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AN EXISTENCE THEOREM FOR RECURSION CATEGORIES

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The notion of a recursion category was introduced ([1], [2]) as an easily characterized and rather general sort of object exhibiting the principal characteristics of classical recursion theory. Examples were adduced to show that these characteristics were to be found in settings with properties quite different from those of the classical theory. These examples however were in general obtained by modifying the classical one, leaving the possibility that the increased generality was, after all, just a matter of inessential variation.

The case, however, is quite otherwise. Our principal result here asserts that within suitable dominical categories are to be found large families of recursion categories. We shall describe here the constructions that lead to this result and point out some examples which illustrate how far we have come from the classical one. Detailed arguments will appear elsewhere.

We assume familiarity with the basic notions relating to dominical categories and recursion categories as set forth in [1] .

1. lwrΣ-dominical categories; isotypes

The categories in which we shall construct recursion subcategories are dominical categories with a number of additional properties:

-- they have binary coproducts, over which the near-product distributes, making them + -dominical;

-- their morphisms have ranges, respected by the near-product, and are thus r-dominical;

-- they satisfy the weak axiom of choice, which asserts that partial monomorphisms have cross-sections, and are accordingly w-dominical.

Those satisfying all of these conditions are wr+ -dominical. We shall want to consider two additional properties:

-- they have, in addition to + , countably infinite coproducts $\Sigma_n X_n = X_0 + X_1 + \cdots$, over which the near-product distributes, such categories being Σ-dominical;

-- they are locally indecomposable, making them l-dominical.

The last property is defined as follows. A domain $\varepsilon \in \mathrm{Dom}X$ is decomposable if $\varepsilon = \varepsilon' \cup \varepsilon''$, $\varepsilon' \cap \varepsilon'' = 0$, $\varepsilon' \neq 0 \neq \varepsilon''$. A dominical category is locally indecomposable (l-dominical) if every domain is the supremum of its indecomposable subdomains.

A dominical category satisfying all of these conditions is lwrΣ-dominical.

Recursion categories are, inter alia, categories in which any two objects are isomorphic. In ([1] , [2]) such categories were referred to, following a precedent, as "semigroupoids." This terminology now seems unfortunate. We shall, rather, call them isotypical categories or isotypes.

If a Σ-dominical category $\mathbb{C}$ contains an $X \approx X \times X$, a fortiori, if $\mathbb{C}_T$ has a terminal object 1, we shall see that $\mathbb{C}$ contains many isotypical dominical subcategories. If $\mathbb{C}$ is lwrΣ-dominical we shall see that many among these are recursion categories.

The conditions we have just enumerated may not seem familiar. They are nonetheless satisfied by many familiar categories, whose variety may be suggested by the following examples:

-- the category PCat of small categories, having as morphisms functors defined on subcategories

-- for any small category $\mathbb{K}$ the category $\mathrm{PSets}^{\mathbb{K}}$ of functors $\mathbb{K} \longrightarrow \mathrm{Sets}$ and natural transformations defined on subfunctors

-- for any field k the category Pcoalg_k of commutative coalgebras over k and coalgebra homomorphisms defined on subcoalgebras.

2. $B^{\#}$- categories

The structure of an lwrΣ-dominical category $\mathbb{C}$ is only that of a dominical category: the prefix refers to properties. A key to our construction is the conversion of some of these properties into structure.

The near product is already structural. It consists in a bifunctor $\times$: $\mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$, together with natural transformations p_0, p_1 , the projections, and Δ , the diagonal, on $\mathbb{C}_T$, the category of total morphisms, having some additional naturality properties on $\mathbb{C}$. The binary coproduct, we now demand, is to be a bifunctor $+$: $\mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ together with natural transformations i_0 , i_1 , the injections, and ∇, the codiagonal, on $\mathbb{C}$.

