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AN INTRODUCTION TO NAIVE SET THEORY THROUGH THE NOTION OF NORMAL FILTER.

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This is a short course on naive set theory at the first year graduate level. The emphasis lays on the notion of a normal filter. All the material presented here is by now classical.

Section I is devoted to closed unbounded sets which constitute the first historically known example of a normal filter. Most of its content can be traced back to the prehistory of the field.

Section II uses the machinery presented in section I to construct pathological Abelian groups. This work was done by P. Eklof in the early 70's and has been continued by S. Shelah in his celebrated solution of the Whitehead problem.

Section III introduces the notion of a normal ideal. There , it is used to prove an important theorem on closed unbounded sets due to R. Solovay.

Section IV touches upon real-valued measurable cardinals. The material presented there is also due to Solovay.

Section III and IV deal with topics which play an important role in today's combinatorial set theory. Finally section V collects some facts about measurable cardinals which follow directly from the previous sections. The only important omission we can think of is Silver's theorem on the continuum hypothesis at singular cardinals. Our excuse for not including it here is that some authors seem to think that it should be taught at the undergraduate level (clearly, this view is highly questionable).

Prerequisites. Ordinal and cardinal numbers. Cofinality $cf(\alpha)$ of a cardinal number α. Some basic facts about Abelian groups and filters in a Boolean algebra.

I - Closed unbounded sets.

1. Definition. Let k be an infinite cardinal number. A set $C \subseteq k$ is closed unbounded (in k) if it is unbounded below k and closed in the sense that $\alpha \in C$ whenever $\alpha < k$ and $C \cap \alpha$ is unbounded below α.

Examples. For $k = \omega$, C is closed unbounded iff C is infinite. If $k > \omega$, k and the set of limit ordinals below k are closed unbounded. If $cf(k) > \omega$ and $f: k \to k$, the set of $\alpha < k$ which are closed under f (i.e. $f(\xi) < \alpha$ whenever $\xi < \alpha$) is closed unbounded.

2. Let $cf(k) > \omega$ and $C, C' \subseteq k$ be closed unbounded; given $\alpha < k$, pick $\alpha_o \in C - \alpha$, $\alpha'_o \in C' - \alpha_o$, $\alpha_1 \in C - \alpha'_o$, $\alpha'_1 \in C' - \alpha_1, \ldots$. Then $\beta = \sup\{\alpha_i : i < \omega\} = \sup\{\alpha'_i : i < \omega\}$ is $< k$ since $cf(k) > \omega$. Also, both $C \cap \beta$ and $C' \cap \beta$ are unbounded below β. So $\beta \in C \cap C'$. This shows that $C \cap C'$ is unbounded below k. $C \cap C'$ is obviously closed, so it is closed unbounded. Elaborating on this argument, we obtain the

Definition and fact. Let $cf(k) > \omega$. Then the collection of closed unbounded sets is closed under intersections indexed by sets of cardinality $< cf(k)$. Hence, it is a filter basis and the resulting filter

$D(k) = \{X \subseteq k : \exists C \subseteq k$ such that $C \subseteq X$ and C is closed unbounded$\}$ is $cf(k)$ - complete in the sense that it is closed under intersections as above. $D(k)$ is called the <u>closed unbounded filter</u>.

3. When k is regular, we obtain a little more. Let $(X_\xi)_{\xi<k}$ be a collection of subsets of k; the diagonal intersection of (X_ξ), ΔX_ξ, is the set of $\alpha<k$ such that $\forall \xi<\alpha \ \alpha \in X_\xi$.

   <u>Fact</u>. If $k>\omega$ is regular, then $D(k)$ is closed under diagonal intersection.

