The Lambda Calculus, its Syntax and Semantics 40 years later

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The problem of determining when two programs are equivalent is central in computer science, e.g., it is necessary to verify that the optimizations performed by a compiler actually preserve the meaning of the input program. For λ-calculi, Morris proposed in his PhD thesis [9] to consider two λ-terms M and N as equivalent when they are *contextually equivalent* with respect to some fixed set $\mathcal{O}$ of *observables*. Let us call $\mathcal{T}_{\mathcal{O}}$ the observational theory with observables $\mathcal{O}$:

$$\mathcal{T}_{\mathcal{O}} \vdash M = N \iff \forall C[\,]\,.\,[\; C[M] \in \mathcal{O} \iff C[N] \in \mathcal{O} \;]$$

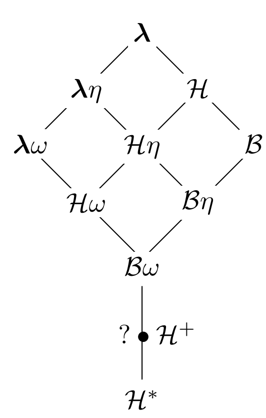
The problem of this definition is that the quantification over all possible contexts is difficult to handle in practice. Therefore, many researchers over the years undertook a quest for characterizing observational equivalences both semantically, by defining fully abstract denotational models, and syntactically, by introducing several kinds of extensional equivalences on Böhm trees (that are possibly infinite trees representing program executions).

The observational theory $\mathcal{H}^* := \mathcal{T}_{\mathrm{SOL}}$ where the observables are the solvable (equivalently, head-normalizable) λ-terms is by far the most well studied theory of λ-calculus — it is the theory of Scott's $\mathcal{D}_\infty$ model [11] and equates two λ-terms exactly when their Böhm trees are equal up to possibly infinite η-expansions. Curiously, the first extensional observational theory that has been defined in the literature is not $\mathcal{H}^*$, but rather $\mathcal{H}^+ := \mathcal{T}_{\mathrm{NF}}$ where the observables are the β-normalizable λ-terms. This theory has been little studied in the literature, although a fully abstract filter model has been defined by Coppo *et al.* in [4] and it is well known that two λ-terms are equal in $\mathcal{H}^+$ whenever their Böhm trees coincide up to finite η-expansions.

It should now be clear that observational theories and extensional equivalences are tightly connected. Now, the λ-calculus admits a notion of extensionality *a priori* stronger than η:

$$(\omega\text{-rule}) \qquad \forall \text{ closed } \lambda\text{-term } P\,.\, MP = NP \quad\Longrightarrow\quad M = N$$

so, it is natural to wonder how this rule compares with the notions of extensionality above. Classic results establish that neither $\boldsymbol{\lambda}$ (i.e., the theory of β-equivalence), $\mathcal{H}$ (the least theory equating all unsolvables), $\mathcal{B}$ (the theory of Böhm trees) nor their extensional versions $\boldsymbol{\lambda}\eta$, $\mathcal{H}\eta$ and $\mathcal{B}\eta$ do satisfy the (ω)-rule. Given a theory $\mathcal{T}$, it is however possible to define $\mathcal{T}\omega$ as the least theory satisfying the ω-rule.
In Barendregt's book [1] a "kite" shaped diagram depicts all strict inclusion relations among these theories (see the figure on the right, where $\mathcal{T}_1$ is above $\mathcal{T}_2$ whenever $\mathcal{T}_1 \subsetneq \mathcal{T}_2$). In the seventies Barendregt raised the question of determining the position of $\mathcal{H}^+$ in this diagram, and in 1978 Sallé conjectured that $\mathcal{B}\omega \subsetneq \mathcal{H}^+$. The problem remained open for almost 40 years. In 2016 Breuvart *et al.* proved that $\mathcal{B}\omega \subseteq \mathcal{H}^+$ [3]. Sallé's conjecture has been refuted by Intrigila *et al.* in 2017 by showing that $\mathcal{B}\omega$ and $\mathcal{H}^+$ actually coincide [6]. As a byproduct, we obtain a characterization of all degrees of extensionality in $\mathcal{B}$ [7].



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During Manzonetto's *Habilitation à diriger des recherches* dissertation [8], Barendregt heard about the solution of Sallé's conjecture and realized with satisfaction that most of the problems he proposed in his first book were nowadays solved. However, Hyland complained that the results were difficult to access since scattered in the literature, often written in long PhD thesis, and/or cryptic articles. Examples of these are the following:

1. **Invertibility and bijectivity.** In set theory it is easy to verify that a function $f : X \to X$ is bijective if and only if it is invertible. Now, given a λ-theory $\mathcal{T}$, every combinator F can be considered as function $F : \Lambda^o \to \Lambda^o$ (modulo $\mathcal{T}$). Assuming that F is a bijection, can one conclude from this that F is $\mathcal{T}$-invertible? In other words, is there a λ-term $G \in \Lambda^o$ such that $F \circ G =_{\mathcal{T}} G \circ F =_{\mathcal{T}} \mathsf{I}$? For $\mathcal{T} = \boldsymbol{\lambda}$ the answer is positive as shown by Böhm and Dezani. The invertibility problem for $\mathcal{T} = \boldsymbol{\lambda}\eta$ was raised in [1, Ex. 21.4.9]. More than 10 years later, in his PhD thesis, Folkerts [5] showed that this correspondence does hold.

2. **The range property fails for $\mathcal{H}$.** Given a λ-theory $\mathcal{T}$, one defines the *range of a λ-term F modulo $\mathcal{T}$* as $\mathsf{Range}_{\mathcal{T}}(F) = \{[FM]_{\mathcal{T}} \mid M \in \Lambda^o\}$. In many theories $\mathcal{T}$, for all $F \in \Lambda^o$, $\mathsf{Range}_{\mathcal{T}}(F)$ is either a singleton or infinite. For $\mathcal{T} = \mathcal{H}$ this property was resisting. Several steps were made attempting to prove the range property for $\mathcal{H}$. In his PhD thesis, Andrew [10] cleverly used all these hints to construct a diabolic 'tunnel' having only 2 survivors and refute the conjecture: there is a λ-term with range modulo $\mathcal{H}$ of cardinality 2.

Barendregt and Manzonetto accepted Hyland's challenge and agreed the time had come to write a 'sequel' of [1], where these results could be collected, sometimes simplified, and uniformly presented. The resulting monograph, called "A Lambda Calculus Satellite", is now complete, entered recently the proof-checking phase, and should be published in the fall [2].

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