}_{\leq r}\alpha'$; hence, by induction hypothesis, in $\underline{K}_1$

there is α' such that $\alpha' \Vdash B$ and $\alpha' \nVdash C$, with $\alpha \leq \alpha'$. This contradicts $\alpha \Vdash B \to C$ in $\underline{K}_1$.□

For every $r \geq 0$ and finite set v of propositional variables, we call $\overset{v}{\underset{\leq r}{\equiv}}$ *-equivalence classes* the classes generated by $\overset{v}{\underset{\leq r}{\equiv}}$ over $\bigcup_{0\leq r'\leq r} \mathfrak{R}^{r'}_{v}$; further, we denote by $[\alpha]^{v}_{\leq r}$ such classes (where α is some representative element of the class).

Definition 12: Given Kripke models $\underline{K}_1$ and $\underline{K}_2$, let $\alpha \in \underline{K}_1$ and $\beta \in \underline{K}_2$. Assume further that α and β belong to two $\overset{v}{\underset{\leq r}{\equiv}}$-equivalence classes. We write $\alpha \leq^{v}_{r} \beta$ if there is an element α' such that $\alpha \leq \alpha'$ in $\underline{K}_1$ and $\alpha' \overset{v}{\underset{\leq r}{\equiv}} \beta$. We say that β is a $\overset{v}{\underset{\leq r}{\equiv}}$ *-immediate successor of* α iff:

(a) $\alpha \leq^{v}_{r} \beta$ and $\beta \not\leq^{v}_{r} \alpha$;

(b) for every model $\underline{K}_3$ (possibly with $\underline{K}_3 = \underline{K}_1$ or $\underline{K}_3 = \underline{K}_2$) and every element $\gamma \in \underline{K}_3$, if $\alpha \leq^{v}_{r} \gamma \leq^{v}_{r} \beta$, then we have that $\gamma \overset{v}{\underset{\leq r}{\equiv}} \alpha$ or $\gamma \overset{v}{\underset{\leq r}{\equiv}} \beta$.

It is easy to see that α and β belong to the same $\overset{v}{\underset{\leq r}{\equiv}}$-equivalence class iff $\alpha \leq^{v}_{r} \beta$ and $\beta \leq^{v}_{r} \alpha$.

Proposition 24: Let $r \geq 0$, and let v be a finite set of propositional variables. Let α and β be elements of two Kripke models $\underline{K}_1$ and $\underline{K}_2$ such that α and β belong to $\bigcup_{0\leq r'\leq r} \mathfrak{R}^{r'}_{v}$ and $\alpha \not\leq^{v}_{r} \beta$. Then there exists a wff $H_{\alpha,\beta}$, containing only the variables of v, such that $\alpha \Vdash H_{\alpha,\beta}$ and $\beta \nVdash H_{\alpha,\beta}$ in their own respective models.

Proof: The proof is by induction on $\rho = \max(r_\alpha, r_\beta)$, with $\rho \leq r$. If $\rho = 0$, let $H_{v(\alpha)}$ be the wff given by the conjunction of the wff's in $v(\alpha)$. Let $H_{\alpha,\beta} = H_{v(\alpha)}$. Since $\rho = 0$, α and β are v-final elements. Moreover, $\alpha \not\leq^{v}_{0} \beta$ entails $v(\alpha) \neq v(\beta)$. Therefore, $\alpha \Vdash H_{\alpha,\beta}$ and $\beta \nVdash H_{\alpha,\beta}$ in their own respective models. Now

assuming that the proposition holds for every v-rank less than ρ, we prove it for ρ $(\rho \leq r)$. We have six possible cases:

CASE 1: $\beta \leq^{v}_{r} \alpha$, and β has just one $\overset{v}{\underset{\leq r}{\equiv}}$-class of $\overset{v}{\underset{\leq r}{\equiv}}$-immediate successors.

In this case, let β' be any $\overset{v}{\underset{\leq r}{\equiv}}$-immediate successor of β. Then β and β' have distinct v-atomic forcings. For, if this were not the case, β and β' would belong to the same $\overset{v}{\underset{\leq r}{\equiv}}$-equivalence class, thus contradicting the definition of $\overset{v}{\underset{\leq r}{\equiv}}$-immediate successor. Thus, $\alpha \Vdash H_{v(\alpha)}$ and $\beta \nVdash H_{v(\alpha)}$. Setting $H_{\alpha,\beta} = H_{v(\alpha)}$, we have the induction step for Case 1.

CASE 2: $\beta \leq^{v}_{r} \alpha$, and β has at least two $\overset{v}{\underset{\leq r}{\equiv}}$-classes of $\overset{v}{\underset{\leq r}{\equiv}}$-immediate successors.

Let γ be a $\overset{v}{\underset{\leq r}{\equiv}}$-immediate successor of β such that $\gamma \leq^{v}_{r} \alpha$, and let δ be a $\overset{v}{\underset{\leq r}{\equiv}}$-immediate successor of β such that $[\delta]^{v}_{\leq r} \neq [\gamma]^{v}_{\leq r}$. The v-ranks of δ and γ are less than ρ. Then, by induction hypothesis, there is a wff $H_{\gamma,\delta}$ such that $\gamma \Vdash H_{\gamma,\delta}$ and $\delta \nVdash H_{\gamma,\delta}$ in their own respective models. Since $\beta \leq^{v}_{r} \delta$ and $\gamma \leq^{v}_{r} \alpha$, it follows that $\beta \nVdash H_{\gamma,\delta}$ and $\alpha \Vdash H_{\gamma,\delta}$ in their own respective models. Thus, setting $H_{\alpha,\beta} = H_{\gamma,\delta}$, we have proved the induction step for the present case.

CASE 3: $\beta \not\leq^{v}_{r} \alpha$ (and $\alpha \not\leq^{v}_{r} \beta$), $r_\beta = \rho$, and $r_\alpha < \rho$.

In this case, there exists a $\overset{v}{\underset{\leq r}{\equiv}}$-immediate successor γ of β such that $\alpha \not\leq^{v}_{r} \gamma$ (otherwise, $r_\beta \leq r_\alpha$ or $\beta \leq^{v}_{r} \alpha$). Then, by induction hypothesis, there is a wff $H_{\alpha,\gamma}$ such that $\alpha \Vdash H_{\alpha,\gamma}$ and $\gamma \nVdash H_{\alpha,\gamma}$. Let us set $H_{\alpha,\beta} = H_{\alpha,\gamma}$. Then, of course, $\alpha \Vdash H_{\alpha,\beta}$ and $\beta \nVdash H_{\alpha,\beta}$ in their own respective models.

CASE 4: $\beta \not\leq^{v}_{r} \alpha$ (and $\alpha \not\leq^{v}_{r} \beta$), $r_\alpha = \rho$, and $r_\beta < \rho$.

As in Case 3, we can find a wff $K_{\beta,\alpha}$, containing only the variables of v, such that $\beta \Vdash K_{\beta,\alpha}$ and $\alpha \nVdash K_{\beta,\alpha}$. For every $\overset{v}{\underset{\le r}{\equiv}}$-immediate successor α' of α, we have $\alpha' \not\le^{v}_{r} \beta$ (otherwise, $\alpha \le^{v}_{r} \beta$, which is impossible). Let $\alpha'_1,\ldots,\alpha'_n$ be representative elements of the distinct $\overset{v}{\underset{\le r}{\equiv}}$-classes of $\overset{v}{\underset{\le r}{\equiv}}$-immediate successors of α. Since, for every i with $1\le i\le n$, $\alpha'_i \not\le^{v}_{r} \beta$, by induction hypothesis there are wff's $H_{\alpha'_i,\beta}$ such that $\alpha'_i \Vdash H_{\alpha'_i,\beta}$ and $\beta \nVdash H_{\alpha'_i,\beta}$ in their own respective models. Let now $H_{\alpha,\beta}=K_{\beta,\alpha}\to \bigvee_{1\le i\le n} H_{\alpha'_i,\beta}$. We prove that $\alpha \Vdash H_{\alpha,\beta}$ and $\beta \nVdash H_{\alpha,\beta}$. For, $\beta \Vdash K_{\beta,\alpha}$, but $\beta \nVdash \bigvee_{1\le i\le n} H_{\alpha'_i,\beta}$. Moreover, $\alpha \nVdash K_{\beta,\alpha}$ while, by Proposition 23, every $\overset{v}{\underset{\le r}{\equiv}}$-immediate successor of α in the model of α forces $\bigvee_{1\le i\le n} H_{\alpha'_i,\beta}$.

<u>CASE 5</u>: $\beta \not\le^{v}_{r} \alpha$ (and $\alpha \not\le^{v}_{r} \beta$), $r_\alpha=r_\beta=\rho$, and there exists a $\overset{v}{\underset{\le r}{\equiv}}$-immediate successor β' of β such that there is no $\overset{v}{\underset{\le r}{\equiv}}$-immediate successor α' of α with $\alpha' \le^{v}_{r} \beta'$.

Obviously, β' has v-rank less than ρ. Then there is at least one $\overset{v}{\underset{\le r}{\equiv}}$-immediate successor α''_1 of α such that $\beta' \not\le^{v}_{r} \alpha''_1$ (for otherwise, β' would have at least v-rank ρ, which is impossible). Let $\alpha''_2,\ldots,\alpha''_m$ be representative elements of the distinct $\overset{v}{\underset{\le r}{\equiv}}$-classes of $\overset{v}{\underset{\le r}{\equiv}}$-immediate successors of α not belonging to $[\alpha''_1]$ (if any). By our hypotheses, $\alpha''_1 \not\le^{v}_{r} \beta'$, $\alpha''_2 \not\le^{v}_{r} \beta',\ldots,\alpha''_m \not\le^{v}_{r} \beta'$. By induction hypothesis, there are wff's $H_{\alpha''_1,\beta'},\ldots,H_{\alpha''_m,\beta'}$ such that $\beta' \nVdash H_{\alpha''_i,\beta'}$ and $\alpha''_i \Vdash H_{\alpha''_i,\beta'}$. Moreover, again by induction hypothesis, there is a wff H_{β',α''_1} such that $\beta' \Vdash H_{\beta',\alpha''_1}$ and $\alpha''_1 \nVdash H_{\beta',\alpha''_1}$. Let $H_{\alpha,\beta}=H_{\beta',\alpha''_1}\to H_{\alpha''_1,\beta'}\vee\ldots\vee H_{\alpha''_m,\beta'}$. We prove that

$\beta' \nVdash H_{\alpha,\beta}$. Indeed we have $\beta' \Vdash H_{\beta',\alpha''_1}$, while $\beta' \nVdash H_{\alpha''_1,\beta'}\vee\ldots\vee H_{\alpha''_m,\beta'}$. Since $\beta \le^{v}_{r} \beta'$, we get $\beta \nVdash H_{\alpha,\beta}$ in its own model. On the other hand, $\alpha \Vdash H_{\alpha,\beta}$, since $\alpha \nVdash H_{\beta',\alpha''_1}$ and, by Proposition 23, every $\overset{v}{\underset{\le r}{\equiv}}$-immediate successor of α forces $H_{\alpha''_1,\beta'}\vee\ldots\vee H_{\alpha''_m,\beta'}$ in the model of α.

<u>CASE 6</u>: $\beta \not\le^{v}_{r} \alpha$ (and $\alpha \not\le^{v}_{r} \beta$), $r_\alpha=r_\beta=\rho$ and, for every $\overset{v}{\underset{\le r}{\equiv}}$-immediate successor β' of β, there is a $\overset{v}{\underset{\le r}{\equiv}}$-immediate successor α' of α such that $\alpha' \le^{v}_{r} \beta'$.

If $v(\alpha)=v(\beta)$, there is a $\overset{v}{\underset{\le r}{\equiv}}$-immediate successor α' of α such that, for every $\overset{v}{\underset{\le r}{\equiv}}$-immediate successor β' of β, $\beta' \not\le^{v}_{r} \alpha'$ (for otherwise, we would deduce $\alpha \overset{v}{\underset{\le r}{\equiv}} \beta$, which is impossible). Proceeding as in the previous case, we can find a wff $H_{\beta,\alpha}$ such that $\beta \Vdash H_{\beta,\alpha}$ and $\alpha \nVdash H_{\beta,\alpha}$. On the other hand, let $\alpha'_1,\ldots,\alpha'_m$ be a complete sequence of representative elements of the $\overset{v}{\underset{\le r}{\equiv}}$-equivalence classes of $\overset{v}{\underset{\le r}{\equiv}}$-immediate successors of α. Since $\alpha'_1 \not\le^{v}_{r} \beta,\ldots,\alpha'_m \not\le^{v}_{r} \beta$, and since the v-rank of $\alpha'_1,\ldots,\alpha'_m$ is $<\rho$, as in Cases 1-3 one can find wff's $H_{\alpha'_1,\beta},\ldots,H_{\alpha'_m,\beta}$ such that, for every i, $\alpha'_i \Vdash H_{\alpha'_i,\beta}$ and $\beta \nVdash H_{\alpha'_i,\beta}$ in their own respective models. Let $H_{\alpha,\beta}=H_{\beta,\alpha}\to H_{\alpha'_1,\beta}\vee\ldots\vee H_{\alpha'_m,\beta}$. It is easy to see that $\alpha \Vdash H_{\alpha,\beta}$ and $\beta \nVdash H_{\alpha,\beta}$. If $v(\alpha)\neq v(\beta)$, we proceed as follows. If there is a variable $p\in v$ such that $\alpha \Vdash p$ and $\beta \nVdash p$, we set $H_{\alpha,\beta}=p$; if there is a variable $p\in v$ such that $\alpha \Vdash \neg p$ and $\beta \nVdash \neg p$, we set $H_{\alpha,\beta}=\neg p$. Otherwise, there must be a variable $p\in v$ such that $\beta \Vdash p$ and $\alpha \nVdash p$, or $\beta \Vdash \neg p$ and $\alpha \nVdash \neg p$: in the first case, we set $p^*=p$; in the second case, we set $p^*=\neg p$. Let $H_{\alpha'_1,\beta},\ldots,H_{\alpha'_m,\beta}$ be as above. Setting $H_{\alpha,\beta}=p^*\to H_{\alpha'_1,\beta}\vee\ldots\vee H_{\alpha'_m,\beta}$, it is easy to prove that $\alpha \Vdash H_{\alpha,\beta}$ and $\beta \nVdash H_{\alpha,\beta}$ holds in their own respective models. This concludes the proof.□

Given any set $v=\{p_1,\dots,p_n\}$ of propositional variables, the set of formulas of the form $\neg B$ containing only the variables of v is divided into a finite set of equivalence classes $[\neg B]_v$ by intuitionistic biimplication. By a *v-complete set of negated formulas* we mean any (finite) set $\{\neg C_1,\dots,\neg C_h\}$ satisfying the following conditions:
1) for every equivalence class $[\neg B]_v$, there is an i, $1\leq i\leq h$, such that $\neg C_i\in[\neg B]_v$; 2) for every i,j with $1\leq i,j\leq h$ and $i\neq j$, $[\neg C_i]_v\neq[\neg C_j]_v$.

