Game semantics for constructive modal logic.

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Semantics is the area of logic concerned with specifying the meaning of the logical constructs. We distinguish between two main kind of semantic approaches to logic. The first, the model-theoretic approach, is concerned with specifying the meaning of formulas in terms of truth in some model. The second, the denotational semantic approach, is concerned with specifying the meaning of proofs of the logic under a compositional point of view. Proofs are interpreted as mathematical objects called denotation, and the meaning of composed proofs is obtained by composing denotations. One of the desired features of denotational models is full completeness: in a fully complete model, every morphism is the interpretation of some proof. Reasoning about the property of full complete models allows one to have a syntax-free characterization of the property of proofs. We say that a denotational model is *concrete* if its elements are not obtained by the quotient on proofs induced by cut-elimination. Game semantics [6, 5, 1, 2] is a form of denotational semantics in which proofs are interpreted as winning strategies for two player games.

In this presentation, we focus on denotational semantics for modal logics. Modal logics are, traditionally, an extension of *classical logic* making use of unary connectives, called *modalities*, that qualify the truth of a judgement. More precisely, modal logics are obtained by extending classical logic with a modality operator $\Box$ (together with its dual operator $\Diamond$), which are usually interpreted as *necessity* (respectively *possibility*).

Beginning with Simpson's work [10], intuitionistic and constructive modal logics have aroused growing interest. In particular, during the last three decades the proof theory of constructive modal logics has been developed considerably providing proof systems by means of sequent calculi [8, 11], natural deduction and λ-calculus [9, 7, 4].

The subject of our talk will be the basic constructive modal logic: the constructive version of the modal logic K (called CK) [4]. The formulas of CK are written using the connectives $\supset$ and $\wedge$ and the modalities $\Box$ and $\Diamond$. A complete sequent calculus system for this logic is obtained by adding the following two rules to a standard sequent calculus system for minimal logic

$$\frac{A_1, \ldots, A_n \vdash C}{\Box A_1, \ldots, \Box A_n \vdash \Box C}\ \mathsf{K}^{\Box} \qquad\qquad \frac{A_1, \ldots, A_n, B \vdash C}{\Box A_1, \ldots, \Box A_n, \Diamond B \vdash \Diamond C}\ \mathsf{K}^{\Diamond}$$

In particular, we present a concrete denotational semantics for CK (introduced in [3]). Our semantics is a game semantics. We present winning strategies that correspond to proofs of CK, we show that our winning strategies can be composed, and that —furthermore— our semantics is fully complete: each modal winning strategy is the interpretation of some sequent calculus proof.

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