   Proof. Clearly, it is enough to show that the collection of closed unbounded sets has this property. Let $(C_\xi)_{\xi<k}$ be a family of closed unbounded sets and $\alpha<k$. Pick $\alpha_0 \in \bigcap_{\xi<\alpha} C_\xi - \alpha$,

   $$\alpha_1 \in \bigcap_{\xi<\alpha_0} C_\xi - \alpha_0, \alpha_2 \in \bigcap_{\xi<\alpha_1} C_\xi - \alpha_1, \ldots$$

   Put $\beta = \sup\{\alpha_i : i<\omega\}$. Clearly $\beta>\alpha$ and $\beta \in \Delta C_\xi$. This shows that ΔC_ξ is unbounded below k. As usual, ΔC_ξ is obviously closed.

4. We are now finished with the basics on closed unbounded sets. Since we intend to work hard on them, it will be helpful to look for different ways to look at them. A set $X \subseteq k$ is <u>stationary</u> iff $k - X \notin D(k)$ or equivalently iff it meets every closed unbounded set. A function $f : X \to k$ where

$X \subseteq k$ is <u>regressive</u> iff $\forall \alpha \in X \ f(\alpha)<\alpha$. Fact 3 can now be restated as follows.

<u>Fodor's Lemma</u>. Let $k>\omega$ be regular. If $X \subseteq k$ is stationary, then every regressive function $f: X \to k$ is constant on a stationary subset of X.

Proof. Suppose for a contradiction that a regressive function $f: X \to k$ is constant on no stationary subset of X. Let $X_\xi = \{\alpha<k : f(\alpha) \neq \xi\}$. Clearly $X_\xi \in D(k)$. Therefore $\Delta X_\xi \in D(k)$. So $\Delta X_\xi \cap X \neq \emptyset$. Pick $\alpha \in \Delta X_\xi \cap X$. Then

$$\forall \xi<\alpha \quad f(\alpha) \neq \xi$$
$$f(\alpha) < \alpha$$

a contradiction.

5. Let $cf(k)>\omega$. We would like to know if $D(k)$ is an ultrafilter. The answer is: never. In other words there always exists a stationary set which does not belong to $D(k)$. Three cases may arise:

   case i. $cf(k) > \omega_1$. $\{\alpha<k : cf(\alpha) = \omega\}$ is such a set;

   case ii. $k = \omega_1$. The existence of such a set will be shown in § III;

   case iii. $\omega_1 = cf(k) < k$. The existence of the desired set readily follows from case ii.

6. <u>The Boolean algebra</u> $P(k)/D(k)$. Let $cf(k)>\omega$ and for $X\subseteq k$, let $\tilde{X}$ be the class of X modulo $D(k)$, i.e. the set of $Y\subseteq k$ which are equal to X on a closed unbounded set. Also let $P(k)/D(k)$ be the set of these $\tilde{X}$'s endowed with the structure induced by the boolean operations on $P(k)$. $P(k)/D(k)$ is well defined and is also a Boolean algebra. Notice that

$X\in D(k)$ iff $\tilde{X}=1$

X is stationary iff $\tilde{X}>0$

$D(k)$ is an ultrafilter iff $P(k)/D(k) = \{0,1\}$

(we remind the reader that we are planing to show that this is never the case).
Using diagonal intersection it is easy to show that $P(k)/D(k)$ is complete, although we shall not use this fact.

II <u>AN APPLICATION TO ABELIAN GROUPS</u>.

In this section, a group will always be an abelian group. If A and B are groups, we shall mean by $A\subseteq B$ that A is a subgroup of B. The rank of a free group is defined to be the cardinality of a maximal set of independent generators. Finally F_ω denotes the free group of rank ω.
We remind the reader that, in the category of abelian groups, free=projective. This means that if $A\subseteq B$ and B/A is free, then A is a direct summand of B.

1. <u>Reminder</u>. Let $(A_i)_{i<\omega}$ be a sequence of free groups such that $A_i\subseteq A_{i+1}$ and each factor group A_{i+1}/A_i is free. Then $A=\bigcup A_i$ is free.
This tells us how to build a countably generated free group using finitely generated ones. We shall now state a result allowing us to construct free groups of cardinality ω_1 using countable ones.