The distributivity is characterized by yet another piece of structure, viz. a natural transformation d on $\mathbb{C}_T$ together with the two equations,

$$d(X\times i_0 \quad X\times i_1) = \mathrm{id}\ , \quad (X\times i_0 \quad X\times i_1)d = \mathrm{id}$$

where of course

$$(X\times i_0 \quad X\times i_1) = \nabla(X\times i_0+X\times i_1):X\times Y+X\times Z \longrightarrow X\times(Y+Z) \quad .$$

We shall say that $\mathbb{C}_T$, provided with this structure, viz. the bifunctors $\times$, $+$ and the seven natural transformations $p_0, \cdots, d$ is a B-category. To this we shall want to add additional structure expressing not the existence of the infinite coproduct Σ but rather a consequence of this, that of the free semigroup $X^{\#} = X+X\times X+ \cdots$. This gives us an additional functor $\#: \mathbb{C} \longrightarrow \mathbb{C}$, restricting to $\mathbb{C}_T$, some, but not all, of whose properties are to be expressed by equations involving seven additional natural transformations on $\mathbb{C}_T$.

Briefly, these are the following:

-- the associative multiplication m of $X^{\#}$;

-- the injection j of X in $X^{\#}$;

-- the homomorphism "removal of parentheses" $e:X^{\#\#}\longrightarrow X^{\#}$;

-- the isomorphisms $X^{\#} \approx X+X\times X^{\#}$, $X^{\#} \approx X^{\#}\times X+X$, assured by natural transformations l, r ;

-- the "diagonal parametrization" par: $X\times Y^{\#}\longrightarrow (X\times Y)^{\#}$, which satisfies $p_1{}^{\#}\mathrm{par} = p_1: X\times Y^{\#}\longrightarrow Y^{\#}$;

-- the isomorphism $(X+Y)^{\#} \approx X^{\#}+Y^{\#}+(X^{\#}\times Y^{\#})^{\#}+(X^{\#}\times Y^{\#})^{\#}\times X^{\#}$ $+Y^{\#}\times (X^{\#}\times Y^{\#})^{\#}+Y^{\#}\times (X^{\#}\times Y^{\#})^{\#}\times X^{\#}$, expressed by w.

The last is familiar as the partition of words in X+Y in terms of their first and last letters. The equations relating these may be deduced from their brief descriptions.

A category supplied with all three functors $\times$, $+$, $\#$, and these fourteen structure transformations $p_0, \ldots$, w is a $B^{\#}$-category. A

subcategory closed under all this structure is a $B^{\#}$-subcategory. In such a subcategory $\times$ and $+$ are again product and coproduct, but $\#$, although its values are semigroups, need not have free semigroups as values, since the properties guaranteed by the structure m, ... , w do not assure the freeness of $X^{\#}$, that is to say the adjointness of $\#$ to the forgetful functor. We shall describe them rather as formally-free semigroups and say that a $B^{\#}$-category has a formally-free semigroup structure.

3. Uniformly generated $B^{\#}$-isotypes

An object X of a $B^{\#}$-category $\mathbb{B}$ is isotypical if $X \approx X\times X \approx X+X \approx X^{\#}$. For such an X the category $\mathbb{B}[X]$, the smallest full subcategory of $\mathbb{B}$ containing X and closed under $\times$, $+$, $\#$, is an isotypical $B^{\#}$-subcategory of $\mathbb{B}$. With regard to the existence of isotypical X we may make the following remarks. Suppose the functor $\#$ in $\mathbb{B}$ comes from an infinite coproduct, as in the case $\mathbb{B} = \mathbb{C}_T$, where $\mathbb{C}$ is Σ-dominical. Then

(i) If $\mathbb{B}$ has a terminal object 1 and $X \approx X+X$ (e.g. if X is any countable copower $Y+Y+ \cdots$) then $1+X+X^2+ \cdots$ is isotypical;

(ii) If $X \approx X\times X$ then the countable copower $X+X+ \cdots$ is isotypical.

Isotypical objects, in other words, are not uncommon.

Now, inasmuch as a $B^{\#}$-category $\mathbb{B}$ is characterized by its structure, it is clear that any class W of morphisms in $\mathbb{B}$ is contained in a smaller $B^{\#}$-subcategory $B^{\#}W$, the $B^{\#}$-subcategory generated by W.

