Estratto da

C. Bernardi (a cura di), *Atti degli incontri di logica matematica* Siena 7-8-9 gennaio 1982, 15-16-17 aprile 1982, 4-5-6 giugno 1982.

Disponibile in rete su http://www.ailalogica.it

On Martin - Löf's Theory of Types

Helmut Schwichtenberg
Universität München

Per Martin - Löf has developed a rather general theory of types (cf [6] , [7]). The present paper is the result of some reflection on the basic concepts of this theory. We define a formal system of type theory with the intention to express these concepts as clear as possible. Then we begin with the development and the metamathematical investigation of this formal system.

The formal system is described in §1. It differs from Martin - Löf's theory of types (in [7]) in a number of points. The main difference is that we have tried to avoid any unnecessary generality. So we distinguish between types X in the usual sense (built up from N , G by $\rightarrow$, x and +) and formulas (better: problems) A (built up from equations $\ulcorner r = s \in X \urcorner$ by $\rightarrow , \wedge , \vee , \forall x \in X$ and $\exists x \in X$) ; any formula A has a type τA associated with it. For instance, $\tau(\forall x \in X)A = X \rightarrow \tau A$ and $\tau r = s \in X = G$. We derive judgements of the form $r \in X$, $r = s \in X$ and $r \in A$, to be read: r is an object of type X , r and s are (extensionally) equal objects of type X , and r realizes A (r solves the problem A) . For any form of a type / formula, there are introduction and elimination rules. The introduction rules determine what the objects / realizations (solutions) of the type / formula (problem) are. The elimination rules express that the objects / realizations (solutions) given by the introduction rules are the only ones. A somewhat similar restriction of Martin - Löf's system has also been considered by Diller in [4].

Another noteworthy difference between Martin - Löf's theory of types and ours is that we have no finite types N_n and in particular no

N_o , which serves Martin - Löf to identify the logic coming out of his system as intuitionistic. In our system, however, there would be no closed term r_o such that $r_o \in N_o$ is well-formed (the reason for this is that any term r determines uniquely the type X such that $r \in X$ is well-formed), and hence no judgement serving as "absurdity" is available. Hence the logic inherent in our formulation of the theory of types is minimal logic.

A final minor difference is that we have replaced the recursion and induction rules by their infinitary version. This is necessary to make our normalization argument in §3 go through. To get again a formal system of limited proof-theoretic strength (and in our case of the strength of Heyting arithmetic HA), it is necessary to make the usual restrictions on infinitary derivations.

In §2 we show that a number of expected derived rules are valid in our formal system. This is done up to a point where it can be seen that and how the usual system of Heyting arithmetic in all finite types with extensionality and the axiom of choice can be embedded. The arguments used here are routine.

In §3 we finally come to the metamathematical investigation of our formal system. To put this work into perspective let us first recall that by work of Beeson (cf [2], [3]) it is possible to define a model of our system in HA . Then the technique of Beeson developed in [1] can be used to show that our system is conservative over HA (see also [5]). Hence a good metamathematical analysis is already available. However, it still seems to be of interest to get some more direct and more informative metamathematical results. This is particularly obvious when we view the formal system as a programming language as stressed by Martin - Löf in [7]. Then in a derived judgement $r \in X$ the r may

be viewed as a program which, when executed, yields an object of type X . It suffices to look at such judgements, since from a derivation of $r \in A$ one easily (cf Lemma 2.1 below) obtains a derivation of $r \in \tau A$, and the "computational part" of the solution r of the problem A is already captured by this derivation. Now a good metamathematical analysis should give some information on possible simplifications and generally on the complexity of the program r .

We will proof in §3 the following normalization theorem: Whenever we have a derivation $d \vdash r \in X$ of finite rank, then we can find a normal derivation (i.e., without properly or permutively convertible subderivations) $d^* \vdash r^* \in X$ such that $\vdash r = r^* \in X$ and $|d^*| \leq$ the least ε - number $> |d|$. In particular, in the normal derivation $d^* \vdash r^* \in X$ (and hence in the "program" r^*) only subtypes of the type X occur. Also, when the usual restrictions of the infinitary rules are made, it can be seen that the "program" r^* determines a $< \varepsilon_0$ - recursive object of type X (cf [8]). The method of proof for the normalization theorem is rather standard and basically goes back to Tait [9]. However, some care is necessary to treat the permutative conversions.

§1 A formal system of type theory

Types are N (the type of natural numbers) and G (consisting of a single object $\checkmark$, being thought of as the only possible realization of an equation), and if X, Y are types, then so are $X \to Y$, $X \times Y$ and $X + Y$. Our formal system will derive judgements of the form

$r \in X$	to be read	r is an object of type X.
$r = s \in X$		r, s are equal objects of type X.
A formula		A is a formula (or, somewhat better: A is a problem).
$r \in A$		r realizes A.

Judgements will be denoted by $U\ V\ W$. A judgement may contain free variables; these must be bound by a context

$$x_1 \in X_1, \ldots, x_m \in X_m, u_1 \in A_1, \ldots, u_n \in A_n .$$

Here A_i may contain $x_1, \ldots, x_m, u_1, \ldots, u_{i-1}$ free. In the statement of the rules below, the same context should always be added to the premisses and to the conclusion. Only those parts of the context which are either cancelled or introduced by the rule are mentioned explicitely. In the statement of the axioms no context is shown for brevity. More completely, the axiom N^+ below should read

$$\frac{A_1 \text{ formula } (\underset{\sim}{x} \in \underset{\sim}{X}) \quad \ldots\ldots \quad A_n \text{ formula } (\underset{\sim}{x} \in \underset{\sim}{X}, u_1 \in A_1, \ldots, u_{n-1} \in A_{n-1})}{\underline{i} \in N(\underset{\sim}{x} \in \underset{\sim}{X}, u_1 \in A_1, \ldots, u_n \in A_n)}$$

and similarly for the other axioms. Also, the rule for an introduction of an assumption below should read

$$\frac{A_1 \text{ formula } (\underset{\sim}{x} \in \underset{\sim}{X}) \quad \ldots\ldots \quad A_n \text{ formula } (\underset{\sim}{x} \in \underset{\sim}{X}, u_1 \in A_1, \ldots, u_{n-1} \in A_{n-1})}{u_i \in A_i(\underset{\sim}{x} \in \underset{\sim}{X}, u_1 \in A_1, \ldots, u_i \in A_i, \ldots, u_n \in A_n)}$$

If in a judgement with context $U(\ldots . u \in A \ldots .)$ the variable u occurs neither in U nor in other parts of the context, then we

shall also write $U(\ldots .\ A \text{ true } \ldots .)$.