2. Let A be a group of cardinality ω_1.
A <u>filtration</u> (of A) is a family $(A_\xi)_{\xi<\omega_1}$ of countable subgroups of A such that $A_\xi\subseteq A_{\xi+1}$, $A_\eta=\bigcup_{\xi<\eta} A_\xi$ whenever $\eta<\omega_1$ is a limit ordinal (and $\bigcup A_\xi = A$). Clearly, every group of cardinality

ω_1 admits a filtration. We are now in a position to state the promised result.

Theorem. Let A be a group of cardinality ω_1. A is free iff it admits a filtration (A_ξ) such that A_0 is free and every factor group $A_{\xi+1}/A_\xi$ is free.

Proof. Trivial.

3. This can be restated in terms of equivalence classes modulo $D(\omega_1)$. A group of cardinality ω_1 is ω_1-free iff every countable subgroup is free. Let A be an ω_1-free group; pick a filtration (A_ξ) of A and put $E = \{\xi < \omega_1 : \exists \eta > \xi \ \ A_\eta / A_\xi \text{ is not free}\}$. Let $\tilde{E}(A) = \tilde{E}$. Then $\tilde{E}(A)$ is independent of the choice of (A_ξ). To see this, pick another filtration $(B_\xi)_\xi$ of A and put $F = \{\xi < \omega_1 : \exists \eta > \xi \ \ B_\eta / B_\xi \text{ is not free}\}$. Let $C = \{\xi < \omega_1 : A_\xi = B_\xi\}$. Clearly, C is closed unbounded. Let $\xi \in C$, $\xi \in E$, $\eta > \xi$ be such that A_η/A_ξ is not free and $\eta' > \eta$ be in C. Clearly $A_{\eta'}/A_\xi$ is not free But $A_{\eta'} = B_{\eta'}$. So $\xi \in F$. By symmetry, $E \cap C = F \cap C$, as desired. Using the above notion, we are now in a position to state.

Theorem 2 reformulated. Let A be ω_1-free. Then A is free iff $\tilde{E}(A) = 0$.

4. An obvious question is: is every ω_1-free group free? The answer is a very definite no. The rest of

this section will be devoted to a proof of this fact.

5. Lemma. Let F be a free group and F/B be a presentation of the additive group of rational numbers $\mathbb{Q}$. Then B is not finitely generated.

Proof. Suppose by contradiction that B is finitely generated. Let $a_0, \dots, a_{n-1}$ be a list of free generators of F such that every generator of B is a linear combination of the a_i's. Let F_0 be the subgroup of F generated by the a_i's. $F \cong F_0 \oplus F_1$. By the structure theorem for finitely generated abelian groups, F is not finitely generated, so $F_1 \neq 0$. Also

$$\mathbb{Q} \cong F/B \cong F_0/B \oplus F_1$$

a contradiction.

6. Theorem. (P. Eklof). Let $E \subseteq \omega_1$. There exists an ω_1-free group A such that $\tilde{E}(A) = \tilde{E}$.

Proof. We can assume without loss of generality that E contains no successor ordinal.
We shall construct by recursion on α a filtration $(A_\alpha)_{\alpha<\omega_1}$ such that

i. A_α is free of rank ω

ii. $A_\eta/A_{\xi+1}$ is free of rank ω whenever $\xi+1 < \eta < \alpha$

iii. $A_{\xi+1}/A_\xi$ is free iff $\xi \notin E$.

From ii and iii, it will follow that

$$A_\beta/A_\alpha \text{ is free iff } \alpha \notin E, \text{ whenever } \alpha < \beta < \omega_1 .$$

Then $A = \cup A_\alpha$ will be as desired.

Four cases may arise

case a. α is 0. Put $A_\alpha = F_\omega$.

case b. α is a limit ordinal. Put $A_\alpha = \cup\{A_\xi : \xi < \alpha\}$.

