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A fuzzy logical approach to linguistic vagueness.

Some observations on the *Vagueness-as-Closeness* definition.

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To my grandfather Enzo

Io non sarei tanto drastico: penso che siamo sempre alla caccia di qualcosa di nascosto o di solo potenziale o ipotetico, di cui seguiamo le tracce che affiorano sulla superficie del suolo. Credo che i nostri meccanismi mentali elementari si ripetono dal Paleolitico dei nostri padri cacciatori e raccoglitori attraverso tutte le culture della storia umana. La parola collega la traccia visibile alla cosa invisibile, alla cosa assente, alla cosa desiderata o temuta, come un fragile ponte di fortuna gettato sul vuoto.
Per questo il giusto uso del linguaggio per me è quello che permette di avvicinarsi alle cose (presenti o assenti) con discrezione e attenzione e cautela, col rispetto di ciò che le cose (presenti o assenti) comunicano senza parole.

Italo Calvino - Lezioni americane

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Chapter 1

Introduction

The aim of this dissertation is trying an exploration of one of the most intriguing and, at the same time, puzzling aspect of ordinary language: its vagueness. Of course it is a deep theme, which can be treated from numerous points of view, and through a huge amount of instruments; however, we have choosen to deal with the problem of linguistic vagueness, through the tools provided by *fuzzy logic*. In particular, we examine a theory called *Fuzzy Plurivaluationism*, developed by the philosopher Nicholas J.J. Smith and based on a definition of semantic vagueness known as the *Vagueness-as-Closeness definition*.

There are two essential steps in our path: the first is represented by the issue of defining linguistic vagueness in a precise way, which allows us to try a formal approach. The last step concerns the interpretation of this definition and of the logical results, from the point of view of linguistic usage.

This dissertation is not animated by a contextual aim. Conversely, although I think that a detailed work based on a specific question is dutiful, in order to exclude approximation, nevertheless, an overview about the sense of the work, is fundamental. In this sense, I intend to present this introduction as a *zoom*: from an overall look, to a detailed explanation of the theme. I think it would be an interesting way to highlight the most significant facets that are involved in this theory, and, at the same time, to reveal the "underground rivers" that run under our choice of dealing with Smith's theory.

To illustrate the perspective from which this work arises, we may use the Nietzschean image of a philosopher, who is, from the beginning, forced to think on the icy and lonely heights, where the air is rarefied:

> Wer die Luft meiner Schriften zu athmen weiss, weiss, dass es eine Luft der Höhe ist, eine starke Luft. Man muss für sie geschaffen sein, sonst ist die Gefahr keine kleine, sich in ihr zu erkälten. [...] Philosophie, wie ich sie bisher verstanden und gelebt habe, ist das freiwillige Leben in Eis und Hochgebirge - das Aufsuchen alles Fremden und Fragwürdigen im Dasein, alles dessen, was durch die Moral bisher in Bann gethan war.[1]

An authentic philosopher is already on the top of the mountain. So, what is the strenght of our attempt, that is the philosophical reason of analising the vagueness in ordinary language? Well, I think that the winning strategy is starting by defining what is the common ground of philosophical problems, and in this sense, we must clarify from the beginning that, in my opinion, all the open philosophical questions are essentially problems of *assigning meanings*.

But let pass to the choice of the logical tool. If we consider the process of construction of meanings as a play of symbolic forms, we can interpret math, or rather, some algebraic concepts, as *forms*, in the ancient sense of εἶδος. As George Boole writes: "The mathematics we have to construct are the mathematics of the human intellect. Nor are the form and character of the method, apart from all regard to its interpretation, undeserving of notice."[2], and just on this wake, some mathematical notions - like "set", "class", "membership", "domain", and so on - would be legitimate tools for a semantic research about ordinary language. A semantic research which involves the complex wholeness of human conceptualisation, "juggling" with all its possible forms.

It is just the necessity of a precise definition of the logic of the natural language, that makes interesting an analysis for instance in the direction of polyvalent logical systems, particularly those based on special functions – called *t-norms* – on which it has been possible to build some semantics to assign evaluations and interpretations to ordinary language. In a logical-mathematical semantic research, in fact, we have the aim of building some

[1]Nietzsche F., *Ecce homo: Wie man wird, was man ist,* [1888], Deutscher Taschenbuch Verlag, (2005). English translation by Anthony M. Ludovici [1911], Dover Pubn Inc, (2004), : "He who knows how to breathe the air of my writings knows that it is an air of the heights, a bracing air. One must be made for it, otherwise the danger is no small one of catching cold in it. [...] Philosophy, as I have understood and lived it hitherto, is the voluntary living among ice and high mountains — the seeking-out of all things curious and questionable in existence, everything that has been put under a ban by morality hitherto."

[2]Boole G., *The Mathematical Analysis of Logic, being an Essay towards a Calculus of Deductive Reasoning*, [1847], Cambridge University Press, New York, (2009), 7.

systems of reference that allow us the right of modifying them, due to the results which represent a possible description of linguistic usage. However, the most significant aspect seems to be a philosophical *praxis*, which could be characterized as a construction or as a description, but that remains inevitably a pillar of the specific logical tool under exhamination. So – more generally- the philosophical activity is conceived as a practical attitude, in the sense that it draws it strenght from the intention of the agent, from which must not be separated.

Now, after having explained how we are led to a semantic research through the logical tools, this is the place to specify what we intend with the term *semantic vagueness of ordinary language*. In fact, this aspect finds its complete insertion in the statement that all the philosophical problems are essentially troubles about the *assignment of meaning*.

In ordinary language, we often find ourselves into situation where the assignment of a property to an object does not appear easily determinable, just because these properties seems not to be univocally characterizable trough a precise meaning. Moreover, the difficulty of determining the truth value of a sentence, also depends on the speakers' context of utterance, therefore I see these features as signals of considering the theme of semantic vagueness as an aspect of the assigment of meanings. However, the presence of this sort of sentences does not justify automatically the choice of a multi-valued logical approach: actually, the ways of studying semantic vagueness are manifold, and in this work we will try to substain the reasons of this guidance, which does remain, however, a possibility.

To zoom more, I have choosen to examine Smith's Fuzzy Plurivaluationism, which is based on the vagueness-as-closeness definition. To achieve this goal, we need first of all to specify the two fundamental "areas" that we try to keep as a reference background: (i) what is the *nature* of vagueness and of the many-valued interpretation about sentences, and then (ii) what could be the *number* of these interpretations, if one or more. These two "areas" could represent an interesting way to examine this argument.

In detail, this work is divided into two parts, that mirror respectively the two areas mentioned above: in chapters 1-4 we will tackle into question the nature of vagueness, emphasizing on where the problem of semantic vagueness arises, and what is the idea of vagueness implied by a degree-theory, that legitimates our proposal. In the middle of this part, there will also find space a detailed exposition of the logical tools under examination. In fact, I try an "experimental way", in the sense that I have added in this first part, among Smith's arguments, two chapter which are devoted to a close explanation of these fuzzy logical systems. Actually, the author does not examine the fuzzy framework in detail in his papers about vagueness, however, in my opinion it is a fundamental step, in order to achieve the goal of this survey, to be an example of a method of philosophical investigation, *tested* by logical instruments.

In chapter 5, instead, we will study some of the most significant logical-philosophical interpretations about the many-valued approach to ordinary language. Particularly, we will call into question a degree form of Plurivaluationism, which seems to be a entitled consequence of a degree approach to vagueness, about the number of many-valued semantics allowed about meanings.

This analysis is based on Smith's papers, like [24] , [25] , [26] ,[27] ,[28] , and especially on his main book: *Vagueness and degrees of truth* [Oxford, 2009].

To conclude, I would stress that the underlying path of the analysis aims to be *implicitly bidirectional*: on the one hand, this survey begins from the interest about many-valued logics (in their broader sense), and tries to prove that degrees of truth cannot be integrated with key developments in philosophy of language, outside the theme of vagueness. On the other side, we proceed from the philosophical problems linked with vagueness, to a legitimation of a bridge with fuzzy logical systems.

Finally, Chapter 7 is devoted to explain my *thesis*.

In a few words, I think that Smith's theory is an intriguing way to explore the huge theme of semantic vagueness, notwithstanding its intrinsic weaknesses. However, in my opinion, even if Smith's attempt of using a logical tool to examine this argument is legitimate, it works only because it covers a narrow domain of syntactic objects, and due to the analysis and the employment of only some parts of the fuzzy logical framework. Nevertheless, I am persuaded that these weaknesses (and their consequences) do not represent a final attack to Smith's position, the prelude of its sinking.

Rather, I suggest that the main objections should be answered through a refinement which follows three directions: extending the definition of vagueness-as-closeness beyond predicates, making some distinctions amongst the different categories of elements of discourse (we will further see in what sense), and finally, by a closer examination of the consequences of assuming a "worldly vagueness".

