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INTERMEDIATE LOGICS

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Part I: PROPOSITIONAL LOGICS

A. PRELIMINARIES.

1. Positive, minimal and intuitionistic propositional logics.

Let $\mathfrak{F}$ be the set of propositional formulas built up from the variables $p_o, p_1, \dots$ and the connectives $\neg, \wedge, \vee, \rightarrow$: further let $\mathfrak{F}^+$ be the set of negation-free formulas.

$\mathbb{P}$ (the positive logic) is the smallest subset of $\mathfrak{F}^+$ which contains :

A1.1 $p \rightarrow (q \rightarrow p)$ A1.2 $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

A2.1 $p \wedge q \rightarrow p$ A2.2 $p \wedge q \rightarrow q$ A2.3 $(p \rightarrow q) \rightarrow ((p \rightarrow r) \rightarrow (p \rightarrow q \wedge r))$

A3.1 $p \rightarrow p \vee q$ A3.2 $q \rightarrow p \vee q$ A3.3 $(p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow (p \vee q \rightarrow r))$

and is closed under modus ponens (MP) and substitution rule (MS). $\overline{\mathbb{P}}$ (the minimal logic) is the smallest subset of $\mathfrak{F}$ which contains:

A1.1 – A3.3 as well as A4.1 $(p \rightarrow q) \rightarrow ((p \rightarrow \neg q) \rightarrow \neg p)$

and is closed under MP and MS.

$\overline{\overline{\mathbb{P}}}$ (the intuitionistic logic) is the smallest subset of $\mathfrak{J}$ which contains:

A1.1 – A4.4 as well as A4.2 $p \to (\neg p \to q)$

and is closed under MP and MS.

We recall some of the most popular alternative axiomatizations.

Instead of A1.2 : $(p \to q) \to ((p \to (q \to r)) \to (p \to r))$

" " A1.2 : $(p \to (p \to q)) \to (p \to q)$ and $(p \to q) \to ((q \to r) \to (p \to r))$

" " A2.3 : $p \to (q \to p \wedge q)$

" " A2.1-3: $(p \to (q \to r)) \to (p \wedge q \to r)$ and $(p \wedge q \to r) \to (p \to (q \to r))$

" " A4.1 : $(p \to \neg q) \to (q \to \neg p)$

" " A4.1 : $(p \to q) \to (\neg q \to \neg p)$ and $p \to \neg\neg p$

" " A4.1 : $(p \to q) \to (\neg q \to \neg p)$ and $(p \to \neg p) \to \neg p$

" " A4.2 : $\neg p \to (\neg\neg p \to p)$

" " A4.2 : $p \vee q \to (\neg p \to q)$

In subsequent discussion it will be useful to remember that the following formulas are in $\mathbb{P}$:

$$\begin{cases} (p \vee q \to r) \leftrightarrow (p \to r) \wedge (q \to r) \\ (p \to q \wedge r) \leftrightarrow (p \to q) \wedge (p \to r) \end{cases}$$

$$\left.\begin{cases} (p \wedge q \to r) \leftarrow (p \to r) \vee (q \to r) \\ (p \to q \vee r) \leftarrow (p \to q) \vee (p \to r) \end{cases}\right\} \text{not, however, } \to$$

$$(p \vee q \to r \wedge s) \leftrightarrow (p \to r) \wedge (p \to s) \wedge (q \to r) \wedge (q \to s)$$

$$(p \wedge q \to r \vee s) \leftarrow (p \to r) \vee (p \to s) \vee (q \to r) \vee (q \to s) \} \text{not, however} \to$$

$$(p \wedge q \to r \vee s) \leftarrow (p \to r) \vee (q \to s) \quad \} \quad \text{not, however } \to$$

and also that following formulas are in $\overline{\mathbb{P}}$:

$$\neg\neg\neg\alpha \leftrightarrow \neg\alpha \qquad \neg\neg(\alpha \vee \neg\alpha) \qquad \neg\alpha \to (\alpha \to \neg\beta)$$

$$\begin{cases} \neg(\alpha \vee \beta) \leftrightarrow \neg\alpha \wedge \neg\beta \\ \neg(\alpha \wedge \beta) \leftarrow \neg\alpha \vee \neg\beta \; \} \end{cases} \text{not, however, } \to \text{ (also not in } \overline{\overline{\mathbb{P}}}\text{!)}$$

and that following formula is in $\overline{\overline{\mathbb{P}}} - \overline{\mathbb{P}}$

$$\neg(\alpha \to \beta) \to \neg\neg\alpha$$

Finally remember that minimal logic and intuitionistic logic may be obtained by dropping $\neg$, assuming the constant $\bot$, defining $\neg p =_{df} p \to \bot$ and, respectively, making no assumption on $\bot$ and making the assumption $\bot \to p$.

2. Relational propositional semantics (Kripke-semantics).

Def. 1 (1) $\langle P, \leq \rangle$ is a partial order iff for

$$\text{all } i,j,k \in P: \begin{cases} i \leq i \\ i \leq j \wedge j \leq k \to i \leq k \\ i \leq j \wedge j \leq i \to i = j \end{cases}$$

(2) $Q \subseteq P$ is open (in the partial order $\langle P, \leq \rangle$) iff for all $i,j \in P$: $i \in Q \wedge i \leq j \to j \in Q$.

(3) $\langle P, \leq, Q \rangle$ is a special partial order iff $\langle P, \leq \rangle$ is a partial order and Q is open.

(4) $\langle P, \leq, Q \rangle$ is a scotian partial order iff $\langle P, \leq, Q \rangle$ is a special partial order and $Q = \emptyset$.

In what follows we always assume $P \neq \emptyset$.

Def. 2 A realization of (the propositional language) $\mathcal{L}$ on the partial order $\langle P, \leq \rangle$ (on the special partial order $\langle P, \leq, Q \rangle$) is a mapping ρ of the set of propositional variables into the set of open subsets of $\langle P, \leq \rangle$ $(P, \leq, Q)$.

Def. 3 If ρ is a realization of $\mathcal{L}$ on $\langle P, \leq \rangle$ (on $\langle P, \leq, Q \rangle$), the relation "$\rho$ forces α at i"- to be written: $\rho_i \vDash \alpha$ or, simply, $i \vDash \alpha$ - between $i \in P$ and $\alpha \in \mathfrak{Z}^+$ $(\alpha \in \mathfrak{Z})$ is so defined by recursion on α:

(1) $i \vDash p$ iff $i \in \rho(p)$

(2) $i \vDash \alpha \wedge \beta$ iff $i \vDash \alpha$ and $i \vDash \beta$

(3) $i \vDash \alpha \vee \beta$ iff $i \vDash \alpha$ or $i \vDash \beta$

(4) $i \vDash \alpha \rightarrow \beta$ iff $\forall_{i \leq j}\, j$ (if $j \vDash \alpha$, then $j \vDash \beta$)

(5) $i \vDash \neg \alpha$ iff $\forall_{i \leq j}\, j$ (if $j \vDash \alpha$, then $j \in Q$)

Theor. 1 (1) for all $\alpha \in \mathfrak{Z}^+ (\alpha \in \mathfrak{Z})$, $[i / \rho_i \vDash \alpha]$ is open in $\langle P, \leq \rangle$ (in $\langle P, \leq, Q \rangle$)

(2) if $\langle P, \leq, Q \rangle$ is scotian, then $i \vDash \neg \alpha$ iff $\forall_{i \leq j}\, j \;\; j \nvDash \alpha$.

Def. 4 (1) α holds in ρ $[\rho \vDash \alpha]$ iff $\forall_{i \in P}\, i \;\; \rho_i \vDash \alpha$

(2) α is valid in $\langle P, \leq \rangle$ (in $\langle P, \leq, Q \rangle$) $[P \vDash \alpha]$ iff $\rho \vDash \alpha$ for all ρ on $\langle P, \leq \rangle$ (on $\langle P, \leq, Q \rangle$)

(3) if $\mathfrak{P}$ is a class of (special) partial orders, then $\mathfrak{P} \vDash \alpha$ iff $\forall_{P \in \mathfrak{P}}\, P \;\; P \vDash \alpha$.

Theor. 2 (1) for $\alpha \in \mathfrak{Z}^+$: $\alpha \in \mathbb{P}$ iff α is valid in all partial orders;

(2) for $\alpha \in \mathfrak{Z}$: $\alpha \in \overline{\mathbb{P}}$ iff α is valid in all special partial orders (Segerberg, 1968);

: $\alpha \in \overline{\overline{\mathbb{P}}}$ iff α is valid in all scotian partial orders (Kripke, 1965).

B. INTERMEDIATE PROPOSITIONAL LOGICS.

1. Syntactical characterization.

Def. 1 An intermediate $\left\{\begin{array}{c}\text{negation-free} \\ \text{negative} \\ \_\_\_\_\_\_ \end{array}\right\}$ propositional logic L is a subset of $\left\{\begin{array}{c}\mathfrak{Z}^+ \\ \mathfrak{Z} \\ \mathfrak{Z}\end{array}\right\}$ which contains $\left\{\begin{array}{c}\mathbb{P} \\ \overline{\mathbb{P}} \\ \overline{\overline{\mathbb{P}}}\end{array}\right\}$ and is closed under MP and MS.

$\mathfrak{J}$, $\bar{\mathfrak{J}}$ and $\bar{\bar{\mathfrak{J}}}$ shall be, respectively, the set of negation-free intermediate logics, the set of negative intermediate logics, the set of intermediate logics (always: propositional, of course).

It is easy to see that for $\mathcal{J} \subseteq \mathfrak{J}$, $\bigcap \mathcal{J} \in \mathfrak{J}$ (and that the same holds for $\mathcal{J} \subseteq \bar{\mathfrak{J}}$ or $\mathcal{J} \subseteq \bar{\bar{\mathfrak{J}}}$); so we may define, for $\mathfrak{M} \subseteq \mathfrak{F}^+$ (respectively in $\mathfrak{F}$), $\mathbb{L}_{\mathfrak{J}}(\mathfrak{M}) = \bigcap [L \in \mathfrak{J} / \mathfrak{M} \subseteq L]$ (and similarly $\mathbb{L}_{\bar{\mathfrak{J}}}(\mathfrak{M})$ and $\mathbb{L}_{\bar{\bar{\mathfrak{J}}}}(\mathfrak{M})$) and thus prove

<u>Theor. 1</u> $<\mathfrak{J}, \subseteq>$ is a <u>complete</u> partial order with $\mathrm{Inf}(\mathcal{J}) = \bigcap \mathcal{J}$ and $\sup(\mathcal{J}) = \mathbb{L}_{\mathfrak{J}}(\bigcup \mathcal{J})$ (and analogously for $<\bar{\mathfrak{J}}, \subseteq>$ and $<\bar{\bar{\mathfrak{J}}}, \subseteq>$).

Further:

<u>Theor. 2</u> Defining: $L_1 \circ L_2 = \mathrm{Inf}(\{L_1, L_2\})$; $L_1 + L_2 = \mathrm{Sup}(\{L_1, L_2\})$; $L_1 \rightarrow\!\!\!\!\exists\, L_2 = \mathrm{Sup}[L \mid L \in \mathfrak{J} \wedge L_1 \circ L \subseteq L_2]$; $1 = \mathfrak{F}^+$; $0 = \mathbb{P}$; the algebra $<\mathfrak{J}, +, \cdot, \rightarrow\!\!\!\!\exists, 1, 0>$ is a complete Heyting algebra (and analogously for $\bar{\mathfrak{J}}$ and $\bar{\bar{\mathfrak{J}}}$)

2. <u>Some examples of negation-free intermediate logics</u>.

Disregarding the trivial negation-free inconsistent logic $\mathfrak{F}^+$, three well-renowned elements of $\mathfrak{J}$ are the negation-free reducts of intuitionistic, Dummett's

and classical logic respectively. The first one is precisely $\mathbb{P}$; the second one, to be called $\mathbb{B}$, can be obtained by adjoining to $\mathbb{P}$ one of the following interesting schemata.