Axioms and rules of the formal system

N^+ $\qquad \underline{i} \in N \qquad$ for any numeral $\underline{i}$

N^- : recursion

$$\frac{t \in N \qquad \ldots\ldots\ r_i \in X\ \ldots\ldots}{Et\langle r_i \rangle \in X}$$

$$\frac{t \in N \qquad \ldots\ldots\ r_i \in A\ \ldots\ldots}{Et\langle r_i \rangle \in A}$$

induction

$$\frac{t \in N \qquad \ldots\ldots\ U(\ulcorner t = \underline{i} \in N \urcorner \text{ true})\ \ldots\ldots}{U}$$

$N^=$: proper conversion

$$\frac{\ldots\ldots\ r_i \in X\ \ldots\ldots}{E\underline{j}\langle r_i \rangle = r_j \in X}$$

compatibility $\qquad \underline{i} = \underline{i} \in N$

$$\frac{t = t' \in N \qquad \ldots\ldots\ r_i = r_i' \in N\ \ldots\ldots}{Et\langle r_i \rangle = Et'\langle r_i' \rangle \in N}$$

G^+ $\qquad \checkmark \in G$

G^- $\qquad \dfrac{t \in G}{t = \checkmark \in G}$

$G^=$: compatibility $\qquad \checkmark = \checkmark \in G$

$\to^+$ for types $\qquad \dfrac{r \in Y(x \in X)}{\lambda x r \in X \to Y}$

$\to^-$ for types $\qquad \dfrac{t \in X \to Y \qquad s \in X}{ts \in Y}$

$\to^{=}$: proper conversion

$$\frac{r \in Y(x \in X) \qquad s \in X}{(\lambda x r)s = r_x[s] \in Y}$$

compatibility

$$\frac{r = r' \in Y \qquad (x \in X)}{\lambda x r = \lambda x r' \in X \to Y}$$

$$\frac{t = t' \in X \to Y \qquad s = s' \in X}{ts = t's' \in Y}$$

extensionality

$$\frac{r \in X \to Y}{\lambda x r x = r \in X \to Y}$$

$\times^{+}$

$$\frac{r \in X \qquad s \in Y}{(r,s) \in X \times Y}$$

$\times^{-}$

$$\frac{t \in X \times Y}{pt \in X} \qquad \frac{t \in X \times Y}{qt \in Y}$$

$\times^{=}$: proper conversion

$$\frac{r \in X \qquad s \in Y}{p(r,s) = r \in X}$$

$$\frac{r \in X \qquad s \in Y}{q(r,s) = s \in Y}$$

compatibility

$$\frac{r = r' \in X \qquad s = s' \in Y}{(r,s) = (r',s') \in X \times Y}$$

$$\frac{t = t' \in X \times Y}{pt = pt' \in X}$$

$$\frac{t = t' \in X \times Y}{qt = qt' \in Y}$$

$+^{+}$

$$\frac{r \in X}{i_Y r \in X + Y} \qquad \frac{s \in Y}{j_X s \in X + Y}$$

$+^{-}$: definition by cases

$$\frac{t \in X + Y \qquad r \in Z(u \in X) \qquad s \in Z(v \in Y)}{D_{uv} trs \in Z}$$

$$\frac{t \in X + Y \qquad r \in C(u \in X, \ulcorner t = i_Y u \in X + Y \urcorner \text{ true}) \qquad s \in C(v \in Y, \ulcorner t = j_X v \in X + Y \urcorner \text{ true})}{D_{uv} trs \in C}$$

proof by cases

$$\frac{t \in X + Y \qquad U(u \in X, \ulcorner t = i_Y u \in X + Y \urcorner \text{ true}) \qquad U(v \in Y, \ulcorner t = j_X v \in X + Y \urcorner \text{ true})}{U}$$

$+^{=}$: proper conversion

$$\frac{r \in X \qquad r' \in Z(u \in X) \qquad s' \in Z(v \in Y)}{D_{uv}(i_Y r) r's' = r'_u[r] \in Z}$$

$$\frac{s \in Y \qquad r' \in Z(u \in X) \qquad s' \in Z(v \in Y)}{D_{uv}(j_X s) r's' = s'_v[s] \in Z}$$

compatibility

$$\frac{r = r' \in X}{i_Y r = i_Y r' \in X + Y}$$

$$\frac{s = s' \in Y}{j_X s = j_X s' \in X + Y}$$

$$\frac{t = t' \in X+Y \qquad r = r' \in Z(u \in X, \ulcorner t = i_Y u \in X+Y \urcorner \text{ true}) \qquad s = s' \in Z(v \in Y, \ulcorner t = j_X v \in X+Y \urcorner \text{ true})}{D_{uv} trs = D_{uv} t'r's' \in Z}$$

$\ulcorner\ \urcorner$

$$\frac{r \in X \qquad s \in X}{\ulcorner r = s \in X \urcorner \text{ formula}}$$

$\ulcorner\ \urcorner^{+}$

$$\frac{r = s \in X}{\checkmark \in \ulcorner r = s \in X \urcorner}$$

$\ulcorner\ \urcorner^{-}$

$$\frac{t \in \ulcorner r = s \in X \urcorner}{r = s \in X}$$

$\to$ $$\frac{B \text{ formula} \quad (A \text{ true})}{A \to B \text{ formula}}$$

$\to^{+}$ for formulas $$\frac{r \in B \qquad (u \in A)}{\lambda u r \in A \to B}$$

$\to^{-}$ for formulas $$\frac{t \in A \to B \qquad s \in A}{ts \in B}$$

$\wedge$ $$\frac{A \text{ formula} \qquad B \text{ formula}}{A \wedge B \text{ formula}}$$

$\wedge^{+}$ $$\frac{r \in A \qquad s \in B}{(r,s) \in A \wedge B}$$

$\wedge^{-}$ $$\frac{t \in A \wedge B}{pt \in A} \qquad\qquad \frac{t \in A \wedge B}{qt \in B}$$

$\vee$ $$\frac{A \text{ formula} \qquad B \text{ formula}}{A \vee B \text{ formula}}$$

$\vee^{+}$ $$\frac{r \in A \qquad B \text{ formula}}{i_{\tau B} r \in A \vee B}$$

$$\frac{s \in B \qquad A \text{ formula}}{j_{\tau A} s \in A \vee B}$$

Here τA is the type of A, defined by $\tau(\ulcorner r = s \in X \urcorner) = G$, $\tau(A \to B) = \tau A \to \tau B$, $\tau(A \wedge B) = \tau A \times \tau B$, $\tau(A \vee B) = \tau A + \tau B$, $\tau((\forall x \in X)A) = X \to \tau A$, $\tau((\exists x \in X)A) = X \times \tau A$.