We show that A_α is free. Pick a sequence $(\alpha_i)_{i<\omega}$ such that $\alpha_i < \alpha$ and $\sup \alpha_i = \alpha$. $A_\alpha = \cup A_{\alpha_i+1}$.
So, by ii, A_α is free. The rest is obvious.

case c. $\alpha = \alpha'+1$ and either α' is not a limit ordinal or $\alpha' \notin E$. Put $A_\alpha = A_{\alpha'} \oplus F_\omega$. Clearly this will do the job.

case d. $\alpha = \alpha'+1$ and $\alpha \in E$ (so α' is limit). Let F_ω/B be a presentation of $\mathbb{Q}$. By lemma 5, B is not finitely generated. Since B is a subgroup of a free group, it is free. So, it can be written as $B = \bigcup_{i<\omega} B_i$ where $B_o \cong F_\omega$, $B_i \subseteq B_{i+1}$ and each B_{i+1}/B_i is isomorphic to F_ω.

Write $A_{\alpha'} = \bigcup_{i<\omega} A_{\alpha'_i+1}$ where $\alpha'_i < \alpha'$ and $\alpha' = \sup \alpha'_i$. From i, it follows that $A_{\alpha'_o}$ is free of rank ω. From ii, it follows that $A_{\alpha'_{i+1}+1}/A_{\alpha'_i+1}$

is also isomorphic to F_ω.

Putting things together, we obtain a commutative diagram which can be complete as shown.

$$\begin{tikzcd}
A_{\alpha'_o+1} \arrow[r, hook] \arrow[d, leftrightarrow, "f_o"'] & A_{\alpha'_1+1} \arrow[r, hook] \arrow[d, leftrightarrow, "f_1"'] & \cdots & A_{\alpha'} \arrow[r, hook, dotted] \arrow[d, leftrightarrow, "\cup f_i"'] & X \arrow[d, leftrightarrow, dotted] \\
B_o \arrow[r, hook] & B_1 \arrow[r, hook] & \cdots & B \arrow[r, hook, dotted] & F_\omega
\end{tikzcd}$$

Put $A_\alpha = X$. Clearly, A_α is free and $A_\alpha/A_{\alpha'} \cong \mathbb{Q}$ is not free. Also, ii follows from the corresponding property of B. This finishes the proof of the theorem.

<u>Corollary</u>. The exists an ω_1-free group A which is not free.

<u>Proof</u>. By 3, any A such that $\tilde{E}(A)=1$ will do.

<u>Question</u>. How many are there non isomorphic ω_1-free groups?

By the above theorem there are at least as many such groups as the cardinality of $P(k)/D(k)$.

This cardinality will be investigated in section III.

III. NORMAL FILTERS.

This section is devoted to showing that $P(k)/D(k)$ has cardinality 2^k when k is uncountable and regular. In order to deal with the situation within a somewhat more general framework, we introduce the notion of a normal filter. This will turn out to be useful later on. By a filter on an infinite cardinal number k, we mean a filter in the Boolean algebra $P(k)$. Given a filter F on k, we denote by F^* the ideal dual to F i.e. $\{X \subseteq k : k - X \in F\}$.

1. Normal filters. Let k be an infinite cardinal number. $\{k - \alpha : \alpha < k\}$ has the finite intersection property. So it is a filter basis.

The resulting filter is called the Fréchet filter on k.

Let k be an infinite cardinal number and F be a filter on k. F is normal iff F contains the Fréchet filter and is closed under Δ (see I.3 for a definition).

Examples. If k is regular and uncountable, $D(k)$ is a normal filter. If k is regular and uncountable and $A \subseteq k$ is stationary, then $F = \{X \subseteq k : \exists$ closed unbounded C such that $A \cap C \subseteq X\}$ is a normal filter.

Facts. Let k be an infinite cardinal number and F be a normal filter on k. Then

i. F is k-complete

ii. If $X \subseteq k$ does not belong to F^* and $f : X \to k$ is regressive, there exists a set $X_o \subseteq X$ not belonging to F^* such that f is constant on X_o (see I.4 for a proof).

iii. k is uncountable and regular.

Fact ii has a converse.

