Embeddings of metric Boolean algebras in $\mathbb{R}^N$

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A metric Boolean algebra (see e.g. [1, 2, 3]) consists of a Boolean algebra $\mathbf{A}$, equipped with a strictly positive (finitely-additive) probability measure[1] $m\colon \mathbf{A} \to [0,1]$, which makes $(\mathbf{A}, d_m)$ a metric space, where the distance between any two points $a, b \in A$ is defined as:

$$d_m(a,b) \coloneqq m((a \wedge b') \vee (a' \wedge b)).$$

From a geometrical point of view, it is natural to wonder under which conditions a metric Boolean algebra $(\mathbf{A}, d_m)$, or some of its relevant subspaces, can be isometrically embedded in $\mathbb{R}^N$ (equipped with the Euclidean distance), for a given positive integer N. Actually, for $|A| > 2$, there is no such embedding. However, under the assumption that $\mathbf{A}$ is finite (or, more generally, atomic), it makes sense to restrict the question to the subspace $\mathsf{At}(\mathbf{A})$ of its atoms.

A classical result by Morgan [5] states that a metric space (X,d) embeds in $\mathbb{R}^N$ if and only if it is flat and has dimension less or equal to N, where (X,d) is flat if the determinant of the matrix $M(\vec{x}_n)$, whose generic entry is $M_{ij} = \frac{1}{2}(d(x_0,x_i)^2 + d(x_0,x_j)^2 - d(x_i,x_j)^2)$, is non-negative for every n-simplex (namely every choice of $n+1$ points $\vec{x}_n = \{x_0, \ldots, x_n\}$ in X) and the dimension of (X,d) is the greatest N (if exists) such that there exists a N-simplex with positive determinant.

Given a finite metric Boolean algebra $\mathbf{A}$ with $\mathsf{At}(\mathbf{A}) = \{a_0, a_1, \ldots, a_k\}$, it is easily checked that the matrix $M(\vec{x}_n) = \{M_{ij}\}$, $2 \le n \le k$ (introduced in Morgan's theorem) has generic entry

$$M_{ij} = (x_0 + x_i)^2 \delta_{ij} + (x_0^2 + x_0 x_1 + x_0 x_j - x_i x_j)(1 - \delta_{ij}),$$

where $x_\alpha = m(a_\alpha)$ (thus $x_\alpha > 0$, for every $\alpha \in \{0, 1, \ldots, k\}$). Therefore the form of the determinant can be simplified according to the following.

Lemma 1. *Let $M(\vec{x}_n)$, $2 \le n \le k$ be the matrix associated to a finite metric atomic Boolean algebra $\mathbf{A}$ with $k+1$ atoms. Then*

$$\det(M(\vec{x}_n)) = 2^{n-1} \left[\left(\sum_{\alpha=0}^{n} x_0 \cdot \cdots \cdot \hat{x}_\alpha \cdot \cdots \cdot x_n \right)^2 - (n-1) \left(\sum_{\alpha=0}^{n} x_0^2 \cdot \cdots \cdot \hat{x}_\alpha^2 \cdots \cdot x_n^2 \right) \right],$$

where $\hat{x}_i$ means that x_i has to be omitted.

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[1] Recall that a *strictly positive* (finitely additive) probability measure over a Boolean algebra $\mathbf{A}$ is a map $m\colon \mathbf{A} \to [0,1]$ such that:

1. $m(\bot) = 1$,
2. $m(a \vee b) = m(a) + m(b)$, for every $a, b \in A$ such that $a \wedge b = \bot$,
3. $m(a) > 0$, for every $a \in A$, $a \neq \bot$.

It follows, for instance, that the space $(\mathsf{At}(\mathbf{A}), d_m)$ of the $k+1$ atoms of a finite metric Boolean algebra such that $m(a_i) = \frac{1}{k+1}$ (for every $a_i \in \mathrm{At}(\mathbf{A})$) embeds in $\mathbb{R}^k$ with the Euclidean metric and that $\det(M(\vec{x}_2)) > 0$.

Upon indicating by $\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ the space of the (finitely additive) probability measures m such that $(\mathrm{At}(\mathbf{A}), d_m)$ admits an isometric embedding into some Euclidean space $\mathbb{R}^N$, in virtue of Morgan's theorem one has

$$\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A})) = \bigcap_{n=3}^{k} C_n \cap \Pi_k,$$

where $C_n = \{\vec{x} \in \mathbb{R}_+^{k+1} \mid \det M(\vec{x}_n) \geq 0\}$, with $3 \leq n \leq k$ and Π_k is the interior of the standard k-simplex (or probability simplex) of $\mathbb{R}^{k+1}$, namely

$$\Pi_k = \{\vec{x} \in (0,1)^{k+1} \mid \sum_{\alpha=0}^{k} x_\alpha = 1\}.$$

We are interesting in solving the following.

Problem. Study the topology of $\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ with the topology induced by $(0,1)^{k+1} \subset \mathbb{R}_+^{k+1}$.

In order to get a solution, we first analyze the topology of C_n.

Lemma 2. *For each $3 \leq n \leq k$, the space $C_n \cong H_n \times \mathbb{R}_+^{k-n}$ where H_n is a solid half-hypercone in $\mathbb{R}_+^{n+1}$.*

The solution to the above presented problem is given by the following.

Theorem 3. *Let $k \geq 3$. Then:*

1. *$\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ is contractible.*
2. *$\mathcal{M}(\mathrm{At}(\mathbf{A})) \setminus \mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ is simply-connected (not contractible).*

In the final part of the talk, we will draw some considerations on the significance of our results for probability theory and on their possibile extensions to metric MV-algebras (MV-algebras equipped with a faithful state [6, 4]).

References

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