$\vee^{-}$: definition by cases

$$\frac{t \in A \vee B \qquad r \in Z(u \in A) \qquad s \in Z(v \in B)}{D_{uv} trs \in Z}$$

$$\frac{t \in A \vee B \qquad r \in C(u \in A, \ulcorner t = i_{\tau B} u \in \tau(A \vee B) \urcorner \text{ true}) \qquad s \in C(v \in B, \ulcorner t = j_{\tau A} v \in \tau(A \vee B) \urcorner \text{ true})}{D_{uv} trs \in C}$$

proof by cases

$$\frac{t \in A \vee B \qquad U(u \in A, \ulcorner t = i_{\tau B} u \in \tau(A \vee B) \urcorner \text{ true}) \qquad U(v \in B, \ulcorner t = j_{\tau A} v \in \tau(A \vee B) \urcorner \text{ true})}{U}$$

$\forall$ $$\frac{A \text{ formula} \quad (x \in X)}{(\forall x \in X)A \text{ formula}}$$

$\forall^{+}$ $$\frac{r \in A \quad (x \in X)}{\lambda x r \in (\forall x \in X)A}$$

$\forall^{-}$ $$\frac{t \in (\forall x \in X)A \qquad s \in X}{ts \in A_x[s]}$$

$\exists$ $$\frac{A \text{ formula} \quad (x \in X)}{(\exists x \in X)A \text{ formula}}$$

$\exists^{+}$ $$\frac{r \in X \qquad s \in A_x[r] \qquad A \text{ formula} \quad (x \in X)}{(r,s) \in (\exists x \in X)A}$$

$\exists^{-}$ $$\frac{t \in (\exists x \in X)A}{pt \in X} \qquad\qquad \frac{t \in (\exists x \in X)A}{qt \in A_x[pt]}$$

Rules of permutation conversion

NN $$\frac{t \in N \quad \ldots .\ r_i \in N \ \ldots . \quad \ldots .\ s_j \in X \ \ldots .}{E(Et\langle r_i\rangle)\langle s_j\rangle = Et\langle Er_i\langle s_j\rangle\rangle \in X}$$

$N \to$ $$\frac{t \in N \quad \ldots . r_i \in X \to Y \ \ldots . \qquad s \in X}{(Et\langle r_i\rangle)s. = Et\langle r_i s\rangle \in Y}$$

$N \times$ $$\frac{t \in N \quad \ldots .\ r_i \in X \times Y \ \ldots .}{p(Et\langle r_i\rangle) = Et\langle pr_i\rangle \in X}$$

$$\frac{t \in N \quad \ldots\ldots\ r_i \in X \times Y \ \ldots\ldots}{q(Et\langle r_i\rangle) = Et\langle qr_i\rangle \in Y}$$

N+ $$\frac{t \in N \quad \ldots\ r_i \in X + Y \ \ldots \quad r \in Z\ (u \in X) \quad s \in Z\ (v \in Y)}{D_{uv}(Et\langle r_i\rangle)rs = Et\langle D_{uv} r_i rs\rangle \in Z}$$

+N $$\frac{t \in X + Y \quad r \in N\ (u \in X) \quad s \in N\ (v \in Y) \quad \ldots\ r_i \in Z \ \ldots}{E(D_{uv}trs)\langle r_i\rangle = D_{uv}t(Er\langle r_i\rangle)(Es\langle r_i\rangle) \in Z}$$

+→ $$\frac{t \in X + Y \quad r \in Z \to Z'\ (u \in X) \quad s \in Z \to Z'\ (v \in Y) \quad s' \in Z}{(D_{uv}trs)s' = D_{uv}t(rs')(ss') \in Z'}$$

+× $$\frac{t \in X + Y \quad r \in Z \times Z'\ (u \in X) \quad s \in Z \times Z'\ (v \in Y)}{p(D_{uv}trs) = D_{uv}t(pr)(ps) \in X}$$

$$\frac{t \in X + Y \quad r \in Z \times Z'\ (u \in X) \quad s \in Z \times Z'\ (v \in Y)}{q(D_{uv}trs) = D_{uv}t(qr)(qs) \in Y}$$

++ $$\frac{t \in X+Y \quad r \in Z+Z'(u \in X) \quad s \in Z+Z'(v \in Y) \quad r' \in \overline{Z}(u' \in Z) \quad s' \in \overline{Z}(v' \in Z')}{D_{u'v'}(D_{uv}trs)r's' = D_{uv}t(D_{u'v'}rr's')(D_{u'v'}sr's') \in \overline{Z}}$$

Reflexivity $\quad x = x \in X \quad (x \in X)$

Symmetry $$\frac{r = s \in X}{s = r \in X}$$

Transitivity $$\frac{r = s \in X \quad s = t \in X}{r = t \in X}$$

Introduction of assumptions

$x \in X \quad (x \in X)$

$$\frac{A \text{ formula}}{u \in A \quad (u \in A)}$$

§2 Development of the formal system

2.1 Lemma Let a derivation of a judgement $r \in X$ or $r \in A$ be given. Thean one can find another derivation of $r \in X$ or $r \in \tau A$ with each part $u_i \in A_i$ of the original context replaced by $u_i \in \tau A_i$, and the new derivation is not longer than the given one.

Proof: Cancel all parts of the given derivation ending with a judgement of the form $r = s \in X$ or A formula. Then replace all judgements or the form $r \in A$, including those occurring in contexts, by $r \in \tau A$. It is easy to see that this procedure yields the required derivation.

2.2 Equalitiy Lemma for Terms

$\vdash r = s \in X,\ \vdash t[r] \in Y \Rightarrow \vdash t[r] = t[s] \in Y$

Proof: By induction on the length of the given derivation of $t[r] \in Y$, using Lemma 2.1. If t is just a variable to be substituted by r, then Y is the same as X and $t[r] = t[s] \in Y$ is the same as $r = s \in X$. So we may exclude this case in the following. We treat some typical cases for the last rule used to derive $t[r] \in Y$.

N^+ $\underline{i} \in N$. We have to derive $\underline{i} = \underline{i} \in N$, which is obtained by $N^=$ compatibility.

N^- recursion: $$\frac{t[r] \in N \quad \ldots\ldots\ r_i[r] \in X \ \ldots\ldots}{Et[r]\langle r_i[r]\rangle \in X}$$

By induction hypothesis and $N^=$ compatibility we have

$$\frac{t[r] = t[s] \in N \quad \ldots\ldots\ r_i[r] = r_i[s] \in X \ \ldots\ldots}{Et[r]\langle r_i[r]\rangle = Et[s]\langle r_i[s]\rangle \in X}$$

The second rule of N^- recursion has a conclusion $Et\langle r_i\rangle \in A$ and hence cannot have been applied.