Definition 13: Θ *is a completed wff* iff there is a sequence $v, N, N_1,\dots,N_s$ such that the following conditions are satisfied:
1) $v=\{p_1,\dots,p_n\}$ is the set of propositional variables of Θ;
2) $N=\{\neg C_1,\dots,\neg C_h\}$ is a a v-complete set of negated formulas;
3) $N_1,\dots,N_s$ are the nonempty subsets of N;
4) for every j with $1\leq j\leq s$, D_{N_j} is the disjunction of all the formulas of N_j;
5) $Z_\Theta=D_{N_1}\wedge\dots\wedge D_{N_s}$;
6) Z_Θ is a subformula of Θ.

It is always possible to extend any wff Θ to a completed wff Θ_C containing the same variables and having Θ as a subformula. Just set $\Theta_C=\Theta\wedge Z_\Theta$.

Given a natural number $r\geq0$, consider the set of $\overset{v}{\equiv}_{\leq r}$ -equivalence classes. By Proposition 21, the number of such classes is finite. Let $[\alpha^*]^v_{\leq r}$ and $[\beta^*]^v_{\leq r}$ be two such classes, and let $\alpha\in[\alpha^*]^v_{\leq r}$ and $\beta\in[\beta^*]^v_{\leq r}$. If $\alpha\not\leq^v_r\beta$, then, by Proposition 24, there exists a wff $H_{\alpha,\beta}$, containing only the variables of v, such that $\alpha\Vdash H_{\alpha,\beta}$ and $\beta\nVdash H_{\alpha,\beta}$ in their own respective models. For any two $\overset{v}{\equiv}_{\leq r}$ -equivalence classes $[\alpha^*]^v_{\leq r}$ and $[\beta^*]^v_{\leq r}$ such that $\alpha\in[\alpha^*]^v_{\leq r}$, $\beta\in[\beta^*]^v_{\leq r}$ and $\alpha\not\leq^v_r\beta$, let us consider the wff $H_{\alpha,\beta}$. Observe that such a wff is independent of the choice of the elements in $[\alpha^*]^v_{\leq r}$ and $[\beta^*]^v_{\leq r}$. Let $\mathfrak{H}$ be the set of wff's $H_{\alpha,\beta}$ obtained in this way, and let Θ^v_r be the conjunction of all the formulas of $\mathfrak{H}$.

Definition 14: We say that a formula Θ is *v-extensively completed up to* r iff Θ is completed, and Θ contains Θ^v_r as a subformula.

Any formula Θ can be extended to a formula Θ' which is v-extensively completed up to r, has Θ as a subformula, and contains the same propositional variables as Θ. For, we can set $\Theta'=\Theta\wedge Z_\Theta\wedge\Theta^v_r$, where Z_Θ and Θ^v_r are as in Definitions 13 and 14.

7: The logics $\overline{\overline{\mathbb{L}}}^T$.

Consider a path $\{L_i\}^T_{i\geq1}$ of the tree T: for each $m\geq1$ there are n_m and k_m such that the logic L_m in the path is a (n_m,k_m) logic, for a suitable class of posets $\mathfrak{F}_{L_m}$. Let $(CONE(L_m)_{n_m})$ be the axiom-schema related to the logic L_m of the path $\{L_i\}^T_{i\geq1}$. Since, for every $m\geq1$, n_m is uniquely determined in terms of m and $\{L_i\}^T_{i\geq1}$, for the sake of simplicity we denote by $(CONE(L_m))$ the axiom schema $(CONE(L_m)_{n_m})$, and let $\mathfrak{F}_{CONE(L_m)}$ denote the class of posets $\mathfrak{F}_{CONE(L_m)_{n_m}}$.

Definition 15: Let $\{L_i\}^T_{i\geq1}$ be a path of the tree T, and let $\overline{\overline{L}}^T=INT+\bigcup_{m\geq1}\{(CONE(L_m))\}$, with L_m ranging over the path. We call $\overline{\overline{L}}^T$ the *second logic associated with* $\{L_i\}^T_{i\geq1}$. We also set $\mathfrak{F}_{\overline{\overline{L}}^T}=\bigcap_{m\geq1}\mathfrak{F}_{CONE(L_m)}$.

Proposition 25: For every path $\{L_i\}^T_{i\geq1}$ of the tree T, $\overline{\overline{L}}^T\subseteq\mathfrak{L}(\mathfrak{F}_{\overline{\overline{L}}^T})$.
<u>Proof</u>: Immediate from Proposition 4.□

Proposition 26: Let $\overline{\overline{L}}^T$ be the second logic associated with the path $\{L_i\}^T_{i\geq1}$ of the tree T, let L be a logic such that $L\supseteq\overline{\overline{L}}^T$ and let, for each $m\geq2$, $A(p_1,\dots,p_m)$ be a wff containing only the propositional variables $p_1,\dots,p_m$. Also, let

L_h be the (n_h,k_h) logic of the path $\{L_i\}^T_{i\geq 1}$ such that $n_{h-1}<2^m\leq n_h$, where L_{h-1} is a (n_{h1},k_{h-1}) logic. Finally, let $\underline{P}^{fin}=\langle P^{fin},\leq,\Gamma*\rangle$ be the underlying poset of the model $\mathfrak{C}_L(\Gamma)^{fin}/A$. If $\underline{P}^{fin}\in\mathfrak{F}_{CONE(L_h)}$ then, for every $j>h$, $\underline{P}^{fin}\in\mathfrak{F}_{CONE(L_j)}$.

Proof: If A is a wff containing only the variables $p_1,\ldots,p_m$, then the model $\mathfrak{C}_L(\Gamma)^{fin}/A$ has at most 2^m final elements: such final elements correspond to the 2^m distinct classical interpretations of the variables $p_1,\ldots,p_m$.

Let $\Gamma*$ be the root of $\underline{P}^{fin}$; since $\underline{P}^{fin}\in\mathfrak{F}_{CONE(L_h)}$ and $|Fin(\Gamma*)|\leq n_h$, there is an element $\Gamma*'$ such that $\Gamma*\leq\Gamma*'$, $Fin(\Gamma*')=Fin(\Gamma*)$ and, if $\underline{P}'$ is the principal subordering generated by $\Gamma*'$ in $\underline{P}^{fin}$, then $\underline{P}'\in\mathfrak{F}_{L_h}$. Now, $\mathfrak{F}_{L_h}\subseteq\mathfrak{F}_{RI(L_h)}$ and $\mathfrak{F}_{L_h}\subseteq\mathfrak{F}_{LE(L_h)}$, whence $\underline{P}'\in\mathfrak{F}_{L_{h+1}}$. It follows that $\underline{P}^{fin}\in\mathfrak{F}_{CONE(L_{h+1})}$. Similary, we can now prove that $\underline{P}^{fin}\in\mathfrak{F}_{CONE(L_{h+2})}$. Proceeding in this way, for every $j>h$ we get that $\underline{P}^{fin}\in\mathfrak{F}_{CONE(L_j)}$.$\square$

By the previous proposition, to prove that $\underline{P}^{fin}\in\bigcap_{m\geq 1}\mathfrak{F}_{CONE(L_m)}$ it is sufficient to show that $\underline{P}^{fin}\in\bigcap_{1\leq m\leq h}\mathfrak{F}_{CONE(L_m)}=\mathfrak{F}_{CONE(L_h)}$.

For every $m\geq 1$, if we consider the logic L_m of the path $\{L_i\}^T_{i\geq 1}$, then the parameter k_m such that L_m is an (n_m,k_m) logic (for $\mathfrak{F}_{L_m}$), reaches its maximum value when $L_m=RI^{m-1}(GS)$, where $RI^0(GS)=GS$ and $RI^{i+1}(GS)=RI(RI^i(GS))$. We denote by $max_m(k_m)$ such a maximum value.

Proposition 27: Let $\{L_i\}^T_{i\geq 1}$ be a path of the tree T, and let $\overline{\overline{L}}^T$ be the second logic associated with the path. Let L be a logic such that $L\supseteq\overline{\overline{L}}^T$, let $A(p_1,\ldots,p_n)$ $(n\geq 2)$ be a wff containing only the variables $p_1,\ldots,p_n$, and let Γ be an L-saturated set. Denote by L_h be the logic of the path $\{L_i\}^T_{i\geq 1}$ such that $n_{h-1}<2^n\leq n_h$. Suppose Θ is a wff such that $\mathcal{V}_\Theta=\{p_1,\ldots,p_n\}$, $\Theta=A(p_1,\ldots,p_n)\wedge\Theta'$, and Θ' is $\mathcal{V}_\Theta$-extensively completed up to $\bar{r}$, with $\bar{r}=max_h(k_h)$. Then the underlying poset of $\mathfrak{C}_L(\Gamma)^{fin}/\Theta$ belongs to $\bigcap_{1\leq j\leq h}\mathfrak{F}_{CONE(L_j)}$.

Proof: By induction on j, we prove that the underlying poset $\underline{P}^{fin}$ of $\mathfrak{C}_L(\Gamma)^{fin}/\Theta$ belongs to $\mathfrak{F}_{CONE(L_j)}$. At the basis step, we will prove that $\underline{P}^{fin}\in$

$\mathfrak{F}_{CONE(GS)}$. Assume the contrary. Then, denoting the ordering relation of $\underline{P}^{fin}$ by $\leq_3$, as in §5, there are elements $\Gamma*'$, $\Gamma*^f_1$, $\Gamma*^f_2$ of $\underline{P}^{fin}$ such that :

(a) $Fin(\Gamma*')\supseteq\{\Gamma*^f_1,\Gamma*^f_2\}$ and, for every $\Gamma*''$ such that $\Gamma*'\leq_3\Gamma*''$ and $Fin(\Gamma*'')\supseteq\{\Gamma*^f_1,\Gamma*^f_2\}$, $\underline{P}^{fin}_{\Gamma*''}\notin\mathfrak{F}_{GS}$.

By the finiteness of $\underline{P}^{fin}$, we can assume without loss of generality that $\Gamma*'$ satisfies the following additional property:

(b) for every $\Gamma*''$ such that $\Gamma*'\leq_3\Gamma*''$ and $\Gamma*'\neq\Gamma*''$, $Fin(\Gamma*'')\not\supseteq\{\Gamma*^f_1,\Gamma*^f_2\}$.

There are two possible cases:

CASE A: $|Fin(\Gamma*')|\geq 3$ (in $\underline{P}^{fin}$).

In this case, denoting by $\Vdash_3$ the forcing of $\mathfrak{C}_L(\Gamma)^{fin}/\Theta$, we can find two wff's A_1 and A_2 of $Sf_{\wedge,\rightarrow,\neg}(\Theta)$ such that $\Gamma*^f_1\Vdash_3 A_1$, $\Gamma*^f_1\Vdash_3\neg A_2$, $\Gamma*^f_2\Vdash_3 A_2$ and $\Gamma*^f_2\Vdash_3\neg A_1$. For every final element $\Gamma*^f$ such that $\Gamma*^f\in Fin(\Gamma*')$ and $\Gamma*^f\notin\{\Gamma*^f_1,\Gamma*^f_2\}$, $\Gamma*^f\Vdash_3\neg A_1$ and $\Gamma*^f\Vdash_3\neg A_2$. Let $(FIN_2)'$ be the instance of the axiom-schema (FIN_2) given by the wff's A_1 and A_2. Let $(DE_2)'$ be any instance of the axiom-schema (DE_2). We set $(GS)'=(FIN_2)'\wedge(DE_2)'$. We prove that $\Gamma*'\Vdash(GS)'\rightarrow\neg A_1\vee\neg A_2$ (where $\Vdash$ denotes the forcing of $\mathfrak{C}_L(\Gamma)$).

Let Γ'' be any element of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma*'\leq\Gamma''$ (where $\leq$ is the ordering relation of $\mathfrak{C}_L(\Gamma)$) and $\Gamma''\Vdash(GS)'$. Then $\Gamma*'\not\equiv_\Theta\Gamma''$. As a matter of fact, since $\Gamma*'$ does not force $\neg A_1$, $\neg A_1\rightarrow\neg A_2$ and $\neg A_1\rightarrow\neg\neg A_2$ in $\mathfrak{C}_L(\Gamma)^{fin}/\Theta$, and since $\{\neg A_1,\neg A_1\rightarrow\neg A_2,\neg A_1\rightarrow\neg\neg A_2\}\subseteq Sf_{\wedge,\rightarrow,\neg}(\Theta)$, we have that $\Gamma*'$ does not force $\neg A_1$, $\neg A_1\rightarrow\neg A_2$ and $\neg A_1\rightarrow\neg\neg A_2$ in $\mathfrak{C}_L(\Gamma)$; but $\Gamma*'\equiv_\Theta\Gamma''$ implies that Γ'' does not force $\neg A_1$, $\neg A_1\rightarrow\neg A_2$ and $\neg A_1\rightarrow\neg\neg A_2$ in $\mathfrak{C}_L(\Gamma)$, from which one has $\Gamma''\nVdash(FIN_2)'$, and a fortiori $\Gamma''\nVdash(GS)'$, a contradiction. Thus, by definition of $\mathfrak{C}_L(\Gamma)^{fin}/\Theta$, there is an immediate successor $\Gamma*^s$ of $\Gamma*'$ in $\underline{P}^{fin}$ such that $\Gamma*^s\subseteq_\Theta\Gamma''$. From (b), we get that either $\Gamma*^f_1\notin Fin(\Gamma*^s)$, or $\Gamma*^f_2\notin Fin(\Gamma*^s)$. Hence, $\Gamma*^s\Vdash_3\neg A_1$, or $\Gamma*^s\Vdash_3\neg A_2$. Therefore, since $\{\neg A_1,\neg A_2\}\subseteq Sf_{\wedge,\rightarrow,\neg}(\Theta)$, we have $\Gamma''\Vdash\neg A_1$ or, $\Gamma''\Vdash\neg A_2$. Since Γ'' is any element of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma*'\leq\Gamma''$ and $\Gamma''\Vdash(GS)'$, it follows that $\Gamma*'\Vdash(GS)'\rightarrow\neg A_1\vee\neg A_2$.