Summing up, even if Smith's position is usually known in literature as "fuzzy plurivaluationism", I think that the authentic crux of the matter is rather the vagueness-as-closeness definition: without assuming this sort of definition, fuzzy plurivaluationism does not hold. Therefore, if we want to improve fuzzy plurivaluationism, we must start by an appropriate modification on the *vagueness-as-closeness definition*.

Part I

SMITH ON VAGUENESS

Chapter 2

Defining vagueness

As we have said in the introduction, we are dealing with the *nature* of the many-valued approach to vagueness, which depends on how philosophical problems are formulated.

In particular, the open philosophical problems we will analize here, are inherently problems of *assignment of meaning*, related to the *use* of the language. The background is indeed that there is a parallelism between meaning and use, and so it is interesting to focus on how we can build semantical concepts, while we are describing our use of ordinary language .

In detail, Smith's aim is to consider how the speakers can assign meanings to propositions, whereas they are using vague *predicates*.

Actually, vagueness is a phaenomenon which interests terms belonging to different lexical categories, for example:

- adjectives ("high", "young", "orange", "pollute" . . .)
- adverbs ("shamefully", "quickly", . . .)
- substantives ("chair", "mountain", "old person", . . .)

or, semantically, vagueness may concern properties and/or objects.

Therefore - returning to the question of the assigments of meanings - the first problem that arises is if they are uniquivocally determined.

In this sense, the first thing to specify better is what kind of language we are dealing with. In this respect, it seems significant to consider the distinction between the "ideal language philosophers" and the "philosophers of ordinary language", that has been emerged in the middle of the last century. Very briefly, this bipolarity is manifested functionally as a fork of trends: on the one hand, we have a "constructive tendence" in which the work of the ideal linguist is essentially to build a vocabulary in order to clarify a specific ideal language. On the other side, the "commonlinguist" has a descriptive viewpoint of the *use of the language*, and then he tries to describe linguistic usage.

About this second statement, let us do some examples, to make more clear when people *use* vague predicates. Let us imagine a boy who has to colour his drawing, and imagine that the teacher says him to use a *dark-blue* pencil. Yet, the boy takes a dark blue pencil from his pencilcase, and colours the paint: he is sure that the teacher will praise him. But unespectedly, she scolds him.

Now, imagine a man who enters into a barbershop, asking to the coiffeur to cut his hair, without making him look like a *bald* man. After a few minutes, the coiffeur has finished, he is pleased with himself. He asks the man if he wants to cut his hair more, but the man is shocked and angry: he feels already like a bald man!

These two examples apparently different, instead, express the same problem: whether the boy in the first example, or the coiffeur in the last, are confused, because they are sure to have understood the interlocutor's will. So, what's the gap? From a descriptive viewpoint, based on the use of the language, the answer lies on the statement that both predicates "being dark blue" and "being bald" are sorts of vague predicates, because "dark blue" and "baldness" are properties not univocally definable. In fact, all the protagonists of our scenes are certain to have grasped the concept expressed by the other people, and actually they do. The problem is that their understanding of the meaning respectively of "dark blue" and "baldness", is based on how they *use* - or *would use* - these predicates.

At last, we can consider another example, which differs slightly from the previous, because it involves predicates covering an area of the human knowledge, which by its nature departs from the everyday uses of language: the scientific language, for example those used in Ecological reasearch.[1].

Even in this context, the vagueness of natural language characterizes human understanding, and in detail the representation of natural phaenomena, like for instance the evaluation of ecological conditions. Let us consider, for instance, the following passage:

> Concepts such as *poor* ecological status, *significant* impact, *good* ecosystem health, etc., are immediately and clearly understandable by everyone, and at the same time absolutely vague when they have to be

[1]For a complete description and a close examination of these ideas, see [13] .

translated into a set of rules or quantified into a numerical value.[2]

It emerges that vague predicates like "is significant" or "is good", could embroil also other fields of human investigation, and for this reason it is interesting to include them - even as examples - within the aspect of vagueness in the philosophy of language. Indeed, it seems to me that considering this facet is intriguing, because it is an example of how the entire human understanding - therefore, even science - is determined by linguistic usage, i.e. by the meaning that people assign to the words, expecially here to predicates.

I recognize of course that it is a huge theme, which would be treated deeper elsewhere; anyway, in this work we will content ourselves to keep it as a reference of the fact that the vagueness of the language could cover different areas of knowledge.

To sum up, the examples considered above lead us to two main philosophical consequences: first, that the researcher's activity is - as underlined in the introduction - conceived as a *practical* attitude, in the sense of the ancient *praxis,* that is the moral action that derives its value from the intention of the agent, and from which, therefore, can not be separated. A fascinating linguistic research, just because distant from being isolated and unsaddled from social and scientific effort, but rather aware of its dominating role in different human contexts, apparently far.

Lastly, it is an apted example of the background thesis of this dissertation, which - once again - consists in the statement that problems which arise concerning human knowledge, are primarily linguistic problems.

To conclude, after having specified the context on which we intend to move, we could keep it as a backbone for the problems we are going to examine, and we can start with the analysis of the problem concerning the *nature* of vagueness linked to the many-valued approach.

2.1 Where is vagueness?

The first significant question is where vagueness may be localized. It is not a secondary inquiry, because asking where is vagueness is already a wondering about what vagueness is.

In literature, in this sense there are two main perspectives: the first recognizes vagueness in the relatioship between language and the world, the second sees it in the world itself. In other words, we can read vagueness as a *semantic* or *metaphysical* phenomenon.

[2] [13],117.

Indeed, these prospects assume a common specific point of view, that we call *semantic realism.* Semantic realism we consider in this circumstance, is defined as the idea that, when we speak vaguely, we are playing with three different "characters": our language, the world and semantical relations among them: vagueness lies in one of these places. Furthermore, semantic realism implies that relations of reference are real, although it does not mean automatically - as we will see further - that there is a unique determinate reference.

About this last statement, it is indeed a deep and open question yet, that we will try to find an answer at the end of the dissertation.

Let's start with *vagueness-in-language* approach. It is based on the idea of *no predication without correspondence*, which means that vagueness concerns the difficulty to express correctly and univocally this relationship with ordinary language. From this point of view, concepts and objects have specific properties because the world is determined and precise. Each concept has an univocal meaning, and if we were able to express ourselves precisely, we would describe perfectly the concepts we are dealing with. The problem for the theorists of this perspective, rises just when one tries to describe the world, because despite their best efforts, they fail to grasp and to express exactly the original meanings.

On the other hand, *worldly vagueness* means that properties and objects themselves are vague. The difference between this position and the previous is that here it is inherently impossible to represent the meaning of a proposition or of a predicate, because it does not depend on our representational capacities.

To sum up, hereinafter - expecially at the end of the work - we will try to argue that if we assume a many-valued approach, we are led to support this last thesis.

Therefore, we could say that there are two main questions involved linked together with the many-valued approach: where locate vagueness and what could be a correct definition of vagueness; if we aim to legitimate this approach we must take into account these two elements.

2.2 Which vagueness?

The first thing to underline is that, if we want to determine what is the correct theory of vagueness we must give a *fundamental* definition of vagueness.

In particular, if we have fundamental definition of a property, a phenomenon or an object, this definition must be essentially *useful,* which means that it must be able to account how vague language is used by the speakers.

So, we begin focusing on a particular notion - called *closeness* - in order to build a definition of vagueness based on this concept, and we will try to show that this definition satisfies both the requirements. To do this, we will follow essentially the arguments of the viewpoint provided by the philosopher Nicholas J.J. Smith, expecially we will take into account some papers published in the last ten years[3].

Finally, in the third chapter it will be possible to justify a link between vagueness-as-closeness definition and the many-valued approach.[4]By this way, we will also see why some of the following definitions could not be complete in order to define a notion of vagueness.

The first of these definitions is *the borderline case idea*. From this point of view, a predicate P is defined to be vague if there is not a perfectly sharp dividing line between the cases to which P applies, the cases to which P does not apply, and the borderline cases.[5]

For instance, if we consider the predicate "is polluted", we have the following definitions:

I. If x has a level of concentration of benzene in the air less than $3\mu g/m^3$, then "x is polluted" is false.

II. If x has a level of concentration of benzene in the air more than $10\mu g/m^3$, then "x is polluted" is true.

However, the predicate "is polluted" has a borderline case: all zones having a concentration of benzene in the air between $3\mu g/m^3$ and $10\mu g/m^3$.