$(\alpha \to \beta) \vee (\beta \to \alpha)$ $\quad$ $((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma)$

$(\alpha \wedge \beta \to \gamma) \to (\alpha \to \gamma) \vee (\beta \to \gamma)$ $\quad$ $(\alpha \to \beta \vee \gamma) \to (\alpha \to \beta) \vee (\alpha \to \gamma)$

$(\alpha \vee \beta \to \alpha) \vee (\alpha \vee \beta \to \beta)$ $\quad$ $(\alpha \to \alpha \wedge \beta) \vee (\beta \to \alpha \wedge \beta)$

$((\alpha \to \beta) \to \beta) \wedge ((\beta \to \alpha) \to \alpha) \to \alpha \vee \beta$

$(\alpha \wedge \beta \to \gamma \vee \delta) \to (\alpha \to \gamma) \vee (\alpha \to \delta) \vee (\beta \to \gamma) \vee (\beta \to \delta)$

$(\alpha \to \beta) \vee ((\alpha \to \beta) \to \beta)$ $\quad$ $(\alpha \to \beta) \vee ((\alpha \to \beta) \to \alpha)$

Among the most interesting properties of $\mathbb{B}$ can be mentioned:

1) in $\mathbb{B}$ the $\vee$ is definable by $\alpha \vee \beta \leftrightarrow ((\alpha \to \beta) \to \beta) \wedge ((\beta \to \alpha) \to \alpha)$

2) in $\mathbb{B}$ every $\alpha \in \mathfrak{F}^+$ can be equivalently written as con junction of disjunctions (and, of course, viceversa) of ' falling implications' (i.e. implications of the form $(\ldots((p_0 \to p_1) \to p_2) \to \ldots \to p_n)$.

The third one, to be called $\mathbb{T}$, can be obtained by adjoining to $\mathbb{B}$ (in fact to $\mathbb{P}$) one of the following interesting schemata.

$\alpha \vee (\alpha \to \beta)$ $\quad$ $((\alpha \to \beta) \to \alpha) \to \alpha$ $\quad$ $((\alpha \to \beta) \to \gamma) \to ((\alpha \to \gamma) \to \gamma)$

$((\alpha \to \beta) \to \beta) \to \alpha \vee \beta$ $\quad$ $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$ $\quad$ $(\alpha \to \gamma) \vee (\gamma \to \beta)$

$(\alpha \wedge \beta \to \gamma \vee \delta) \to (\alpha \to \gamma) \vee (\beta \to \delta)$.

The inclusion diagram is of course the following: $\mathbb{P} \to \mathbb{B} \to \mathbb{T}$. Two interesting refinements of this diagrams can be obtained by considering the two sequences of logics $\{\mathbb{A}_n\}_{n\in\omega}$ $\{\mathbb{T}_n\}_{n\in\omega}$ which are obtained by adjoining to $\mathbb{P}$ the following schemata:

$$\mathbb{A}_n : \bigvee_{i=0}^{n+1} (\alpha_i \to \bigvee_{j\neq i}^{n+1} \alpha_j)$$

$$\mathbb{T}_n : \alpha_0 \vee \bigvee_{i=0}^{n} (\alpha_i \to \alpha_{i+1}) \qquad \text{(Segerberg,1968)}$$

It is: $\bigcap_{n\in N} \mathbb{A}_n = \mathbb{P}$; $\bigcap_{n\in N} \mathbb{T}_n = \mathbb{B}$; $\mathbb{A}_0 = \mathbb{B}$; $\mathbb{T}_0 = \mathbb{T}$.

And the inclusion diagramm (all inclusions are proper!) is the following:

$$\mathbb{P} \to \ldots \to \mathbb{A}_n \to \ldots \to \mathbb{A}_1 \to \mathbb{B} \ldots \to \mathbb{T}_n \to \ldots \to \mathbb{T}_1 \to \mathbb{T}$$

Two other interesting sequences of logics, to be called $\mathbb{B}_n$ and $\mathbb{C}_n$ respectively, are obtained by adjoining to $\mathbb{P}$ the following schemata:

$$\mathbb{B}_n : \bigwedge_{i=0}^{n+1} ((\alpha_i \to \bigvee_{i\neq j} \alpha_j) \to \bigvee_{i\neq j} \alpha_j) \to \bigvee_{i=0}^{n+1} \alpha_i$$

(Gabbay-De Jongh,1974)

$$\mathbb{C}_n : \begin{cases} \mathbb{C}_0 = \alpha_1 \vee (\alpha_1 \to \alpha_0) \\ \mathbb{C}_{k+1} = \alpha_{k+2} \vee (\alpha_{k+2} \to \mathbb{C}_k) \end{cases}$$

It is $\mathbb{B}_0 = \mathbb{B}$; $\mathbb{C}_0 = \mathbb{T}$; $\cap \mathbb{B}_n = \mathbb{P}$; $\cap \mathbb{C}_n = \mathbb{P}$ and the inclusion diagram is as follows

$$\begin{tikzcd}
 & & & & & \mathbb{C}_n \arrow[r, dotted] \arrow[d] & \mathbb{C}_2 \arrow[r] \arrow[d] & \mathbb{C}_1 \arrow[d] & \\
\mathbb{P} \arrow[urrrrr, dotted] \arrow[r, dotted] \arrow[dr, dotted] & \mathbb{A}_n \arrow[r, dotted] & \mathbb{A}_2 \arrow[r] & \mathbb{A}_1 \arrow[r] & \mathbb{B} \arrow[r, dotted] & \mathbb{T}_n \arrow[r, dotted] & \mathbb{T}_2 \arrow[r] & \mathbb{T}_1 \arrow[r] & \mathbb{T} \\
 & \mathbb{B}_n \arrow[u] \arrow[r, dotted] & \mathbb{B}_2 \arrow[u] \arrow[r] & \mathbb{B}_1 \arrow[u] & & & & &
\end{tikzcd}$$

moreover $\mathbb{T}_n = \mathbb{C}_n + \mathbb{B}$.

A sequence equivalent to $\mathbb{C}_n$ is

$$\mathbb{N}_n : \begin{cases} \mathbb{N}_0 = ((\alpha_1 \to \alpha_0) \to \alpha_1) \to \alpha_1 \\ \mathbb{N}_{k+1} = (((\alpha_{k+2} \to \mathbb{N}_k) \to \alpha_{k+2}) \to \alpha_{k+2}) \end{cases}$$

(Troelstra 1965,Nagata 1966)

3. Some examples of (negative) intermediate logics.

When negation is considered there are two kinds of logics which deserve attention. The first one is that obtained by simply adjoining the negation-free schemata considered in $\mathfrak{J}$ to $\overline{\mathbb{P}}$ or $\overline{\overline{\mathbb{P}}}$ .The resulting logics,for given name $\mathbb{X}$ in $\mathfrak{J}$ will be denoted by $\overline{\mathbb{X}}$ or $\overline{\overline{\mathbb{X}}}$ respectively.So, for example, $\overline{\overline{\mathbb{B}}}$ is Dummett's logic and $\overline{\overline{\mathbb{T}}}$ is classical logic. Besides these,there are however interesting logics in $\overline{\mathfrak{J}}$ or $\overline{\overline{\mathfrak{J}}}$ which may be obtained from negative logics by adjoining schemata which concern nagation. Within the most famous schemata of this kind we only recall:

(1) Kreisel-Putnam schema: $(\neg\alpha\to\beta\vee\gamma)\to(\neg\alpha\to\beta)\vee(\neg\alpha\to\gamma)$

(2) Weak TND schemata: one of the following schemata:
$\neg\alpha\vee\neg\neg\alpha$; $(\neg\alpha\to\beta)\to((\neg\neg\alpha\to\beta)\to\beta)$; $\neg(\alpha\wedge\beta)\to\neg\alpha\vee\neg\beta$; $(\neg\alpha\to\neg\beta)\vee(\neg\beta\to\neg\alpha)$; $(\neg\neg\alpha\to\alpha)\to\neg\alpha\vee\alpha$;

(3) Strong TND schemata: one of the following schemata:
$\alpha\vee\neg\alpha$, $(\alpha\to\beta)\to((\neg\alpha\to\beta)\to\beta)$; $(\neg\alpha\to\alpha)\to\alpha$; $(\alpha\to\beta)\to\neg\alpha\vee\beta$.

[remark that (3) adjoined to $\overline{\overline{\mathbb{P}}}$ gives, of course, classical logic but adjoined to $\overline{\mathbb{P}}$ gives a logic, first investigated by Curry, the negation-free reduct of which is still $\mathbb{P}$ (Pearson)]. Further remark that $\overline{\overline{\mathbb{P}}}+(1)\subseteq\overline{\overline{\mathbb{P}}}+(2)$ and of course $\overline{\mathbb{P}}+(1)$, $\overline{\mathbb{P}}+(2)\subseteq\overline{\mathbb{B}}$; $\overline{\mathbb{P}}+(3)\subseteq\overline{\mathbb{T}}$.

Another interesting schema is $\neg(\alpha\to\beta)\to\neg\neg\alpha$ (or equivalently: $\neg\alpha\to\neg\neg(\alpha\to\beta)$) which adjoined to $\overline{\mathbb{P}}$ gives a logic which is both in $\overline{\overline{\mathbb{P}}}$ and in $\overline{\mathbb{T}}$.

At last, remark that $\overline{\overline{\mathbb{P}}}+(2)\equiv\overline{\mathbb{P}}+(\neg\neg\alpha\to\alpha)\leftrightarrow\alpha\vee\neg\alpha$.

An interesting sequence (due to Smoriński, 1973) is the following S_n $(n>0)$:

$$S_n:\quad \bigwedge_{0\le i<j\le n}\neg\neg(\alpha_i\vee\alpha_j)\to\bigvee_{i=0}^{n}(\neg\alpha_i\to\bigvee_{i\neq j}\neg\alpha_j).$$

Remark: $\overline{\mathbb{P}}+S_1\equiv\overline{\mathbb{P}}+\neg\alpha\vee\neg\neg\alpha$ (the same for $\overline{\overline{\mathbb{P}}}$); $\bigcap_{n\in N}\overline{\mathbb{P}}+S_n=\overline{\mathbb{P}}$ (the same for $\overline{\overline{\mathbb{P}}}$).

4. Relational semantic for intermediate logics.

To avoid tedious repetitions and notational complications we will, in general, speak of "logics" and "partial orders". If no special warning is made, following definitions and theorems are to be understood as separate referring to three couples: < negation free intermediate logics - partial orders >; <negative intermediate logics - special partial orders > ; < intermediate logics - scotian partial orders > .

Def. 1 1) If $\mathcal{P}$ is a class of partial orders, then
$\mathbb{L}^*(\mathcal{P})=[\alpha\mid\mathcal{P}\models\alpha]$

2) If $\mathfrak{M}$ is a set of formulas, then $\mathbb{M}(\mathfrak{M})=$
$=[P\mid P\models\mathfrak{M}]$

From Def.1 it follows:

Theor. 1 $\mathbb{L}^*$ and $\mathbb{M}$ determine a Galois connection between sets of formulas and classes of partial orders i.e.

1.1 $\mathfrak{M}\subseteq\mathbb{L}^*(\mathbb{M}(\mathfrak{M}))$ 1.2 $\mathcal{P}\subseteq\mathbb{M}(\mathbb{L}^*(\mathcal{P}))$

2.1 $\mathfrak{M}\subseteq\mathfrak{N}\to\mathbb{M}(\mathfrak{N})\subseteq\mathbb{M}(\mathfrak{M})$ 2.2 $\mathcal{P}\subseteq Q\to\mathbb{L}^*(Q)\subseteq\mathbb{L}^*(\mathcal{P})$

Theor. 2 For every $\mathcal{P}$, $\mathbb{L}^*(\mathcal{P})$ is a logic i.e. a set of formulas including, respectively, $\mathbb{P}$, $\overline{\mathbb{P}}$, $\overline{\overline{\mathbb{P}}}$ and closed under MP and MS.

