Logical Approximations of Qualitative Probability

Paolo Baldi[1]*and Hykel Hosni[1]

University of Milan,
{paolo.baldi,hykel.hosni}@unimi.it

Comparative structures provide a natural bridge between the logical and probabilistic representation of uncertainty, of relevance both for the foundations of probability and statistics [4], and AI[5]. Formally, a comparative structure is a pair $(\mathcal{A}, \preceq)$ where $\mathcal{A}$ is a boolean algebra and $\preceq$ is interpreted as a *qualitative probability (relation)* on $\mathcal{A}$, i.e. we write $\theta \preceq \phi$ to say that θ is no-more-probable-than ϕ, for any $\theta, \phi \in \mathcal{A}$. The relations $\theta \approx \phi$ and $\theta \prec \phi$ are defined from $\preceq$ as usual.

Definition 1 (Comparative structure). $(\mathcal{A}, \preceq)$ *is a* comparative structure *if: 1)* $\preceq$ *is a total preorder over* $\mathcal{A}$*; 2)* $\bot \prec \top$*; 3)* $\alpha \preceq \beta$ *holds, whenever* $\alpha \sqsubseteq \beta$*; 4) If* $\alpha \wedge \gamma = \bot$ *and* $\beta \wedge \gamma = \bot$*, then* $\alpha \preceq \beta$ *if and only if* $\alpha \vee \gamma \preceq \beta \vee \gamma$*.*

Here by $\sqsubseteq$ we denote the lattice order of the Boolean algebra, to be distinguished by $\preceq$. The definition above is essentially due to De Finetti [3], who conjectured that such structures would be representable by usual probability functions. Let us recall that a comparative structure $(\mathcal{A}, \preceq)$ is said to be: *representable*, if there exists a finitely additive probability P such that $\alpha \preceq \beta$ if and only if $P(\alpha) \leq P(\beta)$; *almost representable* in case the former claim only holds in the left-to right direction.

Contrary to De Finetti's conjecture, even almost representability fails to hold for general comparative structures, as shown in 1959 by [7]. Since then, various authors have proposed additional suitable axioms for establishing (almost) representability, e.g. [9, 8, 10, 6]. All the approaches to comparative structures still make, however, strong idealizing assumptions, e.g.:

- $\preceq$ has to contain the classical relation $\sqsubseteq$. But finding out whether $\sqsubseteq$ holds is non-feasible, according to standard assumptions in computational complexity.
- The axioms that need to be added to Definition 1, to obtain representability, either postulate [9] that there are arbitrarily fine-grained events to be compared, or impose conditions which are hard to interpret intuitively [8, 10]

In this work, we address these problems, by developing a sequence of comparative structures, which are meant to be *approximations* of almost representable comparative structures. Each structure in the sequence is not by itself representable, and may be rather seen as a qualitative counterpart of a (bounded) Belief function[1, 11]. We then attain the representability result only in the limit, provided that the sequence satisfies certain conditions.

Our framework is crucially based on Depth-Bounded Boolean Logics [2], instead of classical logic. These logics are centered around the idea of limiting the applications of the bivalence principle, which holds unboundedly for classical logic. In natural deduction-style the principle may be presented as follows:

$$\frac{\begin{array}{cc}[\varphi] & [\neg\varphi] \\ \vdots & \vdots \\ \psi & \psi\end{array}}{\psi}\ (\text{PB})$$

*Speaker.

This means that to infer the formula ψ, it suffices to infer it both under the assumption that φ is the case and under the assumption that $\neg\varphi$ is the case. The square brackets around the formulas φ and $\neg\varphi$ signal that those are pieces of information assumed for the sake of deriving ψ, but not actually held true (they are *discharged*, in natural deduction terminology). We call this type of information *hypothetical*, in contrast to the *actual* information which an agent may hold as her premises.

[2] introduces a logic $\vdash_0$, which does not allow any manipulation of hypothetical information, i.e. any application of PB, and may be defined proof-theoretically by a core set of introduction and elmination rules (Intelim Rules [2]) for each connective, both when occurring positively (as the main connective of a formula) and negatively (in the scope of a negation). The family of Depth-Bounded Boolean Logics $\{\vdash_k\}_{k\in\mathbb{N}}$ is then characterized, for $k > 0$, by allowing, in addition to the rules of $\vdash_0$, at most k nested applications of PB.

Results in [2] show that:

- $\vdash_0\subset\vdash_1\subset\cdots\subset\vdash_k\subset\cdots$, so the depth-bounded consequence relations form a hierarchy;
- $\lim_{k\to\infty}\vdash_k=\vdash$, i.e. at the limit, the hierarchy of depth-bounded boolean logics coincides with classical logic;
- for each k, $\vdash_k$ has a polynomial decision procedure.

These properties make these logics a suitable starting point for addressing the problems of classical comparative structures discussed above. In this work, we shall: define a sequence of bounded comparative structures, based on the sequence of Depth-Bounded Boolean logics; identify the conditions under which our bounded comparative structures are asymptotically (almost) representable by a probability measure and, conversely, those conditions under which a representable qualitative probability structure can be approximated.

References

[1] P. Baldi and H. Hosni. Depth-bounded belief functions. *International Journal of Approximate Reasoning*, 123:26–40, 2020.

[2] M. D'Agostino, M. Finger, and D.M. Gabbay. Semantics and proof-theory of depth bounded Boolean logics. *Theoretical Computer Science*, 480:43–68, 2013.

[3] B. de Finetti. Sul significato soggettivo della probabilità. *Fundamenta Mathematicae*, 17:289–329, 1931.

[4] B. de Finetti. Recent suggestions for the reconciliation of theories of probability. *Proceedings of the Second Berkley Symposium on Mathematical Statistics and Probability*, 1:217–225, 1951.

[5] J. P. Delgrande, B. Renne, and J. Sack. The logic of qualitative probability. *Artificial Intelligence*, 275:457–486, 2019

[6] P.C. Fishburn. Finite Linear Qualitative Probability. *Journal of Mathematical Psychology*, 40(1):64–77, 1996.

[7] C. Kraft, J. Pratt, and A. Seidenerg. Intuitive Probability On Finite Sets. *The Annals of Mathematical Statistics*, 30(2):408–419, 1959.

[8] D. Kranz, R.D. Luce, P. Suppes, and A. Tversky. *Foundations of measurement. Volume 1.* Academic Press, New York, 1971.

[9] L.J. Savage. *The Foundations of Statistics.* Dover, 2nd edition, 1972.

[10] D. Scott. Measurement Structures and Linear Inequalities. (1956):233–247, 1964.

[11] S.K.M. Wong, Y.Y. Yao, P. Bollmann, and H.C. Burger. Axiomatization of qualitative belief structure. *IEEE Transactions on Systems, Man, and Cybernetics*, 21(4):726–734, 1991.