Since $\Gamma*'$ is an L-saturated set, $\Gamma*'$ forces in $\mathfrak{C}_L(\Gamma)$ every instance of the axiom-schema (CONE(GS)). It follows that $\Gamma*'\Vdash\neg A_1\vee\neg A_2$, which implies $\Gamma*'\Vdash\neg A_1$ or $\Gamma*'\Vdash\neg A_2$. Since $\{\neg A_1,\neg A_2\}\subseteq Sf_{\wedge,\rightarrow,\neg}(\Theta)$, we obtain

$\Gamma*' \Vdash_3 \neg A_1$ or $\Gamma*^f \Vdash_3 \neg A_2$. But this is impossible, since $Fin(\Gamma*') \supseteq \{\Gamma*^f_1, \Gamma*^f_2\}$.

<u>CASE B</u>: $Fin(\Gamma*') = \{\Gamma*^f_1, \Gamma*^f_2\}$, and $\Gamma*'$ is not a prefinal element (in $\underline{P}^{fin}$). In such a case there is a nonfinal immediate successor $\Gamma*^s$ of $\Gamma*'$ such that $\Gamma*^s \leq_3 \Gamma*^f_1$, or $\Gamma*^s \leq_3 \Gamma*^f_2$. Let, for the sake of definiteness, $\Gamma*^s \leq_3 \Gamma*^f_1$. Since $\Gamma*' \neq \Gamma*^s$ and $\Gamma*^s$ is not final, we can find two wff's H and K of $Sf_{\wedge,\rightarrow,\neg}(\Theta)$ such that: $\Gamma*' \nVdash_3 H$, $\Gamma*^s \Vdash_3 H$ and $\Gamma*^s \nVdash_3 K \vee \neg K$. Moreover, we can find two wff's A_1 and A_2 of $Sf_{\wedge,\rightarrow,\neg}(\Theta)$ such that $\Gamma*^f_1 \Vdash_3 A_1$, $\Gamma*^f_1 \Vdash_3 \neg A_2$, $\Gamma*^f_2 \Vdash_3 A_2$, and $\Gamma*^f_2 \Vdash_3 \neg A_1$. Let $(GS)' = (FIN_2)' \wedge (DE_2)'$, where $(DE_2)'$ is the instance of (DE_2) given by the wff's H and K, and $(FIN_2)'$ is any instance of (FIN_2). Let Γ'' be an element of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma*' \leq \Gamma''$ and $\Gamma'' \Vdash (GS)'$. Let us assume that $\Gamma'' \equiv_\Theta \Gamma*'$. Then, since $H \in Sf_{\wedge,\rightarrow,\neg}(\Theta)$ and $\Gamma*' \nVdash_3 H$, we deduce $\Gamma'' \nVdash H$. On the other hand, $\Gamma*'$ is a $\leftarrow_\Theta$-terminal element of $\mathfrak{C}_L(\Gamma)$ and $\Gamma*' \leq \Gamma''$. Then there is a Γ^3 in $\mathfrak{C}_L(\Gamma)$ such that $\Gamma'' \leq \Gamma^3$ and $\Gamma^3 \equiv_\Theta \Gamma*^s$. Since $\Gamma*^s \Vdash H$, $\Gamma*^s \nVdash K$ and $\Gamma*^s \nVdash \neg K$ (because $\{H, K, \neg K\} \subseteq Sf_{\wedge,\rightarrow,\neg}(\Theta)$), we have $\Gamma^3 \Vdash H$ and $\Gamma^3 \nVdash K \vee \neg K$. Therefore, $\Gamma'' \nVdash H \vee (H \rightarrow K \vee \neg K)$, whence $\Gamma'' \nVdash (GS)'$, which contradicts the hypothesis $\Gamma'' \Vdash (GS)'$. Thus, $\Gamma'' \not\equiv_\Theta \Gamma*'$. By definition of $\mathfrak{C}_L(\Gamma)^{fin}/\Theta$, there is an immediate successor $\Gamma*''$ of $\Gamma*'$ in $\mathfrak{C}_L(\Gamma)^{fin}/\Theta$ such that $\Gamma*'' \subseteq_\Theta \Gamma''$. From Property (b) it follows that $\Gamma*'' \Vdash_3 \neg A_1$, or $\Gamma* \Vdash_3 \neg A_2$. Since $\{\neg A_1, \neg A_2\} \subseteq Sf_{\wedge,\rightarrow,\neg}(\Theta)$, $\Gamma'' \Vdash \neg A_1 \vee \neg A_2$. Since Γ'' is any element of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma*' \leq \Gamma''$ and $\Gamma'' \Vdash (GS)'$, it follows that $\Gamma*' \Vdash (GS)' \rightarrow \neg A_1 \vee \neg A_2$. This gives rise to a contradiction, as in CASE A. Thus, neither CASE A nor CASE B can hold, and $\underline{P}^{fin} \in \mathcal{F}_{CONE(GS)}$.

<u>Induction step</u>. We inductively assume that, for every j, $1 \leq j \leq m-1$ and $m \leq h$, we have $\underline{P}^{fin} \in \mathcal{F}_{CONE(L_j)}$. Assume, on the contrary, that $\underline{P}^{fin} \notin \mathcal{F}_{CONE(L_m)}$. Then there are elements $\Gamma*', \Gamma*^f_1, \ldots, \Gamma*^f_{m*}$, with $2 \leq m* \leq n_m$ (n_m being the maximum number of final elements that posets of $\mathcal{F}_{L_m}$ can have) such that:

(a') $Fin(\Gamma*') \supseteq \{\Gamma*^f_1, \ldots, \Gamma*^f_{m*}\}$, and, for every $\Gamma*''$ such that $\Gamma*' \leq_3 \Gamma*''$, if $Fin(\Gamma*'') \supseteq \{\Gamma*^f_1, \ldots, \Gamma*^f_{m*}\}$ then $\underline{P}^{fin}_{\Gamma*''} \notin \mathcal{F}_{L_m}$.

By the finiteness of $\underline{P}^{fin}$, we can further assume that $\Gamma*'$ satisfies the following property:

(b') $\underline{P}^{fin}_{\Gamma*'} \notin \mathcal{F}_{L_m}$ and, for every $\Gamma*''$ such that $\Gamma*' \leq_3 \Gamma*''$ and $\Gamma*' \neq \Gamma*''$, $Fin(\Gamma*'') \not\supseteq \{\Gamma*^f_1, \ldots, \Gamma*^f_{m*}\}$.

Now, L_m can be obtained from L_{m-1} in two different ways: $L_m = RI(L_{m-1})$, or $L_m = LE(L_{m-1})$. We will treat the two cases separately.

<u>CASE 1</u>: $L_m = RI(L_{m-1})$.

Since $\underline{P}^{fin}_{\Gamma*'} \notin \mathcal{F}_{L_m}$, the following four subcases can occur.

<u>SUBCASE 1</u>: $\underline{P}^{fin}_{\Gamma*'} \notin \mathcal{F}_{FIN_{2(n_{m-1})}}$.

For the sake of simplicity, we set $z = 2(n_{m-1})$. Since $\underline{P}^{fin}_{\Gamma*'} \notin \mathcal{F}_{FIN_z}$, there are distinct final elements $\Gamma*^f_1, \ldots, \Gamma*^f_z, \Gamma*^f_{z+1}, \ldots, \Gamma*^f_{z+s}$ of $\underline{P}^{fin}_{\Gamma*'}$, with $s \geq 1$, such that $Fin(\Gamma*') = \{\Gamma*^f_1, \ldots, \Gamma*^f_z, \Gamma*^f_{z+1}, \ldots, \Gamma*^f_{z+s}\}$. We remark that $n_m = 2(n_{m-1}) = z \geq m*$. Hence, we may assume, without loss of generality, that $\{\Gamma*^f_1, \ldots, \Gamma*^f_{m*}\} \subseteq \{\Gamma*^f_1, \ldots, \Gamma*^f_z\}$. Now we can find wff's $A_1, \ldots, A_z$ of $Sf_{\wedge,\rightarrow,\neg}(\Theta)$ such that for every i with $1 \leq i \leq z$, $\Gamma*^f_i \Vdash_3 A_i$ and, for every j with $1 \leq j \leq z+s$ and $j \neq i$, $\Gamma*^f_j \Vdash_3 \neg A_i$. Let $(FIN_z)'$ be the instance of the axiom-schema (FIN_z) given by the wff's $A_1, \ldots, A_z$. It is easy to prove that $\Gamma*' \nVdash_3 (FIN_z)'$. This means that $\Gamma*'$ does not force in $\mathfrak{C}_L(\Gamma)^{fin}/\Theta$ any of the $z+1$ disjoint wff's contained in this instance. These disjoint wff's belong to $Sf_{\wedge,\rightarrow,\neg}(\Theta)$ and so, for every Γ'' of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma'' \equiv_\Theta \Gamma*'$, we have $\Gamma'' \nVdash (FIN_z)'$. Now, let $(A_{L_m})'$ be any instance of the characteristic axiom-schema of L_m including $(FIN_z)'$ among its conjuncts. Let Γ'' be an element of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma*' \leq \Gamma''$ and $\Gamma'' \Vdash (A_{L_m})'$. By the above discussion, we get $\Gamma*' \not\equiv_\Theta \Gamma''$. Then, by definition of $\mathfrak{C}_L(\Gamma)^{fin}/\Theta$, there is an immediate successor $\Gamma*^s$ of $\Gamma*'$ in $\underline{P}^{fin}$ such that $\Gamma*^s \subseteq_\Theta \Gamma''$. From Property (b') and from our choice of $A_1, \ldots, A_z$, it follows that there is an i with $1 \leq i \leq m*$, such that $\Gamma*^s \Vdash \neg A_i$, and hence $\Gamma'' \Vdash \neg A_i$. Since Γ'' is an arbitrary element of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma*' \leq \Gamma''$ and $\Gamma'' \Vdash (A_{L_m})'$, it follows that $\Gamma*' \Vdash (A_{L_m})' \rightarrow \neg A_1 \vee \ldots \vee \neg A_{m*}$. Since $\Gamma*'$ is an L-saturated set, $\Gamma*'$ forces in $\mathfrak{C}_L(\Gamma)$ every instance of the axiom-schema $(CONE(L_m))$. From this we obtain $\Gamma*' \Vdash \neg A_1 \vee \ldots \vee \neg A_{m*}$. It follows that there is an i with $1 \leq i \leq m*$ such that $\Gamma*' \Vdash \neg A_i$. Since $\neg A_i \in Sf_{\wedge,\rightarrow,\neg}(\Theta)$, we get $\Gamma*' \Vdash_3 \neg A_i$, which contradicts the fact that there exists (in $\underline{P}^{fin}$) a $\Gamma*^f \in \{\Gamma*^f_1, \ldots, \Gamma*^f_{m*}\}$ such that $\Gamma*' \leq_3 \Gamma*^f$ and $\Gamma*^f \Vdash_3 A_i$. Thus, SUBCASE 1 cannot hold.

SUBCASE 2: $\underline{P}^{fin}{}_{\Gamma*'}\notin\mathfrak{F}_{CONE(L_{m-1})}$.

This subcase cannot hold. For, by induction hypothesis, $\underline{P}^{fin}\in\mathfrak{F}_{CONE(L_{m-1})}$. A fortiori, $\underline{P}^{fin}{}_{\Gamma*'}\in\mathfrak{F}_{CONE(L_{m-1})}$.

SUBCASE 3: $\underline{P}^{fin}{}_{\Gamma*'}\notin\mathfrak{F}_{DE_{n_{m-1}+k_{m-1}}}$.

For the sake of simplicity, we set $z=n_{m-1}+k_{m-1}$. Hence $\underline{P}^{fin}{}_{\Gamma*'}\notin\mathfrak{F}_{DE_z}$. This means that the depth of $\Gamma*'$ in $\mathfrak{G}_L(\Gamma)^{fin}/\Theta$ is at least z+1. Then, there are some elements $\Gamma*'_1,\Gamma*'_2,\ldots,\Gamma*'_z$ of $\underline{P}^{fin}{}_{\Gamma*'}$ such that:

(1) $\Gamma*'<_3\Gamma*'_z<_3\Gamma*'_{z-1}<_3\ldots<_3\Gamma*'_1$, where $\Gamma*'<_3\Gamma*'_z$ means $\Gamma*'\leq_3\Gamma*'_z$ and $\Gamma*'\neq\Gamma*'_z$, etc.;

(2) $\Gamma*'_1$ is a final element of $\underline{P}^{fin}{}_{\Gamma*'}$;

(3) for every j with $1\leq j\leq z$, $\Gamma*'_j$ has depth exactly equal to j in $\underline{P}^{fin}{}_{\Gamma*'}$.