It is an important observation, because some supporters of vagueness-as-closeness perspective - like Nicholas Smith - argue that it is a sign that this definition cannot be taken as fundamental about vague predicates. In fact, here the problem is a matter of *uncertainly* about the application of the predicate, thus, it is the application of predicates to be vague, not the predicate itself. Therefore, giving rise to borderline cases is like a *mark* of vagueness, rather then a constitutive element of it. [6]

[3] See [24] ,[25] ,[28] and [29].

[4] It is important to note that here emerges the *bidirectionality* mentioned above: vagueness as closeness leads us to the many-valued approach, but at the same time, if we want to start with degree-systems related to vagueness in ordinary language, we see that they need a sort of notion of Closeness.

[5] Here we will deal expecially with predicates and properties.

[6] For a closer examination of this definition - expecially about its weaknesses - see: [19] .

Moreover, in this sense we could not exclude the case in which there would be some precise predicates which have borderline cases simply due to our ignorance about them, or because of our uncertainty as to whether these predicates apply to some objects.

Linked to this idea, we could mention *the blurred boundaries idea.* This concept is similar to the previous, but the difference is that it is briefly representable by a line drawn *around* all the things to which the predicate could be applied.

Actually, there is a huge amount of definitions of vagueness, but a close description of the whole scenario goes beyond the aims of this analysis, which follows essentially Smith's argument. Rather, we will content ourselves to dwell finally - with Smith - on the definition of vagueness as sorites susceptibility.

2.2.1 Sorites-susceptibility

Last but not least, the *Sorites-susceptibility idea.* It needs a special paragraph because it will reveal to be a key-notion to explore the link between vagueness-as-closeness and the many-valued approach.

Who supports this definition, argues that the philosophical problem of vagueness is saying what vagueness is, in order to express the sorites- paradox. In other words, vague predicates are those that give rise to this paradox.

But, although it is easy to recognize, intuitively, that soritical predicates are vague, can we automatically conclude that all vague predicates are soritical?

To answer this question, we must remind how this paradox is classically presented:

1 grain of wheat does not make a heap.

If 1 grain of wheat does not make a heap then 2 grains of wheat do not.

If 2 grains of wheat do not make a heap then 3 grains do not.

$\vdots$

If $9,999$ grains of wheat do not make a heap then $10,000$ do not.

$\therefore 10,000$ grains of wheat do not make a heap.

Indeed, this argument could be represented in many ways: a common form of this paradox,could be, for instance, the following: let P represents a soritical predicate (e.g. "is bald", or "does not make a heap") and let the expression a_n (where n is a natural number) represents a subject expression in the series.

Then, the sorites proceeds by way of a series of conditionals, and can be schematically represented as follows:

Pa_1
If Pa_1 then Pa_2
If Pa_2 then Pa_3
$\vdots$
If Pa_{n-1} then Pa_n

$\therefore Pa_n$(where n can be arbitrarily large).

Another variant is given by replacing the set of conditional premises with a universally quantified premise. So, the sorites paradox is seen as proceeding by the inference pattern known as mathematical induction (where n is a variable ranging over the natural numbers):

Pa_1
$\forall n\,(Pa_n > Pa_{n+1})$

$\therefore\ \forall n Pa_n$

Actually, returning to the question at the beginning of the paragraph, we will see that also this definition of vagueness fails to our requests. In particular, in the fifth chapter - after having explained the main many-valued systems and the legitimacy of the vagueness-as-closeness definition - we will be able to support two claims about the Sorites-susceptibility idea: first, that this idea could not be fundamental, because it does not belong to the definition of vagueness; and then we will prove that, if we suppose that a predicate conforms to Closeness, we can see both why a Sorites paradox for this predicate is persuasive, and also how the paradox is mistaken.

2.2.2 Closeness

Intuitively, for any set S of objects, and any predicate P - vague or not - a competent user of P can distinguish the relationships of closeness or *nearess* or *similarity* between the members of S, in the respects that are relevant to whether something is P. For instance, let think about a pencilcase full of coloured pencils, with several shades of each basic colour (e.g. instead of only "one red", we have scarlet red, vivid red, purple-red, bordeaux, copper-red, mahogany, and so on). So, given any colour predicate P, ordering these pencils means that we have to put those which are closer together, in $P-$relevant respect, and this is quite different from what we do when we select a P coloured pencil.

Another example could be to consider the term "polluted" and the set of all italian basins. As competent users of the term "polluted", we will discern some relationships of closeness or similarity amongst these objects, in pollute-relevant respect. In other words, in the respect that determine whether something is pollute, some lakes - for instance those which are near the factories - are closer to the pollutes lakes than the sources in high mountain.

Now, let us think about a large set of basins, with several gradations of the basic characteristic (e.g polluting by barium, by arsenic, by mercury, by cyanide, by selenium, by asbestos, by dioxin, and so on). We can easily distinguish the task of ordering the basins in a classification (i.e in an increasing order of polluteness), from the task of identifying, for instance, the most carcinogen (or the most mutagenic, etcc) substances. So, given any predicate P, what we are doing when we *order* the basins, is putting which are similar in $P-$relevant respect, close together.

However, although for each type of predicate we can find closeness relationships, we need to distinguish two sorts of similarity relationships that are apparent to competent users and speakers: relationships of *relative* closeness, and relationships of *absolute* closeness.

Let start with relative similarity. For example, in respect relevant to whether something is a heap, the twenty-grain pile of sand is closer to the twenty-one grain pile of sand than is the ten-grain pile of sand.

In general, given a set of objects S and a predicate P, we want to represent the relative closeness relationships on that set, associated with the predicate. A simple and general way to do this, is directly in terms of a three place relation

$x \overset{P}{\leq_z} y$ "x is at least as close to z as y is, in $P-$relevant respect."

However, we can also extend this terminology for binary relations, assuming that this relation is a linear order, which means that it is:

- transitive: $\forall x, y, z, w\,(((x \leq_w y) \wedge (y \leq_w z)) \rightarrow (x \leq_w z))$
- reflexive: $\forall x, y\,(x \leq_y x)$
- antisymmetric: $\forall x, y, z\,(((x \leq_z y) \wedge (y \leq_z x)) \rightarrow x = y)$.

Conversely, absolute closeness is just the notion involved in the Sorites paradox. For example, as far as the polluteness, we could consider a situation in which 65% of samples from the Lake Como are beyond the limits allowed by the law, Lake Garda has 33% of samples outlawed and Lake Iseo has 45% of samples outlawed. Therefore, we can say that in absolute sense Lake Como and Lake Iseo are very close in respects relevant to whether something is pollute.

More generally, given a set of objects and a predicate P, and an associated structure of relative closeness relationships, we could represent this addictional structure of absolute closeness by a two-place relation:

$x \overset{P}{\approx} y$ "x is very close to y, in $P-$relevant respect".

To sum up, if we aim to provide a general theory, which codifies our intuition about closeness of objects, in respect relevant to whether something is P - for a given predicate P - we have first to determine the relevant respects. Then, we have to associate each respect to a numerical scale, giving rise to a vector space, where each object corresponds to a vector whose coordinates are the numbers to which the object is associated on each numerical scale. Now, relative closeness could be extracted via the idea that x is at least as close to z as y is, just in case the distance between x and z is less than, or equal to the distance between y and z. On the other hand, absolute closeness may be extracted via the selection of a particular number d , and here the idea is that x and y are very close just in case the distance between them is less than d.

2.2.3 Vagueness as Closeness

Now, we are able to examine how we can define vagueness as closeness[7].

The closeness picture of vague predicates is the closeness of x and y in $P-$relevant respect, which means the closeness of "$F[x]$" and "$F[y]$" in respect of truth.[8] To be more precise, a predicate P is vague just in case it satisfies the following condition, for any objects x and y:

Closeness. If x and y are very close in $P-$relevant respects, then "Px" and "Py" are VERY CLOSE in respect of truth.

But what does *closeness in respect of truth* mean? Actually, it could be a particular instance of the more general notion of closeness in respect of a property, where the property is *truth;* but indeed it is not like a general predicate, and for this reason it turns out to be a fundamental element for our

[7]Here we will follow the reasoning of N. Smith in *Vagueness and degree of truth*, where the idea of vagueness as closeness is characterized and justified.

[8]Here, $[x]$ and $[y]$ are singular terms which refers respectively to x and y.

proposal, to conceive the range of the truth values as a continuous interval between 0 and 1. This definition will prove to be essential for our goals, so it is better here to postpone this issue, and resume it in due course.

However, there is a fundamental point to underline at the beginning: there is a huge difference between being *close* in respect of truth and being *identical* in respect of truth. In other words, it is important here to rule out that each predicate P is *tolerant*[9].

But if we do not consider now this definition of Tolerance in respect of truth, it turns out to be important here to justify the correctness of the Closeness definition. Indeed, Smith argues that there are two main features that allow it to be a good one.