Proposition. Let k be an infinite cardinal number and F is a filter on k containing the Fréchet filter. If for every subset $X \notin F^*$ of k, every regressive function on X is constant on some subset $X_o \notin F^*$ of X, then F is normal.

Proof. Let $(X_\xi)_{\xi < k}$ be a family of elements of F. Put $X = k - \Delta X_\xi$ and define $f : X \to k$ as

$$f(\alpha) = \text{some } \xi < \alpha \text{ such that } \alpha \notin X_\xi .$$

Suppose $\Delta X_\xi \notin F$. Then $X \notin F^*$ and, since f is regressive, it is constant on some $Y \notin F^*$. Let $f[Y] = \{\eta_o\}$. $Y \cap X_{\eta_o} \neq \emptyset$. Pick some $\alpha \in Y \cap X_{\eta_o}$. $f(\alpha) = \eta_o$ since $\alpha \in Y$ and so $\alpha \notin X_{\eta_o}$.

This is a contradiction.

2. Saturated filters. Let k and λ be cardinal numbers and F be a filter on k. F is λ-saturated

iff there is no family $(X_\xi)_{\xi<\lambda}$ of subsets of k such that for every $\xi<\lambda$, $X_\xi \notin F^*$ and for every distinct $\xi, \eta<\lambda$, $X_\xi \cap X_\eta \in F$.

Clearly, a filter on k is $(2^k)^+$-saturated and if $\lambda'<\lambda$, then λ'-saturation $\Rightarrow \lambda$-saturation.

Finally F is an ultrafilter iff it is 2-saturated.

Fact. Let F be a filter on k. If F is not λ-saturated then the cardinality of $P(k)/F$ is $\geq 2^k$.

Proof. Let $(X_\xi)_{\xi<\lambda}$ be a family witnessing the fact that F is not λ-saturated. The map

$$S \in P(\lambda) \mapsto \text{class of } \bigcup_{\xi\in S} X_\xi \text{ modulo } F \in P(k)/F$$

is one to one.

3. Theorem. If k carries a normal, k-saturated filter then k is weakly inaccessible (i.e. k is uncountable, regular and $\forall\lambda<k \ \lambda^+<k$).

Corollary. $P(\omega_1)/D(\omega_1)$ has cardinality 2^{ω_1}. Also, $\{\alpha<\omega_2 : cf(\alpha)=\omega\}$, $\{\alpha<\omega_2 : cf(\alpha)=\omega_1\}$ can be split into ω_2 disjoint stationary sets (see examples in §1).

Proof of the theorem. Let F be a normal, k-saturated filter on k.

Claim : If $f: X\to k$ is regressive and $X\notin F^*$ then f is almost bounded in the sense that there exists $A\in F$ such that the range of f on $X\cap A$ is bounded below k.

Proof of the claim. Let $E=\{\xi<k : f^{-1}(\xi)\notin F^*\}$.

Since F is normal, $E\neq\emptyset$ and $X_0 = \bigcup\{f^{-1}(\xi):\xi\notin E\}\in F^*$.

Since F is k-saturated, the cardinality of E is <k.

Let $A=k-X_0$. Clearly $f[X\cap A]$ is bounded.

Now, we know from §1 that k is uncountable and regular. So, it suffices to show that it is not a successor cardinal. Suppose for a contradiction that $k=\lambda^+$. Pick for each limit $\alpha<k$ a sequence $(a_\xi^\alpha)_{\xi<\lambda}$ such that $a_\xi^\alpha<\alpha$ and $\sup\{a_\xi^\alpha : \xi<\lambda\}=\alpha$.

Then $\forall\xi<\lambda \ \alpha\mapsto a_\xi^\alpha$ is regressive. So, by the claim, it is almost bounded and $\forall\xi<k$ we can find $A_\xi\in F$ and $\beta_\xi<k$ such that

$$\forall\alpha\in A_\xi \quad a_\xi^\alpha<\beta_\xi .$$

Let $A=\bigcap\{A_\xi : \xi<\lambda\}$ and $\beta=\sup\{\beta_\xi : \xi<\lambda\}$. Then $A\in F$ and $\beta<k$. Clearly $A\subseteq\beta+1$, a contradiction.

