Some notes on the rebirth of mathematical logic in Italy

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Mathematical logic has in Giuseppe Peano one of the significant international representatives between the 19th and 20th centuries. When Peano's interests shift to other questions (interlingua), the studies of logic are abandoned in Italy, just at a time when international research in the field was making remarkable advances under the propulsive thrust of Godel's theorems.

In the 1960s, research in mathematical logic was reborn in Italy thanks to influential people like Ludovico Geymonat, Ettore Casari and Roberto Magari, who worked in a multidisciplinary setting grounded on the relationships between Logic, Mathematics and Philosophy.

In this talk we will analyze just a couple of research lines that originated from the first meetings of the CNR Logic group founded by Ludovico Geymonat in the early sixties. So a complete picture of the italian logic in the nineteenth century is far beyond the scope of our contribution.

Diagonalizable algebras. The first topic concerns a line of research conceived by Roberto Magari in the Siena research group during the 1970s. We refer to *diagonalizable algebras*, which are now called *Magari algebras* after the proposal of George Boolos after Magari's death.
In 1973 Magari began researches whose aim was to describe, in algebraic terms, the features that make Peano arithmetic (PA) self-referential. To pursue its aim, Magari considers the Lindenbaum Algebra of PA: a Boolean algebra whose elements are equivalence classes of formulas. We recall that the theorems of PA can be represented in PA by a formula $Th(x)$ whose meaning is *it exists a proof of the formula whose Gödel number is* x. The starting point in the construction of diagonalizable algebras is the introduction of the τ operator in order to translate the provability predicate $Th(x)$ in an algebraic framework. Recalling that every formula α of PA is encoded by a Gödel number $\ulcorner\alpha\urcorner$ (Gödelization), the τ operator is defined as follows: let $[\alpha]$ the equivalence class of all the formulas that are logically equivalent to α, then

$$\tau[p] = [Th(\ulcorner p\urcorner)]$$

Using the properties of $Th(x)$ it is possible to prove that τ is well defined and satisfies suitable properties such as $\tau(x \cdot y) = \tau x \cdot \tau y$ and $\tau x \leq \tau\tau x$.
The next step toward the diagonalizable algebras is the algebraic translation of the diagonalization lemma:

Lemma 1 (Diagonalization)**.** *Let $\alpha(x)$ be a formula of PA with exactly one free variable x. Then it exists a formula p such that:*

$$\vdash_{PA} p \leftrightarrow \alpha(\ulcorner p\urcorner)$$

The algebraic translation of the diagonalization lemma is accomplished by assuming that in a diagonalizable algebra every polynomial (i.e. every expression $f(x)$ built up starting from the constants 0, 1, using the operators $+, \cdot, '$ and τ) with a variable x whose occurrences are under the scope of τ, has a fixed point

$$(*) \qquad \text{for all } f(x) \text{ it exists } g_f \text{ such that } f(g_f) = g_f$$

The work of Magari and its group then focused on the study of the ties between (∗) and the properties of τ. This study was developed and elaborated by the young collaborators of Magari, in search of an adequate axiomatization to the point of highlighting the links with other fields of logic (modal logic) and international research.

The importance of the Magari algebraic approach lies in reaching the *right degree of abstraction*. This has the double effect of both leading to new results and casting new light on the topic of provability. The analysis of these works by the Magari research group will be an opportunity to recall the cultural environment of the Siena school, one of the most important and fruitful contexts for mathematical logic in Italy between the seventies and the eighties.

Logic and Computer Science. The other theme of our talk is the relationship between mathematical logic and theoretical computer science in Italy between the 70s and 80s. Without pretending to give a complete picture of the work of Italian logicians in this area, we limit ourselves to sketching out a few ideas to explain how these two subjects, initially distant, came to convergence towards the end of the eighties.

Between the sixties and the seventies Corrado Böhm and a group of young scholars trained around his charismatic figure achieved essential results in research on λ-calculus [2], obtaining international recognition and starting a fruitful exchange of ideas and techniques with foreign logicians of the calibre of Kreisel, Barendregt, Hindley, Seldin. Barendregt, for instance, devoted an entire chapter of its text [1] to Böhm's results. This exchange was almost absent in the world of Italian logic.

We tried to investigate this phenomenon and it seems to us that some research themes, in particular in the context of proof theory and constructive mathematics, created fertile ground for the encounter between mathematical logic and theoretical computer science. We will therefore analyze some parts of Carlo Cellucci's early work in proof theory and, in particular, the normalization theorem for natural deduction in which Cellucci inserts an original result for controlling the length of proofs by means of iterated exponentiation [3]. The normalization theorem is one of the links between proof theory and typed λ-calculus and, therefore, with theoretical computer science through the Curry-Howard correspondence.

In addition to these common topics between logic and theoretical computer science, other reasons emerge that will push the two worlds to converge during the eighties. Among them is the growing enthusiasm for information technology in society and education and the privileged role that logic played in the foundation of information technology.

Furthermore, in the first half of the 1980s, there are many scientific opportunities that see, side by side, logicians and computer scientists. For example, the SILFS conference which takes place in San Gimignano in 1983 and the mathematical logic meetings organized by the Scuola di Specializzazione of Siena in 1985 and 1986 and devoted to topics of Proof Theory, Intuitionism and Logic in Computer Science. Finally the foundation of AILA in 1987 - an endpoint for our work - is an act of institutionalization of the Italian logical community in which theoretical computer science is formally recognised as a part of it.

References

[1] Barendregt, H. P. *The Lambda Calculus. It's syntax and semantics.* NorthHolland Publishing Co., Amsterdam, revisited edition, 1984.

[2] Böhm, C. *Alcune proprietà delle forme β-η-normali nel λ-K-calcolo.* Pubblicazioni dell'IAC, 696:1-19, 1968.

[3] Cellucci, C. *Teoria della Dimostrazione.* Bollati Boringhieri, 1978.

[4] Magari, R. *Significato e verità nell'aritmetica peaniana.* Ann. Mat. Pura Appl. (4), 103, 1975.