Now, if $\Gamma*''$ has depth r in $\underline{P}^{fin}{}_{\Gamma*'}$, then the $\mathcal{V}_\Theta$-rank of $\Gamma*''$ (in $\mathfrak{G}_L(\Gamma)^{fin}/\Theta$) is r-1. Also, by our choice of $\bar{r}$, we have $z\leq\bar{r}$, because $\bar{r}=\max_h(k_h)$ and $m\leq h$. Let j be such that $1\leq j\leq z$. Consider the distinct $\overset{v}{\underset{\leq\bar{r}}{\equiv}}$-equivalence classes $[\tilde{\Gamma}_1]^v{}_{\leq\bar{r}},\ldots,[\tilde{\Gamma}_t]^v{}_{\leq\bar{r}}$ (where $v=\mathcal{V}_\Theta$) such that, for every i with $1\leq i\leq t$, $\Gamma*'_j\not\leq^v{}_{\bar{r}}\tilde{\Gamma}_i$. We already know that the number of such classes is finite. By Proposition 24, we can find, for every i with $1\leq i\leq t$, a wff $\tilde{H}_{j,i}$ containing only the variables of $v=\mathcal{V}_\Theta$ such that, for every α with $\alpha\in[\Gamma*'_j]^v{}_{\leq\bar{r}}$, $\alpha\Vdash\tilde{H}_{j,i}$ in its own model, and, for every α with $\alpha\in[\tilde{\Gamma}_i]^v{}_{\leq\bar{r}}$, $\alpha\nVdash\tilde{H}_{j,i}$ in its own model. Now, let $K_j=\tilde{H}_{j,1}\wedge\ldots\wedge\tilde{H}_{j,t}$. Then, for every α such that $\alpha\in[\Gamma*'_j]^v{}_{\leq\bar{r}}$, $\alpha\Vdash K_j$ in its own model, while, for every i and every element α such that $\alpha\in[\tilde{\Gamma}_i]^v{}_{\leq\bar{r}}$, $\alpha\nVdash K_j$ in its own model. By our choice of Θ (to the effect that Θ is extensively completed up to $\bar{r}$), we can assume, without loss of generality, that K_j belongs to $Sf_{\wedge,\rightarrow,\neg}(\Theta)$ and that $\Gamma*'\nVdash_3 K_z$ in $\mathfrak{G}_L(\Gamma)^{fin}/\Theta$. As a matter of fact, if $\Gamma*'\Vdash_3 K_z$, choose a $K*\in Sf_{\wedge,\rightarrow,\neg}(\Theta)$ such that $\Gamma*'\nVdash_3 K*$ and $\Gamma*'_z\Vdash_3 K*$, and use the wff $K'_z=K_z\wedge K*\in Sf_{\wedge,\rightarrow,\neg}(\Theta)$ instead of K_z. Now, consider any element Γ'' of $\mathfrak{G}_L(\Gamma)$ such that $\Gamma*'\leq\Gamma''$ and $\Gamma*'\equiv_\Theta\Gamma''$. Since $\Gamma*'$ is a $\leftarrow_\Theta$-terminal element, there is Γ_z in $\mathfrak{G}_L(\Gamma)$ such that $\Gamma''\leq\Gamma_z$ and $\Gamma*'_z\equiv_\Theta\Gamma_z$.

Since the depth of $\Gamma*'_z$ in $\underline{P}^{fin}$ is z, the state $\Gamma*'_z$, regarded as an element of $\mathfrak{G}_L(\Gamma)^{fin}/\Theta$, has a v-rank $z-1<\bar{r}$. On the other hand, we do not know whether Γ_z, as an element of the (possibly infinite) model $\mathfrak{G}_L(\Gamma)$, has a v-rank at all, or has a v-rank $\leq\bar{r}$. Let us suppose that Γ_z either has no v-rank or has a v-rank $>\bar{r}$. Then, by Proposition 22, there is Γ'_z in $\mathfrak{G}_L(\Gamma)$ such that $\Gamma_z\leq\Gamma'_z$, and Γ'_z (in $\mathfrak{G}_L(\Gamma)$) has the same v-rank as $\Gamma*'_z$ (in $\mathfrak{G}_L(\Gamma)^{fin}/\Theta$). Since $K_z\in Sf_{\wedge,\rightarrow,\neg}(\Theta)$ and $\Gamma*'_z\Vdash_3 K_z$, we have $\Gamma*'_z\Vdash K_z$ in $\mathfrak{G}_L(\Gamma)$. Since $\Gamma*'_z\equiv_\Theta\Gamma_z$, it follows that $\Gamma_z\Vdash K_z$ in $\mathfrak{G}_L(\Gamma)$. Hence, from $\Gamma_z\leq\Gamma'_z$, we get $\Gamma'_z\Vdash K_z$ in $\mathfrak{G}_L(\Gamma)$. Since Γ'_z has a v-rank $\leq\bar{r}$, it follows that $\Gamma*'_z\leq^v{}_{\bar{r}}\Gamma'_z$. Indeed, if $\Gamma*'_z\not\leq^v{}_{\bar{r}}\Gamma'_z$, then $\Gamma'_z\nVdash K_z$ in $\mathfrak{G}_L(\Gamma)$, since, taking j=z, Γ'_z is in one of the equivalence classes $[\tilde{\Gamma}_1]^v{}_{\leq\bar{r}},\ldots,[\tilde{\Gamma}_t]^v{}_{\leq\bar{r}}$). Thus, since $\Gamma*'_z\leq^v{}_{\bar{r}}\Gamma'_z$ and Γ'_z has the same v-rank as $\Gamma*'_z$, we get that $\Gamma*'_z\overset{v}{\underset{\leq\bar{r}}{\equiv}}\Gamma'_z$. We have proved that, if Γ_z has no v-rank or has a v-rank $>\bar{r}$, then there is Γ'_z in $\mathfrak{G}_L(\Gamma)$ such that $\Gamma*'_z\overset{v}{\underset{\leq\bar{r}}{\equiv}}\Gamma'_z$ and $\Gamma_z\leq\Gamma'_z$. On the other hand, if Γ_z has a v-rank $\leq\bar{r}$, then $\Gamma*'_z\overset{v}{\underset{\leq\bar{r}}{\equiv}}\Gamma_z$. As a matter of fact, since $\Gamma*'_z\equiv_\Theta\Gamma_z$, we cannot have $\Gamma*'_z\not\leq^v{}_{\bar{r}}\Gamma_z$, for otherwise, $\Gamma*'_z\Vdash K_z$ and $\Gamma_z\nVdash K_z$ with $K_z\in Sf_{\wedge,\rightarrow,\neg}(\Theta)$. For a similar reason, since Θ is extensively completed up to $\bar{r}$, we cannot have $\Gamma_z\not\leq^v{}_{\bar{r}}\Gamma*'_z$. Thus, whether or not Γ_z has a v-rank $\leq\bar{r}$, we see that in $\mathfrak{G}_L(\Gamma)$ there is a Γ'_z such that $\Gamma''\leq\Gamma_z\leq\Gamma'_z$ and $\Gamma*'_z\overset{v}{\underset{\leq\bar{r}}{\equiv}}\Gamma'_z$. By definition of $\overset{v}{\underset{\leq\bar{r}}{\equiv}}$, there is Γ'_{z-1} such that $\Gamma'_z\leq\Gamma'_{z-1}$ and $\Gamma'_{z-1}\overset{v}{\underset{\leq\bar{r}}{\equiv}}\Gamma*'_{z-1}$. Here, $\Gamma*'_{z-1}$ is regarded as an element of $\mathfrak{G}_L(\Gamma)^{fin}/\Theta$, and Γ'_{z-1} as an element of $\mathfrak{G}_L(\Gamma)$. Indeed, in $\mathfrak{G}_L(\Gamma)^{fin}/\Theta$ one has that $\Gamma*'_z\leq_3\Gamma*'_{z-1}$ and the v-rank of $\Gamma*'_{z-1}$ is less than the v-rank of $\Gamma*'_z$. We then see that, if $\Gamma*'\equiv_\Theta\Gamma''$, then in $\mathfrak{G}_L(\Gamma)$ there are elements $\Gamma'_z,\ldots,\Gamma'_1$ such that: $\Gamma''<\Gamma'_z<\Gamma'_{z-1}<\ldots<\Gamma'_1$ and $\Gamma*'_z\overset{v}{\underset{\leq\bar{r}}{\equiv}}\Gamma'_z$, $\Gamma*'_{z-1}\overset{v}{\underset{\leq\bar{r}}{\equiv}}\Gamma'_{z-1},\ldots,\Gamma*'_1\overset{v}{\underset{\leq\bar{r}}{\equiv}}\Gamma'_1$. Thus, for every i with $1<i\le z$, $\Gamma'_i \Vdash K_i$ in $\mathfrak{C}_L(\Gamma)$ and, for every j with $1\le j<i$, $\Gamma'_i \nVdash K_j$ in $\mathfrak{C}_L(\Gamma)$.

Now, let $(DE_z)'$ be the instance of (DE_z) corresponding to the wff's $K_1,\dots,K_z$. By the above discussion, it follows that, for every element Γ'' of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma^{*\prime}\le\Gamma''$ and $\Gamma^{*\prime}\equiv_\Theta\Gamma''$, $\Gamma''\nVdash(DE_z)'$ in $\mathfrak{C}_L(\Gamma)$. Furthermore, let $(A_{L_m})'$ be any instance of the characteristic axiom-schema of L_m containing $(DE_z)'$ as one of its conjuncts. Then, for every $\Gamma^{*\prime}\le\Gamma''$ such that $\Gamma^{*\prime}\equiv_\Theta\Gamma''$, $\Gamma''\nVdash(A_{L_m})'$ in $\mathfrak{C}_L(\Gamma)$.

Finally, let $A_1,\dots,A_{m^*}$ be wff's of $Sf_{\wedge,\rightarrow,\neg}(\Theta)$ such that, for every i with $1\le i\le m^*$, $\Gamma^{*f}_i\Vdash_3 A_i$, and, for every Γ^{*f} in $\underline{P}^{fin}$ such that $\Gamma^{*f}\neq\Gamma^{*f}_i$ and $\Gamma^{*f}\in Fin(\Gamma^{*\prime})$, $\Gamma^{*f}\Vdash_3\neg A_i$. In particular, for every j with $1\le j\le m^*$ and $j\neq i$, $\Gamma^{*f}_j\Vdash_3\neg A_i$. If Γ'' is an element of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma^{*\prime}\le\Gamma''$ and $\Gamma''\Vdash(A_{L_m})'$, we have $\Gamma^{*\prime}\not\equiv_\Theta\Gamma''$. Then, by definition of $\mathfrak{C}_L(\Gamma)^{fin}/\Theta$, there is an immediate successor Γ^{*s} of $\Gamma^{*\prime}$ in $\underline{P}^{fin}$ such that $\Gamma^{*s}\subseteq_\Theta\Gamma''$. From (b') and the properties of $A_1,\dots,A_{m^*}$, it follows that there is an i with $1\le i\le m^*$ such that $\Gamma^{*s}\Vdash\neg A_i$ in $\mathfrak{C}_L(\Gamma)$. Hence $\Gamma''\Vdash\neg A_i$ in $\mathfrak{C}_L(\Gamma)$. Since Γ'' is an arbitrary element of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma^{*\prime}\le\Gamma''$ and $\Gamma''\Vdash(A_{L_m})'$, we get that $\Gamma^{*\prime}\Vdash(A_{L_m})'\rightarrow\neg A_1\vee\dots\vee\neg A_{m^*}$ in $\mathfrak{C}_L(\Gamma)$. Since $\Gamma^{*\prime}$ is an L-saturated set, $\Gamma^{*\prime}$ forces every instance of $(CONE(L_m))$ in $\mathfrak{C}_L(\Gamma)$. It follows that $\Gamma^{*\prime}\Vdash\neg A_1\vee\dots\vee\neg A_{m^*}$ in $\mathfrak{C}_L(\Gamma)$ and, arguing as in the previous cases, we obtain a contradiction. Hence SUBCASE 3 cannot hold.

SUBCASE 4: $\underline{P}^{fin}_{\Gamma^{*\prime}}\notin\mathfrak{F}_{DIFFIN_{n_{m-1}},2(n_{m-1})-1}$.

For the sake of simplicity, we set $z=n_{m-1}$. One can easily prove that there is an instance $(DIFFIN_{z,2z-1})'$ of $(DIFFIN_{z,2z-1})$, only containing the variables $p_1,\dots,p_n$, such that $\Gamma^{*\prime}\nVdash_3(DIFFIN_{z,2z-1})'$. We note that any instance of $(DIFFIN_{z,2z-1})$ is a negatively saturated formula (see § 1) and that (being Θ an extensively completed wff) every instance of such a schema with wff's of $Sf_{\wedge,\rightarrow,\neg}(\Theta)$ is intuitionisically equivalent to a wff of $Sf_{\wedge,\rightarrow,\neg}(\Theta)$. Assume for simplicity that $(DIFFIN_{z,2z-1})'$ is such a wff. This is no loss of generality. It follows that, for every Γ'' of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma^{*\prime}\le\Gamma''$ and $\Gamma^{*\prime}\equiv_\Theta\Gamma''$, $\Gamma''\nVdash(DIFFIN_{z,2z-1})'$. Now, let $(A_{L_m})'$ be any instance of the characteristic axiom-schema of L_m containing $(DIFFIN_{z,2z-1})'$ among its conjuncts. Then, for

every Γ'' such that $\Gamma^{*\prime}\le\Gamma''$ and $\Gamma^{*\prime}\equiv_\Theta\Gamma''$, we get $\Gamma''\nVdash(A_{L_m})'$. Let $A_1,\dots,A_{m^*}$ be wff's of $Sf_{\wedge,\rightarrow,\neg}(\Theta)$ such that, for every i with $1\le i\le m^*$, $\Gamma^{*f}_i\Vdash_3 A_i$, and, for every final state $\Gamma^{*f}\neq\Gamma^{*f}_i$ of $Fin(\Gamma^{*\prime})$ in $\underline{P}^{fin}_{\Gamma^{*\prime}}$, $\Gamma^{*f}\Vdash_3\neg A_i$. Let Γ'' be an element of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma^{*\prime}\le\Gamma''$ and $\Gamma''\Vdash(A_{L_m})'$. By the above remarks, we have $\Gamma^{*\prime}\not\equiv_\Theta\Gamma''$. Then, by definition of $\mathfrak{C}_L(\Gamma)^{fin}/\Theta$, there is an immediate successor Γ^{*s} of $\Gamma^{*\prime}$ in $\underline{P}^{fin}$ such that $\Gamma^{*s}\subseteq_\Theta\Gamma''$. From (b') and the properties of $A_1,\dots,A_{m^*}$, it follows that there is an i with $1\le i\le m^*$, such that $\Gamma^{*s}\Vdash\neg A_i$. Hence, $\Gamma''\Vdash\neg A_i$. Since Γ'' is an arbitrary element of $\mathfrak{C}_L(\Gamma)$ such that $\Gamma^{*\prime}\le\Gamma''$ and $\Gamma''\Vdash(A_{L_m})'$, we get that $\Gamma^{*\prime}\Vdash(A_{L_m})'\rightarrow\neg A_1\vee\dots\vee\neg A_{m^*}$. But $\Gamma^{*\prime}$ is an L-saturated set, and so $\Gamma^{*\prime}$ forces every instance of $(CONE(L_m))$ in $\mathfrak{C}_L(\Gamma)$. It follows that $\Gamma^{*\prime}\Vdash\neg A_1\vee\dots\vee\neg A_{m^*}$, which is a contradiction. Thus, SUBCASE 4 cannot hold.

