Towards parallel reduction for Lineal Logic Proof-nets

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Abstract

This work sits at the interface between Proof Theory and Theoretical Computer Science in the framework of Curry-Howard [3] correspondence. In particular, we will show how to approach a notion of parallel reduction for Linear Logic Proof-nets.

Curry-Howard correspondence establishes a strong relationship that allows to see formal proofs as programs. In some sense logic becomes a specification language expressing the behaviour of programs: a language that can describe a system at a much higher level than a programming language. The Curry-Howard correspondence has also motivated the creation of new logical systems allowing a finer analysis of programs: it is the case of Linear Logic,introduced in the 80s by Jean-Yves Girard [2]. In classical and intuitionistic logic, using structural rules, all formulas can be duplicated or deleted in the cut elimination procedure. The objects that the formulas represent are "eternal" resources in the sense that they can be used without limit. Just as a lemma can be used in a proof as many times as desired. On the other hand, Linear Logic is adapted to a computational point of view. Formulas are no more "eternal truths". In LL the structural rules are controlled by the introduction of two new connectives ! and ?, called exponentials. Among other novelties, Linear Logic introduced a new syntax for proofs/programs called Proof- nets that abandon the traditionnal representations by means of trees in favour of a representation by graphs caracterised by a topological condition. The general principle is to represent logical rules as links between the active formulas of the premises and the conclusion formula(s), without any a priori relation to a context. The cut elimination procedure is defined as a local interaction between logical rules and no longer by global transformations on the proof trees. The conciseness of the formalism is both an asset and a difficulty for the mathematician: one leaves the purely syntactic framework where everything is a tree, the notion of the last rule, which previously served as a "handle" by which to catch the demonstration object, is lost. This partly explains why some apparently essential results of the theory were formally established only very late. The most striking example is the proof of strong normalization of the complete system of second-order linear logic: a proof was quickly sketched in Girard's founding paper, but it was not until the work of Pagani and Tortora [4] that a complete proof was available.

Our goal is to lay the groundwork for revisiting this result implementing recent advances in Proof-net theory. In particular we will mobilize the definition of simultaneous reduction of parallel cuts introduced by Chouquet and Vaux Auclair [1]. This technique generalizes the approach of Tait and Martin-Löf to establish the confluence of the λ-calculus. A step of parallel-beta-reduction on a *lambda*-term t allows to reduce simultaneously an arbitrary number of redexes available in t. The fact that this reduction is strongly confluent implies that the system is Church-Rosser.

Currently a definition of parallel reduction on Proof-nets is available for a small fragment where exponential formulas are not present, namely MLL.

Extending the definition to MELL presents some challenges. In particular, the introduction of exponential connectives makes it necessary to reintroduce in Proof-nets some level of sequencing. This is achieved by means of objects called boxes, which are used to delineate a part of the proof-net graph. When eliminating the exponential cut, i.e. where one of the two formulas active in the cut is of type !A, may result in duplication or deletion of boxes. That is why the rewriting process is no longer local in nature: hence defining

parallel reduction becomes a delicate operation. We will highlight the critical cases through some examples and show how these difficulties can be overcome.

References

[1] Jules Chouquet and Lionel Vaux Auclair. An application of parallel cut elimination in multiplicative linear logic to the taylor expansion of proof nets, 2020.

[2] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–101, 1987.

[3] William Alvin Howard. The formulae-as-types notion of construction. In Haskell Curry, Hindley B., Seldin J. Roger, and P. Jonathan, editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus, and Formalism.* Academic Press, 1980.

[4] Michele Pagani and Lorenzo Tortora de Falco. Strong normalization property for second order linear logic. *Theoretical Computer Science*, 411(2):410–444, 2010.