Def. 2 If L is a logic then we say:

(1) L is valid in $\mathcal{P}$ iff $L \subseteq \mathbb{L}^*(\mathcal{P})$ (i.e. iff $\alpha \in L \rightarrow \mathcal{P} \models \alpha$)

(2) L is complete in $\mathcal{P}$ iff $\mathbb{L}^*(\mathcal{P}) \subseteq L$ (i.e. iff $\mathcal{P} \models \alpha \rightarrow \alpha \in L$)

(3) L is characterized by $\mathcal{P}$ iff $L = \mathbb{L}^*(\mathcal{P})$

(4) L is Kripke-complete iff there is a $\mathcal{P}$ such that $\mathcal{P}$ characterizes L.

(5) L is finitary (or has the finite model property) iff it is characterized by a class of finite partial orders.

(6) L is tabular iff it is characterized by a single finite partial order.

We collect now some semantical characterizations which are known for logics we spoke about up to now. First some definitions:

Def. 1 (1) $i \ll j$ iff $i < j \wedge \forall k\, (i \leq k \leq j \rightarrow i=k \vee k=j)$

(2) $[i\gg\; = [j \mid i \ll j]$

Def. 2 Let $A \subseteq P$, then:

(1) A is a chain in P iff $\forall_{i,j\in A}\, (i \leq j \vee j \leq i)$

(2) A is an antichain in P iff $\forall_{i,j\in A}\, (i \neq j \rightarrow i \nleq j \wedge j \nleq i)$

(3) A is directed in P iff $\forall_{i,j\in A}\, \exists_{k\in A}\, (i \leq k \wedge j \leq k)$

Def. 3 (1) P has height n iff n is the maximal length of chains in P

(2) P has width n iff n is the maximal length of antichains in P

(3) P has local width n iff $\forall_{i\in P}$ (n is the maximal number of elements in $[i\gg$)

(4) P has top width n iff P has n maximal elements.

(5) P is a tree iff $\forall_{i\in P}\, (\langle i]$ is well ordered) and P is principal (where $\langle i] = [j \mid j \leq i]$)

When the partial orders one is considering are special ones (i.e. $\langle P \leq Q \rangle$) then it is often useful to consider the above properties limited to the normal part of P i.e. to $P - Q$. In these cases we use the specification 'normal' as indicated by the following example: 'P has normal height n' means 'n is the maximal length of chains in P-Q'.

A rather involved property of partial orders is the following. Define, first, for $A \subseteq P$, $A^{\leq} = [j \mid \exists i\,(i \in A \wedge i \leq j)]$, $A^{\geq} = [j \mid \exists i\,(i \in A \wedge j \leq i)]$ and then A° by setting: $A^{\circ} = (A^{\leq})^{\geq}$. We then say: A partial order has property $\#$ iff $\forall_{A \subseteq P} A.$ $P - A^{\circ}$ is either empty or has a first element; A partial order has the Kreisel - Putnam property iff all its principal suborders have property $\#$.

5. CHARACTERIZATIONS OF SOME LOGICS IN $\mathfrak{J}$.

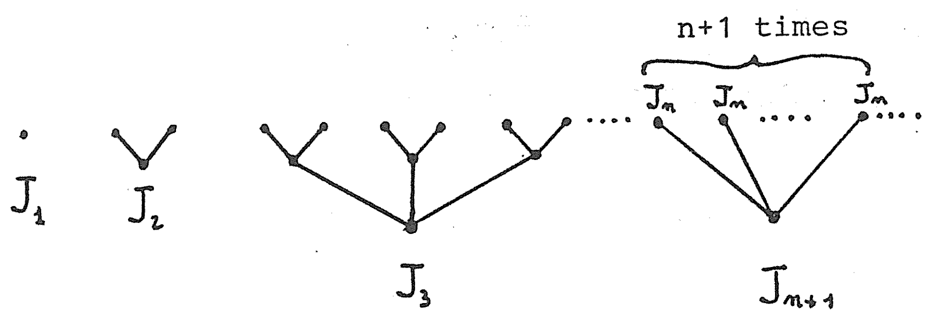
Logic	Classes of partial orders	Classes of finite partial orders
$\mathbb{P}$	Partial orders trees $\{$full binary tree$\}$	finite partial orders finite trees Jaskowski trees $\{J_n\}_{1 \leq n}$
$\mathbb{B}$	chains $\{\omega\}$	finite chains
$\mathbb{T}$		$\{\mathbb{1}\}$ $(\mathbb{1} = \langle \{0\} = \rangle)$
$\mathbb{T}_n$		chains of card $\leq$ n + 1 $\{\mathbb{n} + \mathbb{1}\}$ $(= \langle \{0, \ldots n\} (0 \to 1 \to \ldots \to n) \rangle)$
$\mathbb{C}_n$	partial orders of height $\leq$ n+1 partial orders of height n+1	finite partial orders of height $\leq$ n+1 finite partial orders of height n+1
$\mathbb{A}_n$	partial orders of width $\leq$ n+1	finite partial orders of width $\leq$ n+1
$\mathbb{B}_n$		finite partial orders with local width $\leq$ n+1

Remarks: (1) Characterization of $\mathbb{B}_n$ by local width is not extendible to infinite partial orders (the full binary tree is of local width 2 but $\mathbb{B}_n$ is not positive!)

(2) All logics in the list are finitary.

(3) Only $\mathbb{T}$ and $\mathbb{T}_n$ $(n > 0)$ are tabular.

(4) The Jaskowski trees are:



6. CHARACTERIZATIONS OF SOME LOGICS IN $\bar{\mathfrak{J}}$.

Logic	Classes of special partial orders	Classes of finite special partial orders
$\overline{\mathbb{P}}$	Special partial orders special trees	finite special partial orders finite special trees
$\overline{\mathbb{B}}$	special chains	finite special chains
$\mathbb{T}$		$\{\langle\{0\}=\emptyset\rangle, \langle\{0\}=\{0\}\rangle\}$
$\mathbb{T}_n$		special chains of card $\leq n + 1$
$\overline{\mathbb{C}}_n$	special partial orders of height $\leq n + 1$	
$\overline{\mathbb{P}}_{+\neg\alpha\vee\neg\neg\alpha}$	special partial orders normally directed	

Logic	Classes of special partial orders	Classes of finite special partial orders
$\overline{\mathbb{P}}+\alpha\vee\neg\alpha$	special partial orders with $P-Q=\{0\}$	finite special partial orders with $P-Q=\{0\}$
$\overline{\mathbb{P}}+F$	spec. part. orders with: $\forall_{i\in P-Q}\ \exists_{j\in P}\ (i\leq j\wedge [j\rangle\cap Q=\emptyset)$	

Remarks: (1) F is $\neg(\alpha\to\beta)\to\neg\neg\alpha \quad (+\overline{\mathbb{P}})$

or

$\neg\alpha\to\neg\neg(\alpha\to\beta) \quad (+\overline{\mathbb{P}})$

or

$\neg\neg(\neg\neg\alpha\to\alpha) \quad (+\overline{\mathbb{P}})$

7. CHARACTERIZATIONS OF SOME LOGICS IN $\overline{\overline{\mathfrak{J}}}$.

Logic	Classes of scotian partial orders	Classes of finite scotian partial orders
$\overline{\overline{\mathbb{P}}}$	scotian partial orders scotian trees $\{$ full binary scotian tree $\}$	finite scotian partial orders finite scotian tree scotian Jaskowski trees
$\overline{\overline{\mathbb{T}}}$		$\{$ scotian $\mathbb{1}\}$
$\mathbb{T}_n$		scotian chains of card $\leq n+1$ $\{$ scotian $n+1\}$
$\overline{\overline{\mathbb{C}}}_n$	scotian partial orders of length $\leq n+1$ scotian partial orders of length $n+1$	finite scotian partial orders of length $\leq n+1$ finite scotian partial orders of length $n+1$
$\overline{\overline{\mathbb{A}}}_n$	scotian partial orders of width $\leq n+1$	finite scotian partial orders of width $\leq n+1$

Logic	Classes of scotian partial orders	Classes of finite scotian partial orders
$\overline{\overline{\mathbb{B}}}_n$		finite scotian partial orders of local width $\leq$ $\leq n+1$
$\mathbb{KP}$		finite scotian partial orders of Kreisel-Putnam
$\mathbb{J}$	scotian directed partial orders scotian partial orders with maximum	finite scotian directed partial orders $\Vert$ finite scotian partial orders with maximum
$\mathbb{S}_n$	scotian partial orders with: $\exists x_1,..,x_n \forall y \bigvee_{i=1}^{n} y \leq x_i$ (i.e. maximal top. width n)	finite scotian partial orders with: $\exists x_1,...,x_n \forall y \bigvee_{i=1}^{n} y \leq x_i$

Remarks: (1) $\mathbb{J}$ is $\overline{\overline{\mathbb{P}}} + \neg\alpha \vee \neg\neg\alpha$

(2) $\mathbb{KP}$ is $\overline{\overline{\mathbb{P}}} + (\neg\alpha \rightarrow \beta \vee \gamma) \rightarrow (\neg\alpha \rightarrow \beta) \vee (\neg\alpha \rightarrow \gamma)$

(3) $\overline{\overline{\mathbb{P}}}$, $\overline{\overline{\mathbb{B}}}_n$ and $\mathbb{KP}$ are prime; an other logics in the list are not prime.

From the preceding tables it is immediate to deduce:

1. There are Kripke-Complete Logics.
2. There are finitary logics.
3. There are tabular logics.

But also:

4. There are logics which are finitary but not tabular.

It is then important to stress:

5. There are logics which are not finitary (Fine, 1970; Kuznezov - Gercin, 1970; see Gabbay [1981] pp. 103-105).
7. There are logics which are not Kripke - complete (Shechtmann, 1977).

After this brief survey of some particular intermediate propositional logics we go over to consider one example of investigations of a more general kind.

8. <u>HOMOMORPHISMS AND SUBORDERS</u>.

A homomorphism between (special) partial orders is a map which preserves order (as well as normality and non-normality) i.e.

Def. 1 f is an <u>homomorphism</u> from $\langle P \leq Q \rangle$ into

$\langle P' \leq' Q' \rangle$ iff (1) $f : P \to P'$; (2) $\forall_{i,j\in P} (i \leq j \to f(i) \leq' f(j))$; (3) $\forall_{i\in P} (i \in Q \leftrightarrow f(i) \in Q')$.

<u>Def. 2.</u> A homomorphism f is an <u>epimorphism</u> iff it is <u>onto</u> and it is a <u>monomorphism</u> iff it is injective. If $A \subseteq P$ then $\langle A, \leq\restriction A, Q \cap A\rangle$ is a <u>suborder</u> of $\langle P \leq Q \rangle$.

It turns out that all these notions are too general to show interesting results in relational semantics. The right specialization seems to be that obtained by imposing to the involved maps to be <u>open</u> i.e. preserving openess.

<u>Def. 3</u> A homomorphism (epimorphism, monomorphism) f from $\langle P \leq Q \rangle$ into $\langle P' \leq' Q' \rangle$ is <u>open</u> iff for all $A \subseteq P$, if A is open in P then $f(A) = [f(i) \mid i \in A]$ is open in P'.

Openess of a homomorphism can be elementary characterized:

<u>Theor. 1</u> A homomorphism f from $\langle P \leq Q \rangle$ into $\langle P' \leq' Q' \rangle$ is open iff $\forall_{i\in P} \forall_{j'\in P'} (f(i) \leq' j' \to \exists_{j\in P} (i \leq j \wedge f(j) = j'))$

Proof: ($\Rightarrow$) Let $f(i) \leq' j'$ with $i \in P$ and $j' \in P'$. Consider $f([i\rangle)$ (with $[i\rangle = [j \mid i \leq j]$). As $[i\rangle$ is open in P, $f([i\rangle)$ is open in P' by hypothesis.

As, however, $f(i) \in f([i\rangle)$ and $f(i) \leq' j'$ so $j' \in f([i\rangle)$ i.e. $j' = f(j)$ for some $j \in [i\rangle$ i.e. for some j such that $i \leq j$.