Since Subcases 1-4 cannot hold, we have that, if $L_m=RI(L_m)$, then $\underline{P}^{fin}\in\mathfrak{F}_{CONE(L_m)}$.

CASE 2: $L_m=LE(L_m)$.

The proof is essentially the same as for CASE 1.□

From Propositions 26 and 27 we obtain:

Proposition 28: Let $\overline{\overline{L}}^T$ be the second logic associated with the path $\{L_i\}^T_{i\ge1}$. Let L be a logic such that $L\supseteq\overline{\overline{L}}^T$. Let $A(p_1,\dots,p_n)$ be a wff containing only the variables $p_1,\dots,p_n$ $(n\ge2)$, and let Γ be a L-saturated set. Denote by L_h the logic of the path $\{L_i\}^T_{i\ge1}$ such that $n_{h-1}<2^n\le n_h$, and let Θ be a wff such that $\mathcal{V}_\Theta=\{p_1,\dots,p_n\}$, $\Theta=A(p_1,\dots,p_n)\wedge\Theta'$ and Θ' is a $\mathcal{V}_\Theta$-exstensively completed wff up to $\bar{r}$, with $\bar{r}=max_h(k_h)$. Then the underlying poset of $\mathfrak{C}_L(\Gamma)^{fin}/\Theta$ belongs to $\mathfrak{F}_{\overline{\overline{L}}^T}$.□

As a corollary of Proposition 28, we obtain:

Corollary 1: Let $\overline{\overline{L}}^T$ be the second logic associated with the path $\{L_i\}^T_{i\ge1}$. Then $\mathfrak{L}(\mathfrak{F}_{\overline{\overline{L}}^T})\subseteq\overline{\overline{L}}^T$.□

From Proposition 25 and Corollary 1, we have:

Theorem 5: Let $\overline{\overline{L}}^T$ be the second logic associated with the path $\{L_i\}^T_{i\geq 1}$. Then $\mathcal{L}(\mathcal{F}_{\overline{\overline{L}}^T})=\overline{\overline{L}}^T$.□

Now, we are in a position to prove the main consequence of Proposition 27, from which we will obtain the constructive incompatibility of the two (first) logics associated with two paths of the tree T.

Theorem 6: Let $\{L_i^1\}^T_{i\geq 1}$ and $\{L_i^2\}^T_{i\geq 1}$ be two distinct paths of the tree T. Let $\overline{\overline{L}}^T_1$ be the second logic associated with $\{L_i^1\}^T_{i\geq 1}$, and let $\overline{\overline{L}}^T_2$ be the second logic associated with $\{L_i^2\}^T_{i\geq 1}$. Then $\overline{\overline{L}}^T_1+\overline{\overline{L}}^T_2=\mathcal{L}(\mathcal{F}_{\overline{\overline{L}}^T_1}\cap\mathcal{F}_{\overline{\overline{L}}^T_2})$.

Proof: Let $A(p_1,\dots,p_n)$ $(n\geq 2)$ be a wff containing only the variables $p_1,\dots,p_n$. Let L^1_{h*} be the logic of the path $\{L_i^1\}^T_{i\geq 1}$ such that $n_{h*-1}<2^n\leq n_{h*}$ and let L^2_{h**} be the logic of the path $\{L_i^2\}^T_{i\geq 1}$ such that $n_{h**-1}<2^n\leq n_{h**}$. Let $\bar{r}_1=\max_{h*}(k_{h*})$ and $\bar{r}_2=\max_{h**}(k_{h**})$. We set $\bar{r}=\max(\bar{r}_1,\bar{r}_2)$ (we incidentally remark that $\bar{r}=\bar{r}_1=\bar{r}_2$). Also, let $\Theta=A(p_1,\dots,p_n)\wedge\Theta'$, where Θ' is v-extensively completed up to $\bar{r}$, with $v=\mathcal{V}_\Theta=\{p_1,\dots,p_n\}$. Then Θ and Θ' are v-extensively completed respectively up to $\bar{r}_1$ and up to $\bar{r}_2$. Now, let us consider any logic L including both second logics $\overline{\overline{L}}^T_1$ and $\overline{\overline{L}}^T_2$. Let Γ be any L-saturated set. Then, applying Proposition 27 with reference to the path $\{L_i^1\}^T_{i\geq 1}$ (with the related logic $\overline{\overline{L}}^T_1$) and with reference to the path $\{L_i^2\}^T_{i\geq 1}$ (with the related logic $\overline{\overline{L}}^T_2$) and using Proposition 26, we deduce that $\mathcal{C}_L(\Gamma)^{fin}/\Theta$ is built on a poset belonging to $\mathcal{F}_{\overline{\overline{L}}^T_1}\cap\mathcal{F}_{\overline{\overline{L}}^T_2}$. From this we get that $\mathcal{L}(\mathcal{F}_{\overline{\overline{L}}^T_1}\cap\mathcal{F}_{\overline{\overline{L}}^T_2})\subseteq\overline{\overline{L}}^T_1+\overline{\overline{L}}^T_2$ (we have just to set $L=\overline{\overline{L}}^T_1+\overline{\overline{L}}^T_2$, etc.). Finally,

since $\overline{\overline{L}}^T_1\subseteq\mathcal{L}(\mathcal{F}_{\overline{\overline{L}}^T_1})\subseteq\mathcal{L}(\mathcal{F}_{\overline{\overline{L}}^T_1}\cap\mathcal{F}_{\overline{\overline{L}}^T_2})$ and $\overline{\overline{L}}^T_2\subseteq\mathcal{L}(\mathcal{F}_{\overline{\overline{L}}^T_2})\subseteq\mathcal{L}(\mathcal{F}_{\overline{\overline{L}}^T_1}\cap\mathcal{F}_{\overline{\overline{L}}^T_2})$, we get $\overline{\overline{L}}^T_1+\overline{\overline{L}}^T_2\subseteq\mathcal{L}(\mathcal{F}_{\overline{\overline{L}}^T_1}\cap\mathcal{F}_{\overline{\overline{L}}^T_2})$. Thus, $\overline{\overline{L}}^T_1+\overline{\overline{L}}^T_2=\mathcal{L}(\mathcal{F}_{\overline{\overline{L}}^T_1}\cap\mathcal{F}_{\overline{\overline{L}}^T_2})$.□

Theorem 6 allows to "syntactically overlap" the logics $\overline{\overline{L}}^T_1$ and $\overline{\overline{L}}^T_2$ into the logic $\overline{\overline{L}}^T_1+\overline{\overline{L}}^T_2$, in such a way that the resulting logic can be characterized by a "compound semantics" coinciding with the intersection of the semantics characterizing the two logics. We do not see how to prove an analogous result for the (first) logics $\overline{L}^T$ associated with the paths of the tree T. This is the reason why we have introduced the auxiliary logics $\overline{\overline{L}}^T$.

8: On the cardinality of the set of maximal intermediate constructive propositional logics.

Definition 16: Let $\{L_i^1\}^T_{i\geq 1}$ and $\{L_i^2\}^T_{i\geq 1}$ be two distinct paths of the tree T. We say that L is the *last logic common to* $\{L_i^1\}^T_{i\geq 1}$ and $\{L_i^2\}^T_{i\geq 1}$ iff L belongs to both paths, $RI(L)\in\{L_i^1\}^T_{i\geq 1}$ and $LE(L)\in\{L_i^2\}^T_{i\geq 1}$, or $RI(L)\in\{L_i^2\}^T_{i\geq 1}$ and $LE(L)\in\{L_i^1\}^T_{i\geq 1}$.

Since L_1 belongs to every path of the tree T, we get that, for every pair of distinct paths of the tree T, there exists the (uniquely determined) last logic common to both paths.

Proposition 29: Let $\overline{\overline{L}}^T_1$ be the second logic associated with $\{L_i^1\}^T_{i\geq 1}$, and let $\overline{\overline{L}}^T_2$ be the second logic associated with $\{L_i^2\}^T_{i\geq 1}$, where $\{L_i^1\}^T_{i\geq 1}$ and $\{L_i^2\}^T_{i\geq 1}$ are distinct paths of the tree T. Let L_m be the last logic common to $\{L_i^1\}^T_{i\geq 1}$ and

$\{L^2_i\}^T_{i\geq 1}$, and let L_m be an (n,k) logic with respect to $\mathfrak{F}_{L_m}$. Then, for every poset $\underline{P}\in\mathfrak{F}_{\overline{\overline{L}}^T_1}\cap\mathfrak{F}_{\overline{\overline{L}}^T_2}$, $\underline{P}$ has at most $n+1$ final elements.

<u>Proof</u>: Otherwise, there is a poset $\underline{P}=\langle P,\leq,r\rangle\in\mathfrak{F}_{\overline{\overline{L}}^T_1}\cap\mathfrak{F}_{\overline{\overline{L}}^T_2}$ such that $|Fin(r)|\geq n+2$ (where $n+2\leq 2n$). For the sake of definiteness, suppose that $LE(L_m)\in\{L^1_i\}^T_{i\geq 1}$ and $RI(L_m)\in\{L^2_i\}^T_{i\geq 1}$. From $\underline{P}\in\mathfrak{F}_{\overline{\overline{L}}^T_1}$ we get

(i) $\underline{P}\in\mathfrak{F}_{CONE(LE(L_m))_{2n}}$.

From $\underline{P}\in\mathfrak{F}_{\overline{\overline{L}}^T_2}$, we get

(ii) $\underline{P}\in\mathfrak{F}_{CONE(RI(L_m))_{2n}}$.

From (i) and from $|Fin(r)|\geq n+2$ we see that there is an element $\alpha\in P$ such that $|Fin(\alpha)|\geq n+2$ and $\underline{P}_\alpha\in\mathfrak{F}_{LE(L_m)}$. Since $\underline{P}_\alpha\in\mathfrak{F}_{LE(L_m)}$ implies $\underline{P}_\alpha\in\mathfrak{F}_{DE_{k+1}}$, we have

(iii) α has at most depth $k+1$ in $\underline{P}$.

From (ii) it follows that $\underline{P}_\alpha$ belongs also to $\mathfrak{F}_{CONE(RI(L_m))_{2n}}$. Hence, there is an element $\beta\in P$ such that $\alpha\leq\beta$, $|Fin(\beta)|\geq n+2$, and $\underline{P}_\beta\in\mathfrak{F}_{RI(L_m)}$. But $\underline{P}_\beta\in\mathfrak{F}_{RI(L_m)}$ implies $\underline{P}_\beta\in\mathfrak{F}_{DIFFIN_{n,2n-1}}$, whence, in $\underline{P}_\beta$, there is an element β_1 such that $\beta<\beta_1$ and $|Fin(\beta_1)|=n+1$. Moreover, $\underline{P}_\beta\in\mathfrak{F}_{RI(L_m)}$ implies $\underline{P}_\beta\in\mathfrak{F}_{CONE(L_m)_n}$. Hence $\underline{P}_{\beta 1}\in\mathfrak{F}_{CONE(L_m)_n}$ (because $\beta\leq\beta_1$). Since $|Fin(\beta_1)|=n+1$, there is β_2 such that $\beta_1<\beta_2$, $|Fin(\beta_2)|=n$, and $\underline{P}_{\beta_2}\in\mathfrak{F}_{L_m}$. This implies that β_2 has depth k in $\underline{P}$. As a matter of fact, by induction on $m\geq 1$, one sees that, if the m-th logic L_m is an (n_m,k_m) logic with respect to $\mathfrak{F}_{L_m}$, then the root of any poset of $\mathfrak{F}_{L_m}$ with exactly n_m final elements has a depth $=k_m$. Hence β_1 has a depth $\geq k+1$ in $\underline{P}$, whence

(iv) β has a depth $\geq k+2$ in $\underline{P}$.

Since $\alpha\leq\beta$, from (iii) and (iv) we obtain a contradiction. Then, $|Fin(r)|\geq n+2$ cannot hold.□

From Theorem 6, Proposition 29, and the characterization of the logics FIN_k given in §1, we immediately obtain:

Corollary 2: Let $\overline{\overline{L}}^T_1$ be the second logic associated with $\{L^1_i\}^T_{i\geq 1}$, and let $\overline{\overline{L}}^T_2$ be the second logic associated with $\{L^2_i\}^T_{i\geq 1}$, when $\{L^1_i\}^T_{i\geq 1}$ and $\{L^2_i\}^T_{i\geq 1}$ are two distinct paths of the tree T. Then there is $k\geq 3$ such that $FIN_k\subseteq\overline{\overline{L}}^T_1+\overline{\overline{L}}^T_2$.□

To prove that the second logics associated with two different paths of the tree T cannot be jointly extended to a constructive logic, we prepare:

Proposition 30: Let $k\geq 3$, and let L be a logic such that $FIN_k\subseteq L$. Then L is not a constructive logic.

