Clones and polymorphisms, algebraically

Antonio Bucciarelli and Antonino Salibra*

Institut de Recherche en Informatique Fondamentale, CNRS et Université Paris Cité

Clones are sets of finitary operations on a given set that contain all the projections and are closed under composition. They play an important role in universal algebra, since the set of all term operations of an algebra always forms a clone and in fact every clone is of this form. Comparing clones of algebras is much more appropriate than comparing their basic operations, in order to classify algebras according to different behaviours (see [13, 14]). Clones play another important role in the study of first-order structures. Indeed, the polymorphism clone of a first-order structure, consisting of all finitary functions which preserve the structure, forms a clone. Polymorphism clones carry information about the structures that induce them, and are a powerful tool in their analysis. Clones are also important in theoretical computer science. Many computational problems can be phrased as constraint satisfaction problems (CSPs): in such a problem, we fix a structure A (also called the template or constraint language). The problem $\mathrm{CSP}(A)$ is the computational problem of deciding whether a given conjunction of atomic formulas over the signature of A is satisfiable in A. The seminal discovery in the algebraic approach to CSP is the result of Jeavons [6] that, for a finite structure A, the complexity of $\mathrm{CSP}(A)$ is completely determined by the polymorphism clone of A (e.g. see [1]).

We have introduced in [4] the variety of clone algebras (CA) as a one-sorted algebraic theory of clones. A crucial feature of our approach is connected to the role played by variables in free algebras and projections in clones. In clone algebras these are abstracted out, and take the form of a countable infinite system of fundamental elements (nullary operations) $\mathsf{e}_1, \mathsf{e}_2, \dots, \mathsf{e}_n, \dots$ of the algebra. One important consequence of the abstraction of variables and projections is the abstraction of term-for-variable substitution and functional composition in CAs, obtained by introducing an $(n+1)$-ary operator q_n for every $n \geqslant 0$. Roughly speaking, $q_n(a, b_1, \dots, b_n)$ represents the substitution of b_i for e_i into a for $1 \leqslant i \leqslant n$ (or the composition of a with $b_1, \dots, b_n$ in the first n coordinates of a).

The most natural CAs, the ones the axioms are intended to characterise, are algebras of functions, called functional clone algebras (FCA). The elements of a functional clone algebra are infinitary operations from A^ω into A, for a given set A. In this framework $q_n(f, g_1, \dots, g_n)$ represents the n-ary composition of f with $g_1, \dots, g_n$, acting on the first n coordinates: $q_n(f, g_1, \dots, g_n)(s) = f(g_1(s), \dots, g_n(s), s_{n+1}, s_{n+2}, \dots)$, for every $s \in A^\omega$ and the nullary operators are the projections p_i defined by $p_i(s) = s_i$ for every $s \in A^\omega$. Hence, the universe of a FCA is a set of infinitary operations containing the projection p_i and closed under finitary compositions, called hereafter ω-clone. One of the main results in [4] is the functional representation theorem: every CA is isomorphic to a FCA. Moreover, we have shown that the finite-dimensional CA are the abstract counterpart of the clones of finitary operations, where the dimension of an element in a CA is an abstraction of the notion of arity. Among the applications of CAs, we point out the lattice of equational theories problem stated by Birkhoff [2] in 1946: Find an algebraic characterisation of those lattices which can be isomorphic to a lattice of equational theories. This problem is still open, but work on it has led to many results described in [9, Section 4]. The problem of characterising the lattices of equational theories as the congruence lattices of a class of algebras was tackled by Newrly [7] and Nurakunov [8]. We have proposed an alternative answer to the lattice of equational theories problem. We proved

*Speaker.

that a lattice is isomorphic to a lattice of equational theories if and only if it is isomorphic to the lattice of all congruences of a finite dimensional clone algebra. Unlike in Newrlys and Nurakunovs approaches, we are able to provide the equational axiomatisation of the variety whose congruence lattices are exactly the lattices of equational theories, up to isomorphisms.