($\Leftarrow$) Let A be open in P and consider f(A). Let $i' \in f(A)$ and $i' \leq' j'$. As $i' \in f(A)$, then $i' = f(i)$ for some $i \in A$. Thus $f(i) \leq' j'$. Thus, by hypothesis, $j' = f(j)$ for some $j \in P$ such that $i \leq j$. But A is open and then from $i \in A$ and $i \leq j$ it follows: $j \in A$. But then $f(j) \in f(A)$, i.e. $j' \in f(A)$.

<u>Theor. 2</u> (<u>Open homomorphism lemma</u>). If f is an open homomorphism from $\langle P \leq Q \rangle$ into $\langle P' \leq' Q' \rangle$ and if ρ is a realization on $\langle P \leq Q \rangle$ and ρ' a realization on $\langle P' \leq' Q' \rangle$ such that for all variables p, $i \in \rho(p)$ iff $f(i) \in \rho'(p)$ (for all $i \in P$), then, for all formulas α and all $i \in P$: $\rho_i \models \alpha$ iff $\rho'_{f(i)} \models \alpha$.

Proof: by induction on α. For $\alpha = p$, $\alpha = \beta \wedge \gamma$, $\alpha = \beta \vee \gamma$, trivial. For $\alpha = \beta \to \gamma$: let $\rho_i \not\models \beta \to \gamma$. Then, for some $j \in P$: $i \leq j$ and $\rho_j \models \beta$ and $\rho_j \not\models \gamma$. Thus, by ind. hyp., $\rho'_{f(j)} \models \beta$ and $\rho'_{f(j)} \not\models \gamma$. But $i \leq j \to f(i) \leq' f(j)$. Thus $\rho'_{f(i)} \not\models \beta \to \gamma$. On the other side, let $\rho'_{f(i)} \not\models \beta \to \gamma$. Then, for some $j' \in P'$: $\rho'_{j'} \models \beta$ and $\rho'_{j'} \not\models \gamma$ and $f(i) \leq' j'$. But f is open, thus, by theor. 1, $f(j) = j'$ for some $j \in P$ such that $i \leq j$ and so $\rho'_{f(j)} \models \beta$ and $\rho'_{f(j)} \not\models \gamma$.

Then, by ind.hypothesis, $\rho_j \models \beta$ and $\rho_j \not\models \gamma$. But $i \leq j$, thus $\rho_i \not\models \beta \to \gamma$. For $\alpha = \neg\beta$: let $\rho_i \not\models \neg\beta$, then, for some j such that $j \notin Q$ and $i \leq j$: $\rho_j \models \beta$. Then, by ind. hyp., $\rho'_{f(j)} \models \beta$. From $i \leq j$ it follows $f(i) \leq' f(j)$ and from $j \notin Q$ it follows $f(j) \notin Q'$. But then: $\rho'_{f(i)} \not\models \neg\beta$. On the other side, let $\rho'_{f(i)} \not\models \neg\beta$. Then, for some $j' \notin Q'$, it is $f(i) \leq' j'$ and $\rho'_{j'} \models \beta$. But f is open, and thus from $f(i) \leq' j'$ it follows that for some $j \in P: i \leq j$ and $f(j) = j'$. Thus $\rho'_{f(j)} \models \beta$ and $f(j) \notin Q'$. By ind. hyp., $\rho_j \models \beta$ and from $f(j) \notin Q'$ it follows $j \notin Q$. Being $i \leq j$, we have then: $\rho_i \not\models \neg\beta$.

Theor. 3 (Open epimorphism theorem). If there is an open epimorphism from $\langle P \leq Q \rangle$ onto $\langle P' \leq' Q' \rangle$, then :

$$\mathbb{L}^*(\langle P \leq Q \rangle) \subseteq \mathbb{L}^*(\langle P' \leq' Q' \rangle)$$

Proof. Let $\alpha \in \mathbb{L}^*(P)$ but suppose $\alpha \notin \mathbb{L}^*(P')$. Then There is a realization ρ' on $\langle P' \leq' Q' \rangle$ and an $i' \in P'$ such that $\rho'_{i'} \not\models \alpha$. Let f by any epimorphism from $\langle P \leq Q \rangle$ onto $\langle P' \leq' Q' \rangle$ and define $\rho(p) = [j \mid f(j) \in \rho'(p)]$; $\rho(p)$ is open in P (f is indeed continous -exercise) and so ρ is a realization on $\langle P \leq Q \rangle$. Moreover it satisfies the hypotheses of Theor.2: Thus $\rho_j \models \alpha$ iff $\rho'_{f(j)} \models \alpha$ for all $j \in P$. But f is epi ; then, for the $i' \in P'$ such that $\rho'_{i'} \not\models \alpha$, there is $i \in P$ such

that $f(i)=i'$. And so, from $\rho'_{f(i)} \not\models \alpha$ it follows $\rho_i \not\models \alpha$ But then $\rho \not\models \alpha$ and also: $\alpha \notin \mathbb{L}^*(\langle P \leq Q \rangle)$.

Theor. 4 (Open monomorphism theorem). If there is an open monomorphism from $\langle P \leq Q \rangle$ into $\langle P' \leq' Q' \rangle$ then:

$$\mathbb{L}^*(\langle P' \leq' Q' \rangle) \subseteq \mathbb{L}^*(\langle P \leq Q \rangle)$$

Proof. Let $\alpha \in \mathbb{L}^*(P')$ but suppose $\alpha \notin \mathbb{L}^*(P)$. Then there is a realization ρ on $\langle P \leq Q \rangle$ and an $i \in P$ such that $\rho_i \not\models \alpha$. Let f be a monomorphism from $\langle P \leq Q \rangle$ into $\langle P' \leq' Q' \rangle$ and define, for all p, $\rho'(p) = [f(i) \mid i \in \rho(p)]$. As $\rho(p)$ is open in P and f is open so $\rho'(p)$ is open in P' and thus ρ' is a realization on $\langle P' \leq' Q' \rangle$. Moreover, as f is mono, for all $j \in P$: $j \in \rho(p)$ iff $f(j) \in \rho'(p)$. Then, by Theor.2, for all $j \in P$: $\rho_j \models \alpha$ iff $\rho'_{f(j)} \models \alpha$. As , however for the given $i \in P$: $\rho_i \not\models \alpha$ so $\rho'_{f(i)} \not\models \alpha$. But then $\rho' \not\models \alpha$ and so $\alpha \notin \mathbb{L}^*(\langle P' \leq' Q' \rangle)$

Def. 4 A suborder $\langle P' \leq' Q' \rangle$ of $\langle P \leq Q \rangle$ is principal iff it is $[i\rangle = P'$ for some $i \in P$.

Remark that every principal suborder is an open suborder and that the identity is an open monomorphism from it into the order.

Theor. 5 (Principal suborder theorem).

$$\mathbb{L}^*(\langle P \leq Q\rangle) = \bigcap \left[\mathbb{L}^*(\langle P' \leq' Q'\rangle) \mid \langle P' \leq' Q'\rangle \text{ principal suborder in } \langle P \leq Q\rangle\right].$$

Proof. ($\Rightarrow$) If $\alpha \in \mathbb{L}^*(\langle P \leq Q\rangle)$ and $\langle P' \leq' Q'\rangle$ is any principal suborder in $\langle P \leq Q\rangle$ then $\alpha \in \mathbb{L}^*(\langle P' \leq' Q'\rangle)$ by Theor. 4.

($\Leftarrow$) If $\alpha \notin \mathbb{L}^*(\langle P \leq Q\rangle)$ then, for some ρ and $i \in P$: $\rho_i \not\models \alpha$. Consider $\langle [i\rangle, \leq\restriction[i\rangle, Q \cap [i\rangle\rangle$.

This is an open suborder of $\langle P \leq Q\rangle$ and the identity is an open homomorphism from it into $\langle P \leq Q\rangle$. Define $\rho'(p) = \rho(p) \cap [i\rangle$. Such a set is open in $[i\rangle$ and so ρ' is a realization. Moreover, for $j \in [i\rangle$, $j \in \rho'(p)$ iff $j \in \rho(p)$. Thus, by theor.2, for all $j \in [i\rangle$, and also in particular for i: $\rho'_i \models \alpha$ iff $\rho_i \models \alpha$. But $\rho_i \not\models \alpha$. So not $\rho'_i \models \alpha$ and then $\alpha \notin \mathbb{L}^*(\langle [i\rangle , \leq\restriction[i\rangle, Q \cap [i\rangle)$ i.e. α does not belong to the logic of a principal suborder of $\langle P \leq Q\rangle$.

Def. 5 The disjoint union $\biguplus$ of a family $\wp = \{\langle P_n \leq_n Q_n\rangle\}_{n \in I}$ of partial orders is the partial order $\langle P \leq Q\rangle$ with :

1) $P = \bigcup_{n \in I} P_n \otimes \{n\}$;

2) $Q = \bigcup_{n \in I} Q_n \otimes \{n\}$;

3) $\langle i\ n\rangle \leq \langle j\ m\rangle$ iff $n = m \wedge i \leq_n j$.

Theor. 6 (Disjoint union theory).

$$\mathbb{L}^*(\wp) = \mathbb{L}^*(\biguplus \wp)$$

Proof. By repeated uses of theor.5.

Remark that $\biguplus \wp$ does not in general preserve properties of elements of $\wp$ (e.g. principality, linearity or finiteness).

From Theor. 5 and Theor. 6 we immediately get:

Theor. 7 (Composition and decomposition theorem).

(1) For every partial order P there is a family $\wp$ of principal partial orders such that

$$\mathbb{L}^*(P) = \mathbb{L}^*(\wp) = \bigcap_{P' \in \wp} \mathbb{L}^*(P')$$

(2) For every family $\wp$ of partial orders there is a partial order P such that

$$\bigcap_{P' \in \wp} \mathbb{L}^*(P') = \mathbb{L}^*(\wp) = \mathbb{L}^*(P)$$

As the only homomorphisms, epimorphisms, monomorphisms, and suborders which we need in what follows are all open we will omit this specification. From now on, therefore , 'epimorphism', for example, will always mean 'open epimorphism'.

Remark: it can be proved

(1) each Jaskowski tree is epimorphic image of the full binary tree.

(2) each finite tree is epimorphic image of some Jaskowski tree .

From Theor. 3 and Theor. 4 it immediately follows:

<u>Theor. 8</u> If $\langle P \leq Q \rangle$ is an epimorphic image of a principal suborder of a partial order $\langle P' \leq' Q' \rangle$, then $\mathbb{L}^*(\langle P' \leq' Q' \rangle) \subseteq \mathbb{L}^*(\langle P \leq Q \rangle)$

From a deep result of Jankov, to be soon discussed, it follows that at least for finite principal scotian partial orders, the above statement can be reversed i.e.

<u>Theor. 9</u> If $\langle P \leq \emptyset \rangle$ is a finite principal scotian partial order and $\langle P' \leq' \emptyset \rangle$ is any scotian partial order then:
$\langle P \leq \emptyset \rangle$ is an epimorphic image of a principal suborder of $\langle P' \leq' \emptyset \rangle$ iff $\mathbb{L}^*(\langle P' \leq' \emptyset \rangle) \subseteq$
$\subseteq \mathbb{L}^*(\langle P \leq \emptyset \rangle)$.

<u>Theor. 10</u> (<u>Jankov's lemma</u>). To every principal finite scotian partial order $\langle P \leq \emptyset \rangle$ it is possible to associate in an effective way a formula

α_p such that for every scotian partial order $\langle P' \leq' \emptyset \rangle$ the following conditions are equivalent:

(I) $\langle P \leq \emptyset \rangle$ is an epimorphic image of a principal suborder of $\langle P' \leq' \emptyset \rangle$.

(II) $\mathbb{L}^*(\langle P' \leq' \emptyset \rangle) \subseteq \mathbb{L}^*(\langle P \leq \emptyset \rangle)$

(III) $\alpha_p \notin \mathbb{L}^*(\langle P' \leq' \emptyset \rangle)$

Proof. (Sketch)

1. <u>Construction, for given finite principal scotian partial order $\langle P \leq \emptyset \rangle$, of α_p</u>

(1) to the finitely many elements of P we associate different propositional variables; let p_o be associated to the first element of $\langle P \leq \emptyset \rangle$.

