Algebras of Counterfactual Conditionals

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A *counterfactual conditional* (or simply a counterfactual) is a conditional statement of the form "If [antecedent] were the case, then [consequent] would be the case", where the antecedent is usually assumed to be false. Counterfactuals have been studied in different fields: for instance in the philosophy of language and in linguistics (e.g. [2] and [12]), in artificial intelligence (e.g. [3]), and philosophy (e.g. [8]). The logical analysis of counterfactuals is rooted in the work of Lewis [9, 7] and Stalnaker [13] who have introduced what has become the standard semantics for counterfactual conditionals based on particular Kripke models equipped with a similarity relation among the possible worlds.

In Lewis' language, a counterfactual is formalized as a formula of the kind "$\varphi \mathrel{\Box\!\!\rightarrow} \psi$" which is intended to mean that if φ were the case, then ψ would be the case. Lewis [9] has introduced different logics of counterfactuals arising from his semantics; these logics have been studied from a proof-theoretic perspective by, for instance, Negri and Sbardolini [10] and Lellman and Pattinson [6].

Although the research on counterfactuals and their logic has been prolific, a deep and coherent algebraic investigation of Lewis' logic of counterfactuals is, to the best of the authors' knowledge, still missing. Some steps towards this direction can be found in the work of Nute [11] and Weiss [14], however their approach only shows that (some of the) Lewis logics of counterfactuals are weakly complete with respect to some algebraic structures obtained from Boolean algebras by introducing a binary operator $\star$ that stands for the counterfactual conditional connective.

In the present work, we start filling this gap by providing an equivalent algebraic semantics, in the sense of Blok-Pigozzi [1], for the logics of *global* consequence associated to Lewis' systems **C0**, **C1** and **C2** introduced in [7]. These systems correspond to the systems V, VC, and VCS in [9]. It is worth mentioning that the system **C1** is, in Lewis' own opinion, the "correct logic of counterfactuals conditionals as we ordinarily understand them" (see [7, p.80]).

More precisely, in analogy with modal logics, we start by observing that, to each of Lewis' systems **Ci** (with $0 \leq \mathbf{i} \leq 2$), we can associate two logics, that is, the logic of *global* consequence and the logic of *local* consequence, which differ depending on how one specifies the rule of deduction within conditionals (DWC in [7, p. 80]). Then, for each system **Ci**, we define an associated class of Boolean algebras equipped with a binary operator $\Box\!\!\rightarrow$ that stands for the counterfactual connective. We show that each of these classes of can be axiomatized by means of equations, and is therefore a variety. We then prove completeness for $\mathbf{Ci}_l$ and $\mathbf{Ci}_g$ with respect to their associated class. In particular, it turns out that $\mathbf{Ci}_l$ is the logic preserving degrees of truth of the class of **Ci**-algebras, and that **Ci**-algebras provide an equivalent algebraic semantics for $\mathbf{Ci}_g$ with $\tau = \{x \approx 1\}$ and $\Delta = \{x \to y, y \to x\}$ witnessing the algebraizability of $\mathbf{Ci}_g$. As a consequence of algebraizability, we obtain that all axiomatic extensions of $\mathbf{Ci}_g$ are also algebraizable.

It is worth noticing that said structures do not belong to the framework of *Boolean algebras with operators*, as studied for instance by Jipsen in [5], since $\Box\!\!\rightarrow$ is not additive, in the sense that it does not preserve the Boolean disjunction on the left. Nonetheless, such algebras are well-behaved from the point of view of their structure theory, indeed they are ideal-determined with respect to 1 in the sense of [4]. Thus, congruences are characterized by their 1-blocks, which turn out to be particular lattice filters respecting a further condition involving $\Box\!\!\rightarrow$. Notably, we characterize the structure theory of these algebras in order to study the subdirectly irreducible and directly indecomposable algebras of such class via the description of *congruence elements*, that is, elements whose upset is a congruence filter.

We observe that, while the class of **Ci**-algebras are also an algebraic semantics with respect to $\mathbf{Ci}_l$, it is not the case that they provide an equivalent algebraic semantics for it. Indeed, it can be seen that the congruence filters of the **Ci**-algebras do not correspond to the deductive filters induced by the logic $\mathbf{Ci}_l$. This shows an interesting parallel with the modal logic case, where the class of modal algebras provide an equivalent algebraic semantics for the logic of global consequence $\mathbf{K}_g$, whereas the logic of local consequence $\mathbf{K}_l$, although being complete with respect to the class of modal algebras, is not algebraizable.

Now, it is important to stress that both the global and local consequences of **Ci** admit a possible worlds semantics: a Lewis' model for a **Ci**-logic consists in a tuple $\langle W, \mathcal{S}, v\rangle$ where $\mathcal{S} : W \to \wp(\wp(W))$ and $\mathcal{S}(w)$ is nested, i.e. for each $S, T \in \mathcal{S}(w)$, either $S \subseteq T$ or $T \subseteq S$. Depending on stronger constraints imposed on $\mathcal{S}$, Lewis defines models for each **Ci** system (e.g., **C1**-models are those in which $\mathcal{S}(w)$ is centered, i.e. $\{w\} \in \mathcal{S}(w)$). We provide a completeness result for local and global consequences with respect to Lewis' models, again in parallel with the case of modal logic, in which global an local consequence of **K** corresponds, respectively, to the global and local logical consequence relation over Kripke frames. Given these results, we then aim at studying the duality relations between our algebras of counterfactuals and Lewis' models.

Finally, using our framework, we analyse the connections between $\mathbf{Ci}_l$ and $\mathbf{Ci}_g$. In particular: we observe that $\mathbf{Ci}_l$ and $\mathbf{Ci}_g$ share the same theorems, but differ with respect to the logical consequences; while $\mathbf{Ci}_l$ has the deduction theorem with respect to the classical implication, $\mathbf{Ci}_g$ does not; we investigate whether global and local consequence can be characterized in terms of each other, as in the case of modal logic.

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