On Indeterminism

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There are arguments, like the so-called Church's and Türing's theorems, which limit the expressive power of the first order calculus by asserting its undecidability. Taking any first order formula we cannot always decide whether it is true or false. An upstream idea underlies these arguments seeing like an impasse in the existence of formulas that are neither true nor false. See for example the well-known words " ... there are propositions which are neither true nor false but *indeterminate*" [4, 126]. As well-known *indeterminacy* led Lukasiewicz not only to reject the principle of bivalence and to introduce a third truth value, but also to the necessity of formalizing rejection. Gödel numbering is an arithmetization of syntax which codes provability as a primitive recursive predicate, $Pf(x, v)$ [5, 190-198]. All around this well-known recursive predicate are today widespread a multiplicity of researches and results in many areas of logic and AI. Not equally investigated is the refutability predicate defined by Gödel numbering within the same primitive recursive status. $Rf(x, v)$ can be defined as a recursive predicate meaning that x is the Gödel number of a refutation in *PA* of the formula with Gödel number v [1, 2]. The logical investigation of the links between provability and refutability predicates when defined within the same recursive status seems to display another *way out* from the *indeterminacy*.

Lemma 1. *For any natural number n and for any formula α not both $Rf(n, \ulcorner\alpha\urcorner)$ and $Pf(n, \ulcorner\alpha\urcorner)$.*

As $Pf(x, v)$ and $Rf(x, v)$ are primitive recursive then also their characteristic functions C_{Pf} and C_{Rf} are primitive recursive. Therefore from Lemma 1 we have the following.

Lemma 2. *For any formula α, and n as the Gödel number of a proof in PA of α*

$$\vdash_{PA} \ C_{Pf}(\overline{n}, \overline{\ulcorner\alpha\urcorner}) = \overline{0} \ \wedge \ C_{Rf}(\overline{n}, \overline{\ulcorner\alpha\urcorner}) = \overline{1}$$

Lemma 3. *For any formula α, and n as the Gödel number of a refutation in PA of α*

$$\vdash_{PA} \ C_{Rf}(\overline{n}, \overline{\ulcorner\alpha\urcorner}) = \overline{0} \ \wedge \ C_{Pf}(\overline{n}, \overline{\ulcorner\alpha\urcorner}) = \overline{1}$$

Lemma 4. *For any formula α*

(I) *not both* $\vdash_{PA} \ Pf(\overline{n}, \overline{\ulcorner\alpha\urcorner}) \ \vdash_{PA} \ Rf(\overline{n}, \overline{\ulcorner\alpha\urcorner})$,

(II) *for n as the Gödel number of a refutation in PA of α*

$$\vdash_{PA} \ Rf(\overline{n}, \overline{\ulcorner\alpha\urcorner}) \Longleftrightarrow \neg Pf(\overline{n}, \overline{\ulcorner\alpha\urcorner}),$$

(III) *for n as the Gödel number of a proof in PA of α*

$$\vdash_{PA} \ Pf(\overline{n}, \overline{\ulcorner\alpha\urcorner}) \Longleftrightarrow \neg Rf(\overline{n}, \overline{\ulcorner\alpha\urcorner}).$$

These Lemmas are clarifying and open new perspectives to the incompleteness argument in *PA* [5, 203-204; 206][1, 2]. Applying an unlimited existential quantifier to a predicate whose kernel is recursive yields a recursively enumerable predicate, r.e. predicate, and on the other side every r.e. predicate can be represented in this way. Pf is a primitive recursive predicate then by its characteristic function C_{Pf}, $<\overline{n},\overline{\ulcorner\alpha\urcorner}> \in Pf \leftrightarrow C_{Pf}(\overline{n},\overline{\ulcorner\alpha\urcorner}) = \overline{0}$. Being Pf decidable we can decide whether n is the Gödel number of a proof in *PA* of α or not, namely whether $\vdash_{PA} \alpha$ or $\nvdash_{PA} \alpha$. Respectively, Rf is a primitive recursive predicate then by its characteristic function C_{Rf} $<\overline{n},\overline{\ulcorner\alpha\urcorner}> \in Rf \leftrightarrow C_{Rf}(\overline{n},\overline{\ulcorner\alpha\urcorner}) = 0$, and since Rf is decidable we can decide whether n is the is the Gödel number of a refutation in *PA* of α or not, i.e. whether $\vdash_{PA} \neg\alpha$ or $\nvdash_{PA} \neg\alpha$. But when we state $\exists x Pf(x,\overline{\ulcorner\alpha\urcorner})$ we make an assumption of existence, namely of a r. e. predicate, that can be represented as [3, xxiv]

$$i) \quad Pf(0,\overline{\ulcorner\alpha\urcorner}) \vee Pf(1,\overline{\ulcorner\alpha\urcorner}) \vee Pf(2,\overline{\ulcorner\alpha\urcorner}) \vee \dots .$$

