The deductive system of a 2-category

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Abstract

We introduce judgemental theories and their calculi as a general framework to present and study deductive systems. As an exemplification of their expressivity, we encode both dependent type theory and the calculus of natural deduction as special kinds of judgemental theories. Our efforts allow for new definitions to be given, new relations to be explored, and old results to be recovered.

The purpose of this talk is to present a category-based unified approach that accommodates diverse takes on the topic of *deduction*. One of the motivating examples is to give a theoretical framework in which the two following rules, which stand on very conceptually different grounds, can be compared.

$$\text{(Subs)}\ \frac{\Gamma \vdash a : A \qquad \Gamma.A \vdash B}{\Gamma \vdash B[a]} \qquad\qquad \text{(Cut)}\ \frac{x; \Gamma \vdash \phi \qquad x; \Gamma, \phi \vdash \psi}{x; \Gamma \vdash \psi}$$

One can traditionally be found in type theory [MLS84], the other in proof theory [TS00]: despite their incredibly similar look, and the somehow parallel development of the respective theories in the same notational framework, there are some philosophical differences between the interpretation of the symbols above. Not only that, but the same "⊢" symbol seems to regard only statements of one kind `formula` in the case of (Cut), while it pertains to two - `term` and `type` - in that of (Subs).

Of course one could argue that these different points of view are mostly philosophical, and, in particular, the deep connection between proof theory and type theory has been studied for a while: its development falls under the paradigm that is mostly known as *propositions-as-types*, which [Wad15] gives a thorough presentation of. We believe our theory gives testament to that and, in fact, it gives it a categorical backbone.

Rebooting some ideas from [Jac99], we develop what we call *judgemental theories*. Going back to the example of (Subs) and (Cut), we intuitively see how they both fit the same paradigm, in the sense that we could read both as instances of the following syntactic string of symbols

$$(\triangle)\ \frac{\heartsuit \vdash \blacksquare \qquad \square \vdash \clubsuit}{\heartsuit \vdash \spadesuit}$$

which we usually parse as: *by* $\triangle$, *given* $\heartsuit \vdash \blacksquare$ *and* $\square \vdash \clubsuit$ *we deduce* $\heartsuit \vdash \spadesuit$. Our theory allows for a coherent categorical expression of all such strings of symbols, and shows how a suitable choice of *context* either produces (Subs) or (Cut): it is not about the interpretation of the symbols, just about the relation they are in with one another.

The deductive power of each system is coded by our theories being closed under finite limits, providing a framework that has both the advantage of being very versatile and computationally meaningful. It allows, for example, to give a (first) general definition of type constructor.

If the process of formalization of a given deductive system is purely syntactical, in the sense that we are not interested in what a given judgement or rule should *mean*, only in the symbols involved, the judgemental theory we obtain is often as well behaved as one hopes a categorical semantics would be: if we consider the case study of dependent types, traditional categorical models ([Tay99], [Dyb96], [Car86], [Awo18]) all fit into our paradigm, while for first order logic the judgemental theories we describe are an elaboration of doctrines as in introduced in [Law70] and later developed in [Mak93], [HJP80], [MR13].

Moreover, we discuss the relation between properties that are *internal* (such as modeling substitution) and those that are *external* (such as being a fibration), and see that our framework allows for some external properties to be suitably internalized: for example, this is the case for CE-systems, introduced in [AENR21] to extend contextual categories from [Car86], which we can show to be freely internally equipped with dependent sums.

Remarkably, when encoding rules such as (Subs) and (Cut), we notice that some comonads come into play. Since it would be an interesting development of our research, we hope to elaborate on that.

In summation, we hope to show that judgemental theories provide an interesting framework which can be used to both do calculations inside deductive systems and to compare different ones. The generality they are written in allows for new definitions to be given (such as that of a type constructor), new relations to be explored (such as that of "cut-like" rules), and old results to be recovered (such as cut elimination for natural deduction). The main results we will present are collected in [CDL22].

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