<u>Proof</u>: The logic FIN_k contains all instances of the axiom-schema (FIN_k). A fortiori, L contains all instances of the axiom-schema (FIN_k). Hence, L contains in particular the wff

$$H=\neg p_1\vee(\neg p_1\rightarrow\neg p_2)\vee(\neg p_1\wedge\neg p_2\rightarrow\neg p_3)\vee\ldots\vee(\neg p_1\wedge\ldots\wedge\neg p_{k-1}\rightarrow\neg p_k)\vee$$
$$\vee(\neg p_1\wedge\ldots\wedge\neg p_{k-1}\rightarrow\neg\neg p_k),$$

where $p_1,\ldots,p_k$ are distinct propositional variables. Now, H is a disjunction of $k+1$ (negatively saturated) wff's, none of which is a classical tautology. If L were a constructive logic, then one of these disjuncts would belong to L, hence $L\not\subseteq CL$, a contradiction.□

From Corollary 2 and Proposition 30 we immediately get:

Corollary 3: Let $\{L^1_i\}^T_{i\geq 1}$ and $\{L^2_i\}^T_{i\geq 1}$ be two distinct paths of the tree T, and let $\overline{\overline{L}}^T_1$ and $\overline{\overline{L}}^T_2$ be the second logics associated with these paths. Then there is no constructive logic L such that $\overline{\overline{L}}^T_1+\overline{\overline{L}}^T_2\subseteq L$.□

Again, let us remark that the proof of the previous proposition depends on the fact that $\overline{\overline{L}}^T_1+\overline{\overline{L}}^T_2=\mathfrak{L}(\mathfrak{F}_{\overline{\overline{L}}^T_1}\cap\mathfrak{F}_{\overline{\overline{L}}^T_2})$. Without this result, we can only prove that $\mathfrak{L}(\mathfrak{F}_{\overline{\overline{L}}^T_1}\cap\mathfrak{F}_{\overline{\overline{L}}^T_2})$ does not admit any constructive extension in the set of the constructive logics (but if $\overline{\overline{L}}^{T}_{1}+\overline{\overline{L}}^{T}_{2}\neq\mathfrak{I}(\mathfrak{F}_{\overline{\overline{L}}^{T}_{1}}\cap\mathfrak{F}_{\overline{\overline{L}}^{T}_{2}})$ would hold, then a constructive extension of $\overline{\overline{L}}^{T}_{1}+\overline{\overline{L}}^{T}_{2}$ might exist).

Proposition 31: Let $\overline{L}^{T}$ and $\overline{\overline{L}}^{T}$ be the first logic associated with the path $\{L_i\}^{T}_{i\geq 1}$ of the tree T and the second logic associated with $\{L_i\}^{T}_{i\geq 1}$ respectively. Then $\overline{L}^{T}\supseteq\overline{\overline{L}}^{T}$.

Proof: Let $\underline{P}\in\mathfrak{F}_{\overline{L}^{T}}$. Then there is an integer $i\geq 1$ such that $\underline{P}\in\mathfrak{F}_{L_i}$. Let (n_i,k_i) be the parameters associated with L_i. It easily follows that, for every $j\geq i$, $\underline{P}\in\mathfrak{F}_{CONE(L_j)_{n_j}}$. Further, from $\underline{P}\in\mathfrak{F}_{L_i}$ we deduce that $\underline{P}\in\mathfrak{F}_{CONE(L_{i-1})_{n_{i-1}}}$ and so, inductively, we deduce that, for every h with $1\leq h\leq i-1$, $\underline{P}\in\mathfrak{F}_{CONE(L_h)_{n_h}}$. Hence $\underline{P}\in\bigcap_{i\geq 1}\mathfrak{F}_{CONE(L_i)_{n_i}}$, that is, $\underline{P}\in\mathfrak{F}_{\overline{\overline{L}}^{T}}$. From $\mathfrak{F}_{\overline{\overline{L}}^{T}}\supseteq\mathfrak{F}_{\overline{L}^{T}}$ it follows that $\overline{L}^{T}\supseteq\overline{\overline{L}}^{T}$.□

Proposition 32: Let $\{L^1_i\}^{T}_{i\geq 1}$ and $\{L^2_i\}^{T}_{i\geq 1}$ be two distinct paths of the tree T, and let $\overline{L}^{T}_{1}$ and $\overline{L}^{T}_{2}$ be the first logics associated with these paths. Then there is no constructive logic L such that $\overline{L}^{T}_{1}+\overline{L}^{T}_{2}\subseteq L$.

Proof: Directly from Corollary 3 and Proposition 31.□

Thus, there are $2^{\aleph_0}$ distinct paths of the tree T, and, by Theorem 4, we are able to associate a constructive logic $\overline{L}^{T}$ with every path of the tree T, in such a way that constructively incompatible constructive logics are associated with different paths.

Using Zorn's lemma, we have (see, e.g., [9]):

Proposition 33: For every constructive logic L there is a maximal constructive logic L^* such that $L\subseteq L^*$.□

According to Proposition 33, with every path of the tree T we can associate a maximal constructive logic $\overline{L}^{T*}\supseteq\overline{L}^{T}$, the latter being the first logic associated with the path. By Proposition 32, we deduce that, with any two paths of the tree T, two distinct maximal constructive logics are associated. Now, the set of maximal constructive logics does not contain more logics than the set of the constructive logics, and the latter, in turn, does not contain more logics than the set of logics; further, the cardinality of the set of logics is not greater than the cardinality of the power set $\mathfrak{P}(WFF)$ of the set WFF of wff's. We then have:

Theorem 7: There exist exactly $2^{\aleph_0}$ maximal constructive logics.□

It is not difficult to prove that the second logics $\overline{\overline{L}}^{T}$ associated with the paths of T are constructive logics. Indeed, any two finite and disjoint elements of $\mathfrak{F}_{\overline{\overline{L}}^{T}}$ can be combined into an element of $\mathfrak{F}_{\overline{\overline{L}}^{T}}$, in such a way that the constructiveness of $\overline{\overline{L}}^{T}$ can be guaranteed. Thus, the logics $\overline{\overline{L}}^{T}$ can be directly used to prove Theorem 7, while the logics $\overline{L}^{T}$ are unnecessary to this purpose. However,we believe that neglecting the logics $\overline{L}^{T}$ would have obscured the basic ideas of our proof. On the other hand, the logics $\overline{L}^{T}$ will allow us to obtain a simple proof of the main result of the next section.

9: On the cardinality of the set of maximal intermediate constructive predicate logics.

From the above proof that the set of maximal intermediate constructive propositional logics has the power of continuum, we can easily obtain a proof that the set of maximal intermediate constructive predicate logics has the power of continuum, too. To this purpose, we introduce the appropriate notions.

The predicate language WFF_{pred} will be the set of all formulas built up starting from the propositional connectives $\neg,\vee,\wedge$ and $\rightarrow$, the quantifiers $\forall$ and $\exists$, a denumerable set of individual variables $x_0,x_1,\ldots,x_n,\ldots$ (also represented by symbols such as $x,y,z,v,w,\ldots$) and, for every $n\geq 0$, a denumerable set of n-ary predicate variables $P^n_0,P^n_1,\ldots,P^n_h,\ldots$ (also represented by symbols such as

$P^n, Q^n, R^n, \ldots$, where, for $n=0$, one recovers the propositional variables). Neither constants, nor function symbols will be admitted. By INT_{pred} and CL_{pred} we denote the set of intuitionistically valid and the set of classically valid formulas of WFF_{pred} respectively. An *intermediate predicate logic* will be any set L such that $INT_{pred} \subseteq L \subseteq CL_{pred}$, and L is closed under detachment, substitution and generalization (see, e.g., [16] for background). *Throughout this paper, by a "predicate logic" we shall mean an intermediate predicate logic.* A predicate logic L will be said to be *constructive* (and will be referred to as a *constructive predicate logic*) iff L satisfies the *disjunction property* (defined as in the propositional case) and the *explicit definability property*: for every $A(x) \in WFF_{pred}$, if $\exists x A(x) \in L$ then $A(t) \in L$ for some term t.

Since the only terms available are individual variables, if $\exists x A(x)$ is closed (i.e., it does not contain free individual variables) then the assertion "if $\exists x A(x) \in L$, then $A(t) \in L$ for some term t" amounts to saying " if $\exists x A(x) \in L$, then $\forall x A(x) \in L$". But this is no longer true if $\exists x A(x)$ contains free (individual) variables, i.e., if $\exists x A(x)$ has the form $\exists x A(x, y_1, \ldots, y_m)$, where $m \geq 1$ and $y_1, \ldots, y_m$ are all the free variables of $\exists x A(x)$. In the latter case, however, one can easily prove that the assertion "if $\exists x A(x, y_1, \ldots, y_m) \in L$, then $A(t, y_1, \ldots, y_m) \in L$ for some term t" is equivalent to the assertion "if $\exists x A(x, y_1, \ldots, y_m) \in L$, then $A(y_i, y_1, \ldots, y_m) \in L$ for some i such that $1 \leq i \leq m$". Now, $\exists x A(x) \in L$ is true iff $\exists x (A(x) \wedge B(y)) \in L$ is true for every $B(y) \in L$ such that $y \neq x$ and y is free in B. Therefore we can restrict attention to *open existential formulas*. On the other hand, if L satisfies the disjunction property, then $A(y_i, y_1, \ldots, y_m) \in L$ for $1 \leq i \leq m$ iff $A(y_1, y_1, \ldots, y_m) \vee \ldots \vee A(y_m, y_1, \ldots, y_m) \in L$. Accordingly, we say that a predicate logic L satisfies the *weak explicit definability property* iff, for every open formula of the form $\exists x A(x, y_1, \ldots, y_m)$ such that $\exists x A(x, y_1, \ldots, y_m) \in L$, we have that $A(y_1, y_1, \ldots, y_m) \vee \ldots \vee A(y_m, y_1, \ldots, y_m) \in L$.

Proposition 34: Let L be a predicate logic satisfying both the disjunction property and the weak explicit definability property. Then L is a constructive predicate logic.□

A constructive predicate logic L will be called *maximal* iff there is no constructive predicate logic L' such that $L \subseteq L'$ and $L \neq L'$.

As a natural extension to the predicative case of the notion of semiconstructiveness given for the propositional level in §2, we say that a predicate logic L is *semiconstructive in the predicate logic* L' iff the following two conditions hold:

1) if $A \vee B \in L$, then $A \in L'$ or $B \in L'$;

2) if $\exists x A(x) \in L$, then there is a term t such that $A(t) \in L'$.

Unfortunately, this definition of semiconstructiveness is not useful for our purposes. What we need is the following weaker notion of semiconstructiveness; namely, we say that a predicate logic L is *weakly semiconstructive in the predicate logic* L' iff the above Condition 1) is satisfied, together with the following condition:

2') for every open formula $\exists x A(x, y_1, \ldots, y_m)$, if $\exists x A(x, y_1, \ldots, y_m) \in L$ then $A(y_1, y_1, \ldots, y_m) \vee \ldots \vee A(y_m, y_1, \ldots, y_m) \in L'$.

Since Condition 1) *is not* the same as the disjunction property, from the fact that L is weakly semiconstructive in L' and that, say, $A \vee B \vee C \in L$, we cannot deduce that $A \in L'$ or $B \in L'$ or $C \in L'$. For, it might happen that $A \notin L'$, whence $B \vee C \in L'$; but, since L' is not necessarily weakly semiconstructive in L' itself, it is quite possible that $B \notin L'$ and $C \notin L'$. Thus, we are not able to prove that, if L is weakly semiconstructive in L' and $A(y_1, y_1, \ldots, y_m) \vee \ldots \vee A(y_m, y_1, \ldots, y_m) \in L$, then there is some term t such that $A(t, y_1, \ldots, y_m) \in L'$. In other words, we cannot prove that L is semiconstructive in L' even if L and L' satisfy Condition 1), and L satisfies the weak explicit definability property –which, together with Condition 1) implies Condition 2').

On the other hand, we are interested in sequences of weakly semiconstructive predicate logics, giving rise to constructive predicate logics.

Proposition 35: Let L and L' be two predicate logics such that L is weakly semiconstructive in L'. Then $L' \supseteq L$.□

Proposition 36: Let $\{L_i\}_{i \geq 1}$ be a sequence of predicate logics such that, for every $i \geq 1$, L_{i+1} is weakly semiconstructive in L_i. Then $\overline{L} = \bigcap_{i \geq 1} L_i$ is a constructive predicate logic.

<u>Proof</u>: As in the proof of Proposition 2, one shows that $\overline{L}$ satisfies the disjunction property. Now, let $\exists x A(x, y_1, \ldots, y_m) \in \overline{L}$. Then, for every $i \geq 1$, $\exists x A(x, y_1, \ldots, y_m) \in L_i$, a fortiori $\exists x A(x, y_1, \ldots, y_m) \in L_i$ for every $i \geq 2$. Hence, by

Condition 2'), one has that, for every $i \geq 1$, $A(y_1, y_1, \ldots, y_m) \vee \ldots \vee A(y_m, y_1, \ldots, y_m) \in L_i$. Thus, $A(y_1, y_1, \ldots, y_m) \vee \ldots \vee A(y_m, y_1, \ldots, y_m) \in \overline{L}$, i.e., $\overline{L}$ also satisfies the weak explicit definability. The assertion then follows from Proposition 34. □

Our notations concerning predicate logics will be quite similar to the ones used above for propositional logics. For instance, given two predicate logics L and L', L+L' will be the smallest predicate logic including both L and L'. In other words, L+L' denotes the deductive closure, with respect to detachment and generalization, of $L_1 \cup L_2$. Likewise, for an $\mathcal{H} \subseteq WFF_{pred}$, $INT_{pred}+\mathcal{H}$ will be the deductive closure, with respect to detachment and generalization, of $INT_{pred} \cup \mathcal{H}$. This gives rise to a predicate logic if $\mathcal{H}$ is closed under substitution. Finally, if S is a set of predicate axiom-schemes, $INT_{pred}+S$ will be the smallest predicate logic containing all the schemes of S. Such a predicate logic is the deductive closure, with respect to detachment and generalization, of $INT_{pred} \cup S^*$, where S^* is the set of all instances of schemes of S.

