Profiniteness and spectra of Heyting algebras

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An algebra is said to be *profinite* if it is isomorphic to the inverse limit of an inverse system of finite algebras. Similarly, the *profinite completion* of an algebra $\boldsymbol{A}$ is the inverse limit of the inverse system consisting of the finite algebras of the form $\boldsymbol{A}/\theta$, where θ is a congruence of $\boldsymbol{A}$. It follows that every profinite completion is a profinite algebra, while the converse need not be true in general.

Even though the study of profinite Heyting algebras and completions has recently gained attention [1, 2, 3, 4], the problem of determining whether all profinite Heyting algebras are profinite completions remained open [4]. In this talk we will resolve it, by characterizing the varieties of Heyting algebras whose profinite members are profinite completions. As a consequence, we will be able to exhibit an array of profinite Heyting algebras that cannot be obtained as profinite completions of any Heyting algebra.

To this end, we rely on the following description of profinite Heyting algebras and completions [1, 2]. A poset is said to be *image finite* when its principal upsets are finite. Accordingly, the *image finite part* X_{fin} of a poset X is the subposet of X with universe

$$X_{\text{fin}} \coloneqq \{x \in X : {\uparrow}x \text{ is finite}\}.$$

It follows immediately that X_{fin} is an image finite poset. For the present purpose, the interest of image finite posets is that a Heyting algebra is profinite precisely when it is isomorphic to the algebra of upsets $\mathsf{Up}(X)$ of an image finite poset X. Furthermore, the profinite completion of a Heyting algebra $\boldsymbol{A}$ is isomorphic to $\mathsf{Up}(X)$ where X is the image finite part of the Esakia space that dualizes $\boldsymbol{A}$ in the sense of [7, 9].

Consequently, in order to construct a profinite Heyting algebra that is not a profinite completion, it suffices to exhibit an image finite poset X that is not the image finite part of any Esakia space (in which case, $\mathsf{Up}(X)$ is a profinite Heyting algebra that is not a profinite completion). We shall do this by constructing, for each poset P depicted below, a p-morphic image X of a disjoint union of copies of P that is image finite, but is not the image finite part

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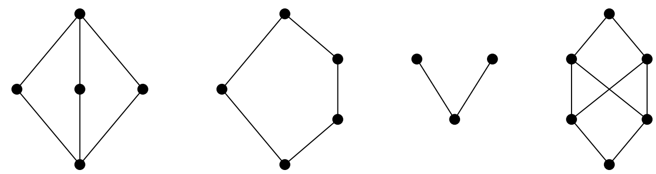
of any Esakia space.



P_1 P_2 P_3 P_4

Since p-morphic images and disjoint unions preserve the validity of formulas, this implies that if the profinite members of a variety K of Heyting algebras are profinite completions, then K must omit the Heyting algebra $\mathsf{Up}(P)$, for every poset P in the above picture.

The converse of this result is also true and constitutes the main result of the talk. To prove it observe that, in view of Jankov's Lemma, there exists a largest variety of Heyting algebras that omits $\mathsf{Up}(P_i)$ for $i = 1, \ldots, 4$. We call its members *diamond Heyting algebras*, because of the shape of their Esakia duals. More precisely, we say that a poset X is a *diamond system* if it satisfies the following conditions:

(i) $\uparrow x$ satisfies Esakia's three point rule for each $x \in X$;

(ii) X has width at most two;

(iii) Principal upsets are upward directed in X;

(iv) For every $\bot, x, y, z, v, \in X$, if $\bot \leqslant x, y \leqslant z, v$, there is $w \in X$ such that

$$x, y \leqslant w \leqslant z, v.$$

Image finite downward directed diamond systems are linear sums of diamonds and lines, whencefrom this terminology.

Diamond algebras and systems are related as follows.

Theorem 1. *The following conditions hold:*

(i) *A Heyting algebra is diamond if and only if the order reduct of its Esakia dual is a diamond system;*

(ii) *If X is a poset and $\mathsf{Up}(X)$ a diamond Heyting algebra, then X is a diamond system;*

(iii) *Every image finite diamond system is the image finie part of the Esakia dual of a diamond Heyting algebra.*

Bearing this in mind, let K be a variety of diamond Heyting algebras and $\boldsymbol{A}$ a profinite member of K. Since $\boldsymbol{A}$ is profinite, it has the form $\mathsf{Up}(X)$ for an image finite poset X that, moreover, is a diamond system in view of Condition (ii) of the above theorem. By Condition (iii) of the same theorem, X is the image finite part of some Esakia space, whence $\mathsf{Up}(X)$ (and, therefore, $\boldsymbol{A}$) is a profinite completion. We conclude that all the profinite members of K are profinite completions. This establishes the remaining part of the main result of the talk.

Theorem 2. *Let K be a variety of Heyting algebras. The profinite members of K are profinite completions if and only if K is a variety of diamond Heyting algebras.*

Corollary 3. *The problem of determining whether the profinite members of a variety of Heyting algebras (which can be presented either by a finite set of equations of by a finite set of algebras) are profinite completions is decidable.*

We close this talk by observing that intermediate logics algebraized by varieties of diamond Heyting algebras form a denumerable set and are both locally tabular [8] and hereditarily structurally complete [6]. Furthermore, they have the infinite Beth definability property [10]. These results have been collected in [5].

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