First, he shows that the predicates we call 'vague', do in fact have the same nature as predicates which conform to the closeness definition.

At this point, it seems to be advantageous to consider a position - expressed by Crispin Wright - in the philosophy of language, which is called *governing view.* This definition has two main characters:

(i) a mastery of a language consists in the internalization of a set of semantic and syntactic rules, that are definitive of that language;

(ii) a mastery of a language can obtain an explicit knowledge of the rules of which they have an implicit understanding.

Wright's idea is that if the governing view is correct, then vague predicates are tolerant.

Tolerance. If a and b are very close in $P-$relevant respect, then "Pa" and "Pb" are IDENTICAL in respect of truth.

It is evident that the difference between the Smith's proposal and the Wright's one lies on the locutions "very close" and "identical".

Well, it is easy to understand - with Smith - that all the considerations in favour of the idea that vague predicates conform to Tolerance, are equally considerations in favour of the idea that vague predicates conform merely to Closeness. In fact, the main leap here is consider Tolerance a special case of Closeness. In other words, Closeness without Tolerance generates no contradictions. Indeed, an important feature of Closeness is giving tolerance intuitions without incoherence.

But this reasoning is supported by examples; in particular, Smith uses the same examples given by Wright, in order to prove that they support

[9]To say that P is *tolerant* is to say that very small differences between objects in $P-$relevant respect never make *any* difference to the application of the predicate P.

the thesis that vague predicates are tolerant, but also the view that vague predicates conform merely to Closeness (and not Tolerance).

But let's do an example. "being a heap" is a predicate of casual observation, so there cannot be a difference of just one grain, between a thing to which this predicate "being a heap" can be applied, and a thing to which this predicate does not apply. In this case, a *negligible* difference (for instance of one grain) between two objects, makes a insignificant difference to the applicability of the word "heap", but - on the contrary - many insignificant differences put together are visible to casual observer. The main point here, once again, is that, in order for a predicate to be *usable* in a context of casual observation, there must not be any difference in application of the predicate to objects that cannot be told apart by casual observation.

Therefore, this example supports the thesis that vague predicates do not conform to Tolerance: they simply conform to Closeness (with Tolerance) .

The second of these features is that vagueness-as-closeness definition could include two of the other definitions of vagueness mentioned in **2.2.**: the borderline cases definition and the blurred boundaries definition. In other words, starting with the Closeness definition, we can obtain also those that before we have proved to be incomplete.

Actually, Smith presents this argument as a kind of feature of the Closeness definition; he does not say expressly that they could be arguments in support of the vagueness as closeness definition. Nevertheless, this last statement seems significant to me, due to the yearning of this definition to be fundamental.

About the borderline cases definition, a predicate which satisfies Closeness admits of borderline cases. In fact, if we consider a predicate P (which conforms to Closeness), and a Sorites series $x_1, ..., x_n$ for P, we can say that Px_1 is true and Px_n is false. But, given Closeness, there could not be an i such that Px_i is true and Px_{i+1} is false. Therefore, there must be sentences Px_i which are neither true, nor false, so the corresponding objects x_i are borderline cases for P.

As far as the blurred boundaries definition, given Closeness, the extension of P amongst this set cannot consist in a sharp line between the elements which belong to P, and which do not.

To take a concrete example, let us consider again the term "pollute" and suppose that it conforms to Closeness. This term does not cut a sharp band out of the landscape: as one moves across the points of the landscape, small steps in pollute-relevant respects can never make for big changes in the truth of the claim that the point one of the landscape we are considering is pollute.

By small steps, one can move from a point which is surely pollute, to one which is certainly not: but there is no sharp boundary between them, that can be crossed in one small step.

The reader may note that it seems we have forgotten the Sorites-susceptibility definition. Indeed, we have already said that in the third chapter we will talk about it and the link with the vagueness-as-closeness definition.

2.2.4 Classify vagueness

The predicates we have hitherto considered, seem to meet Closeness across the entire domain of discourse, but actually - as Smith suggests - in ordinary language we deal also with predicates which do not satisfy this condition.

Consider the following predicates:

(i) "is pollute";

(ii) "has ingested exactly 45mg/kg of arsenic" (abbreviated E);

(iii) "is a high-toxic substance";

(iv) "has ingested greater-than-or-equal-to exactly 45mg/kg of arsenic" (abbreviated O).

First of all, consider (ii). If Alice has ingested exactly 45mg/kg of arsenic, while Bob has ingested 40mg/kg of arsenic, then Alice and Bob are very close in respect relevant to whether a thing has the predicate E. Therefore, "Alice has the predicate E" is true, and "Bob has the predicate E" is false. It is evident that these two sentences are not very similar in respect of truth, so Closeness seems to be violated here, but the reader might have the intuition that E is vague.

The same (odd) thing happens if we consider (iii). If we have the same quantity of cyanide and selenium (whose medial lethal dose for humans are respectively 10mg/kg and 5mg/kg), then these quantities are very close in respect relevant to the application of the predicate "is a high-toxic substance". So, the sentence "cyanide is a high-toxic substance" is true, and the sentence "selenium is a high-toxic substance" is false. Closeness is violated here too and yet, intuitively, the predicate "is a high-toxic substance" is vague.

On the contrary, consider (iv). The predicate also fails to satisfy Closeness, but intuitively it is not vague.

It seems evident that the framework presented until now, cannot explain these situations.

In particular, what are the differences between (ii) and (iii) on the one hand, and (iv) on the other?

The answer is based on the idea that there is a subset of the domain of discourse over which (iv) does not trivially satisfy Closeness. In order to show that, we can consider for instance the subset consisting of people which have ingested either less than 5mg/kg of arsenic, or more than 100mg/kg of arsenic.

We can avoid this problem through three definitions:

- A set S is $P-$*connected* if and only if, for any two objects in S, either they are very close in $P-$relevant respects, or they can be connected by a chain of objects, all of which are in S - with adiacent members of the chain being very close in $P-$relevat respects.

- A set S is $P-$*uniform* if and only if, for every a and b in S, Pa and Pb are very similar in respect of truth. If a set is not $P-$uniform, it is called $P-$*diverse*.

- A predicate satisfies Closeness *over a set* S if and only if, it satisfies Closeness when the initial quantifiers "for any object a and b" in Closeness are taken as ranging only over S.

Given these definitions, it is possible here to enunciate the final definition of vagueness, on which we will base on:

Vagueness as Closeness. A predicate P is vague if and only if, there is *some* $P-$connected, $P-$diverse set S of objects, such that P satisfies Closeness over S.

To conclude this part, we could set up the information obtained, and classify vague predicates in two main groups, which also mirror our intuitions: *totally vague* and *partially vague* predicates.

- A predicate is *totally vague* iff it satisfies Closeness over *every* $P-$connected, $P-$diverse set of objects. An example of a totally vague predicate is (i).

- A predicate is *partially vague* iff it is vague but not totally, and some examples of this group are (ii) and (iii)[10].

[10] About (iv), it is a non-vague predicate, so it is actually out of our interest here. Nevertheless, Smith defines it using his definition of vagueness as Closeness. In particular, he argues that any predicate P which applies equally to everything is non-vague because if P applies equally to everything, then there are no $P-$diverse sets.

2.2.5 Other formulations of Closeness

Now, let us finish with a little gloss: we could provide other formulations of the vagueness-as-Closeness definition.

Suppose that we have a domain of discourse D, and a set T of truth values. Consider a function from D to T which assigns to each object x in D, the truth value of the sentence Px. This function is called the *characteristic function* of the predicate P.

Let $[Px]$ be the value of this characteristic function for P at the object x; and let $\overset{P}{\approx}$ be the relation on D of being very close in $P-$relevant respect. Moreover, let $\approx_T$ be the relation on T of being very close in respect of truth. By this way, the Closeness condition could be stated thus:

$$x \overset{P}{\approx} y \Rightarrow [Px] \approx_T [Py]$$

Thus, we may also state the Closeness definitions as:

Closeness′. If a and b are very close in $P-$relevant respects, then they are very close in respect of P.

This alternative definition is interesting because it reveals that the Closeness definition of vagueness in terms of one-place predicates, can be generalized to many-placed predicates. In particular, the $n-$place predicate R is vague if and only if the $n-$tuples $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ are very close in $R-$relevant respects, then $R(x_1, ..., x_n)$ and $R(y_1, ..., y_n)$ are very close in respect of truth.

For instance, if (petroleum, Mediterranean Sea) and (lead, North Sea) are very close in respect relevant to whether the first mentioned element pollutes the second, then "Petroleum pollutes the Mediterranean Sea" and "Lead pollutes the North Sea" are very close in respect of truth.

But the most important observation we can do, from this alternative definition of Closeness, is that the account applies not just to predicates, but to their wordly counterparts: properties and relations.