Coming to semantical aspects, a *(predicate) frame* $\underline{F}=\langle P,\leq,D\rangle$ will be a triple, where $\langle P,\leq\rangle$ is a poset with last element (the root) and where D is a function associating, with every element $\alpha \in P$, a nonempty set $D(\alpha)$, in such a way that the following condition holds: if $\alpha \leq \beta$ in $\langle P,\leq\rangle$, then $D(\alpha) \subseteq D(\beta)$. A frame $\underline{F}=\langle P,\leq,D\rangle$ is completed into a *predicate Kripke model* $K_{pred}=\langle P,\leq,D,\Vdash\rangle$, by defining on the elements of P the usual forcing relation $\Vdash$ [16,19]. We say that the model $\underline{K}_{pred}=\langle P,\leq,D,\Vdash\rangle$ *is built on the frame* $\langle P,\leq,D\rangle$. Given $A \in WFF_{pred}$ and a predicate Kripke model $\underline{K}_{pred}$, we say that A *holds in* $\underline{K}_{pred}$ iff the root of $\underline{K}_{pred}$ forces in $\underline{K}_{pred}$ the universal closure of A.

It is useful to introduce the notion of an *assignment* (of the individual variables of the language WFF_{pred}) *on a predicate Kripke model* $\underline{K}_{pred}=\langle P,\leq,D,\Vdash\rangle$: such an assignment $\mathfrak{a}$ will be any function defined on the set of the individual variables such that, for every variable x, $\mathfrak{a}(x) \in D(r)$, r being the root of the model. Given a formula $A(y_1,\ldots,y_m)$, a Kripke model $\underline{K}_{pred}=\langle P,\leq,D,\Vdash\rangle$, $\alpha \in P$ and an assignment $\mathfrak{a}$ on $\underline{K}_{pred}=\langle P,\leq,D,\Vdash\rangle$, the notation $\alpha \Vdash_{\mathfrak{a}} A(y_1,\ldots,y_m)$ means that α forces in $\underline{K}_{pred}$ the formula A when $y_1,\ldots,y_m$ are interpreted as

$\mathfrak{a}(y_1),\ldots,\mathfrak{a}(y_m)$ (respectively) in all states of $\underline{K}_{pred}$ (in particular, in all states of the principal subordering generated by α in $\langle P,\leq\rangle$). Of course, if $\alpha \Vdash \forall y_1 \ldots \forall y_m A(y_1,\ldots,y_m)$ in $\underline{K}_{pred}=\langle P,\leq,D,\Vdash\rangle$, then $\alpha \Vdash_{\mathfrak{a}} A(y_1,\ldots,y_m)$ for every assignment $\mathfrak{a}$ on $\underline{K}_{pred}$. Although the converse of this proposition does not hold, still we have:

Proposition 37: A holds in $\underline{K}_{pred}=\langle P,\leq,D,\Vdash\rangle$ iff, for every $\alpha \in P$ and every assignment $\mathfrak{a}$ on $\underline{K}_{pred}|\alpha$, we have $\alpha \Vdash_{\mathfrak{a}} A$ in $\underline{K}_{pred}|\alpha$, where $\underline{K}_{pred}|\alpha$ is the restriction of $\underline{K}_{pred}$ to the states of the principal subordering $\underline{P}_\alpha$ of $\langle P,\leq\rangle$. □

For any Kripke model $\underline{K}_{pred}=\langle P,\leq,D,\Vdash\rangle$ with root r, and any nonempty set $\mathcal{D}$ such that $\mathcal{D} \cap D(\alpha)=\emptyset$ for every $\alpha \in P$, we define for each $d \in D(r)$ the Kripke model $\underline{K}'_{pred}$ and the function f as follows:

1) $\underline{K}'_{pred}=\langle P,\leq,D',\Vdash'\rangle$;
2) P and $\leq$ are as in $\underline{K}_{pred}$;
3) for every $\alpha \in P$, $D'(\alpha)=D(\alpha) \cup \mathcal{D}$;
4) for every $\alpha \in P$ and every $d' \in D'(\alpha)$, $f(d')=d'$ if $d' \notin \mathcal{D}$, and $f(d')=d$ if $d' \in \mathcal{D}$;
5) for every n-ary predicate variable P^n, every $\alpha \in P$ and every n-tuple $\langle d'_1,\ldots,d'_n\rangle$ of elements of $D'(\alpha)$, $\alpha \Vdash' P^n(d'_1,\ldots,d'_n)$ (in $\underline{K}'_{pred}$) iff $\alpha \Vdash P^n(f(d'_1),\ldots,f(d'_n))$ (in $\underline{K}_{pred}$).

The above construction allows to "multiplicate" an element d of the domain of the root of a Kripke model: the element d splits into the elements of the elements of the set $\{d\} \cup \mathcal{D}$). By iterating this construction, i.e., by splitting the elements $d_1,\ldots,d_n$ into the elements of the sets $\{d_1\} \cup \mathcal{D}_1,\ldots,\{d_n\} \cup \mathcal{D}_n$ respectively, one can easily prove the following:

Proposition 38: For every Kripke model $\underline{K}_{pred}=\langle P,\leq,D,\Vdash\rangle$, assignment $\mathfrak{a}$ on $\underline{K}_{pred}$ and finite set v of individual variables, there is a Kripke model $\underline{K}^*_{pred}=\langle P,\leq,D^*,\Vdash^*\rangle$ and an assignment $\mathfrak{a}^*$ on $\underline{K}^*_{pred}$ such that:

1) P and $\leq$ are as in $\underline{K}_{pred}$;
2) for every $A \in WFF_{pred}$ such that only the individual variables of v occur free in A, and for every $\alpha \in P$, $\alpha \Vdash_{\mathfrak{a}} A$ (in $\underline{K}_{pred}$) iff $\alpha \Vdash^*_{\mathfrak{a}^*} A$ (in $\underline{K}^*_{pred}$);

3) for every $y \in v$ and $y' \in v$ such that $y \neq y'$, $\mathfrak{A}^*(y) \neq \mathfrak{A}^*(y')$.□

If $\underline{F}=\langle P,\leq,D\rangle$ is a frame and $A \in WFF_{pred}$, A will be said to *hold in* F iff for every Kripke model $\underline{K}_{pred}=\langle P,\leq,D,\Vdash\rangle$ built on $\underline{F}$, A holds in $\underline{K}_{pred}$. If Φ is a nonempty class of frames and $A \in WFF_{pred}$, we say that A *holds in* Φ iff for every $\underline{F} \in \Phi$, A holds in $\underline{F}$. We also let $\mathcal{L}_{pred}(\Phi)$ denote the set of formulas $A \in WFF_{pred}$ such that A holds in Φ. As is well known, for every nonempty class Φ of frames, $\mathcal{L}_{pred}(\Phi)$ is closed under generalization, substitution and detachment, and includes INT_{pred}, but need not be a predicate logic. In other words, $\mathcal{L}_{pred}(\Phi)$ need not be included in CL_{pred}. More precisely, we have [16]:

Proposition 39: For every nonempty class Φ of frames, $\mathcal{L}_{pred}(\Phi)$ is a predicate logic iff there is $\underline{F}=\langle P,\leq,D\rangle \in \Phi$ together with $\alpha \in P$ such that $D(\alpha)$ is infinite.□

If $\underline{F}=\langle P,\leq,D\rangle$ is a frame, we say that F *is built on the poset* $\langle P,\leq\rangle$. If $\mathcal{F}$ is a non empty class of posets, $\Phi(\mathcal{F})$ will denote the class of all frames $\underline{F}=\langle P,\leq,D\rangle$ such that the underlying poset $\underline{P}=\langle P,\leq\rangle$ of $\underline{F}$ belongs to $\mathcal{F}$. For any nonempty class $\mathcal{F}$ of posets, by $\mathcal{L}_{pred}(\mathcal{F})$ we will denote $\mathcal{L}_{pred}(\Phi(\mathcal{F}))$. Since the choice of the functions D in the frames of $\Phi(\mathcal{F})$ is arbitrary, from Proposition 39 we get:

Proposition 40: For every nonempty class $\mathcal{F}$ of posets, $\mathcal{L}_{pred}(\mathcal{F})$ is a predicate logic.□

Let us consider the binary tree T defined in §4, where every node of T is a propositional logic L characterized by a class $\mathcal{F}_L$ of posets such that $L=\mathcal{L}(\mathcal{F}_L)$. We define the infinitary binary tree T^* as the tree obtained by replacing every node $\mathcal{L}(\mathcal{F}_L)$ of T by the node $\mathcal{L}_{pred}(\mathcal{F}_L)$. If L^*_1 and L^*_2 are two nodes of T^*, we set $L^*_2=RI(L^*_1)$ iff $L_2=RI(L_1)$, where L_1 and L_2 are the nodes of T corresponding to L^*_1 and L^*_2, respectively. In a similar way we define $LE(L^*)$, for any node L^* of T^*. Also, by a path $\{L^*_i\}^{T^*}_{i\geq 1}$ of the tree T^* we mean an infinite sequence $L^*_1, L^*_2, \ldots, L^*_n, \ldots$ of nodes of T^* such that the

corresponding sequence $L_1, L_2, \ldots, L_n, \ldots$ of nodes of T is a path $\{L_i\}^{T}_{i\geq 1}$ of the tree T defined in §4.

Proposition 41: Let $\{L^*_i\}^{T^*}_{i\geq 1}$ be a path of the tree T^*. Then, for every predicate logic L^*_j of the path, L^*_{j+1} is weakly semiconstructive in L^*_j.
Proof: It suffices to show that, for every node L^* of the tree T^*, $LE(L^*)$ and $RI(L^*)$ are weakly semiconstructive in L^*. We analyze only the case corresponding to the logics L^* and $RI(L^*)$, since the other case is similar. Let $L^{*\prime}=RI(L^*)$ and $\mathcal{F}'$ and $\mathcal{F}$ be the classes of posets corresponding to L'^* and to L^* respectively, i.e., $L^{*\prime}=\mathcal{L}_{pred}(\mathcal{F}')$, and $L^*=\mathcal{L}_{pred}(\mathcal{F})$. We have to show:
a) $A \vee B \in L^{*\prime}$ implies $A \in L^*$ or $B \in L^*$;
b) $\exists x A(x,y_1,\ldots,y_m) \in L^{*\prime}$ implies $A(y_1,y_1,\ldots,y_m) \vee \ldots \vee A(y_m,y_1,\ldots,y_m) \in L^*$.
Every formula of the form $H \vee K$ is intuitionistically equivalent to $(H \wedge \Theta(z_1,\ldots,z_k)) \vee (K \wedge \Theta(z_1,\ldots,z_k))$, where $\Theta(z_1,\ldots,z_k)$ is a formula of INT_{pred} containing free the individual variables $z_1,\ldots,z_k$ and where we can take the variables $z_1,\ldots,z_k$ in such a way that all the free variables of $H \vee K$ are among $z_1,\ldots,z_k$. Therefore we can assume, without loss of generality, that both A and B are open formulas and contain the same free variables $y_1,\ldots,y_m$, in symbols, $A=A(y_1,\ldots,y_m)$ and $B=B(y_1,\ldots,y_m)$. To prove a), let $A(y_1,\ldots,y_m) \vee B(y_1,\ldots,y_m) \in L^{*\prime}$, while $A(y_1,\ldots,y_m) \notin L^*$ and $B(y_1,\ldots,y_m) \notin L^*$. Then, by Proposition 37, there are two Kripke models $\underline{K}^1_{pred}=\langle P_1,\leq_1,D_1,\Vdash_1\rangle$ and $\underline{K}^2_{pred}=\langle P_2,\leq_2,D_2,\Vdash_2\rangle$ and two assignments $\mathfrak{A}_1$ and $\mathfrak{A}_2$, respectively on $\underline{K}^1_{pred}$ and on $\underline{K}^2_{pred}$, such that $\underline{P}_1=\langle P_1,\leq_1\rangle \in \mathcal{F}$, $\underline{P}_2=\langle P_2,\leq_2\rangle \in \mathcal{F}$, $r_1 \nVdash_{1\,\mathfrak{A}_1} A(y_1,\ldots,y_m)$, and

$r_2 \nVdash_{2\,\mathfrak{A}_2} B(y_1,\ldots,y_m)$, where r_1 and r_2 are the roots of $\underline{K}^1_{pred}$ and of $\underline{K}^2_{pred}$ respectively. We may safely assume that $P_1 \cap P_2$ is empty and, by Proposition 38, that $\mathfrak{A}_k(y_i) \neq \mathfrak{A}_k(y_j)$ for each $1\leq k\leq 2$, and for any two i and j such that $i \neq j$ and $1\leq i,j\leq m$. Again we can safely assume that $D(r_1) \cap D(r_2)=\{d_1,\ldots,d_m\}$ and, for every h with $1\leq h\leq m$, $d_h=\mathfrak{A}_1(y_h)=\mathfrak{A}_2(y_h)$. As shown in the proof of Proposition 12, there is a poset $\underline{P}'=\langle P',\leq'\rangle$ such that $\underline{P}' \in \mathcal{F}'$, $r_1 \in P'$, $r_2 \in P'$, $\underline{P}'_{r_1}=\underline{P}_1$ and $\underline{P}'_{r_2}=\underline{P}_2$. Let $\underline{F}'=\langle P',\leq',D'\rangle$ be the frame built on $\underline{P}'$ such that $D'(\alpha)=D_1(\alpha)$ for $\alpha \in P_1$, $D'(\beta)=D_2(\beta)$ for $\beta \in P_2$, and $D'(\gamma)=\{d_1,\ldots,d_m\}$ for $\gamma \notin P_1$ and $\gamma \notin P_2$. Let

$\underline{K}'_{pred}=\langle P',\leq',D',\Vdash'\rangle$ be any Kripke model built on $\underline{F}'$ such that $\Vdash'$ coincides with $\Vdash_1$ in the submodel $\underline{K}^1_{pred}=\langle P_1,\leq_1,D_1,\Vdash_1\rangle$, $\Vdash'$ coincides with $\Vdash_2$ in the submodel $\underline{K}^2_{pred}=\langle P_2,\leq_2,D_2,\Vdash_2\rangle$ and $\Vdash'$ is defined in an arbitrary way compatible with the previous requirements on all states $\gamma\in P'$ such that $\gamma\notin P_1$ and $\gamma\notin P_2$. Let $\mathfrak{A}'$ be any assignment on $\underline{K}'_{pred}$ such that $\mathfrak{A}'(y_h)=d_h$ for every h with $1\leq h\leq m$, and $\mathfrak{A}'(x)\in\{d_1,\dots,d_m\}$ for every variable x different from $y_1,\dots,y_m$. Then, if r' is the root of $\underline{K}'_{pred}$, one has that $r'\nVdash'_{\mathfrak{A}'}A(y_1,\dots,y_m)$ and $r'\nVdash'_{\mathfrak{A}'}B(y_1,\dots,y_m)$, which, by Proposition 37, contradicts the fact that $r'\Vdash'_{\mathfrak{A}'}A(y_1,\dots,y_m)\vee B(y_1,\dots,y_m)$, because $\underline{P}'\in\mathcal{F}'$.