Recently, the second author has developed in [11] a new general framework for algebras and clones, called universal clone algebra. Algebras and clones of finitary operations are to universal algebra what infinitary algebras and clone algebras are to universal clone algebra. He has presented a method to codify algebras and clones into infinitary algebras and clone algebras, respectively, and has provided concrete examples showing that general results in universal clone algebra, when translated in terms of algebras and clones, give new versions of known theorems in universal algebra. This methodology is applied to Birkhoff's HSP theorem and to the recent topological versions of Birkhoff's theorem (see [3, 5, 12]).

The Galois connection relating clones and first order structures introduced above extends to the infinitary case. An infinitary first-order structure is just a set of infinitary relations on a given set A, and it turns out that the set of functions from A^ω into A preserving an infinitary first-order structure is a functional clone algebra. As a matter of fact, in the infinitary case the connection is richer: we define the notions of weak and strong polymorphism, both extending the one of the finitary case. We show that for any set of infinitary relations the set of weak polymorphisms is the topological closure of the set of strong ones, with respect to the pointwise convergence topology of the space of functions from A^ω into A. Accordingly, we have two notions of invariance: an infinitary relation $R \subseteq A^\omega$ is weakly (resp. strongly) invariant for a function $\varphi : A^\omega \to A$ if φ is a weak (resp. strong) polymorphism of R. Calling $t1$ and $t2$ polymorphisms the weak and strong ones, respectively, the problem arises of comparing the ω-clones of the form $Pol^i(Inv^j(\mathcal{C}))$, for a given functional clone $\mathcal{C}$ and $1 \leqslant i, j \leqslant 2$, and the structures $Inv^i(Pol^j(\mathcal{R}))$, for a given set $\mathcal{R}$ of infinitary relations and $1 \leqslant i, j \leqslant 2$. Some of these problems are settled, and some others are open. Moreover, we show that if $\mathcal{C}$ is a functional clone algebra closed under q_ω, then $\mathcal{C}$ is topologically closed iff $Pol^1(Inv^2(\mathcal{C})) = C$ iff $Pol^1(Inv^1(\mathcal{C})) = C$. Applications of this new theory to the study of first order structures and CSP is left for future work.

References

[1] Barto, L.: The constraint satisfaction problem and universal algebra. The Bulletin of Symbolic Logic **21**(3), (2015)

[2] Birkhoff, G.: Universal algebra. Proc. First Canadian Math. Conf., Univ. Toronto Press, (1946)

[3] Bodirski, M., Pinsker, M.: Topological Birkhoff. Transactions of the AMS, **367**(4), (2015)

[4] Bucciarelli, A., Salibra, A.: An algebraic theory of clones. Algebra Univers. **83**(2), 14 (2022)

[5] Gehrke, M., Pinsker, M.: Uniform Birkhoff. J. Pure and Applied Algebra, **222**(5), (2018)

[6] Jeavons, P.: On the algebraic structure of combinatorial problems. Theor. Comput. Sci., **200**(1-2), 185–204 (1998)

[7] Newrly N.: Lattices of equational theories are congruence lattices of monoids with one additional unary operation, Algebra Universalis **30**, 217–220 (1993)

[8] Nurakunov A.M.: Equational theories as congruences of enriched monoids, Algebra Univers. **58**, 357–372 (2008)

[9] McNulty, G.F.: A Field Guide to Equational Logic. J. Symbolic Computation **14**, 371–397 (1992)

[10] Neumann, W.D.: Representing varieties of algebras by algebras. J. Austral. Math. Soc. **11**, (1970)

[11] Salibra, A.: Universal clone algebra. (2022) http://arxiv.org/abs/2203.14054v2

[12] Schneider, F.M.: A uniform Birkhoff theorem. Algebra Universalis, **78**, 337–354 (2017)

[13] Szendrei, A.: Clones in Universal Algebra. Les Presses de l'Université de Montréal (1986)

[14] Taylor, W.: Abstract clone theory. In: Algebras and Orders, Kuwer Academic Publisher, (1993)