4. Theorem 3 can be generalized. We first need some extra definitions. Clearly, if α is an ordinal of cofinality $>\omega$, then $D(\alpha)$, when defined as in section I, is still a filter. A set X is stationary in an ordinal α of cofinality $>\omega$ iff it not in $D^*(\alpha)$.

Let k be an uncountable (preferably weakly inaccessible) cardinal. Then for $A\subseteq k$, $M(A)$ is defined to be the set

$$\{\alpha<k : cf(\alpha)>\omega \text{ and } A\cap\alpha \text{ is stationary in } \alpha\}.$$

M is called the Mahlo operation. A filter F on k is a Mahlo filter iff it closed under M.

Theorem. (Solovay). If k is an infinite cardinal number and F is a normal, k-saturated filter, then F is a Mahlo filter.

Sketch of proof. Suppose F is not Mahlo. Let $A \in F$ be such that $M(A) \notin F$. Then we can find $B \notin F^*$ such that

$$\forall \alpha \in B \quad cf(\alpha) = \omega \text{ or } A \cap \alpha \subseteq D^*(\alpha).$$

It can be shown that the $\alpha \in B$ of cofinality ω can be left out. So, we imitate the proof of Theorem 3 By substituting a set $C_\alpha \in D(\alpha)$ such that $A \cap \alpha \cap C_\alpha = \emptyset$ for the sequence $(a^\alpha_\xi)_{\xi < \lambda}$.

Corollary (Solovay) Let k be a regular cardinal $> \omega$ Then D(k) is not k-saturated and so P(k)/D(k) has cardinality 2^k.

Proof. Every closed unbounded set contains a poinc of cofinality ω. So D(k) is not a Mahlo filter.

IV. REAL VALUED MEASURABLE CARDINALS.

We shall now use the material presented in section III to investigate real valued measurable cardinals.

1. A non trivial measure on k is a map $\mu : P(k) \rightarrow [0,1]$ such that $\mu(\{\alpha\}) = 0$ for every $\alpha < k$, $\mu(k) = 1$ and $\mu(\cup X_i) = \Sigma \mu(X_i)$ for every finite, pairwise disjoint family (X_i) of subsets of k. Notice that

$$F_\mu = \{X \subseteq k : \mu(X) = 1\}$$

is a filter and that it is non principal in the sense that $\{\alpha\} \in F^*_\mu$ for every $\alpha < k$.

Fact Let μ be a non trivial measure. Then F_μ is ω_1-saturated.

Proof. Suppose $(X_\xi)_{\xi < \omega_1}$ is a family of subsets of k such that for any two distinct $\xi, \eta < \omega_1$, $X_\xi \cap X_\eta$ has measure 0. Clearly, at most n X_ξ's have measure $\geq \frac{1}{n}$. So μ vanishes on all but countable many X_ξ's. We say that μ is λ-additive iff F_μ is λ-complete. Remark: ω_1-additive= σ-additive. Vitali's theorem tells us that $2^\omega = [0,1]$ carries no ω_1-additive translation invariant measure. Does this fact remain true when we drop the condition that the measure be translation invariant? In view of the above mentioned fact, this question is related to the following

Problem. Let k be an uncountable cardinal number. May it carry a k-complete, k-saturated, non principal filter?

2. Theorem. (Solovay). Let k be an uncountable cardinal number. If k carries a k-complete, k-saturated non principal filter, then k carries a normal, k-saturated filter.

To prove this important result, we need some preliminaries. Throughout this paragraph, k will be an uncountable cardinal number and F will be a k-complete, k-saturated, non principal filter on k. Let $X, Y \subseteq k$ and $f : X \to k$, $g : Y \to k$. Then

i. g is almost bounded iff there exists $A \in F$ such that $g[X \cap A]$ has cardinality $< k$.

ii. f is F^*-to-one iff $f^{-1}(\xi) \in F^*$ for every $\xi < k$.

iii. $g < f$ iff $X \subseteq Y$ and $g(\xi) < f(\xi)$ for every $\xi < k$.

iv. f is incompressible iff $X \notin F^*$ and f is F^*-to-one every $g < f$ is almost bounded.