* * *

To conclude, in this chapter we have considered a particular definition of vagueness, known as Vagueness-as-Closeness definition, formulated by Nicholas Smith. This definition is significant among others because it helps us to review the question of linguistic vagueness as a matter of degrees of truth. We have shown why the Closeness definition should be a fundamental and useful definition of vagueness, unlike those based on the borderline cases, the blurred boundaries idea and the sorites susceptibility. We have been able to provide a definition of vagueness based on Closeness, and then, we have classified vague predicates, specifying when a predicate is totally vague, and when it is not.

Furthermore, this vagueness-as-closeness definition could say something about where we could locate vagueness, and what could be the philosophical scenario of this issue. In fact, even if the argument of this dissertation is a question about if a fuzzy approach to linguistic vagueness is legitimate, the overall aim of this work remains of course - as underlined at the beginning of this chapter - where is vagueness and what kind of entities we must relate to, when we think about linguistic vagueness.

Chapter 3

Fuzzy systems: the syntactic framework

In this chapter, we will acquaint some many-valued systems. *Many-valued logics* are non-classical logics, which are similar to the classical one because they accept the principle of truth-functionality, namely, that the truth of a compound sentence is determined by the truth values of its component sentences (and so, remains unaffected when one of its component sentences is replaced by another sentence with the same truth value). However, they differ from classical logic by the fundamental fact that they do not restrict the number of truth values to only two: they allow for a larger set of truth degrees.

Many-valued logic as a separate subject was created by the Polish logician and philosopher Jan Łukasiewicz (1920), and developed first in Poland. The outcome of these investigations are in fact the Łukasiewicz systems, and a series of theoretical results concerning these logics.

Essentially parallel to the Łukasiewicz approach, the American mathematician Post (1921) introduced the basic idea of additional truth degrees, and applied it to problems of the representability of functions. Later on, Gödel (1932) tried to understand intuitionistic logic in terms of many truth degrees. The result was the family of Gödel systems, and an achievement, namely, that intuitionistic logic does not have a characteristic logical matrix, with only finitely many truth degrees.

The 1950s saw (i) an analytical characterization of the class of truth degree functions definable in the infinite valued propositional Łukasiewicz system by McNaughton (1951), (ii) a completeness proof for the same system by Chang (1958, 1959) introducing the notion of MV-algebra and a more traditional one by Rose/Rosser (1958), as well as (iii) a completeness proof for the infinite valued propositional Gödel system by Dummett (1959). The 1950s also saw an approach of Skolem (1957) toward proving the consistency of set theory in the realm of infinite valued logic.

In the 1960s, Scarpellini (1962) made clear that the first-order infinite valued Łukasiewicz system (L_∞) is not (recursively) axiomatizable. Hay (1963) as well as Belluce/Chang (1963) proved that the addition of one infinitary inference rule, leads to an axiomatization of L_∞. And Horn (1969) presented a completeness proof for first-order infinite valued Gödel logic.

Besides these developments inside pure many-valued logic, Zadeh (1965) started an application oriented approach toward the formalization of vague notions, by generalized set theoretic means, which soon was related by Goguen (1968/69) to philosophical applications, and which later on inspired also a lot of theoretical considerations inside Many-valued logics.

Multi-valued logic is closely related to *fuzzy logic*, although it is fundamental not to superimpose them.

The notion of fuzzy subset was introduced by Zadeh as a formalization of vagueness; i.e., the phenomenon that a predicate may apply to an object not absolutely, but to a certain degree. In fact, as in multi-valued logic, fuzzy logic admits truth values different from "true" and "false". As an example, usually the set of possible truth values is the whole interval $[0, 1]$.

More precisely, there are two approaches to fuzzy logic. The first one is very closely *linked with multi-valued logic tradition* (Hajek school). So, a set of designed values is fixed, and it enables us to define an entailment relation. The deduction apparatus is defined by a suitable set of logical axioms and inference rules.

Another approach (Goguen, Pavelka and others) is devoted to defining a deduction apparatus, in which approximate reasonings are admitted. Such an apparatus is defined by a appropriate fuzzy subset of logical axioms and by a suitable set of fuzzy inference rules. In the first case, the logical consequence operator gives the set of logical consequences of a given set of axioms. In the latter, the logical consequence operator gives the fuzzy subset of logical consequences, of a given fuzzy subset of hypotheses.

It is evident that due to che context we will aim to explore - vagueness in ordinary language - we assume the notion of "fuzzy logic" expressed by the first approach.

As regard the synopsis of the chapter, we will present first the BL logic which is a many-valued system formulated at the end of the XX century by Petr Hajek, and its most important extensions: Łukasiewicz logic, Gödel logic and Product logic, which have been framed independently during the first half of the last century. In particular, here we will focus on the *syntactic* point of view; however, some semantic notions cannot be ignored at the beginning, for a purpose of completeness.

3.1 Propositional calculi

3.1.1 Preliminaries

The aim of a many-valued propositional system is to generalize classical propositional logic, considering as the set of values the real range $[0, 1]$ instead of the set $\{0, 1\}$. In this unit interval, 1 represents absolute truth and 0 absolute falsity.

Furthermore, this set of degrees of truth is linearly ordered, which means that it is equepped with a linear order.[1]

Convention. Henceforth, in this chapter we will use the expression *degrees of truth* in place of *truth values,* reserving the last locution to classical contexts.

Language. A propositional language consists of:

- a set of *propositional constants*: $c, d,...;$
- a set of *propositional variables:* $x, y, ...;$
- *connectives:*$\vee, \wedge, \neg, ...;$
- *truth constants:* $\bar{0}$ and $\bar{1}$;
- *formulas*: propositional variables and propositional constants are formulas; if α, β are formulas, then $(\alpha \to \beta)$,$(\alpha \wedge \beta)$,$(\alpha \vee \beta)$,$(\neg\alpha)$ are formulas. There are no other formulas.

3.1.2 Few semantic drops.

The main systems of *Many-valued logics* often come as families, which comprise uniformly defined finite-valued, as well as infinite-valued systems.

[1] A linear order is a relation on the set X, having the properties:

- Reflexivity: $\forall a \in X, a \leq a$
- Antisimmetriy: $\forall a, b \in X$ if $a \leq b$ and $b \leq a$, then $a = b$
- Transitivity: $\forall a, b, c \in X$ if $a \leq b$ and $b \leq c$, then $a \leq c$
- Totality: $\forall a, b \in X$ $a \leq b$ or $b \leq a$

In particular, the sets of degrees of truth we consider in this work (related to the notion of vagueness-as-closeness)[2] could be defined by a logical matrix, which has :

- the finite set $W_m = \{k/(m-1), 0 \leq k \leq m-1\}$ of rationals within the real unit interval;
- or the whole unit interval $W_\infty = [0,1] = \{x \in R, 0 \leq x \leq 1\}$ as the truth degree set.

If we consider the last of these two sets - the whole unit interval $[0,1]$ - we could talk about *t-norms based systems*, in which each $n-$ary connective has a corresponding characteristic truth function f_c such that:

$$f_c\,[0,1]^n \rightarrow [0,1]\,.$$

Moreover, it must be introduced the semantic concept of *evaluation.*

An evaluation of propositional variables is a mapping e assigning to each propositional variable p its truth value $e(p) \in [0,1]$.

Definition 1 We can extend the notion of evaluation to all formulas as follows:

$e(\bar{0}) = 0;$
$e(\alpha \rightarrow \beta) = (e(\alpha) \Rightarrow e(\beta))$ (for each α, β formulas);
$e(\alpha \otimes \beta) = (e(\alpha) * e(\beta))$ (for each α, β formulas);
$e(\alpha \wedge \beta) = \min(e(\alpha), e(\beta))$ (for each α, β formulas);
$e(\alpha \vee \beta) = \max(e(\alpha), e(\beta))$ (for each α, β formulas).

3.1.2.1 Operations on $[0,1]$.

We define some operations on the intervals, which simulate conjunction, disjunction, implication and negation.

Conjunction Conjunction $(\wedge)$ is semantically interpreted with the truth function $f_\wedge$, which by convention we write as $*$ and that is called *t-norm.*

A *t-norm* is a binary operation $*$ on $[0,1]$ such that, for each x, y, z belonging to $[0,1]$, the following conditions are satisfied:

(i) $*$ is commutative: $x * y = y * x$

(ii) $*$ is associative: $(x * y) * z = x * (y * z)$

[2]See further 5.2.

(iii) $*$ is monotonic:

- if $x_1 \leq x_2$ then $x_1 * y \leq x_2 * y$
- if $y_1 \leq y_2$ then $x * y_1 \leq x * y_2$

(iv) $1 * x = x$, $0 * x = 0$

Definition 2 Given a binary function g , we can say that $x \in Dom(g)$ is an *idempotent* of g if $g(x, x) = x$.