(2) for each $A \subseteq P$ we define a formula γ_A by

$$\gamma_A = \begin{cases} \bigwedge_{i \in A} p_i \rightarrow \bigvee_{j \notin A} p_j & , \text{ if } \emptyset \neq A \neq P \\ \bigvee_{i \in P} p_i & , \text{ if } A = \emptyset \\ \neg \bigwedge_{i \in P} p_I & , \text{ if } A = P \end{cases}$$

(3) Letting $\mathfrak{a} = [\langle i] \mid i \in P]$, we define:

$$\alpha'_p = \bigwedge_{A \notin \mathfrak{a}} \gamma_A \wedge \bigwedge_{i \leq j} (\gamma_{\langle j]} \rightarrow \gamma_{\langle i]}) \qquad \alpha''_p = \bigvee_{i \in P} \gamma_{\langle i]}$$

and then : $\alpha_p = \alpha'_p \rightarrow \alpha''_p$

2. <u>It is shown $P \not\models \alpha_p$</u>. To this aim we first define a realization ρ on $\langle P \leq \emptyset \rangle$ by:

$$\rho(p) = \begin{cases} [i\rangle & , \text{ if } p = p_i \text{ and } i \in P \\ \emptyset & , \text{ otherwise.} \end{cases}$$

and then show that, for O initial element in $\langle P \leq \emptyset \rangle$, it holds:

$$\rho_O \models \alpha'_p \quad \text{and} \quad \rho_O \not\models \alpha''_p$$

3. <u>It is shown that if $\langle P' \leq' \emptyset \rangle$ is any principal scotian partial order and ρ' any realisation on it such that (O' being its first element) both $\rho'_{O'} \models \alpha'_p$ and $\rho'_{O'} \not\models \alpha''_p$, then $\langle P \leq \emptyset \rangle$ is an epimorphic image of $\langle P' \leq' \emptyset \rangle$.</u>

To this aim it is first shown that for each $k \in P'$ there is one (and only one) $i \in P$ such that $[j \in P \mid \rho'_k \models p_j] = \langle i]$, and then shown that the f, which to every $k \in P'$ associates the only one $i \in P$ for which $[j \in P \mid \rho'_k \models p_j] = \langle i]$ holds, is an epimorphism from $\langle P' \leq' \emptyset \rangle$ onto $\langle P \leq \emptyset \rangle$.

4. <u>From 3. it follows that if $\langle P' \leq' \emptyset \rangle$ is any scotian partial order such that no of its principal sub-orders goes epimorphically onto $\langle P \leq \emptyset \rangle$, then</u> $P' \models \alpha_p$.

If, indeed α_p would not be valid on $\langle P' \leq' \emptyset \rangle$

then, by Theor. 5, it would be also invalid in some of its principal suborders which however, by 3) would go epimorphically onto $\langle P \leq \emptyset \rangle$.

5. The equivalences (I)$\longleftrightarrow$(II)$\longleftrightarrow$(III) are now proved:

(I) $\longrightarrow$ (II) Theor. 8.

(II) $\longrightarrow$ (III) As seen under 2), $P \not\models \alpha_p$ so if $\mathbb{L}^*(P') \subseteq \mathbb{L}^*(P)$, then $P' \not\models \alpha_p$.

(III) $\longrightarrow$ (I) If $\alpha_p \notin \mathbb{L}^*(P')$, then, by 4., there is a principal suborder of $\langle P' \leq' \emptyset \rangle$ which goes epimorphically onto $\langle P \leq \emptyset \rangle$.

It is useful to note:

<u>Theor. 10</u> If $\langle P \leq \emptyset \rangle$ is a finite principal scotian partial order different from $\mathbb{1} = \langle \{o\} = \emptyset \rangle$, then its Jankov formula $\alpha_p \in \bar{\bar{T}}$ (i.e. is a classical tautology).

Proof. If, indeed, $\langle P \leq \emptyset \rangle$ is a finite principal scotian partial order different from $\mathbb{1}$ then it cannot be epimorphic image of principal suborders of $\mathbb{1}$. Then, by Jankov's Lemma, $\alpha_p \in \mathbb{L}^*(\mathbb{1})$.

We now make two main applications of Jankov's Lemma.

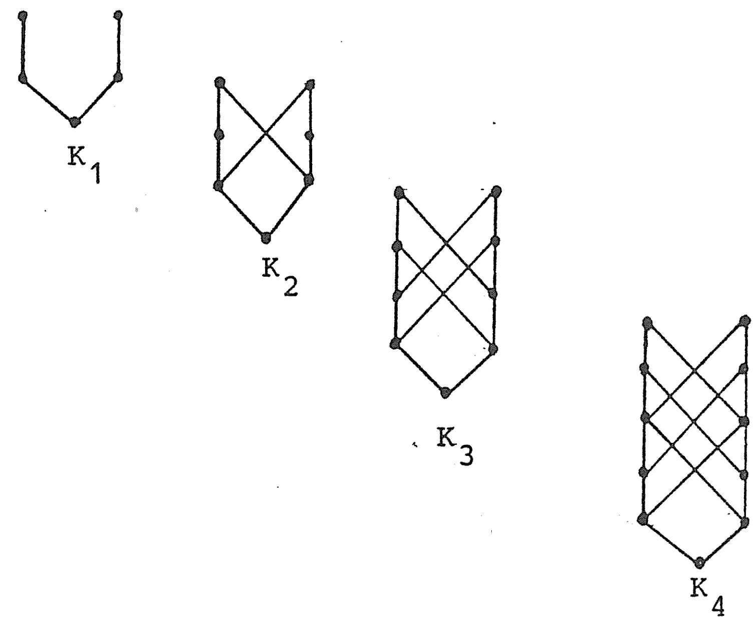
Theor. 11 (<u>Splitting theorem</u>). Let L_o be any semantical scotian logic and $\mathbb{F}(L_o)$ be the set of all semantical scotian logics L such that $L_o \subseteq L$. Let $\langle P \leq \emptyset \rangle \neq \mathbb{1}$ be a finite principal scotian partial order such that $\mathbb{L}^*(P) \in \mathbb{F}(L_o)$ and let L_1 be $\mathbb{L}^*(P)$. Then there is one and only one semantical scotian logic L_2 such that, for each semantical logic $L \in \mathbb{F}(L_o)$: either $L \subseteq L_1$ or $L_2 \subseteq L$ and not both.

Proof. Let L_o, $\mathbb{F}(L_o)$ and $\langle P \leq \emptyset \rangle$ be as in the hypotheses with $L_1 = \mathbb{L}^*(\langle P \leq \emptyset \rangle)$ and assume $L_1 \in \mathbb{F}(L_o)$. Let α_p be the Jankov's formula for $\langle P \leq \emptyset \rangle$. Consider $L_o + \alpha_p$. This is a logic although not necessarily + a semantical one. Define L_2 as the least semantical logic which estends $L_o + \alpha_p$. (Remark that this definition has sense because intersection over semantical logics is a semantical logic and, by Theor. 10, $L_o + \alpha_p$ is in any case contained in $\mathbb{L}^*(\mathbb{1})$ = classical logic). Now: let $L \in \mathbb{F}(L_o)$. Then either $\alpha_p \in L$ or $\alpha_p \notin L$. If $\alpha_p \in L$ then, $L_o + \alpha_p \subseteq L$ and so $L_2 \subseteq L$. If $\alpha_p \notin L$, then, by Jankov's lemma, $L \subseteq L_1$. So if $L \in \mathbb{F}(L_o)$, then: $L \subseteq L_1$ or $L_2 \subseteq L$. But not both, because of α_p. Moreover this logic L_2 is uniquely determined. Let, indeed, be L_3 a semantical logic such that $L_3 \in \mathbb{F}(L_o)$ and $L \subseteq L_1$ or $L_3 \subseteq L$ (but not both) for all $L \in \mathbb{F}(L_o)$. As $L_3 \subseteq L_3$, it must be $L_3 \nsubseteq L_1$. On the other side, from

above we know $L_3 \subseteq L_1$ or $L_2 \subseteq L_3$. So $L_2 \subseteq L_3$. But again, if for all L: either $L \subseteq L_1$ or $L_3 \subseteq L_1$, then either $L_2 \subseteq L_1$ (which is impossible) or $L_3 \subseteq L_2$. So $L_2 = L_3$.

Remark that, as L_2 contains α_p which is not in L_1, it is $L_2 \nsubseteq L_1$, but it is possible that $L_1 \subseteq L_2$. If, however, $L_1 \subseteq L \subseteq L_2$ then $L_1 = L$ or $L_2 = L$ because in any case: $L \subseteq L_1$ or $L_2 \subseteq L_1$.

The second application of Jankov's lemma is as follows. It is possible to define a denumerable sequence $\{K_n\}_{n \in N}$ of finite scotian principal partial orders such that any two members of the sequence are not epimorphic image of any principal suborder of the other. The sequence is as follows:



Let α_n be the Jankov's formula for K_n.
By Jankov's lemma we have $\alpha_n \notin \mathbb{L}^*(K_n)$ and $\alpha_n \in \mathbb{L}^*(K_m)$ for $m \neq n$.
For M a set of natural numbers define: $L_M = \mathbb{L}^*(\{K_n\}_{n \notin M})$.
From preceding remarks we have, for all $n \in N$: $\alpha_n \in L_M$ iff $n \in M$. So if M and M' are two different sets of natural numbers then $L_M \neq L_{M'}$. As there are $2^{\aleph_0}$ different sets of natural numbers, we have:

Theor. 12 (Jankov's Theorem).

There are $2^{\aleph_0}$ different (semantical) logics.

Many refinements and strengthenings of this theorem are possible.

Part. II: PREDICATE LOGICS

A. PRELIMINAIRES

1. Positive, minimal and intuitionistic predicate logics.

Let $\mathfrak{F}$ be the set of formulas built up from the individual variables: $x_0, x_1, \ldots$; for each $n \geq 0$, the the n-ary predicate variables: $P_0^n, P_1^n, \ldots$; the connectives : $\neg, \wedge, \vee, \rightarrow$ and the quantifiers: $\forall, \exists$.
Further let $\mathfrak{F}^+$ be the set of negation-free formulas.

$\mathbb{PQ}$ (the positive (predicate) logic) is the smallest subset of $\mathfrak{F}^+$ which contains A1.1-A3.3 as well as the formulas : Q1 $\forall x\, Px \rightarrow Px$ and Q2 $Px \rightarrow \exists x Px$ and is closed under modus ponens (MP), individual substitution (MV), predicate substitution (MS), alphabetical change of bound variables (RC), posterior generalization (GP): $\dfrac{\alpha \rightarrow \beta(x),\ x \text{ not free in } \alpha}{\alpha \rightarrow \forall x\, \beta(x)}$ and anterior particularization (PA) $\dfrac{\alpha(x) \rightarrow \beta,\ x \text{ not free in } \beta}{\exists x \alpha(x) \rightarrow \beta}$.

$\overline{\mathbb{PQ}}$ (the minimal (predicate) logic) and $\overline{\overline{\mathbb{PQ}}}$ (the intuitionistic (predicate) logic) are defined analogously.

In subsequent discussions it will be useful to remember that following formulas are in $\mathbb{PQ}$:

1.1 $\exists x(Px \wedge Q) \leftrightarrow \exists x Px \wedge Q$ 1.2 $\forall x(Px \wedge Q) \leftrightarrow \forall x Px \wedge Q$

2.1 $\exists x(Px \vee Q) \leftrightarrow \exists x Px \vee Q$ 2.2 $\forall x(Px \vee Q) \leftarrow \forall x Px \vee Q \}$ not $\rightarrow$

3.1 $\forall x(Px \rightarrow Q) \leftrightarrow (\exists x Px \rightarrow Q)$ 3.2 $\forall x(P \rightarrow Qx) \leftrightarrow (P \rightarrow \forall x Qx)$

$$\left.\begin{array}{l} 4.1\ \exists x(Px \rightarrow Q) \rightarrow (\forall x Px \rightarrow Q) \\ 4.2\ \exists x(P \rightarrow Qx) \rightarrow (P \rightarrow \exists x Qx) \end{array}\right\} \text{not} \leftarrow$$

2. Relational predicate semantics (Kripke-semantics).

A Kripke- frame is a family of non empty sets indexed by a partial order and monotonic with respect tc this order i.e.