By a frame b at an object X of $\mathbb{B}$ we mean a family of isomorphisms

$$b_\times : X \longrightarrow X\times X\ ,\quad b_+ : X \longrightarrow X+X\ ,\quad b_\# : X \longrightarrow X^\#$$

together with their inverses. If, now, W is a set of morphisms in $\mathbb{B}[X]$ then $B^\#(b\cup W)$ is an isotypical $B^\#$-subcategory of $\mathbb{B}[X]$. But in fact we might as well consider subsets $W \subset \mathbb{B}(X,X)$ alone, since $B^\# b$ is already isotypical. $B^\# b$ is of course the isotypical $B^\#$-subcategory (rel b) with 0 generators; if W is finite then $B^\#(b\cup W)$ is finitely generated. A moment's thought, however, shows that then $B^\#(b\cup W) = B^\#(b\cup\{f\})$ for some $f:X\longrightarrow X$.

We shall generalize this notion of finite generation. If $t:X\times X\longrightarrow X$ a morphism $g:X\longrightarrow X$ is an index (rel t) of $f:X\longrightarrow X$ if $fp_1 = t(g\times X)$. If f has an index (rel t) we shall say that t lists f. The set $L_{\mathbb{B}}t$ of such f is the uniform list of t.

A $B^\#$-subcategory $\mathbb{D}$ of $\mathbb{B}$ is uniformly generated if $\mathbb{D} = B^\#(b\cup L_{\mathbb{D}}t)$ for some t in $\mathbb{D}$. It is easy to see that any finitely generated $\mathbb{D}$ is uniformly generated. The converse in general is false: a finitely generated $\mathbb{D}$ is countable; uniformly generated ones need not be. It is also easy to see that the set of uniformly generated $\mathbb{D}$ is directed by inclusion. Furthermore, for fixed t the set of solutions of the equation $\mathbb{D} = B^\#(b\cup L_{\mathbb{D}}t)$ is evidently closed under union, so that there is a largest $B^\#$-subcategory uniformly generated via t.

4. Gödelian categories, Turing completion and the main theorem

An lwr+-dominical category $\mathbb{C}$ in which $\mathbb{C}_T$ has further the structure of a $B^\#$-category is a Gödelian category. Thus any

lwr -dominical category can be given the structure of a Gödelian category.

Lemma 4.1: (Iteration lemma) If $\mathbb{C}$ is a Gödelian category, $f:X\rightarrow X$, $\varepsilon\in \mathrm{Dom}X$ and $f\varepsilon = \varepsilon$ then the supremum $\bigcup_n f^{n*}\varepsilon$ exists and there is a unique morphism $It(f,\varepsilon):X\longrightarrow X$ such that $\mathrm{dom}\, It(f,\varepsilon) = \bigcup_n f^{n*}\varepsilon$ and $It(f,\varepsilon)(f^{n*}\varepsilon) = f^m(f^{n*}\varepsilon)$, $m\geq n$.

This permits us to construct, in a Gödelian category, a kind of analogue of a Turing machine. A Turing datum in $\mathbb{C}$ is a diagram

$$X \xrightarrow{\ u\ } W \xrightarrow{\ v\ } W+Y$$

in $\mathbb{C}_T$. If we set $\tilde f = (f\ \ i_1):W+Y\longrightarrow W+Y$ then $\tilde f(0+Y) = 0+Y\in \mathrm{Dom}(W+Y)$. The Turing development $Tur(u,v)$ of the Turing datum (u,v) is the composition

$$X \xrightarrow{\ i_0u\ } W+Y \xrightarrow{\ It(\tilde f,0+Y)\ } W+Y \xrightarrow{\ (0\ \ Y)\ } Y\ .$$

Lemma 4.3: If $\mathbb{C}$ is Gödelian and $\mathbb{B}$ is a $B^\#$-subcategory of $\mathbb{C}_T$, and if $Tur\,\mathbb{B}$ is the class of all $Tur(u,v)$, where (u,v) is a Turing datum in $\mathbb{B}$, then $Tur\,\mathbb{B}$ is a +-dominical subcategory of $\mathbb{C}$ containing $\mathbb{B}$. Furthermore $(Tur\,\mathbb{B})_T = Tur\,\mathbb{B}\cap\mathbb{C}_T$ and $Tur(Tur\,\mathbb{B})_T = Tur\,\mathbb{B}$.