Definition 3 An element x is *nilpotent* if there is an n such that $x^{*n} = 0$, where $(x^{*n} = \underbrace{x{*}x{*}...{*}x}_{n})$.

REMARKS

(v) A t-norm $*$ is called *continuous* if and only if, as a function from $[0,1]^2$ to $[0,1]$, it is continuous according to the usual definition of continuous function in a interval.[3]

(vi) A t-norm is *Archimedean* if it has not idempotents except 0 and 1.

(vii) An Archimedean t-norm is *strict* if it has no nilpotent elements except 0, otherwise it is *nilpotent*.

Example 1 The following are our most important examples of continuous t-norms:

- *Łukasiewicz t-norm*: $x * y = \max(0, x + y - 1)$,
- *Gödel t-norm*: $x * y = \min(x, y)$,
- *Product t-norm*: $x * y = x \cdot y$.

[3]Let f to be a function such that f: $\mathbb{R}_1 \to \mathbb{R}_2$, f is continuous in $x_0 \in \mathbb{R}_1$ if $\lim_{x \to x_0} f(x) = f(x_0)$. A function is continuous in a interval iff it is continuous in each point of the interval.

Disjunction. The truth function of the disjunction $f_\vee$ is called $t-$ *conorm* and we write it with the symbol $\diamond$.

A t-conorm is a binary (continuous) operation on $[0,1]$, such that for each x, y belong to $[0,1]$:

(i) $\diamond$ is commutative

(ii) $\diamond$ is associative

(iii) $\diamond$ is monotonic

(iv) $1 \diamond x = 1$; $0 \diamond x = x$

Implication. Regarding to the implication, the corresponding truth function is called *residuum*, and it is denoted by the symbol $\Rightarrow$.

Given a (continuous) t-norm, there is a unique operation namely
$x \Rightarrow y = \max\{z \mid x * z \leq y\}$
satisfying the condition such that
$\forall x, y, z \in [0,1]$,$(x * z) \leq y \Leftrightarrow z \leq (x \Rightarrow y)$.

REMARKS

(i) Semantically, given a t-norm and its residuum, we can define on $[0,1]$ min and max respectively as:

- $x \cap y = x * (x \Rightarrow y)$
- $x \cup y = ((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)$

(ii) For each countinuous t-norm $*$ and its residuum $\Rightarrow$, there are the following relationships:

- $x \leq y$ iff $(x \Rightarrow y) = 1$
- If $x \leq y$ then $x = y * (y \Rightarrow x)$
- If $x \leq u \leq y$ and u is *idempotent*, then $x * y = x$.

Example 2 The following are our most important examples of residua of the three continuous t-norms mentioned in **Example 1**:

- *Łukasiewicz implication*: $x \Rightarrow y = 1 - x + y$,

- *Gödel implication*: $x \Rightarrow y = y$,
- *Product implication*: $x \Rightarrow y = \frac{y}{x}$.

Negation. Negation ($\neg$) is represented by a unary function $f_\neg : [0,1] \to [0,1]$ and we will symbolize it with the letter n . This function is:

(i) *nonincreasing*[4]

(ii) classical on boolean elements: $n(0) = 1$, $n(1) = 0$

REMARKS

(iii) A negation is *strict* if in addiction to having the properties (i) and (ii), it is also continuous and decreasing.[5].

(iv) A negation is *involutory* if it is strict and if $n(n(x)) = n$.

(v) About ordering relations, a negation inverts the order of the elements.

3.1.3 The *Basic many-valued logic.*

The *Basic many-valued logic* (henceforth BL) is a logicl formal system, framed by Petr Hajek.

The importance of BL is multiple: on the one hand it has been shown that this formal system can be a suitable context to deal with continuous t-norms, furthermore, BL summarizes all the logical polyvalents systems studied previously, as the logic of Łukasiewicz, the logic of Gödel and the Product logic, which can be obtained as extensions of BL.

3.1.3.1 Propositional calculus

The *BL propositional calculus* is based on the following language:

- Propositional variables;

- Propositional constants $\bar{1}$ and $\bar{0}$;

- Connectives:

    - primitive symbols:

[4]A function is *nonincreasing* on X if $\forall x_1, x_2 \in X, x_1 < x_2 \to f(x_1) \geq f(x_2)$

[5]A function is *decreasing* on X if $\forall x_1, x_2 \in X, x_1 < x_2 \to f(x_1) > f(x_2)$

$\otimes$ (strong conjunction)
$\rightarrow$(implication)

- defined symbols:
$\wedge$(weak conjunction) $\alpha \wedge \beta \equiv \alpha \otimes (\alpha \rightarrow \beta)$
$\neg$(negation) $\neg\alpha \equiv \alpha \rightarrow \bar{0}$
$\vee$(disjunction) $\alpha \vee \beta \equiv ((\alpha \rightarrow \beta) \rightarrow \beta) \wedge ((\beta \rightarrow \alpha) \rightarrow \alpha)$
$\leftrightarrow$(equivalence) $\alpha \leftrightarrow \beta \equiv (\alpha \rightarrow \beta) \otimes (\beta \rightarrow \alpha)$;

- Formulas are defined in the usual way: each propositional variable is a formula; $\bar{0}$ is a formula; if α, β are formulas, then $\alpha \otimes \beta$ and $\alpha \rightarrow \beta$ are formulas.

Given the interval $[0, 1]$ with a *t-norm* $*$ and a *residuum* $\Rightarrow$, the functions associated with connectives $\otimes$ and $\rightarrow$ are respectively

$$f_{\otimes} = * \quad ; \quad f_{\rightarrow} = \Rightarrow$$

3.1.3.1.1. Axiom schemata Petr Hajek [9] has introduced a deductive system *a la* Hilbert where the only deductive rule is Modus Ponens (henceforth *MP*). This system is based on the following axiom schemata:

(A1) $(\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma))$

(A2) $((\alpha \otimes \beta) \rightarrow \alpha)$

(A3) $(\alpha \otimes \beta) \rightarrow (\beta \otimes \alpha)$

(A4) $(\alpha \otimes (\alpha \rightarrow \beta)) \rightarrow (\beta \otimes (\beta \rightarrow \alpha))$

(A5a) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \otimes \beta) \rightarrow \gamma)$

(A5b) $((\alpha \otimes \beta) \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \rightarrow \gamma))$

(A6) $((\alpha \rightarrow \beta) \rightarrow \gamma) \rightarrow (((\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \gamma)$

(A7) $\bar{0} \rightarrow \alpha$

Definition 4 A formula of propositional logic is a *tautology* if the formula itself is always true regardless of which valuation is used for the propositional variables.

Proposition 1. Each axiom's evaluation is a tautology.

Proof.

(A1) We have to prove that $(x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z)) = 1$
For the definition of minimum, we have$(x * (x \Rightarrow y)) = (x \bigcap y) \leq y =$
$= (y * (y \Rightarrow z)) \leq z$
and then $(x * (x \Rightarrow y) * (y \Rightarrow z)) \leq y * (y \Rightarrow z)$.
Instead, from $y * (y \Rightarrow z) \leq z$
we obtain $(x * (x \Rightarrow y) * (y \Rightarrow z)) \leq z$
and for the definition of residuum defined above,
i.e. $x \Rightarrow z \leq y$ iff $z \leq (x \Rightarrow y)$
we have that $(x \Rightarrow y) * (y \Rightarrow z) \leq x \Rightarrow z$.
But the t-norm is commutative so $(y \Rightarrow z) * (x \Rightarrow y) \leq x \Rightarrow z$
and, again, by the definition of the residuum
$(x \Rightarrow y) \leq ((y \Rightarrow z) \Rightarrow (x \Rightarrow z))$.
Finally, from $x \leq y$ iff $x \Rightarrow y = 1$,
then $((x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z))) = 1$.

(A2) We have to prove that $((x * y) \Rightarrow x) = 1$
Trivially, this is true because $(x * y) \leq x$ is a property of t-norms.

(A3) We have to prove that $((x * y) \Rightarrow (y * x)) = 1$
and this is true for the property of simmetry of t-norms.

