A survey on exponential-algebraic closure and quasiminimality

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Call an (uncountable) first-order structure $\mathcal{M}$ *quasi-minimal* if every definable subset of its universe is countable or co-countable. Two easy examples are the complex field $\mathbb{C}_{\mathbb{Z}}$ equipped with a predicate for the integers, or the order $(\omega_1, <)$. While both are unstable from the point of view of usual stability theory, $\mathbb{C}_{\mathbb{Z}}$ has some form of stability. For $A \subseteq \mathbb{C}$, let $\mathrm{cl}(A)$ be the union of all countable A-definable sets in $\mathbb{C}_{\mathbb{Z}}$. Then cl is a closure operator with the exchange property. If we take the class generated by $\mathbb{C}_{\mathbb{Z}}$ under isomorphism, taking substructures, and unions of chains of 'cl-closed' embeddings, we obtain an *uncountably categorical* abstract elementary class, consisting of the models of a sentence in $\mathcal{L}_{\omega_1,\omega}$.

This is an instance of a more general phenomenon: if a quasi-minimal structure $\mathcal{M}$ is sufficiently homogeneous and does not define a partial order containing an ω_1-chain, and its associated operator $\mathrm{cl}_{\mathcal{M}}$ is sufficiently definable, then the class generated by $\mathcal{M}$ is again uncountably categorical and consisting of the models of a sentence in $\mathcal{L}_{\omega_1,\omega}(\exists^{\geq \aleph_1})$ [2, 10].

Finding quasi-minimal structures with the above categoricity properties is surprisingly hard. Zilber conjectured that the complex exponential field, equipped with the complex exponential function, is quasi-minimal and model of an uncountably categorical $\mathcal{L}_{\omega_1,\omega}(\exists^{\geq \aleph_1})$-sentence [12]. Zilber's original sentence states that Schanuel's conjecture is true, and that the cartesian powers of the graph of the exponential function intersects as many algebraic varieties in $(\mathbb{C} \times \mathbb{C}^{\times})^n$ as feasible (a property dubbed *exponential-algebraic closedness*). Bays and Kirby have proved recently that Schanuel's conjecture can be replaced with certain (unconditional) consequences of Ax's theorem, massively improving our chances to prove the quasi-minimality of complex exponentiation [3].

There are now more and more known instances of exponential-algebraic closedness for particular families of algebraic varieties, and for different kinds of exponential functions. For a taster: [4, 5] deal with varieties whose projection to $\mathbb{C}^n$ has dimension n; [1] show the same for *abelian* exponential functions, while [6] obtain analogous results for the modular j-function; [7, 8] find results for 'raising-to-powers' style relations on automorphic functions; [11, 9] give full exponential-algebraic closedness statements for complex raising to powers (this list is not exhaustive!).

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