In other terms, we have a computable function f which lists all the Gödel numbers of proofs in *PA* yielding the set of all the ordered couples $<f(o),\overline{\ulcorner\alpha\urcorner}>, <f(1),\overline{\ulcorner\alpha\urcorner}>, <f(2),\overline{\ulcorner\alpha\urcorner}> \dots$ till to reach that $\overline{n}$ such that effectively $<\overline{n},\overline{\ulcorner\alpha\urcorner}> \in Pf$. Being $\exists x Pf(x,\overline{\ulcorner\alpha\urcorner})$ only r.e. we can generate the Gödel number of a proof in *PA* of α, i.e. $\vdash_{PA} \alpha$, but we could not be able to equally obtain $\nvdash_{PA} \alpha$. Similarly for Rf, with $\exists z Rf(z,\overline{\ulcorner\alpha\urcorner})$ we have a r.e. predicate, such that

$$ii) \quad Rf(0,\overline{\ulcorner\alpha\urcorner}) \vee Rf(1,\overline{\ulcorner\alpha\urcorner}) \vee Rf(2,\overline{\ulcorner\alpha\urcorner}) \vee \dots ,$$

and we can reach $\vdash_{PA} \neg\alpha$, but we could not be able to generate a derivation such that $\nvdash_{PA} \neg\alpha$. Let us consider the well-known "Bew(x)" of Gödel's 1931, $\exists y Pf(y,x)$ in our notation, which assumes the existence of a proof y of x, in the light of our lemmas. There is no doubt that "*n is the Gödel number of a proof in PA of* α" can be considered an example of $\exists y Pf(y,x)$, so that we obtain $Pf(\overline{n},\overline{\ulcorner\alpha\urcorner})$. Accordingly by Lemma (4)(III) $\vdash_{PA}$ $Pf(\overline{n},\overline{\ulcorner\alpha\urcorner}) \Longleftrightarrow \neg Rf(\overline{n},\overline{\ulcorner\alpha\urcorner})$, and from the viewpoint of the effective computability this is equivalent to generating from $i)$

$$iii) \quad \neg Rf(0,\overline{\ulcorner\alpha\urcorner}) \vee \neg Rf(1,\overline{\ulcorner\alpha\urcorner}) \vee \neg Rf(2,\overline{\ulcorner\alpha\urcorner}) \vee \dots .$$

Hence we can attain both $\vdash_{PA} \alpha$ and $\nvdash_{PA} \neg\alpha$. On the other side, for "*n is the Gödel number of a refutation in PA of* α" we can obtain $Rf(\overline{n},\overline{\ulcorner\alpha\urcorner})$. Accordingly by Lemma (4)(II) we have $\vdash_{PA}$ $Rf(\overline{n},\overline{\ulcorner\alpha\urcorner}) \Longleftrightarrow \neg Pf(\overline{n},\overline{\ulcorner\alpha\urcorner})$, from $ii)$

$$iv) \quad \neg Pf(0,\overline{\ulcorner\alpha\urcorner}) \vee \neg Pf(1,\overline{\ulcorner\alpha\urcorner}) \vee \neg Pf(2,\overline{\ulcorner\alpha\urcorner}) \vee \dots ,$$

and we can obtain both $\nvdash_{PA} \alpha$ and $\vdash_{PA} \neg\alpha$. We observe that, being the predicate Rf recursively defined, as much as Pf is, it comes to our aid with respect to Bew(x) and the incompleteness of PA. This is something at all in twilights within Gödel's incompleteness argument. Moreover, the recursive definition of refutability highlights how, the assumption of the existence of a proof y when stating $\exists y Pf(y,x)$, i.e. "Bew(x)", is precisely a codification or formal expression of the *indeterminacy*. Indeterminacy, that in the light of the Lemma (4) has no real reason to exist. As we showed, all the four possible cases $\vdash \alpha, \nvdash \alpha$ and $\vdash \neg\alpha$, $\nvdash \neg\alpha$ are now ruled in PA. Thus even after an assertion of existence of a proof or a refutation, like $\exists y Pf(y,x)$ or $\exists z Rf(z,v)$, *indeterminacy* has no reason to be.

References

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