Lemma a. Let $X \subseteq k$, $X \notin F^*$ and $\varphi : X \to k$. There exists a partition $\{Y, Z\}$ of X such that $\varphi | Y$ is incompressible and $\exists \psi : Z \to k$ such that $\psi < \varphi$.

$\langle Z, \psi \rangle$ is obtain as the direct limit of a family of $\langle Z', \psi' \rangle$'s using Zorn's lemma.

Lemma b. There exists an incompressible function defined on k.

Proof. Le $Z_0 = k$, $\psi_0 =$ identity on k and $Y_0 = \emptyset$. Suppose Z_i, ψ_i, Y_i are defined. For $Z_i = X$ and $\psi_i = \varphi$, choose $Z_{i+1} = Z$, $\psi_{i+1} =$, $Y_{i+1} = Y$ as in Lemma a. Clearly, $\psi_0 > \psi_1 > \dots$. So $\bigcap Z_i = \emptyset$. Therefore $k = \bigcup Y_i$ and $f = \bigcup \psi_i / Y_{i+1}$ is as desired.

Proof of the theorem. Let f be as in lemma b. We claim that $\mathcal{F} = \{X \subseteq k : f^{-1}[X] \in F\}$ is the desired filter. The only non trivial fact to show is that $\mathcal{F}$ is normal. We shall do this by using the proposition in III-1.

Let $X \notin \mathcal{F}^*$ and $\varphi : X \to k$ be regressive. Clearly, dom $\varphi f = f^{-1}(X) \notin F^*$ and $\varphi f < f$. Since f is incompressible, the range of φf is bounded by some $\beta < k$ on some $A \in F$. Also $f^{-1}(X) \cap A \notin F^*$. So

$$f^{-1}(X) \cap A \subseteq \bigcup \{(\varphi f)^{-1}(\eta) : \eta < \beta\} \notin F^*$$

Since F is k-complete, there exists $\eta < \beta$ such that $(\varphi f)^{-1}(\eta) \notin F^*$. Therefore $f^{-1} \varphi^{-1}(\eta) \notin F^*$ and so $\varphi^{-1}(\eta) \notin \mathcal{F}^*$, as desired.

Corollary. Let k be an uncountable cardinal number. If k carries a k-complete, k-saturated, non principal filter, then k is weakly inaccessible.

Proof. Use the theorem above and § III.3.

3. Using the machinery developped in III, we can improve on §2. A cardinal number k is weakly Mahlo iff $k \in M$(class of regular cardinals) or, equivalently, iff there is a stationary set of regular cardinals below k. Clearly, a weakly Mahlo cardinal is weakly inaccessible, but the converse is not true.

Theorem. If k is an uncountable cardinal number carrying a non principal, k-complete, k-saturated filter, then there is a Mahlo cardinal below k.

Proof. By theorem 2, there is a normal, k-saturated filter $\mathcal{F}$ on k. Define

$$A_o = \{\alpha < k : \alpha \text{ is regular}\}$$

We claim that $A_o \in \mathcal{F}$. To see this, suppose for a contradiction that $A_o \notin \mathcal{F}$. The function $f(\alpha) = cf(\alpha)$ is regressive on $k - A_o \notin \mathcal{F}^*$. So it is constant on some $X \notin \mathcal{F}^*$, say $f[X] = \{\lambda\}$. For each $\alpha \in X$, pick a sequence $(a_\xi^\alpha)_{\xi<\lambda}$ such that $a_\xi^\alpha < \alpha$ and $\sup\{a_\xi^\alpha : \xi < \lambda\} = \alpha$. Continuing the proof as that of the theorem in § III.3, we obtain a contradiction.