We shall say that $Tur\,\mathbb{B}$ is the Turing completion of $\mathbb{B}$.

We can now state our main theorem.

Theorem 4.3: If $\mathbb{C}$ is a Gödelian category and $\mathbb{B}$ is a uniformly generated isotypical $B^\#$-subcategory of $\mathbb{C}_T$ then $Tur\,\mathbb{B}$ is a recursion category.

The idea of the proof is simple enough, although the details are somewhat delicate. By hypothesis $\mathbb{B} = B^\#(b\cup L_{\mathbb{B}}t)$ for some frame b at X and $t:X\times X\longrightarrow X$ in $\mathbb{B}$. Using t one constructs explicitly a Turing datum (u, v) such that Tur(u, v) yields a Turing morphism for Tur($\mathbb{B}$).

5. Remarks, examples and open questions

(1) The Category P of sets and partial maps is lwrΣ-dominical and thus has a Gödelian structure. If b is the standard frame at $\mathbb{N}$, i.e. $b_{\times}: \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ is the usual pairing function, $b_{+}: \mathbb{N} \longrightarrow \mathbb{N} + \mathbb{N}$ is the "odd-even" map and $b_{\#}: \mathbb{N} \longrightarrow \mathbb{N}^{\#}$ is Gödel's β then, clearly, for some f, $\mathrm{TurB}^{\#}(b \cup \{f\})$ is equivalent to the standard recursion theory $\mathbb{T}$. Is Tur $\mathbb{B}^{\#}b$, the recursion theory on 0 generators, equal to $\mathrm{TurB}^{\#}(b \cup \{f\})$?

(2) In P let X by any infinite set and choose any frame b at X. If $\mathbb{B} = B^{\#}(b \cup L_{\mathbb{B}}p_0)$ is the largest $B^{\#}$-subcategory uniformly generated by p_0 then $\mathbb{B}$ contains all constants. Thus Tur $\mathbb{B}$ is a recursion category of cardinality as large as that of X. Can it be larger?

(3) For any frame b at x in a Gödelian category $\mathbb{C}$ a preordering on $\mathbb{C}_T(X,X)$ is defined by setting f<g if $\mathrm{TurB}^{\#}(b \cup \{f\}) \subset \mathrm{TurB}^{\#}(b \cup \{g\})$. How is this related to Turing reducibility?

(4) In P(Sets$^{\Delta^{op}}$), the category of simplicial sets and partial simplicial maps, let X be an infinite power of a nontrivial connected simplicial set and let Y = X+X+ $\cdots$. Then Y is isotypical and if $\mathbb{B}$ is any uniformly generated isotypical $B^{\#}$-subcategory of Sets$^{\Delta^{op}}$[Y] then Tur $\mathbb{B}$ is a recursion category.

Now for any total $f: Y \longrightarrow Y$, if ε DomY is a union of connected components then so also is $\overset{*}{f}\varepsilon$. If $f\varepsilon = \varepsilon$ then also $\mathrm{domIt}(f,\varepsilon) = \bigcup_n f^{n*}\varepsilon$ is a union of components. It follows that any domain in Tur $\mathbb{B}$ has again the same property. Thus Tur $\mathbb{B}$ is not r-dominical, since, for example, $\Delta: Y \longrightarrow Y \times Y$ lacks a range.

This appears to be the first example of a recursion category which fails to be r-dominical. In the classical theory it is a matter of indifference whether r.e. sets or partial-recursion functions are taken as the fundamental notion. This example demonstrates that the functions ought to have priority.

References

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2 R. DiPaola and A. Heller, Dominical categories, J. Symbolic Logic (in press).