Def. 1 A Kripke- frame is a system $\langle P \leq V\rangle$, where $\langle P\leq\rangle$ is a non empty partial order and V an homomorphism (with respect to $\leq$ and set-theoretical inclusion) from P into the class of non empty sets. (V is sometimes called the domain function of the frame).

Def. 2 A realization ρ of $\mathcal{L}$ on a Kripke - frame $\langle P \leq V\rangle$ is a family of maps $\{\rho_i\}_{i\in P}$ indexed by P such that for every n-ary predicate variable P^n:

(1) for all $i\in P$: $\rho_i(P^n) \subseteq V_i^n$ (we define: $V_i^o = \{\text{truth}\}$);

(2) for all $i,j\in P$: if $i\leq j$, then $\rho_i(P^n) \subseteq \rho_j(P^n)$.

Def. 3 If $\langle P \leq V\rangle$ is a Kripke-frame and $i\in P$ an i-interpretation on $\langle P \leq V\rangle$ is a map from the individual variables of $\mathcal{L}$ into V_i. The set of all i-interpretations will be denoted by $\overline{V}_i$.

Remark that if $\sigma\in\overline{V}_i$ and $i\leq j$, then $\sigma\in\overline{V}_j$ and that if $\sigma\in\overline{V}_i$, $i\leq j$ and $\varphi\in V_j$ then there is one and only one $\tau\in\overline{V}_j$ such that:

$$\tau(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ \varphi & \text{otherwise} \end{cases};$$

such a τ will be denoted by $\sigma(\substack{x\\ \varphi})$.

The preceding notions can be obviously extended to the case of special Kripke-frames i.e. Kripke-frames indexed by special partial orders.

Def. 4 If $\langle P \leq V\rangle$ ($\langle P \leq Q\ V\rangle$) is a Kripke-frame (a special Kripke-frame), $\rho=\{\rho_i\}$ a realization on it and σ an i-interpretation, then the relation $\rho_i^\sigma \models \alpha$ (the valuation ρ_i^σ forces α) is so defined by recursion on α:

$$\left.\begin{array}{l}\left.\begin{array}{ll}
(1) & \rho_i^\sigma \models P^o \text{ iff } \rho_i(P^o) = \text{truth}; \\
 & \rho_i^\sigma \models P^n x_1 .. x_n \text{ iff } \langle\sigma(x_1),..,\sigma(x_n)\rangle \in \rho_i(P^n) \\
(2) & \rho_i^\sigma \models \alpha\wedge\beta \text{ iff } \rho_i^\sigma\models\alpha \text{ and } \rho_i^\sigma\models\beta \\
(3) & \rho_i^\sigma \models \alpha\vee\beta \text{ iff } \rho_i^\sigma\models\alpha \text{ or } \rho_i^\sigma\models\beta \\
(4) & \rho_i^\sigma \models \alpha\rightarrow\beta \text{ iff } \forall_{i\leq j}\, j\ (\rho_j^\sigma\models\alpha \Rightarrow \rho_j^\sigma\models\beta) \\
(5) & \rho_i^\sigma \models \exists x\alpha \text{ iff } \exists_{\varphi\in V_i}\varphi\ \rho_i^{\sigma(\substack{x\\ \varphi})}\models\alpha \\
(6) & \rho_i^\sigma \models \forall x\alpha \text{ iff } \forall_{i\leq j}\, j\ \forall_{\varphi\in V_j}\varphi\ \rho_j^{\sigma(\substack{x\\ \varphi})}\models\alpha
\end{array}\right\}\alpha\in\mathcal{F}^{+} \\
(7)\ \ \rho_i^\sigma \models \neg\alpha \text{ iff } \forall_{i\leq j}\, j\ (j\models\alpha \Rightarrow j\in Q)
\end{array}\right\}\alpha\in\mathcal{F}$$

Theor. 1 If ρ is a realization on $\langle P \leq Q\ V\rangle$ then:

(1) for all α, i, j and $\sigma\in\overline{V}_i$: if $i\leq j$, then $\rho_i^\sigma\models\alpha \Rightarrow \rho_j^\sigma\models\alpha$

(2) if $\langle P \leq Q\ V\rangle$ is scotian, then: $\rho_i^\sigma\models\neg\alpha$ iff $\forall_{i\leq j}\, j\ \rho_j^\sigma \not\models\alpha$

(3) if, for all $i,j \in P$, $V_i = V_j$, then $\rho_i^{\sigma} \models \forall x\alpha$ iff $\forall_{\varphi \in V_i} \rho_i^{\sigma\binom{x}{\varphi}} \models \alpha$.

Def. 5 (1) α holds in ρ ($\rho \models \alpha$) iff $\forall_{i \in P} \; \forall_{\sigma \in \bar{V}_i} \; \rho_i^{\sigma} \models \alpha$

(2) α is valid in $\langle P \leq V \rangle$ ($P,V \models \alpha$) iff, for all ρ on $\langle P \leq V \rangle$, $\rho \models \alpha$.

(3) If $\mathfrak{K}$ is a class of Kripke-frames, then $\mathfrak{K} \models \alpha$ iff, for all $\langle P \leq V \rangle \in \mathfrak{K}$, $P,V \models \alpha$.

(4) α is absolutely valid in $\langle P \leq \rangle$ iff α is valid in $\langle P \leq V \rangle$ for all domain functions V.

Theor.2 (1) for $\alpha \in \mathfrak{J}^{+}$: $\alpha \in \mathbb{P}\mathbb{Q}$ iff α is absolutely valid in all partial orders.

(2) for $\alpha \in \mathfrak{J}$: (a) $\alpha \in \overline{\mathbb{P}\mathbb{Q}}$ iff α is absolutely valid in all special partial orders. (b) $\alpha \in \overline{\overline{\mathbb{P}\mathbb{Q}}}$ iff α is absolutely valid in all scotian partial orders.

B. INTERMEDIATE PREDICATE LOGICS.

1. Syntactical characterization.

Def. 1 A negation-free intermediate predicate logic is a subset of $\mathfrak{J}^{+}$ which contains $\mathbb{P}\mathbb{Q}$ and is closed under MP,MV,MS,RG,GP and PA.

Negative intermediate predicate logics and intermediate predicate logics are defined analogously starting from $\overline{\mathbb{P}\mathbb{Q}}$ and $\overline{\overline{\mathbb{P}\mathbb{Q}}}$ respectively.

Remark. Classical propositional logic (as well as

its negation-free reduct $\mathbb{T}$) are syntactically complete and so the only immediate predecessors of trivial logics $\mathfrak{J}$ and $\mathfrak{J}^{+}$ respectively. This is not true in the predicate case, where, for example, the logic obtained by adjoining all instances of $\exists xPx \rightarrow \rightarrow \forall xPx$ (1-element logic) is not inconsistent. If one wants to exclude such logics from intermediate logics one has only to modify in an obvious way preceding definition. As we are not interested here in general 'structural' properties of the system of intermediate logics, but in the study of some particular examples of them, this question is irrelevant. Our main concern will be with negation-free intermediate logics (briefly: logics) which arise by some "natural" strengthenings of the assumptions about connectives and/or quantifiers.

Let $\mathbb{X}$ be some of the propositional logics considered up to now. By $\mathbb{X}\mathbb{Q}$ we then denote the predicate logic obtained by $\mathbb{P}\mathbb{Q}$ by adjoining the axioms of $\mathbb{X}$. Thus $\mathbb{B}\mathbb{Q}$ and $\mathbb{T}\mathbb{Q}$ will be respectively the negation-free reducts of Dummett's logic and of classical logic. We now consider a first group of formulas which are not in $\mathbb{P}\mathbb{Q}$

D : $\forall x(Px \vee Q) \longrightarrow \forall x \; Px \vee Q$

H_1 : $(P \rightarrow \exists x \; Qx) \rightarrow \exists x(P \rightarrow Qx)$

H_2 : $(\forall xPx \rightarrow Q) \rightarrow \exists x(Px \rightarrow Q)$

H : $(\forall xPx \rightarrow \exists yQy) \rightarrow \exists x \; \exists y \; (Px \rightarrow Qy)$

and call (for given basis $\mathcal{X}$) $\mathcal{X}\mathbb{D}, \mathcal{X}\mathbb{H}_1, \mathcal{X}\mathbb{H}_2, \mathcal{X}\mathbb{H}$ the logics which are obtained by $\mathcal{X}\mathbb{Q}$ by adjoining respectively all instances of D, H_1, H_2, H. If, moreover, we adjoin, for example, both (all instances of) D and H_1, then we write $\mathcal{X}\mathbb{D}\mathbb{H}_1$. For sake of simplicity we go over to schematic formulations (for example, $\forall x(\alpha(x) \vee \beta) \rightarrow \forall x\, \alpha(x) \vee \beta$ for $\forall x(Px \vee Q) \rightarrow \forall x Px \vee Q)$ and omit the obvious limitations about the occurrences of free variables.

We collect some simple facts:

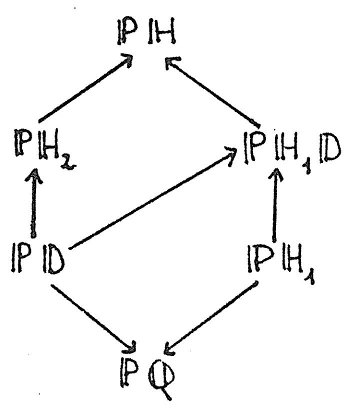
<u>Theor. 1</u> (1) $\mathbb{P}\mathbb{H} \equiv \mathbb{P}\mathbb{H}_1\mathbb{H}_2$.

(2) $\mathbb{P}\mathbb{H}_1 \equiv \mathbb{P}\mathbb{Q} + \exists y(\exists x\alpha(x) \longrightarrow \alpha(y)) \equiv \mathbb{P}\mathbb{Q} +$ $\exists y \forall x\, (\alpha(x) \rightarrow \alpha(y))$

(3) $\mathbb{P}\mathbb{H}_2 \equiv \mathbb{P}\mathbb{Q} + \exists x(\alpha(x) \longrightarrow \forall y\, \alpha(y)) \equiv \mathbb{P}\mathbb{Q} +$ $\exists x \forall y\, (\alpha(x) \rightarrow \alpha(y))$

(4) $\mathbb{P}\mathbb{D} \subseteq \mathbb{P}\mathbb{H}_2$

thus the inclusion diagram is as follows:



It can also be proved :

<u>Theor. 2</u> (1) All arrows in the diagram are proper

(2) Up to transitivity there is no other arrow.

(3) The diagram remains unchanged (with properties (1) and (2)) if is strengthened to $\mathbb{A}_n$ or to $\mathbb{B}_n$ $(n \geq 0)$; in particular to $\mathbb{B}$.

The situation changes when, from $\mathbb{B}$, we go over to $\mathbb{T}_n$. Indeed:

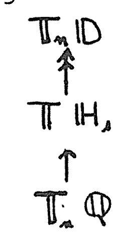
<u>Theor. 3</u> (1) $\mathbb{T}_n\mathbb{H}_1 \subseteq \mathbb{T}_n\mathbb{D}$

(2) $\mathbb{T}_n\mathbb{H}_2 \subseteq \mathbb{T}_n\mathbb{D}$

and so (3) $\mathbb{T}_n\mathbb{D} \equiv \mathbb{T}_n\mathbb{H} \equiv \mathbb{T}_n\mathbb{H}_1\mathbb{H}_2 \equiv \mathbb{T}_n\mathbb{D}\mathbb{H}_1$

(4) $\mathbb{T}_n\mathbb{Q} \not\equiv \mathbb{T}_n\mathbb{H}_1$

Thus the diagram is as follows:



(where $\longrightarrow\!\!\!\!\rightarrow$ indicates an inclusion of which we do not know whether it is proper).

Remarks: (1) of course: $\mathbb{T}_0\mathbb{D} \equiv \mathbb{T}_0\mathbb{Q}$.