Now, by III.4, $\mathcal{F}$ is a Mahlo filter. So $A_1 = M(A_o) \in \mathcal{F}$ and so is $\neq \emptyset$; any $\lambda \in A_1$ is Mahlo.

The theorem tells us that such a k is very large. Since the sequence A_o, A_1 can be continued far into the infinite, the proof tells us that k is even larger.

V. MEASURABLE CARDINALS.

For all we know, a real valued measurable cardinal or a cardinal k carrying a non principal, k-complete, k-saturated filter, although very large, might still be $\leq 2^\omega$. We shall now introduce a new type of large cardinal for which this does not hold.

A cardinal k is strongly inaccessible iff it is regular and $2^\lambda < k$ whenever $\lambda < k$. k is measurable just in case it carries a non principal, k-complete ultrafilter. Clearly ω is strongly inaccessible and measurable.

1. Theorem. A measurable cardinal is strongly inaccessible.

Proof. Let k be an infinite cardinal number and let F be a non principal, k-complete ultrafilter on k. Clearly k must be regular.

Suppose for a contradiction that $\lambda < k \leq 2^\lambda$ for some $\lambda < k$. Abusing language a bit, we can write $k \subseteq 2^\lambda = \{f : f : \lambda \to 2\}$. For $\sigma \in \bigcup\{2^\alpha : \alpha < \lambda\}$, define

$$X_\sigma = \{f \in k : \sigma \subseteq f\}.$$

Now, we define $g \in 2^\lambda$ by recursion on $\xi < \lambda$. Suppose $g|\xi$ is defined and that $X_{g|\xi} \in F$. Let $Y_i = \{f \in k : f \in X_{g|\xi}$ and $f(\xi) = i\}$ $(i=0,1)$. Since $X_{g|\xi} = Y_o \cup Y_1$ and F is an ultrafilter, there is an i such that $Y_i \in F$. Choose $g(\xi)$ such that $Y_i = X_{g|\xi+1}$ (Notice the implicit use of k-completeness).

Let $X = \bigcap\{X_{g|\xi} : \xi < \lambda\}$. By k-completeness $X \in F$. But

$X = \{g\}$. This is a contradiction.

2. We are now going to use the techniques developped in sections III and IV to show that a measurable cardinal is very large. Let k be an uncountable measurable cardinal. A close examination of the proof of theorem IV.2 shows that k carries a normal ultrafilter $\mathfrak{F}$. As in IV.3

$$A_o = \{\alpha < k : \alpha \text{ is regular}\} \in \mathfrak{F} .$$

Now, we claim that:

$$B_o = \{\alpha < k : \alpha \text{ is strongly inaccessible}\} \in \mathfrak{F}.$$

To see this, suppose the contrary; let $f(\alpha)$ = some $\lambda < \alpha$ such that $2^{\lambda} \geq \alpha$. Clearly f is defined on $A_o - B_o \in \mathfrak{F}$. So, it is constant on some $X \in \mathfrak{F}$ such that $X \subseteq A_o$.Let $f[X] = \{\lambda\}$. Then $\forall \alpha \in X \ 2^{\lambda} > \alpha$. So $k \leq 2^{\lambda}$. This contradicts § 1.

By III.4, $\mathfrak{F}$ is a Mahlo filter; so

$$B_1 = M(B_o),\ B_2 = M(B_1), \ldots \in \mathfrak{F}$$

As we have seen in IV.3, this shows that k is very large indeed.

The material presented in section II cames from

P. EKLOF. On the Existence of k-free Abelian Groups. Proc. of the A.M.S. (1975), 65-72.

In section III and IV, we have drawn from

R. SOLOVAY. Real valued measurable cardinals. In: Axiomatic Set Theory, I. A.M.S. Proc. Symp. Pure Math. XIII(1971), 347-428.

For more on the subject, the reader is referred to the Set theory books by Levy, Jeck, Kunen or Drake along with the very readable

P. EKLOF. Set Theoretic Methods in Homological Algebra and Abelian Groups. Séminaires de Mathématiques Superieures, Les Presses de L'Université de Montréal (1980).