The compatibility of the Minimalist Foundations with Homotopy Type Theory

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Homotopy Type Theory (HoTT) [7] is a well-established foundational framework for constructive mathematics with important applications both in mathematics and in computer science. It is an extension of Martin-Löf's type theory with Voevodsky's Univalence Axiom implying the remarkable property that 'isomorphic' sets are propositionally equal. One of the main problems regarding HoTT has been to find a *computational* interpretation proving that its proofs are terminating programs as it is the case for Martin-Löf's Type Theory. The most promising proposal for solving this problem is the interpretation of HoTT in a cubical type theory [1], which recently has been shown to enjoy a normalization result [6].

Our long-term research aim is to understand how this computational interpretation can be adapted to the Minimalist Foundation (MF) and its extensions. MF, first conceived in [5] and then fully formalized in [3], is a two-level type theory consisting of an intensional level, called mTT, an extensional one, called emTT, and an interpretation of the latter in the first. It was designed to be interpretable in the most relevant existing constructive foundations by preserving the meaning of logical and set-theoretical constructors. Therefore, MF enjoys a lot of different interpretations, including the computational interpretations given in [2] and [4].

As a first step of our research program, we want to show that HoTT provides a sufficiently rich setting to interpret both levels of MF. This must be contrasted with what happens in intensional Martin-Löf Type Theory where we can give an interpretation only of the intensional level of MF, as it has been shown in [3].

The new machinery introduced in the context of HoTT, in particular the hierarchy of homotopy levels, the Univalence axiom and the rules for higher inductive types (e.g. quotients), allows us to interpret both the levels of MF by preserving the meaning of all logical and set-theoretical constructors. Therefore, the distinction between propositions and set-constructors that we have in the MF-syntax is preserved by the translation contrary to that in the intensional version of Martin-Löf Type Theory.

The main difficult encountered in pursuing our goal has been to interpret in HoTT the extensional level emTT of MF because emTT-definitional equalities among its types and terms are significantly stronger than those valid in HoTT. We have managed to solve this problem by employing a technique already used in [3] to interpret emTT over the intensional level of MF by interpreting emTT- types and terms into HoTT- types and terms up to a special class of isomorphisms, called *canonical*, by providing a sort of realizability interpretation.

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The resulting interpretation of emTT into HoTT is simpler than that of emTT within mTT in [3] thanks to the presence of higher inductive quotients and the Univalence axiom within HoTT which let us avoid any quotient model construction.

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