N$^-$ induction:
$$\frac{t[r]\in N \qquad \ldots\ldots\ U(\ulcorner t[r]=\underline{i}\in N\urcorner \text{ true})\ \ldots\ldots}{U}$$

Then U must be of the form $t'[r]\in Y$. By induction hypothesis and again N$^-$ induction one gets the required derivation of $t'[r]=t'[s]\in Y$.

Note that rules with equations as conclusion need not be considered.

For G^+ , $\to^+$ for types, $\to^-$ for types, $\times^+$, $\times^-$, $+^+$ and $+^{\bar{=}}$ definition by cases the required derivation is immediately obtained by induction hypothesis and the corresponding compatibility rule. For $+^-$ proof by cases one applies again the induction hypothesis and then the same rule. Most of the following rules need not be considered, since their conclusion has not the required form. It only remains to look at

v$^-$ definition by cases

$$\frac{t[r]\in A\vee B \qquad r'[r]\in Z\ (u\in A) \qquad s'[r]\in Z\ (v\in B)}{D_{uv}t[r]r'[r]s'[r]\in Z}$$

Now by Lemma 2.1 we obtain from the derivations of the premisses

$$t[r]\in \tau A+\tau B \qquad r'[r]\in Z\ (u\in\tau A) \qquad s'[r]\in Z\ (v\in\tau B)$$

The induction hypothesis and $+^=$ compatibility yield the required derivation.

$\exists^-$
$$\frac{t[r]\in(\exists x\in X)A}{pt[r]\in X}$$

Again by Lemma 2.1 we have a derivation of $t[r]\in X\times\tau A$. Now apply the induction hypothesis and then $\times^=$ compatibility.

Introduction of assumptions $x\in X\ (x\in X)$

The required derivation $x=x\in X\ (x\in X)$ is obtained by the reflexivity axiom.

2.3 Inversion Lemma for Formulas

$\vdash \ulcorner r=s\in X\urcorner$ formula	$\Rightarrow$	$\vdash r\in X$, $\vdash s\in X$
$\vdash A\to B$ formula	$\Rightarrow$	$\vdash B$ formula (A true)
$\vdash A\wedge B$ formula	$\Rightarrow$	$\vdash A$ formula, $\vdash B$ formula
$\vdash A\vee B$ formula	$\Rightarrow$	$\vdash A$ formula, $\vdash B$ formula
$\vdash (\forall x\in X)A$ formula	$\Rightarrow$	$\vdash A$ formula $(x\in X)$
$\vdash (\exists x\in X)A$ formula	$\Rightarrow$	$\vdash A$ formula $(x\in X)$

This is trivially seen by induction on the length of the given derivation.

2.4 Lemma a. $\vdash r\in A \Rightarrow \vdash A$ formula

b. $\vdash U(\underset{\sim}{x}\in\underset{\sim}{X},u_1\in A_1,\ldots,u_n\in A_n) \Rightarrow \vdash A_i$ formula $(\underset{\sim}{x}\in\underset{\sim}{X},u_1\in A_1,\ldots,u_{i-1}\in A_{i-1})$

The proof is easy, by induction on the length if the given derivation. For a. one has to use 2.3.

2.5 Lemma $\vdash r=s\in X \Rightarrow \vdash r\in X$, $\vdash s\in X$

This is again proved by induction on the length of the given derivation, and is essentially straightforward; however, one has to make use of all preceding Lemmas 2.1 - 2.4. We just comment on some cases.

$\to^=$ proper conversion:
$$\frac{r\in Y\ (x\in X) \qquad s\in X}{(\lambda xr)s=r_x[s]\in Y}$$

To obtain a derivation of $r_x[s]\in Y$, one has to cancel in the given derivation of $r\in Y\ (x\in X)$ all parts $x\in X$ of the contexts and to substitute all free occurrences of x by s . An introduction of an assumption $x\in X\ (x\in X)$ then becomes $s\in X$; above these nodes one has to substitute the given derivation of $s\in X$. For the reflexivity axiom $x=x\in X\ (x\in X)$ one needs a corrollary to the Equality Lemma 2.2, namely that from $\vdash s\in X$ one can conclude

$\vdash s = s \in X$. Note that in this case the new derivation will be longer in general than any of the given ones.

$+^=$ compatibility, third rule: Here one needs Lemma 2.1 in order to know that in the derivations obtained by induction hypothesis the parts $\ulcorner t = i_Y u \in X + Y \urcorner$ true and $\ulcorner t = j_X v \in X + Y \urcorner$ true of the context are not needed.

$\ulcorner\urcorner^-$:

$$\frac{t \in \ulcorner r = s \in X \urcorner}{r = s \in X}$$

By Lemma 2.4 a one has $\vdash \ulcorner r = s \in X \urcorner$ formula, and by Lemma 2.3 it follows $\vdash r \in X$ and $\vdash s \in X$, as required.

2.6 Equality Lemma for Formulas

$\vdash r = s \in X , \quad \vdash A[r]$ true $\Rightarrow$ $\vdash A[s]$ true

Here we have written $\vdash B$ true instead of $\vdash t \in B$; we shall do so whenever we are not interested in the particular realization t of a formula B . Also, we shall write $\vdash B$ true $(A$ true$)$ for $\vdash t \in B \ (u \in A)$ whenever we are not interested in t and in particular in its dependence on u .

Proof: By induction on the number of logical symbols in A .

$\ulcorner\urcorner$: $\vdash \ulcorner t[r] = t'[r] \in Y \urcorner$ true

$\vdash t[r] = t'[r] \in Y$ by $\ulcorner\urcorner^-$

Using $\vdash r = s \in X$, Lemma 2.5 and the Equality Lemma 2.2 for Terms, we obtain

$\vdash t[r] = t[s] \in Y$

$\vdash t'[r] = t'[s] \in Y$.