(A4) We want prove that $(((x * (x \Rightarrow y)) \Rightarrow (y * (y \Rightarrow x))) = 1$
In this case we have two options:

1) $\quad \text{if } x \leq y, \quad (x * 1) \Rightarrow x, \quad (x \Rightarrow x) = 1$

2) $\quad \text{if } y \leq x, \quad y \Rightarrow (y * 1), \quad (y \Rightarrow y) = 1$

(A5a) We have to prove that $(x \Rightarrow (y \Rightarrow z)) \Rightarrow ((x * y) \Rightarrow z) = 1$
and *viceversa* (A5b) $((x * y) \Rightarrow z) \Rightarrow (x \Rightarrow (y \Rightarrow z)) = 1$.
Starting with $(x \Rightarrow (y \Rightarrow z))$, we can take a t
such that $t \leq (x \Rightarrow (y \Rightarrow z))$.
For the definition of residuum $z \leq (x \Rightarrow y)$ iff $(z * x) \leq y$
we obtain $t \leq (x \Rightarrow (y \Rightarrow z))$ iff $(t * x) \leq (y \Rightarrow z)$ iff $(t * x * y) \leq z$
iff $t \leq ((x * y) \Rightarrow z)$.
Now, $t \leq (x \Rightarrow (y \Rightarrow z))$ iff $t \leq ((x * y) \Rightarrow z)$,
and in this way we obtain $(x \Rightarrow (y \Rightarrow z)) \Rightarrow ((x * y) \Rightarrow z) = 1$,
and finally $((x * y) \Rightarrow z) \Rightarrow (x \Rightarrow (y \Rightarrow z)) = 1$.

(A6) We have to prove that $((x \Rightarrow y) \Rightarrow z) \Rightarrow (((y \Rightarrow x) \Rightarrow z) \Rightarrow z) = 1$

There are two cases:
1) $x \leq y$ that means $(x \Rightarrow y) = 1$.
In this case we have $(1 \Rightarrow z) \Rightarrow (((y \Rightarrow x) \Rightarrow z) \Rightarrow z)$
and $z \Rightarrow (((y \Rightarrow x) \Rightarrow z) \Rightarrow z)$.
We know that $z \leq (y \Rightarrow x) \cup z$,
so we obtain the rule
$x \cup y = ((x \Rightarrow y) \Rightarrow y) \bigcap ((y \Rightarrow x) \Rightarrow x)$.
This last statement is easy to prove because $x \leq y$
(but otherwise the situation would be totally symmetric),
which means that we have on one hand $x \cup y = max(x, y) = y$,
and $((x \Rightarrow y) \Rightarrow y) \bigcap ((y \Rightarrow x) \Rightarrow x) = (1 \Rightarrow y) \bigcap ((y \Rightarrow x) \Rightarrow x) =$
$= y \cap ((y \Rightarrow x) \Rightarrow x) = \min(y, ((y \Rightarrow x) \Rightarrow x))$on the other hand.
Moreover, $(y \Rightarrow x) \leq x$ then $(y \Rightarrow x) \Rightarrow x = 1$.
In this way we obtain $y \leq ((y \Rightarrow x) \Rightarrow x)$ and $min(y, ((y \Rightarrow x) \Rightarrow x) = 1$
which means that $x \cup y = ((x \Rightarrow y) \Rightarrow y) \bigcap ((y \Rightarrow x) \Rightarrow x) = y$
and $(y \Rightarrow x) \cup z = (((x \Rightarrow y) \Rightarrow z) \Rightarrow z) \bigcap (((y \Rightarrow x) \Rightarrow z) \Rightarrow z)$
(where $z \leq (y \Rightarrow x) \cup z$, thus $z \leq (((x \Rightarrow y) \Rightarrow z) \Rightarrow z) \cap$
$\bigcap (((y \Rightarrow x) \Rightarrow z) \Rightarrow z) \leq (((y \Rightarrow x) \Rightarrow z) \Rightarrow z))$.
Now, from $z \leq (((y \Rightarrow x) \Rightarrow z) \Rightarrow z)$
we can conclude that $z \Rightarrow (((y \Rightarrow x) \Rightarrow z) \Rightarrow z) = 1$
2) $y \leq x$, that is $(y \Rightarrow x) = 1$
$((x \Rightarrow y) \Rightarrow z) \Rightarrow ((1 \Rightarrow z) \Rightarrow z)$
$((x \Rightarrow y) \Rightarrow z) \Rightarrow (z \Rightarrow z)$
but being $((x \Rightarrow y) \Rightarrow z) \leq 1$
we have proved that $((x \Rightarrow y) \Rightarrow z) \Rightarrow 1 = 1$.

(A7) We want prove that $\bar{0} \Rightarrow x = 1$ and it is always true beacause $\bar{0} \leq x$.

3.1.3.1.2 Theorems

Theorem 1 These are some theorems of BL:[6]

$$(1_{BL}) \qquad \alpha \rightarrow (\beta \rightarrow \alpha)$$

$$(2_{BL}) \qquad (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma))$$

$$(3_{BL}) \qquad \alpha \rightarrow \alpha$$

[6]Here there are two things implied:
(i) MP preservs tautologies (if $x = 1$ and $x \Rightarrow y= 1$ then $1 \Rightarrow y = 1$. Whereas $1 \Rightarrow y = y$ so $y = 1$.)
(ii) if we prove that $\alpha \rightarrow \beta$ and $\beta \rightarrow \gamma$, then we can prove that $\alpha \rightarrow \gamma$.

(4_{BL}) $\alpha \to (\beta \to (\alpha \otimes \beta))$

(5_{BL}) $(\alpha \otimes (\alpha \to \beta)) \to \beta$

(6_{BL}) $(\alpha \to \beta) \to ((\alpha \otimes \gamma) \to (\beta \otimes \gamma))$

(7_{BL}) $((\alpha \to \beta) \otimes (\beta \to \gamma)) \to (\alpha \to \gamma)$

(8_{BL}) $((\alpha_1 \to \beta_1) \otimes (\alpha_2 \to \beta_2)) \to ((\alpha_1 \otimes \alpha_2) \to (\beta_1 \otimes \beta_2))$

(9_{BL}) $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$

Proof.

(1_{BL}) $\alpha \to (\beta \to \alpha)$
from (A2) $(\alpha \otimes \beta) \to \alpha$
from (A5b) $((\alpha \otimes \beta) \to \alpha) \to (\alpha \to (\beta \to \alpha))$
using MP we obtain $(\alpha \to (\beta \to \alpha))$.

(2_{BL}) $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$
for (A1) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
$((\beta \otimes \alpha) \to (\alpha \otimes \beta)) \to (((\alpha \otimes \beta) \to \gamma) \to ((\beta \otimes \alpha) \to \gamma))$
for (A3) $(\beta \otimes \alpha) \to (\alpha \otimes \beta)$
and for MP$((\alpha \otimes \beta) \to \gamma) \to ((\beta \otimes \alpha) \to \gamma)$
for (A5a) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \otimes \beta) \to \gamma)$
for comment (ii)$(\alpha \to (\beta \to \gamma)) \to ((\beta \otimes \alpha) \to \gamma)$
for (A5b) $((\beta \otimes \alpha) \to \gamma) \to (\alpha \to (\beta \to \gamma))$
and again for comment (ii) we obtain $(\alpha \to (\beta \to \gamma)) \to (\alpha \to (\beta \to \gamma))$.

(3_{BL}) $\alpha \to \alpha$
putting α in place of γ in (2_{BL}),
and taking β as any axiom,
we have$(\alpha \to (\beta \to \alpha)) \to (\beta \to (\alpha \to \alpha))$
for MP with (1_{BL}) $(\beta \to (\alpha \to \alpha))$.
Since β is an axiom, for MP we have $(\alpha \to \alpha)$.

(4_{BL}) $\alpha \to (\beta \to (\alpha \otimes \beta))$
for (3_{BL}), $(\alpha \otimes \beta) \to (\alpha \otimes \beta)$
for (A5b) and putting γ in place of $\alpha \otimes \beta$,
we obtain $((\alpha \otimes \beta) \to (\alpha \otimes \beta)) \to (\alpha \to (\beta \to (\alpha \otimes \beta))$
for MP, $\alpha \to (\beta \to (\alpha \otimes \beta)$.

(5_{BL}) $(\alpha \otimes (\alpha \to \beta)) \to \beta$

for (3_{BL}) $(\alpha \to \beta) \to (\alpha \to \beta)$
using (2_{BL}) and placing $\alpha \to \beta$ instead of α, α instead of β
and β instead of γ,
we obtain$((\alpha \to \beta) \to (\alpha \to \beta)) \to (\alpha \to ((\alpha \to \beta) \to \beta))$
for MP $(\alpha \to ((\alpha \to \beta) \to \beta))$
for (A5a) putting $\alpha \to \beta$ in place of β and β in place of γ ,
we can write$(\alpha \to ((\alpha \to \beta) \to \beta)) \to ((\alpha \otimes (\alpha \to \beta)) \to \beta)$
for MP$(\alpha \otimes (\alpha \to \beta)) \to \beta$.