(2) in $\mathbb{P}\mathbb{H}$ every negation free formula has a prenex normal form.

Define: $\exists^{*}x\,\alpha(x) = \forall y(\forall x(\alpha(x) \longrightarrow \alpha(y)) \longrightarrow \alpha(y))$ and

let E be the schema:

$\exists^{*}x\,\alpha(x) \longrightarrow \exists x\alpha$ (remark that in $\mathbb{PQ}$ it is provable $\exists x\,\alpha(x) \longrightarrow \exists^{*}x\,\alpha(x)$)

<u>Theor. 4</u> $\mathbb{PH}_1 \equiv \mathbb{PQ} + E$.

i.e. assuming schema $\mathbb{H}_1$ is equivalent to assume definability of $\exists$ by means of $\forall$ and $\rightarrow$ through:

$\exists x\alpha \longleftrightarrow \forall y(\forall x(\alpha(x) \rightarrow \alpha(y)) \rightarrow \alpha(y))$.

We now consider a second group of schemata which are weakings of the H-schemata:

H_2^2 $\forall x((\alpha(x) \rightarrow \beta) \rightarrow \gamma) \longrightarrow ((\forall x\,\alpha(x) \rightarrow \beta) \rightarrow \gamma)$

H_2^1 $\forall x((\alpha(x) \rightarrow \beta) \rightarrow \beta) \rightarrow ((\forall x\,\alpha(x) \rightarrow \beta) \rightarrow \beta)$

H_2^0 $\forall x((\alpha(x) \rightarrow \forall y\,\alpha(y)) \rightarrow \forall y\,\alpha(y)) \rightarrow \forall y\,\alpha(y)$

H_1^1 $((\exists x\,\alpha(x) \rightarrow \beta) \rightarrow \gamma) \rightarrow \exists x((\alpha(x) \rightarrow \beta) \rightarrow \gamma)$

H_1^0 $((\exists x\,\alpha(x) \rightarrow \beta) \rightarrow \beta) \rightarrow \exists x((\alpha(x) \rightarrow \beta) \rightarrow \beta)$

Remark: $\exists y\,\alpha(y) \rightarrow \exists x((\alpha(x) \rightarrow \exists y\,\alpha(y)) \rightarrow \exists y\,\alpha(y))$ is already provable in $\mathbb{PQ}$.

<u>Theor. 5</u> (1) $\mathbb{PH}_2^0 \subseteq \mathbb{PH}_2^1 \subseteq \mathbb{PH}_2^2 \subseteq \mathbb{PH}_2$

(2) $\mathbb{PH}_1^0 \subseteq \mathbb{PH}_1^1 \subseteq \mathbb{PH}_2$

but also: (3) $\mathbb{PH}_1^1 \subseteq \mathbb{PH}_1$, and thus : $\mathbb{PH}_1^0 \subseteq \mathbb{PH}_1^1 \subseteq \mathbb{PH}_1$

further: (4) $\mathbb{PH}_2 \subseteq \mathbb{PH}_2^2$, and thus: $\mathbb{PH}_2 \equiv \mathbb{PH}_2^2$ (the $\mathbb{H}_2$-property has so an equivalent $\exists$-free formulation)

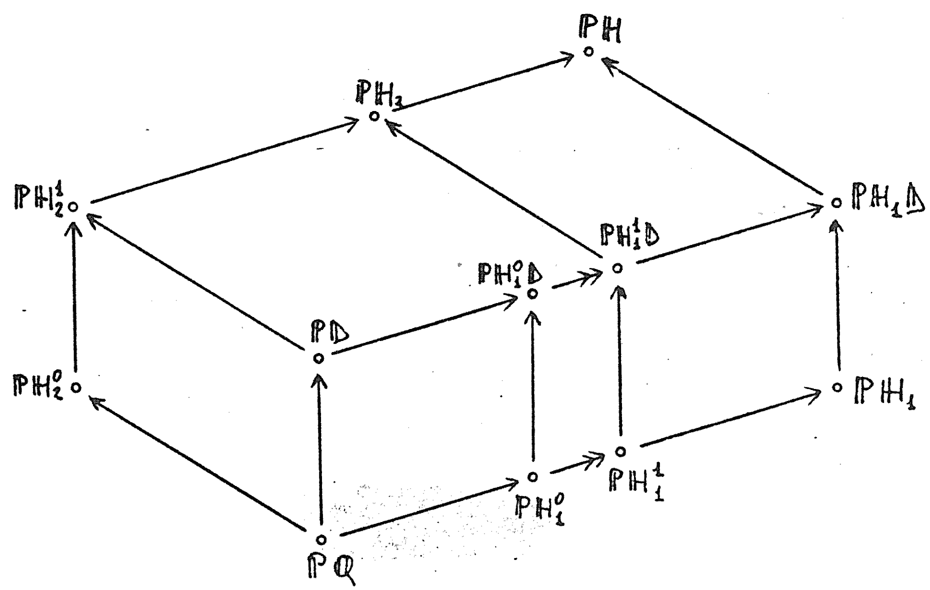
(5) $\mathbb{PD} \subseteq \mathbb{PH}_2^1$ and $\mathbb{PH}_2^1 \subseteq \mathbb{PH}_2^0\mathbb{D}$; thus $\mathbb{PH}_2^1 \equiv$

$\equiv \mathbb{PH}_2^0\mathbb{D}$.

(6) $\mathbb{PH}_2 \subseteq \mathbb{PH}_1^0\,\mathbb{H}_2^1$ and thus $\mathbb{PH}_2 \equiv \mathbb{PH}_1^0\mathbb{H}_2^1$

$\equiv \mathbb{PH}_1^0\,\mathbb{H}_2^0\,\mathbb{D}$.

The resulting diagram is as follows:



Theor. 6 (1) all arrows (with the possible exceptions of $\twoheadrightarrow$) are proper .

(2) up to transitivity, there is no other arrow

(3) the diagram is not changed (with properties (1) and (2)) if $\mathbb{P}$ is strengthened to $\mathbb{B}_n$ or $\mathbb{A}_n$ for $n > 0$.

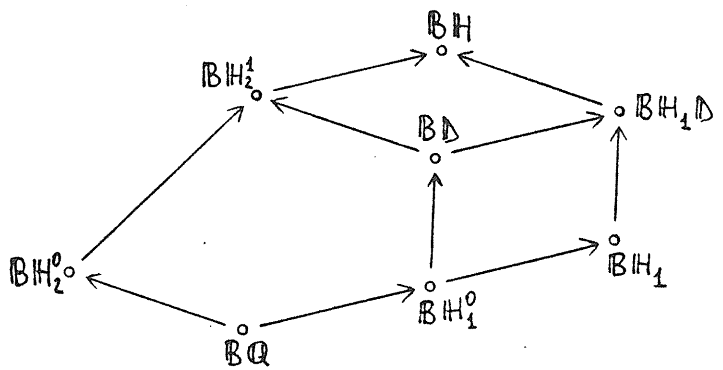
However:

Theor. 7 (1) $\mathbb{B}\mathbb{H}_2 \subseteq \mathbb{B}\mathbb{H}_2^1$ and thus $\mathbb{B}\mathbb{H}_2^1 \equiv \mathbb{B}\mathbb{H}_2^2 \equiv \mathbb{B}\mathbb{H}_2$

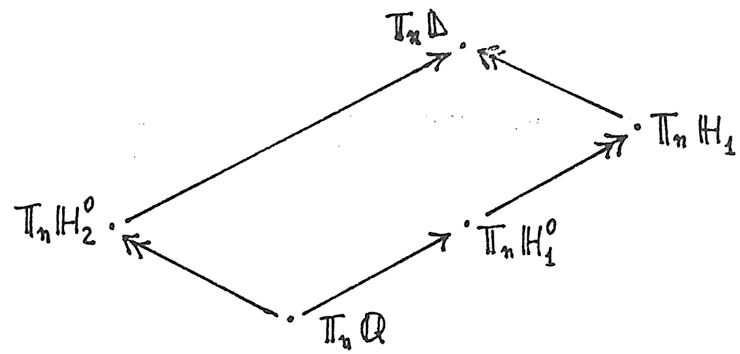
(2) $\mathbb{B}\mathbb{H}_1^1 \subseteq \mathbb{B}\mathbb{H}_1^0$ and thus $\mathbb{B}\mathbb{H}_1^0 \equiv \mathbb{B}\mathbb{H}_1^1$,
$\mathbb{B}\mathbb{H}_1^0\mathbb{D} \equiv \mathbb{B}\mathbb{H}_1^1\mathbb{D}$

(3) $\mathbb{B}\mathbb{H}_1^0 \subseteq \mathbb{B}\mathbb{D}$ and thus $\mathbb{B}\mathbb{H}_1^0\mathbb{D} \equiv \mathbb{B}\mathbb{H}_1^1\mathbb{D} \equiv \mathbb{B}\mathbb{D}$.

The resulting diagram is as follows (with all arrows proper).



Going over to the $\mathbb{T}_n$ the diagram becomes as follows (for $n > 0$, of course).



Besides the two main groups of schemata about which we spoke, some other group has been investigated; in particular weakenings of D (see later) and so-called "strong schemata" i.e. schemata whose "finite version" is already an axiomatization for $\mathbb{T}$. To make an example: $(\forall x\, \alpha(x) \to \exists y\, \beta(y)) \to \exists x(\alpha(x) \to \beta(x))$ whose "finite version" $((p \wedge q \to r \vee s) \to (p \to r) \vee (q \to s))$ adjoined to $\mathbb{P}$ gives $\mathbb{T}$.

2. Relational semantics for intermediate predicate logics.

The whole terminological and notational apparatus which has been introduced under the corresponding heading in the propositional case can be in an obvious way restated and reformulated for the new situation

One point, however, should be stressed: whereas in the propositional case the main tool at disposal for the identification of semantically determined logics was classification of partial orders by means of order-theoretical properties (e.g.: linearity, maximal height, maximal width, and so on) in the new case we can also consider properties of domain functions (e.g.: constance, almost constance, and so on).

The first problem which arises in relational semantics for intermediate predicate logics is, of course, that about the extendibility to the predicate logic $\mathbb{X}\mathbb{Q}$ of those characterizations which are known for the propositional logic $\mathbb{X}$. It turns out that the situation is not so simple as one could at first glance expect.

Indeed, it is in general rather easy to see that the "validity part" of propositional characterizations can be translated into a "validity theorem" for the corresponding predicate logic. Thus, in particular it can be seen that:

<u>Theor. 1</u> (1) $\mathbb{P}\mathbb{Q}$ is absolutely valid in all partial orders (and, a fortiori, in all finite partial orders);

(2) $\mathbb{B}\mathbb{Q}$ is absolutely valid in all chains (all finite chains);

(3) $\mathbb{T}\mathbb{Q}$ is absolutely valid in $\mathbb{1}$;

for all n>0 (4) $\mathbb{C}_n\mathbb{Q}$ is absolutely valid in all partial orders of height $\leq n+1$ (...finite...);

(5) $\mathbb{A}_n\mathbb{Q}$ is absolutely valid in all partial orders of width $\leq n+1$ (..finite..)

(6) $\mathbb{B}_n\mathbb{Q}$ is absolutely valid in all finite partial orders of local width $\leq n+1$ (but <u>not</u> in all partial orders of local width $\leq n+1$).

As for as the "completeness part" is concerned, however, the situation is very different. At present time the only completely satisfactory results are still the well-known facts:

<u>Theor. 2</u> (1) $\mathbb{P}\mathbb{Q}$ is complete for the class of all partial orders.

(2) $\mathbb{T}\mathbb{Q}$ is complete for $\{\mathbb{1}\}$.

To illustrate the kind of difficulties one faces when trying to adapt to the predicational case those procedures which are successful in getting completeness for the propositional case we briefly discuss the most simple case: that of $\mathbb{B}\mathbb{Q}$.