By transitivity and symmetry of equality we have

$\vdash t[s] = t'[s] \in Y$

$\vdash \ulcorner t[s] = t'[s] \in Y \urcorner$ true

$\rightarrow$: $\vdash A[r] \rightarrow B[r]$ true

$\vdash A[r]$ true $(A[s]$ true$)$ by induction hypothesis

$\vdash B[r]$ true $(A[s]$ true$)$

$\vdash B[s]$ true $(A[s]$ true$)$ by induction hypothesis

$\vdash A[s] \rightarrow B[s]$ true

$\wedge$: $\vdash A[r] \wedge B[r]$ true

$\vdash A[r]$ true , $\vdash B[r]$ true

$\vdash A[s]$ true, $\vdash B[s]$ true by induction hypothesis

$\vdash A[s] \wedge B[s]$ true

$\vee$: $\vdash A[r] \vee B[r]$ true

$\vdash A[s]$ true $(A[r]$ true$)$ by induction hypothesis

$\vdash A[s] \vee B[s]$ true $(A[r]$ true$)$

Similarly,

$\vdash A[s] \vee B[s]$ true $(B[r]$ true$)$

Using $\vee^-$ definition by cases one obtains

$\vdash A[s] \vee B[s]$ true

$\forall$: $\vdash (\forall x \in X)A[r]$ true

$\vdash x \in X \ (x \in X)$

$\vdash A[r]$ true $(x \in X)$

$\vdash A[s]$ true $(x \in X)$ by induction hypothesis

$\vdash (\forall x \in X)A[s]$ true

$\exists$: $\vdash (\exists x \in X)A[r]$ true, i.e. $\vdash t \in (\exists x \in X)A[r]$

$\vdash A[pt, r]$ true

$\vdash A[pt, s]$ true by induction hypothesis

$\vdash (\exists x \in X)A[s]$ true

2.7 Canonical Realization for Negative A

Let A be negative, i.e. with $\vee$, $\exists$ only in premisses of implications. Then one can find r_A without free variables such that

$$\vdash r \in A \Rightarrow \vdash r_A \in A .$$

The proof is by induction on the number of logical symbols in A .

$\ulcorner r = s \in X \urcorner$: $\vdash t \in \ulcorner r = s \in X \urcorner$

$\vdash r = s \in X$

$\vdash \checkmark \in \ulcorner r = s \in X \urcorner$

Hence, $r_A :\equiv \checkmark$ for $A \equiv \ulcorner r = s \in X \urcorner$.

$A \to B$: $\vdash r \in A \to B$

$\vdash ru \in B \quad (u \in A)$

$\vdash r_B \in B \quad (u \in A)$ by induction hypothesis

$\vdash \lambda u r_B \in A \to B$

Hence, $r_{A \to B} :\equiv \lambda u r_B$.

$A \wedge B$: $\vdash r \in A \wedge B$

$\vdash pr \in A$, $\vdash qr \in B$

$\vdash r_A \in A$, $\vdash r_B \in B$ by induction hypothesis

$\vdash (r_A , r_B) \in A \wedge B$

Hence, $r_{A \wedge B} :\equiv (r_A , r_B)$.

$(\forall x \in X)A$: $\vdash r \in (\forall x \in X)A$

$\vdash rx \in A \quad (x \in X)$

$\vdash r_A \in A \quad (x \in X)$ by induction hypothesis

$\vdash \lambda x r_A \in (\forall x \in X)A$

Hence, $r_{(\forall x \in X)A} :\equiv \lambda x r_A$.

Remark: By 2.7 we have $\vdash r_A \in A \ (u \in A)$ for A negative. Hence in any derived judgement with $u \in A$ in the context we may assume that u has no free occurrence in any other part of the judgement, including the rest of the context. So we may write an assumption $u \in A$ for A negative in the form A true .

Remark: When we add a rule

$$\frac{r \in X \times Y}{(pr,qr) = r \in X \times Y}$$

to our formal system, then we can prove $\vdash r = r_A \in \tau A$.

2.8 ω - Rule

$\vdash t \in N$, $\vdash A[\underline{i}]$ true , for all $i \Rightarrow \vdash A[t]$ true

Proof: From $\vdash A[\underline{i}]$ true we obtain by the Equality Lemma for Formulas and the Remark following 2.7 $\vdash A[t]$ true ($\ulcorner t = \underline{i} \in N \urcorner$ true) . Now N^- induction yields $\vdash A[t]$ true .

2.9 Embedding of HA^ω + AC + EXT

Let A be derivable in HA^ω + AC + EXT from $A_1, \dots, A_n$, with free variables $x_1, \dots, x_m$. We may assume that $A, A_1, \dots, A_n$ are formulas in our formal system (since there is a natural translation). Then we can derive

$$\vdash A \text{ true } (\underset{\sim}{x} \in \underset{\sim}{X}, A_1 \text{ true}, \dots, A_n \text{ true}, C_1 \text{ true}, \dots, C_n \text{ true}) ,$$

where the C_i are of the form

$$\bot \to B$$

$$(\forall x \in N) \ulcorner x' \neq 0 \in N \urcorner$$

$$(\forall x,y \in N)(\ulcorner x' = y' \in N \urcorner \to \ulcorner x = y \in N \urcorner) .$$

Here $\bot$ is a new atomic formula, $\neg B$ is an abbreviation of $B \to \bot$, and r' denotes the successor, i.e. $r' :\equiv Er\langle 1,2,3,\dots\rangle$.

Proof: The rules of minimal logic are obviously taken care of by our formal system. As an example, let us consider the rule of $\exists$ - elimination. We can assume that we have derived

$$t \in (\exists x \in X)A \quad \text{and} \quad C \text{ true } (x \in X , u \in A)$$

when x , u do not occur free in C . Now we obtain $pt \in X$, $qt \in A_x[pt]$.

Substituting pt for x and qt for u in the second derivation, we get C true , as required.

Intuitionistic logic as well as the Peano axioms except induction are taken care of by the form of our embedding. To obtain induction we apply the ω-rule 2.8. It remains to be seen that extensionality and the axiom of choice can be derived.

EXT : $(\forall x,y \in X \to Y)((\forall z \in X)\ulcorner xz = yz \in Y\urcorner \to \ulcorner x = y \in X \to Y\urcorner)$

We argue informally. So let $x,y \in X \to Y$, and $xz = yz \in Y$ for all $z \in X$. Then by $\overset{=}{\to}$ compatibility we have $\lambda zxz = \lambda zyz \in X \to Y$. By $\overset{=}{\to}$ extensionality we have $\lambda zxz = x \in X \to Y$ and $\lambda zyz = y \in X \to Y$, and hence $x = y \in X \to Y$, as required.

AC : $(\forall x \in X)(\exists y \in Y)A \to (\exists z \in X \to Y)(\forall x \in X)A_y[zx]$

Again we argue informally. So let $u \in (\forall x \in X)(\exists y \in Y)A$, and $x \in X$. Then $ux \in (\exists y \in Y)A$ and hence $p(ux) \in Y$, $q(ux) \in A_y[p(ux)]$. So $\lambda xp(ux) \in X \to Y$, which is the z we are looking for. To see this, let $x \in X$. Then $(\lambda xp(ux))x = p(ux) \in Y$, and from $q(ux) \in A_y[p(ux)]$ we can conclude by the Equality Lemma for Formulas that $A_y[(\lambda xp(ux))x]$ is true.

