Geometry of Super-Łukasiewicz Logics

Antonio Di Nola[1] , Giacomo Lenzi[1] , and Gaetano Vitale[1]

University of Salerno, Fisciano (SA), Italy
{adinola,gilenzi,gvitale}@unisa.it

In this paper we focus on *MV-algebras*, and in particular on the varieties of MV-algebras generated by a single non-simple MV-chain.

Recall that MV-algebras are algebraic structures which generalize Boolean algebras in the sense that the sum is not idempotent. In the same way as Boolean algebras are the algebraic counterpart of two-valued logic, MV-algebras are the algebraic counterpart of Łukasiewicz many-valued logic. Hence MV-algebras are useful for reasoning about vague or fuzzy properties, like "being tall", "being young", etc. MV-algebras are also used in applications, for instance economy, immunology and artificial intelligence. Like in Boolean algebras, the value 1 represents the true event and the value 0 represents the false event. MV-algebras then have other values representing intermediate events. Unlike Boolean algebras, MV-algebras can have *infinitesimals*. We can say that an infinitesimal element represents an "almost false event", and a co-infinitesimal element, the negation of an infinitesimal, represents an "almost true event". More generally we can consider elements infinitely near a fixed truth value.

In [11] we have an example of appearance of infinitesimals in logic: in first order fuzzy logic, the formulas which are true but not provable are coinfinitesimals in the Lindenbaum algebra.

In this work we want to deal with infinitesimals in the field of many-valued logic. We study some varieties of MV-algebras generated by one chain, which are the algebraic counterpart of some axiomatic extensions of Łukasiewicz logic. Our claim is that, in these varieties, infinitesimals occur only in a "controlled" sense, so MV-algebras in these varieties behave similarly to MV-algebras without infinitesimals (that is the semisimple ones).

In [2] and [9] we have another crucial application of infinitesimals: every MV-algebra embeds in a power of ultrapower of $[0,1]$, and the difference between $[0,1]$ and its ultrapowers lies in the extra infinitesimals.

In [5] Komori introduces super-Łukasiewicz logics, providing an algebraic description. An axiomatic description of these algebras, which are non-simple MV-chains, is presented in [3]. We characterize and describe free objects in varieties generated by these algebras with ad hoc domains and in the spirit of universal algebraic geometry, providing categorical dualities between algebraic structures and geometrical objects. Literature shows the relevance of one chain generated varieties, non-simple MV-chains, free MV-algebras and dualities, e.g. see [1, 4, 8, 10].

Among the fundamental structures considered in this paper are the MV-algebras

$$K_m = \Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (m-1, 0))$$

and

$$\Delta_m = \Gamma(\mathbb{Z} \times_{lex} \mathbb{R}, (m-1, 0)),$$

where $\times_{lex}$ is the lexicographic product of groups and Γ is Mundici's functor (see [6]).

Given a set X of MV-algebras, we denote by $V(X)$ the variety generated by X.

In [7], page 11, it is explained that "Rational polyhedra are the genuine algebraic varieties of the formulas of Łukasiewicz logic". As a consequence, $[0,1]$ is important since rational polyhedra are subsets of powers of $[0,1]$. When one moves from Łukasiewicz logic to extensions of this logic, it is not clear whether we have a strict analogy. We propose the MV-algebra Δ_m, which consists of finitely many copies of the real half-line.

In this paper we give:

- a family of functors T_m between MV-algebras and $V(K_m)$-algebras, sending finitely generated free MV-algebras into finitely generated free $V(K_m)$-algebras;
- a Nullstellensatz in $V(K_m)$;
- a categorical duality between a class of so-called restricted $V(K_m)$-algebras (equivalently, finitely presented $V(K_m)$-algebras) and a class of rational cones;
- a Wójcicki-type theorem;
- some logical and category-theoretic results on m-simple and m-semisimple MV-algebras and related issues;
- a McNaughton type theorem for a certain logic named Luk_m.

References

[1] Lawrence Peter Belluce and Antonio Di Nola. Simplicial structures in mv-algebras and logic. *The Journal of Symbolic Logic*, 72(2):584–600, 2007.

[2] Antonio Di Nola. Representation and reticulation by quotients of MV-algebras. *Ricerche di matematica*, 40(2):291–297, 1991.

[3] Antonio Di Nola and Ada Lettieri. Equational characterization of all varieties of MV-algebras. *Journal of Algebra*, 221(2):463–474, 1999.

[4] Antonio Di Nola and Ada Lettieri. One chain generated varieties of MV-algebras. *Journal of Algebra*, 225(2):667–697, 2000.

[5] Yuichi Komori. Super-Łukasiewicz propositional logics. *Nagoya Mathematical Journal*, 84:119–133, 1981.

[6] Daniele Mundici. Mapping abelian l-groups with strong unit one-one into MV-algebras. *Journal of Algebra*, 98(1):76–81, 1986.

[7] Daniele Mundici. *Advanced Łukasiewicz calculus and MV-algebras*. Springer Netherlands, 2011.

[8] Philippe Niederkorn. Natural dualities for varieties of MV-algebras, i. *Journal of mathematical analysis and applications*, 255(1):58–73, 2001.

[9] Antonio Di Nola. MV-algebras in the treatment of uncertainty. In *Fuzzy Logic*, pages 123–131. Springer, 1993.

[10] Giovanni Panti. Varieties of MV-algebras. *Journal of Applied Non-Classical Logics*, 9(1):141–157, 1999.

[11] Bruno Scarpellini. Die nichtaxiomatisierbarkeit des unendlichwertigen prädikatenkalküls von Łukasiewicz. *The Journal of Symbolic Logic*, 27(2):159–170, 1962.