To prove b), let $\exists xA(x,y_1,\dots,y_m)\in L^{*'}$ and $A(y_1,y_1,\dots,y_m)\vee\dots\vee A(y_m,y_1,\dots,y_m)\notin L^*$. Then, by Proposition 37, there is a Kripke model $\underline{K}_{pred}=\langle P,\leq,D,\Vdash\rangle$, and an assignment $\mathfrak{A}$ on $\underline{K}_{pred}$ such that $r\nVdash_{\mathfrak{A}}A(y_1,y_1,\dots,y_m)\vee\dots\vee A(y_m,y_1,\dots,y_m)$, where r is the root of $\underline{K}_{pred}$. In other words, $r\nVdash_{\mathfrak{A}}A(y_1,y_1,\dots,y_m),\dots, r\nVdash_{\mathfrak{A}}A(y_m,y_1,\dots,y_m)$. Let $\{d_1,\dots,d_n\}=\{\mathfrak{A}(y_1),\dots\mathfrak{A}(y_m)\}$, where $1\leq n\leq m$. As in the proof of the above point a), we can safely assume that $\mathfrak{A}(y_1)\neq\dots\neq\mathfrak{A}(y_m)$, i.e., $n=m$, although this is not necessary here. Let $\underline{P}'=\langle P',\leq'\rangle$, where $P'=P\cup\{r'\}$ for some $r'\notin P$. Let $\alpha\leq'\beta$ if α and β are such that $\alpha\leq\beta$ in $\langle P,\leq\rangle$. Let $r'\leq\gamma$ for every $\gamma\in P'$. Let $\underline{F}'=\langle P',\leq',D'\rangle$ be the frame built on $\underline{P}'$ such that $D'(\varepsilon)=D(\varepsilon)$ for $\varepsilon\neq r'$, and $D'(\varepsilon)=\{d_1,\dots,d_n\}$ for $\varepsilon=r'$. Let $\underline{K}'_{pred}=\langle P',\leq',D',\Vdash'\rangle$ be any Kripke model built on $\underline{F}'$ such that $\Vdash'$ coincides with $\Vdash$ for $\varepsilon\neq r'$, and $\Vdash'$ is defined in any way compatible with the previous requirements for $\varepsilon=r'$. Also, let $\mathfrak{A}'$ be any assignment coinciding with $\mathfrak{A}$ on $\{y_1,\dots,y_m\}$ and such that $\mathfrak{A}'(x)\in\{d_1,\dots,d_n\}$ for each individual variable x such that $x\notin\{y_1,\dots,y_m\}$. Then one readily sees that $\underline{P}'\in\mathcal{F}'$, whence, by Proposition 37, $r'\Vdash'_{\mathfrak{A}'}\exists xA(x,y_1,\dots,y_m)$ immediately follows. On the other hand we have $r'\nVdash'_{\mathfrak{A}'}A(y_1,y_1,\dots,y_m),\dots, r'\nVdash'_{\mathfrak{A}'}A(y_m,y_1,\dots,y_m)$, since $r'\leq'r$ and $r\nVdash'_{\mathfrak{A}'}A(y_1,y_1,\dots,y_m),\dots, r\nVdash'_{\mathfrak{A}'}A(y_m,y_1,\dots,y_m)$. One can now see that $r'\nVdash'_{\mathfrak{A}'}\exists xA(x,y_1,\dots,y_m)$, since $d_1,\dots,d_n$ are the only elements of $D'(r')$, a contradiction.□

Let $\{L^*_i\}^{T^*}_{i\geq 1}$ be any path of the tree T^* and let $\overline{L}^{T^*}_{pred}=\bigcap_{i\geq 1}L^*_i$. In analogy with Definition 6, we call $\overline{L}^{T^*}_{pred}$ the *predicate logic associated with* $\{L^*_i\}^{T^*}_{i\geq 1}$. As an immediate consequence of Propositions 36 and 41, we have:

Corollary 4: Let $\{L^*_i\}^{T^*}_{i\geq 1}$ be any path of the tree T^*, and let $\overline{L}^{T^*}_{pred}$ be the predicate logic associated with $\{L^*_i\}^{T^*}_{i\geq 1}$. Then $\overline{L}^{T^*}_{pred}$ is a constructive predicate logic.□

Recall that any predicate logic L_{pred} contains a propositional logic L. In other words, L is the set of formulas of L_{pred} only containing 0-ary predicate symbols (propositional variables). We call L the *propositional logic contained in* L_{pred}. The following propositions are immediate:

Proposition 42: If L_{pred} is a constructive predicate logic, and L is the propositional logic contained in L_{pred}, then L is a constructive logic.□

Proposition 43: Let $\{L^*_i\}^{T^*}_{i\geq 1}$ be any path of the tree T^*, let $\{L_i\}^{T}_{i\geq 1}$ be the corresponding path of the tree T, and let $\overline{L}^{T^*}_{pred}$ and $\overline{L}^{T}$ respectively be the predicate logic associated with $\{L^*_i\}^{T^*}_{i\geq 1}$, and the (first) logic associated with $\{L_i\}^{T}_{i\geq 1}$. Then $\overline{L}^{T}$ is the propositional logic contained in $\overline{L}^{T^*}_{pred}$.□

From Propositions 32, 42 and 43 we get:

Corollary 5: Let $\{L^{*1}_i\}^{T^*}_{i\geq 1}$ and $\{L^{*2}_i\}^{T^*}_{i\geq 1}$ be two distinct paths of the tree T^*, and let $\overline{L}^{T^*}_{1pred}$ and $\overline{L}^{T^*}_{2pred}$ be the predicate logics associated with these paths. Then there is no constructive predicate logic L_{pred} such that $\overline{L}^{T^*}_{1pred}+\overline{L}^{T^*}_{2pred}\subseteq L_{pred}$.

Proof: Let us assume that there is a constructive predicate logic L_{pred} such that $\overline{L}^{T^*}_{1pred}+\overline{L}^{T^*}_{2pred}\subseteq L_{pred}$. Let $\overline{L}^{T}_{1}$, $\overline{L}^{T}_{2}$ and L respectively be the propositional logic contained in $\overline{L}^{T^*}_{1pred}$, the propositional logic contained in $\overline{L}^{T^*}_{2pred}$ and the propositional logic contained in L_{pred}. Then, by Proposition 42,

L is a constructive (propositional) logic and, by Proposition 43, there are two distinct paths $\{L^1_i\}^T_{i\geq 1}$ and $\{L^2_i\}^T_{i\geq 1}$ of the tree T such that $\overline{L}^T_1$ and $\overline{L}^T_2$ are the (first) logics associated with them. Since $\overline{L}^{T^*}_{1pred}+\overline{L}^{T^*}_{2pred}\subseteq L_{pred}$, one has $\overline{L}^T_1\subseteq L$ and $\overline{L}^T_2\subseteq L$, whence $\overline{L}^T_1+\overline{L}^T_2\subseteq L$, where L is a (propositional) constructive logic, and $\overline{L}^T_1$ and $\overline{L}^T_2$ are associated with distinct paths of the tree T. This contradicts Proposition 32.□

Disregarding the purely propositional formulas contained in predicate logics, the constructive incompatibility of the predicate logics associated with two different paths of the tree T^* can be proved, for instance, as follows. Choose an individual variable x and associate, with every propositional variable p_i, the atomic formula $P^1_i(x)$, where P^1_i is a unary predicate variable, different propositional variables being associated with different atomic formulas. Under this correspondence one can associate, with every propositional logic L, a set $\tilde{L}$ of predicate formulas obtained by replacing, in every $A\in L$, any occurrence of a propositional variable by an occurrrence of the corresponding atomic formula. Let $\overline{L}^{T^*}_{1pred}$ and $\overline{L}^{T^*}_{2pred}$ be the two predicate logics associated with two different paths of T^*. Let $\overline{L}^T_1$ and $\overline{L}^T_2$ be the propositional logic contained in $\overline{L}^{T^*}_{1pred}$ and the propositional logic contained in $\overline{L}^{T^*}_{2pred}$. Let $L=\overline{L}^T_1+\overline{L}^T_2$, and let $\tilde{L}$ be the set of predicate formulas associated with the propositional logic L by applying the above substitution. Then $\tilde{L}\subseteq\overline{L}^{T^*}_{1pred}+\overline{L}^{T^*}_{2pred}$, and there is a formula $A\vee B\in\tilde{L}$ such that, for every predicate logic L_{pred} containing $\overline{L}^{T^*}_{1pred}+\overline{L}^{T^*}_{2pred}$, $A\notin L_{pred}$ and $B\notin L_{pred}$.

As in the propositional case, using Zorn's lemma we can prove that, for every constructive predicate logic L_{pred}, there is a maximal constructive predicate logic L'_{pred} such that $L_{pred}\subseteq L'_{pred}$. Hence, from Corollary 5, we get:

Theorem 8: There exist exactly $2^{\aleph_0}$ maximal constructive predicate logics.□

The nodes of the tree T^* are predicate logics, and are semantically characterized by classes of frames. In contrast to the treatment given in the propositional case, for these predicate logics we have not proved completeness

theorems yielding their recursive enumerability (nor it seems to be possible to state such completeness theorems). We can, however, improve our constructive incompatibility results in the following sense: we can replace the nodes of the tree T^* with recursively enumerable predicate logics contained in them, thus obtaining an infinitary binary tree T^{**}. For every path $\{L^{**}_i\}^{T^{**}}_{i\geq 1}$ of the tree T^{**}, the intersection $\overline{L}^{T^{**}}$ of the logics of the path, i.e., the predicate logic associated with the path, is a constructive predicate logic. Further,for any two different paths of the tree T^{**}, the corresponding predicate logics are constructively incompatible.

More precisely, we say that a formula $A\in WFF_{pred}$ is a *negatively saturated predicate formula* iff every occurrence in A of a predicate variable or of a quantifier is in the scope of a negation. Let L be any logic of the propositional tree T and let (A_L) be the characteristic axiom-schema of L. By $NS_{pred}(\{(A_L)\})$ we denote the set of negatively saturated predicate formulas obtained by replacing in every instance of (A_L) every occurrence of any propositional variable by an occucurrence of a negatively saturated predicate formula. It turns out that $NS_{pred}(\{(A_L)\})$ is closed under substitution. Also, for every predicate logic L^* of the predicate tree T^*, by $NS(L^*)$ we denote the set of negatively saturated predicate formulas belonging to L^*. It turns out that $NS(L^*)$ is closed under substitution. Finally, letting (K) be Kuroda' s axiom-schema $\forall x\neg\neg A(x)\rightarrow\neg\neg\forall xA(x)$ (see, e.g., [5]), the tree T^{**} will be obtained from the tree T^* upon replacing every predicate logic L^* of T^* by the recursively enumerable predicate logic $L^{**}=INT_{pred}+\{(K)\}+NS_{pred}(\{(A_L)\})$, where L is the (propositional) logic of T corresponding to L^*. In this way, three one-to-one correspondences are defined, the first between the elements L of T and the elements L^* of T^*, the second between the elements L of T and the elements L^{**} of T^{**}, and the third between the elements L^* of T^* and the elements L^{**} of T^{**}.

From the above results and using the techniques of the above quoted P. Miglioli, *Nota su logiche proposizionali costruttive massimali* (see §0), combining syntactical and semantical tools, we can prove the following:

-If L^* is any predicate logic of T^* and L^{**} is the corresponding predicate logic of T^{**}, then $L^{**}=INT_{pred}+\{(K)\}+NS(L^*)\subseteq L^*$.

-Let $\{L^{**}_i\}^{T^{**}}_{i\geq 1}$ be any path of the tree T^{**}; let $\overline{L}^{T^{**}}$ be the corresponding predicate logic (i.e., $\overline{L}^{T^{**}}$ is the intersection of the elements of the path). Let $\overline{NS}(\{L^{**}_i\}^{T^{**}}_{i\geq 1})$ be the set of negatively saturated predicate formulas belonging to all the elements of the path. Then $\overline{L}^{T^{**}}$ is a constructive predicate logic, and $\overline{L}^{T^{**}} = INT_{pred} + \{(K)\} + \overline{NS}(\{L^{**}_i\}^{T^{**}}_{i \geq 1})$.

-If $\overline{L}^{T^{**}}_1$ and $\overline{L}^{T^{**}}_2$ are the predicate logics associated with two different paths of T^{**}, then $\overline{L}^{T^{**}}_1$ and $\overline{L}^{T^{**}}_2$ are constructively incompatible.

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