§3 Normalization

Theorem Let a derivation $d \vdash r \in X\ (\underset{\sim}{x} \in \underset{\sim}{X})$ of finite rank be given. Then we can find a normal derivation $d^* \vdash r^* \in X\ (\underset{\sim}{x} \in \underset{\sim}{X})$ such that $\vdash r = r^* \in X\ (\underset{\sim}{x} \in \underset{\sim}{X})$ and such that the length $|d^*|$ is $\leq$ the least ε-number $> |d|$.

We first explain the (standard) notions used in the statement of the theorem. Call the rules with index $+$ introduction rules and those with index $-$ elimination rules. A derivation is called properly convertible if it ends with an introduction rule immediately followed by an elimination rule. More precisely, the conclusion of the introduction rule must be the main premiss of the elimination rule, i.e. the leftmost premiss in the statement of the rule. A derivation is called permutatively convertible if it ends with a critical elimination rule immediately followed by an elimination rule. Here an elimination rule is called critical if it is one of the rules N^-, $+^-$ or v^-; these rules are distinguished since their side premisses (i.e. the premisses other than the main premiss) are of the same form as the conclusion. A derivation is called normal if it contains neither properly nor permutatively convertible subderivations. By the rank of a derivation we mean the least number bigger than the level of the main premisses of all its properly convertible subderivations (i.e., of their elimination rules at the end). These main premisses are of the form $r \in X$ or $r \in A$. The level of such a judgement is defined to be the level of the type X or of the type τA, where the level LX of a type X is defined by $LN = LG = 0$, $L(X \to Y) = L(X \times Y) = L(X+Y) = \max(LX, LY) + 1$.

The length $|d|$ of a derivation d is defined to be the least ordinal bigger than the lengths of the derivations of the premisses. Hence we have with an obvious notation for derivations (e.g. $Ed\langle a_i\rangle$ denotes the derivation ending with an application of N^- recursion, where d is the derivation of the premiss $t \in N$ and the a_i are derivations of the premisses $r_i \in X$ or $r_i \in A$) :

$$|Ed\langle a_i\rangle| = \sup(|d|+1, \ldots\ldots |a_i|+1 \ldots\ldots)$$
$$|db| = \max(|d|+1, |b|+1)$$
$$|pd| = |d| + 1$$
$$|D_{uv}dab| = \max(|d|+1, |a|+1, |b|+1)$$

For the proof of the theorem first note that by the proof of Lemma 2.1 we may assume that d contains no judgements of the form $r = s \in X$

or A formula or $r \in A$; this applies also to contexts. Hence, the rules N^- induction and $+^-$ proof by cases are not used in d .

The proof of the theorem now proceeds in 3 steps. In step 2 (which needs a preparatory step 1) it is shown how to eliminate all permutatively convertible subderivations. Step 3 describes a method to reduce the rank of a derivation without permutatively convertible subderivations by 1.

Step 1 Let a derivation $d \vdash r \in X$ be given, as described above. Assume that the last rule in d is an elimination rule. We shall define a new derivation $d^- \vdash r^- \in X$ with the following property. If immediately above the last elimination rule in d there is a block of critical elimination rules, then in d^- the last elimination rule is permuted with all these critical elimination rules. Further, we have $\vdash r = r^- \in X$ and

$$|(Ed\langle a_i\rangle)^-| \leq (\sup|a_i|) + |d| + 1$$
$$|(db)^-| \leq |b| + |d| + 1$$
$$|(pd)^-| \leq |d| + 1$$
$$|(qd)^-| \leq |d| + 1$$
$$|(D_{uv}dab)^-| \leq \max(|a|, |b|) + |d| + 1$$

The definition of d^- is by recursion on some ordinal $\alpha(d)$ to be defined below. We let d^- be d except when d ends with a critical elimination rule immediately followed by an elimination rule. In this case we permute these two rules, shifting the elimination rule to the side premisses of the critical rule. We shall not write out all cases, but just two typical ones:

$$((Ed\langle a_i\rangle)b)^- := Ed\langle (a_ib)^-\rangle$$
$$(E(D_{uv}dab)\langle a_i\rangle)^- := D_{uv}d(Ea\langle a_i\rangle)^-(Eb\langle a_i\rangle)^-$$

For to see that this is a well-founded recursion we define an ordinal $\alpha(d)$ as follows. If d ends with an elimination rule with main premiss a and side premisses $b_0, b_1, \ldots$, then

$$\alpha(d) := \sup(\alpha(b_0), \alpha(b_1), \ldots) + \alpha(a) + 1 .$$

Otherwise, $\alpha(d)$ is defined by the same recursion equation as $|d|$ e.g. $\alpha(\lambda xa) = \alpha(a) + 1$. With this definition of $\alpha(d)$ it is obvious that, for instance, $\alpha((Ed\langle a_i\rangle)b) > \alpha(a_ib)$.

From the definition it is clear that d^- has the following property: If d ends with an elimination rule, and if no immediate subderivation of d has a permutatively convertible part, then d^- contains no permutatively convertible subderivations.

We first show that $\vdash r = r^- \in X$, by induction on $\alpha(d)$, where d is the given derivation $d \vdash r \in X$. For simplicity we give the proof informally, but in such a way that it can be formalized easily. We only treat the two cases written out above.

1. We have to show

$$(Et\langle r_i\rangle)s = ((Et\langle r_i\rangle)s)^- \in Y , \text{ i.e.}$$
$$(Et\langle r_i\rangle)s = Et\langle (r_is)^-\rangle \in Y .$$

Now by the rule $N\rightarrow$ for permutative conversion, we have

$$(Et\langle r_i\rangle)s = Et\langle r_is\rangle \in Y ,$$

and $r_is = (r_is)^- \in Y$ holds by induction hypothesis.

2. We have to show

$$E(D_{uv}trs)\langle r_i\rangle = D_{uv}t(Er\langle r_i\rangle)^-(Es\langle r_i\rangle)^- \in Z .$$

By the rule $+N$ for permutative conversion we have

$$E(D_{uv}trs)\langle r_i\rangle = D_{uv}t(Er\langle r_i\rangle)(Es\langle r_i\rangle) \in Z ,$$

and the required equation is obtained by induction hypothesis.