In the propositional case one goes out from the canonical partial order built up from the prime extensions (on the same language) of a given prime theory S_o. That this order is a chain is inferred as follows. If for some S_1, S_2 ($S_o \subseteq S_1, S_2$) in this order it were $S_1 \notin S_2$ and $S_2 \notin S_1$, then, for some α and β it should be: $\alpha \in S_1 - S_2$ and $\beta \in S_2 - S_1$; but then (the underlying logic being $\mathbb{B}$) $(\alpha \to \beta) \vee \vee (\beta \to \alpha) \in S_o$ and, as S_o is prime, either $\alpha \to \beta \in S_o$ or $\beta \to \alpha \in S_o$. In the first case, however, $\beta \in S_1$ and in the second, $\alpha \in S_2$; against our hypothesis. This kind of argument cannot be replayed in the case of the canonical Kripke frame built up from the prime and rich extensions of a given prime and rich theory S_o on a tower of denumerably many languages each of which contains a denumerable set of individual constants which are not in the preceding language; the point is that it may well be $\alpha \in S_1 - S_2$ and $\beta \in S_2 - S_1$ without that this implies (because of the linguistical levels) that $(\alpha \to \beta) \vee (\beta \to \alpha) \in S_o$. Some natural modifications do not better the situation (at least as for as we know).

Nevertheless we have some interesting negative informations about completeness of $\mathbb{X}\mathbb{Q}$ which follow from the following

<u>Theor.3</u> H_2^o (i.e. all instances of) is absolutely valid in all dually well-founded partial orders (i.e. partial orders without infinite ascending chains).

As all finite partial orders are obviously dually well-founded, from Theor. 3 it follows that H_2^o is in any case absolutely valid in all finite partial orders and thus:

<u>Test for finitariness</u>: If an intermediate predicate logic is finitary then it must prove (any instance of) H_2^o.

But it can be proved:

<u>Theor. 4</u> $\mathbb{B}\mathbb{H}_2^o$ is a proper extension of $\mathbb{B}\mathbb{Q}$.

Thus, using some obvious inclusions:

<u>Theor. 5</u> $\mathbb{P}\mathbb{Q}$, $\mathbb{B}\mathbb{Q}$ and, for all n, $\mathbb{A}_n\mathbb{Q}$ and $\mathbb{B}_n\mathbb{Q}$ are <u>not</u> finitary.

Three remarks:

(1) As we do not know whether the inclusion $\mathbb{T}_n\mathbb{Q} \subseteq \mathbb{T}_n\mathbb{H}_2^o$ $(n > 0)$ is proper we do not know whether $\mathbb{T}_n\mathbb{Q}$ is not finitary.

(2) $\mathbb{B}_n\mathbb{Q}$ $(n > 0)$ is surely not complete for the only class of frames for which we have a validity theorem (<u>finite</u> partial orders of local width $\leq n+1$)

(3) Quite independently of the fact to be reversible in a completeness theorem a validity theorem may be useful in proving independence results; for proving not inclusion of a certain schema S in a

logic L, indeed, it is sufficient to construct a countermodel to an instance of S on a Kripke-frame for which validity of L is known.

A second problem which arises in relational semantics for intermediate predicate logics is connected with those frames which have a constant domain function. Already 1965 Kripke remarked that schema D is valid in all frames having a constant domain function (briefly: frames with constant domains). It is easy to transform all preceding validity results about $\mathbb{X}\mathbb{Q}$ (S. Theor.1) in validity results for $\mathbb{X}\mathbb{D}$ with respect to the corresponding classes of frames with constant domains.

End of the sixties S. Görnemann , D. Klemke and D. Gabbay proved independently

<u>Theor. 6</u> $\mathbb{P}\mathbb{D}$ is characterized by the class of all frames with constant domains.

Modifying Klemke's construction, P. Minari (1981) has also proved:

<u>Theor. 7</u> (1) $\mathbb{B}\mathbb{D}$ is characterized by the class of all chains with constant domains.

(2) $\mathbb{T}_n\mathbb{D}$ is characterized (for $n \geqslant 0$) by the class of all chains of cardinality $\leqslant n+1$ with constant domains.

Remark that, on account of the provable fact that $\mathbb{B}\mathbb{H}_2^o$ is not included in $\mathbb{B}\mathbb{D}$, by preceding test of finitariness we have, in any case:

<u>Theor. 8</u> $\mathbb{P}\mathbb{D}$, $\mathbb{B}\mathbb{D}$ and, for all $n > 0$, $\mathbb{A}_n\mathbb{D}$ and $\mathbb{B}_n\mathbb{D}$ are not finitary.

Thus, in particular, the characterizations of $\mathbb{P}\mathbb{D}$ and $\mathbb{B}\mathbb{D}$ cannot be sharpened to the corresponding classes of finite partial orders with constant domains.

A natural question which arises in this context is that about those logics which are determined by frames whose domain function has only two, three, different values.

P. Minari has found a sequence of schemata $D_o, D_1, \ldots$ such that (1) $D_o = D$; (2) for all $n > 0$: D_n is valid in all chains with at most n+1 different domains but is not valid in all chains with at least n+2 different domains.

As for as the relational semantics of the H-schemata are concerned the main things we know are:

<u>Theor. 9</u> (1) H_2^o is absolutely valid in all dually well-founded frames (Theor. 3).

(2) H_1 is valid in all well-ordered frames with constant domain.

From this, easily

Theor. 10 (1) $\mathbb{P}\mathbb{H}_2^o$ is absolutely valid in all dually well founded frames.

(2) $\mathbb{P}\mathbb{H}_2^1$ ($\equiv \mathbb{P}\mathbb{H}_2^o \mathbb{D}$) is valid in all dually well-founded frames with constant domains.

(3) $\mathbb{B}\mathbb{H}_2$ ($\equiv \mathbb{B}\mathbb{H}_2^1 \equiv \mathbb{B}\mathbb{H}_2^o \mathbb{D}$) is valid in all dually well-ordered frames with constant domains.

(4) $\mathbb{B}\mathbb{H}_1\mathbb{D}$ is valid in all well-ordered frames with constant domains.

(5) $\mathbb{B}\mathbb{H}$ is valid in all finite chains.

We do not know if some or all of these statements can be sharpened into a characterization.

About H_1, however, we have a rather surprising result:

Theor. 11 $\mathbb{P}\mathbb{H}_1$, $\mathbb{B}\, H_1$ and, for all $n > 0$, $\mathbb{A}_n \mathbb{H}_1$, $\mathbb{B}_n \mathbb{H}_1$ are not semantically characterizable (i.e. there is no class of frames whose logics coincides with them).

We can indeed show that in every frame with non constant domain function, there is an instance of H_1 which is not valid in it. [If $< P \leq V >$ is not with constant domain function, then, for some $i > 0$, there is $\varphi \in V_i - V_o$. For P a monadic variable define a realization $\rho = \{\rho_j\}_{j \in P}$ by:

$$\rho_j(P) = \begin{cases} \emptyset, \text{ if } j \not\geq i \\ \\ \{\varphi\}, \text{ otherwise.} \end{cases}$$

It is easy to see: $\rho_o \not\Vdash \exists x(\exists y Py \rightarrow Px)$]. It follows that a characterizing class of frames for $\mathbb{P}\mathbb{H}_1$ ($\mathbb{B}\mathbb{H}_1$ and so on) can only contain frames with constant domains. As already remarked, however, D holds in all frames with constant domains; thus, if $\mathfrak{P}$ characterizes $\mathbb{P}\mathbb{H}_1$ ($\mathbb{B}\mathbb{H}_1, \ldots$) (all instances of) D must be provable in $\mathbb{P}\mathbb{H}_1$ ($\mathbb{B}\mathbb{H}_1, \ldots$). Already 1959, however, Umezawa has proved, by algebraic means, $\mathbb{B}\mathbb{H}_1 \not\supseteq \mathbb{B}\mathbb{D}$ (and so, a fortiori, $\mathbb{P}\mathbb{H}_1 \not\supseteq \mathbb{P}\mathbb{D}$ and so on).

Thus Kripke semantics reveals in the case of the predicate logics a really surprising weakness : it cannot characterize so a "natural" logic as $\mathbb{P}\mathbb{H}_1$: the logic of "ε-exposition", the logic of "independence of premises", the logic of "existence definability".

Recently P. Minari has remarked that $\mathbb{T}_n\mathbb{Q}$ ($n > 0$), if characterizable, must be finitary; thus, by test of finitariness, if $\mathbb{T}_n\mathbb{Q}$ does not prove H_2^o (what, however, we do not know) then $\mathbb{T}_n\mathbb{Q}$ is not chataterizable.

Up to now we have spoken only of what we properly named negation-free logics. When negation is taken into account, disregarding some obvious tranferrings and trivial corollaries only the followings results are known to us:

Theor. 12 (1) $\overline{\overline{\mathbb{P}\mathbb{Q}}} + \forall x \neg\neg\alpha \to \neg\neg\forall x\, \alpha$ is characterized by the class of all $\langle P \leq \emptyset \rangle$ such that $\langle P \leq \rangle$ is a tree for which:

$$\forall_{i \in P}\ \exists_{j \in P}\ (i \leq j \text{ and } \forall_{j \leq k}\ (j=k))$$

(Gabbay 1972-81).

(2) $\overline{\overline{\mathbb{B}\mathbb{D}}} + \neg\forall x\, \alpha \to \exists x \neg\alpha$ is characterized by the class of all $\langle P \leq \emptyset \rangle$ such that V is constant and P is linear and has a maximum (Minari, 1982).

(3) $\overline{\overline{\mathbb{P}\mathbb{D}}} + \neg\forall x\, \alpha \to \exists x \neg\alpha + \neg\alpha \vee \neg\neg\alpha$ is characterized by the class of all $\langle P \leq \emptyset \rangle$ such that V is constant and P has a maximum (Minari, 1982).

Gabbay (1969; 1981) states that $\overline{\overline{\mathbb{P}\mathbb{Q}}} + \neg\alpha \vee \neg\neg\alpha$ is characterized by the class of frames with directed partial order. 1969 it calls it a triviality; 1981 it calls it a difficult exercise; unfortunately up to now we did not solve the exercise.

BIBLIOGRAPHY

Exstensive information (also of bibliographical nature) are to be found in:

H. RASIOWA, An algebraic approach to non-classical logics, North Holland, Amsterdam 1974.

W. RAUTENBERG, Klassische und Nichtclassische Aussagen-logik, Vieweg, Braunschweig 1979.

D. GABBAY, Semantical Investigations in Heyting's Intuitionistic logic, Reidel, Dordrecht 1981.

For the arguments of the last part(and relative bibliography)

E. CASARI - P. MINARI, Negation-free intermediate predicate logics, to appear in Boll. Un. Mat. It.

After these lectures were held and written down we have been informed by the kindness of Prof. H. ONO of important results in the field of intermediate predicate logics. We mention only:

[1] H. ONO, Model extension theorem and Craig's interpolation theorem for intermediate predicate logics.

[2] H. ONO, <u>Some problems of intermediate predicate logics</u>.

[3] Y. KOMORI, <u>Some results on the superintuitionistic predicate logics</u>.

[4] T. NAKAMURA, <u>The disjunction property of some intermediate predicate logics</u>.

[1], [3] and [4] will appear in <u>Reports on Mathematical Logic</u>;
[2] will appear in the <u>Proceedings of the Bronwer-Conference</u>.

As for as problems discussed in the last lecture, at least the following results should be mentioned. Discussing $\forall x \neg\neg Px \to \neg\neg\forall x Px$, ONO introduces in [1] an algebraic model which he attributes to A. Wrónski. It is easy to verify that such a model fournishes the desired proof that $\mathbb{T}_1\mathbb{Q}$ is properly contained in $\mathbb{T}_1\mathbb{H}_2^{\circ}$. So by preceding remarks we get :

Theor.: for all $n > 0$, $\mathbb{T}_n\mathbb{Q}$ is Kripke-incomplete.

Such a theorem follows however easily from a more general theorem in [1] (Theor.3.2), from which also it follows, in particular,

Theor.: for all n, $\mathbb{C}_n\mathbb{Q}$ is Kripke-inclomplete.

Komori proves (Theor. 5.5)

Theor.: $\mathbb{P}\mathbb{H}_2$ is Kripke-incomplete.