Finally we have to verify the estimates given above. These estimates only apply if the given derivation c ends with an elimination rule. Furthermore, we may assume that its main premiss is derived by a critical elimination rule, for otherwise the estimate is trivial. We use induction on $\alpha(c)$ and again shall only treat some typical cases.

<u>NN</u> : Recall that $(E(Ed\langle a_i\rangle)\langle b_j\rangle)^- \equiv Ed\langle(Ea_i\langle b_j\rangle)^-\rangle$. We have to show

$$|Ed\langle(Ea_i\langle b_j\rangle)^-\rangle| \le (\sup_j |b_j|) + |Ed\langle a_i\rangle| + 1 .$$

This is seen as follows

$$
\begin{aligned}
&|Ed\langle(Ea_i\langle b_j\rangle)^-\rangle| \\
&= \sup(|d|+1, \ldots |(Ea_i\langle b_j\rangle)^-|+1 \ldots) \\
&\le \sup(|d|+1, \ldots (\sup_j |b_j|) + |a_i| + 2 \ldots) \quad \text{by induction hypothesis} \\
&\le (\sup_j |b_j|) + \sup(|d|+1, \ldots |a_i|+1 \ldots) + 1 \\
&= (\sup_j |b_j|) + |Ed\langle a_i\rangle| + 1 .
\end{aligned}
$$

<u>N→</u>: Recall that $((Ed\langle a_i\rangle)b)^- \equiv Ed\langle(a_i b)^-\rangle$. We have to show

$$|Ed\langle(a_i b)^-\rangle| \le |b| + |Ed\langle a_i\rangle| + 1 .$$

This is seen as follows

$$
\begin{aligned}
&|Ed\langle(a_i b)^-\rangle| \\
&= \sup(|d|+1, \ldots |(a_i b)^-|+1 \ldots) \\
&\le \sup(|d|+1, \ldots |b| + |a_i| + 2 \ldots) \quad \text{by induction hypothesis} \\
&\le |b| + \sup(|d|+1, \ldots |a_i|+1 \ldots) + 1 \\
&= |b| + |Ed\langle a_i\rangle| + 1 .
\end{aligned}
$$

<u>N×</u> : Recall that $(p(Ed\langle a_i\rangle))^- \equiv Ed\langle(pa_i)^-\rangle$. We have to show

$$|Ed\langle(pa_i)^-\rangle| \le |Ed\langle a_i\rangle| + 1 .$$

This is seen as follows

$$
\begin{aligned}
&|Ed\langle(pa_i)^-\rangle| \\
&= \sup(|d|+1, \ldots |(pa_i)^-|+1 \ldots)
\end{aligned}
$$

$$
\begin{aligned}
&\le \sup(|d|+1, \ldots |a_i|+2 \ldots) \quad \text{by induction hypothesis} \\
&\le \sup(|d|+1, \ldots |a_i|+1 \ldots) + 1 \\
&= |Ed\langle a_i\rangle| + 1 .
\end{aligned}
$$

<u>+N</u> : Recall that $(E(D_{uv}dab)\langle a_i\rangle)^- \equiv D_{uv}d(Ea\langle a_i\rangle)^-(Eb\langle a_i\rangle)^-$. We have to show

$$|D_{uv}d(Ea\langle a_i\rangle)^-(Eb\langle a_i\rangle)^-| \le (\sup_i |a_i|) + |D_{uv}dab| + 1 .$$

This is seen as follows

$$
\begin{aligned}
&|D_{uv}d(Ea\langle a_i\rangle)^-(Eb\langle a_i\rangle)^-| \\
&= \max(|d|, |(Ea\langle a_i\rangle)^-|, |(Eb\langle a_i\rangle)^-|) + 1 \\
&\le \max(|d|, (\sup_i |a_i|) + |a| + 1, (\sup_i |a_i|) + |b| + 1) + 1 \quad \text{by ind. hyp.} \\
&\le (\sup_i |a_i|) + \max(|d|+1, |a|+1, |b|+1) + 1 \\
&= (\sup_i |a_i|) + |D_{uv}dab| + 1
\end{aligned}
$$

<u>Step 2</u> Let again a derivation $d \vdash r \in X$ be given, as described above. We shall define a new derivation $d^\pi \vdash r^\pi \in X$ which has no permutatively convertible subderivations, and such that $\vdash r = r^\pi \in X$ and $|d^\pi| \le 3^{|d|+1}$.

The definition of d^π is by recursion on the length of d. First apply $^\pi$ to the immediate subderivations of d . Then apply the last rule of d again. If the last rule of d is not an elimination rule, then this is already d^π . If it is, then apply $^-$ of step 1 to obtain d^π . So we have, for instance

$$
\begin{aligned}
(Ed\langle a_i\rangle)^\pi &:\equiv (Ed^\pi\langle a_i^\pi\rangle)^- \\
(db)^\pi &:\equiv (d^\pi b^\pi)^- \\
(\lambda xa)^\pi &: \lambda xa^\pi
\end{aligned}
$$

It is clear by the definition that d^π contains no permutatively convertible subderivations.

We now show that $\vdash r = r^{\pi} \in X$, by induction on the length of the given derivation $d \vdash r \in X$. For instance, in the case that the last rule is N^{-} recursion, we have to show

$$Et\langle r_i\rangle = (Et^{\pi}\langle r_i^{\pi}\rangle)^{-} \in X .$$

Now we have $t = t^{\pi} \in X$ and $r_i = r_i^{\pi} \in X$ by induction hypothesis. Using the fact that, generally, $s = s^{-} \in X$, we obtain the reqzired equation. The other cases are treated similarly.

Finally, we have to verify that $|d^{\pi}| < 3^{|d|+1}$. Let us look at some typical cases.

$$\begin{aligned}
&|(Ed\langle a_i\rangle)^{\pi}| \\
&= |(Ed^{\pi}\langle a_i^{\pi}\rangle)^{-}| \\
&\leqslant (\sup_i |a_i^{\pi}|) + |d^{\pi}| + 1 && \text{by step 1} \\
&\leqslant (\sup_i 3^{|a_i|+1}) + 3^{|d|+1} + 1 && \text{by induction hypothesis} \\
&= 3^{\sup_i(|a_i|+1)} + 3^{|d|+1} + 1 \\
&\leqslant 3^{\sup(|d|+1,\ldots\ldots |a_i|+1\ldots\ldots)} \cdot 3 \\
&= 3^{|Ed\langle a_i\rangle|+1}
\end{aligned}$$

$$\begin{aligned}
|(db)^{\pi}| &= |(d^{\pi}b^{\pi})^{-}| \\
&\leqslant |b^{\pi}| + |d^{\pi}| + 1 && \text{by step 1} \\
&\leqslant 3^{|b|+1} + 3^{|d|+1} + 1 && \text{by induction hypothesis} \\
&\leqslant 3^{\max(|d|+1,|b|+1)} \cdot 3 \\
&= 3^{|db|+1}
\end{aligned}$$

$$\begin{aligned}
|(\lambda xa)^{\pi}| &= |\lambda xa^{\pi}| \\
&= |a^{\pi}| + 1 \\
&\leqslant 3^{|a|+1} + 1 && \text{by induction hypothesis} \\
&\leqslant 3^{|\lambda xa|+1}
\end{aligned}$$

<u>Step 3</u> Now let a derivation $d \vdash r \in X$ be given which does not contain permutatively convertible subderivations. Otherwise d should be as described above; in particular, the rank of d should be finite, $\leqslant k + 1$ say. We shall define a new derivation $d' \vdash r' \in X$ of rank $Rd' \leqslant k$ such that $\vdash r = r' \in X$ and $|d'| \leqslant 2^{|d|}$.

It is obvious that from steps 2 and 3 the theorem will follow.

The definition of d' is by recursion on the length of d. It should be clear from the definition that $Rd' \leqslant k$. - <u>Case 1</u> The last rule in d is an elimination with main premiss a and side premisses $b_1, b_2, \ldots.$ We may assume that the last rule in a is not an assumption, for otherwise we can form d' from a and $b_1', \ldots, b_n'$ by the same elimination rule. <u>Case 1.1</u> The level La is $< k$. Let d' be obtained from a' and $b_1', b_2', \ldots$ by the same elimination rule. Since $La < k$ we have $Rd' \leqslant k$. <u>Case 1.2</u> $La = k$. The last rule of a can not be a critical elimination rule, since by assumption d does not contain permutatively convertible subderivations. <u>Case 1.2.1</u> The last rule of a is an introduction. Then the following cases are possible.

$a \equiv \lambda x a_o$, hence $d \equiv (\lambda x a_o)b$.
  Let $d' :\equiv (a_o')_x[b']$. Then $Rd' \leqslant k$, since $Lb < k$.

$a \equiv (a_o, a_1)$, hence $d \equiv p(a_o, a_1)$ or $q(a_o, a_1)$.
  Let $d' :\equiv a_o'$ or $d' :\equiv a_1'$.

$a \equiv i_Y a_o$ or $j_X a_o$, hence $d \equiv D_{uv}(i_Y a_o)bc$ or $D_{uv}(j_X a_o)bc$.
  Let $d' :\equiv b_u'[a_o']$ or $d' :\equiv c_v'[a_o']$.
  Then $Rd' \leqslant k$, since $La_o' < k$.

$a \equiv \underline{j} \in N$, hence $d = Ea\langle b_i\rangle$.
  Let $d' :\equiv b_j'$.

Case 1.2.2 The last rule of a is a non-critical elimination. Then its main premiss has a level $> k$. Since the rank of the whole derivation is $\leq k+1$, the last rule of this main premiss must again be a non-critical elimination, and so on. Finally one must end up with the introduction of an assumption $x \in X$ $(x \in X)$. To form d', apply the ' - operation to all side premisses of these elimination rules.

Case 1.3 $La > k$. Then the last rule in a must again be a non-critical elimination, and we can proceed as in case 1.2.2. Case 2 The last rule in d is an introduction with premisses $a_1, \ldots, a_n$. Let d' be formed from $a'_1, \ldots, a'_n$ by the same introduction rule.

Case 3 d consists of an introduction of an assumption $x \in X$ $(x \in X)$ only. Let $d' :\equiv d$. We now show that $\vdash r = r' \in X$, , by induction on the length of d. In all cases except 1.2.1 this follows immediately from the induction hypothesis and the compatibility rules for $=$. In case 1.2.1, let for instance, $d \equiv (\lambda x a_o)b$ be a derivation of $(\lambda x r)s \in X$. Then $d' \equiv (a'_o)_x[b']$ is a derivation of $r'_x[s'] \in X$. Now by induction hypothesis and $\overset{=}{\to}$ compatibility we have $(\lambda x r)s = (\lambda x r')s' \in X$, and by $\overset{=}{\to}$ proper conversion we have $(\lambda x r')s' = r'_x[s'] \in X$. The other subcases of case 1.2.1 are treated similarly.

Finally we show $|d'| \leq 2^{|d|}$, again by induction on the length of d. It suffices to look at case 1.2.1 with $d \equiv (\lambda x a_o)b$; then other cases are similar or trivial. We then have $d' \equiv (a'_o)_x[b']$ and hence

$$\begin{aligned} |d'| &\leq |b'| + |a'_o| \\ &\leq 2^{|b|} + 2^{|a_o|} \qquad \text{by induction hypothesis} \\ &\leq 2^{\max(|b|,|a_o|)+1} \\ &= 2^{|(\lambda x a_o)b|} \end{aligned}$$

References

[1] M. Beeson, Goodman's Theorem and Beyond. Pacific J. Math. 84 (1979), 1 - 16.

[2] M. Beeson, Recursive Models for Constructive Set Theories. Preprint Nr. 179, November 1980, Dept. of Mathematics, University of Utrecht.

[3] M. Beeson, Foundations of Constructive Mathematics: Metamathematical Studies. Springer, to appear.

[4] J. Diller, Modified Realisation on the Formulae - as - Types Notion. In: To H.B. Curry; Essays on Combinatory Logic, Lambda Calculus and Formalism, edited by J.P. Seldin and J.R. Hindley. Academic Press, New York 1980, 491 - 501.

[5] J. Diller and A.S. Troelstra, Realizability and Intuitionistic Logic. Report 82 - 12, June 1982, Math. Inst., University of Amsterdam.

[6] P. Martin - Löf, An Intuitionistic Theory of Types: Predicative part. In: Logic Colloquium '73, edited by H.E. Rose and J.C. Shepherdson. North-Holland, Amsterdam 1975, 73 - 118.

[7] P. Martin - Löf, Constructive Mathematics and Computer Programming. In: Logic, Methodology, and the Philosophy of Science VI, Proceedings of the International Congress in Hannover 1979, to appear.

[8] H. Schwichtenberg, Proof Theory: Some Applications of Cut-Elimination. In: Handbook of Mathematical Logic, edited by J. Barwise. North-Holland, Amsterdam 1977, 867 - 895.

[9] W.W. Tait, Infinitely Long Terms of Transfinite Type. In: Formal Systems and Recursive Foundations, edited by J. Crossley and M. Dummett. North-Holland, Amsterdam 1965, 176 - 185.