(6_{BL}) $(\alpha \to \beta) \to ((\alpha \otimes \gamma) \to (\beta \otimes \gamma))$
for (5_{BL}) $(\alpha \otimes (\alpha \to \beta) \to \beta$
for $(4_{BL})\beta \to (\gamma \to (\beta \otimes \gamma))$
for (ii) $(\alpha \otimes (\alpha \to \beta)) \to (\gamma \to (\beta \otimes \gamma))$
for (A5b) and putting $\alpha \to \beta$ instead of β
and$(\gamma \to (\beta \otimes \gamma))$ in place of γ, we have
$((\alpha \otimes (\alpha \to \beta)) \to (\gamma \to (\beta \otimes \gamma))) \to (\alpha \to ((\alpha \to \beta) \to (\gamma \to (\beta \otimes \gamma)))$
for (2_{BL}) and putting $\alpha \to \beta$ instead of α,
γ instead of β, and $\beta \otimes \gamma$ instead of γ
we obtain
$((\alpha \to \beta) \to (\gamma \to (\beta \otimes \gamma))) \to (\gamma \to ((\alpha \to \beta) \to (\beta \to \gamma)))$
for (ii) we have $\alpha \to (\gamma \to ((\alpha \to \beta) \to (\beta \to \gamma)))$
for (A5b) placing γ instead of β and$((\alpha \to \beta) \to (\beta \to \gamma))$ instead of γ
we obtain
$\alpha \to (\gamma \to ((\alpha \to \beta) \to (\beta \to \gamma)) \to (\alpha \otimes \gamma) \to ((\alpha \to \beta) \to (\beta \to \gamma))$
for MP$(\alpha \otimes \gamma) \to ((\alpha \to \beta) \to (\beta \to \gamma))$
for (2_{BL}) putting $\alpha \otimes \gamma$ in place of α,
$\alpha \to \beta$ instead of β and$\beta \to \gamma$ in place of γ,
we can write
$(\alpha \otimes \gamma) \to ((\alpha \to \beta) \to (\beta \to \gamma)) \to (\alpha \to \beta) \to ((\alpha \otimes \gamma) \to (\beta \otimes \gamma))$
and for MP, $(\alpha \to \beta) \to ((\alpha \otimes \gamma) \to (\beta \otimes \gamma))$.

(7_{BL}) $((\alpha \to \beta) \otimes (\beta \to \gamma)) \to (\alpha \to \gamma)$
for (A1)$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
for (A5a) and putting $\alpha \to \beta$ in place of α,
$\beta \to \gamma$ in place of β, and $\alpha \to \gamma$ in place of γ
we have
$((\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))) \to$
$\to (((\alpha \to \beta) \otimes (\beta \to \gamma)) \to (\alpha \to \gamma))$
finally, for MP$((\alpha \to \beta) \otimes (\beta \to \gamma)) \to (\alpha \to \gamma)$.

Proposition 2. If it is proved that $\alpha \to \delta$, $\beta \to \gamma$ and $\delta \otimes \gamma \to \chi$,

then we can demonstrate that $\alpha \otimes \beta \to \chi$
Proof.
For (6_{BL}) and placing β instead of δ
we obtain$(\alpha \to \delta) \to ((\alpha \otimes \gamma) \to (\delta \otimes \gamma))$.
For MP we have$(\alpha \otimes \gamma) \to (\delta \otimes \gamma)$
and for (ii) and $(\delta \otimes \gamma) \to \chi$
we have that $(\alpha \otimes \gamma) \to \chi$.
Using (A5b), and because of $\beta \to \gamma$
we have$(\alpha \otimes \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma))$.
Now, for MP$(\alpha \to (\gamma \to \chi))$,
for (2_{BL}) $(\gamma \to (\alpha \to \chi))$,
for (ii) and because of $\beta \to \gamma$,
we obtain that$(\beta \to (\alpha \to \chi)) \to (\alpha \to (\beta \to \gamma)$.
Finally, for (A5a) $(\alpha \otimes \beta) \to \chi$

(8_{BL}) $((\alpha_1 \to \beta_1) \otimes (\alpha_2 \to \beta_2)) \to ((\alpha_1 \otimes \alpha_2) \to (\beta_1 \otimes \beta_2))$
for (6_{BL}) we have:
(a)$(\alpha_1 \to \beta_1) \to ((\alpha_1 \otimes \alpha_2) \to (\beta_1 \otimes \alpha_2))$
(b)$(\alpha_2 \to \beta_2) \to ((\beta_1 \otimes \alpha_2) \to (\beta_1 \otimes \beta_2))$.
For (7_{BL})$[(\alpha \to \beta) \otimes (\beta \to \gamma)] \to (\alpha \to \gamma)$ we obtain
(c)$(((\alpha_1 \otimes \alpha_2) \to (\beta_1 \otimes \alpha_2)) \otimes ((\beta_1 \otimes \alpha_2) \to (\beta_1 \otimes \beta_2))) \to$
$\to ((\alpha_1 \otimes \alpha_2) \to (\beta_1 \otimes \beta_2))$
And if we apply the proposition defined above, to (a) (b) and (c)
we have $((\alpha_1 \to \beta_1) \otimes (\alpha_2 \to \beta_2)) \to ((\alpha_1 \otimes \alpha_2) \to (\beta_1 \otimes \beta_2))$.

(9_{BL}) $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$
for (A1) we have $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \otimes \gamma))$
so$(\alpha \to \beta) \to ((\beta \to \bar{0}) \to (\alpha \to \bar{0}))$.

3.1.3.1.3 Deduction theorem for BL A *theory* T over BL is a set of formulas of BL. A *proof* in a theory T is a sequence $\beta_1, ..., \beta_k$ of formulas, whose each member is either an axiom of BL or a member of T, or follows from some preceding members of the sequence using the rule of modus ponens.

Deduction theorem Let T be a theory and let α and β be formulas. Let us set $\alpha^n = \underbrace{\alpha \otimes \alpha \otimes \alpha \otimes ... \otimes \alpha}_{n}$, we have:

$$T \cup \{\alpha\} \vdash \beta \text{ iff } \exists n \in \mathbb{N} \text{ such that } \mathrm{T} \vdash (\alpha^n \to \beta)\,.$$

Proof.
(i) If $\mathrm{T} \vdash \alpha^n \to \beta$ then $\mathrm{T} \cup \{\alpha\} \vdash \beta$.

If $n = 1$ and $\mathrm{T} \vdash (\alpha \to \beta)$ it is evident that $\mathrm{T} \cup \{\alpha\} \vdash \beta$.
If $n > 1$ and $\mathrm{T} \vdash \alpha^n \to \beta$, then $\mathrm{T} \vdash \left(\alpha \otimes \alpha^{n-1}\right) \to \beta$,
which means that $\mathrm{T} \vdash \alpha \to \left(\alpha^{n-1} \to \beta\right)$.
Hence, $\mathrm{T} \cup \{\alpha\} \vdash \alpha^{n-1} \to \beta$,
and replacing this we get $\mathrm{T} \cup \{\alpha\} \vdash \alpha \to \beta$ and hence $\mathrm{T} \cup \{\alpha\} \vdash \beta$.
(ii) Conversely, assume $\mathrm{T} \cup \{\alpha\} \vdash \beta$
and let $\gamma_1 .... \gamma_k$ be a corresponding $\mathrm{T} \cup \{\alpha\} - proof$ of β.
We prove by induction that, for each $j = 1, ..., k$ there is an n_j
such that $\mathrm{T} \vdash \alpha^{n_j} \to \gamma_j$.
This is clear for γ_j being an axiom of BL or of $\mathrm{T} \cup \{\alpha\}$.
If γ_j results by modus MP from γ_i and $(\gamma_i \to \gamma_j)$.
then, by the induction hypothesis we assume $\mathrm{T} \vdash \alpha^n \to \gamma_i$,
and so $\mathrm{T} \vdash \alpha^m \to (\gamma_i \to \gamma_j)$.
Thus, by (7_{BL}) we obtain $\mathrm{T} \vdash (\alpha^n \otimes \alpha^m) \to (\gamma_i \otimes (\gamma_i \to \gamma_j))$
and then $\mathrm{T} \vdash \alpha^{n+m} \to \gamma_j$.

3.1.4 Logical extensions of BL

Among the logical systems extending BL, the most important are the following three:

(i) Łukasiewicz' infinite-valued system L

(ii) Gödel-Dummett's system G

(iii) Product logic Π

3.1.4.1 Łukasiewicz's infinite-valued system

Let L a propositional formal system based on BL, where the axioms of L are these of BL, to which is added the law of double negation

$$\neg\neg\alpha \to \alpha$$

Definition 5 We have the following operations on L :

- t − norm : $x * y = \max(x + y - 1, 0)$

REMARK: Łukasiewicz's t-norm is *Archimedean* and *nilpotent.*

- t − conorm : $x \diamond y = \min(x + y, 1)$

- residuum : $x \Rightarrow y = 1$ if $x \leq y$

  $x \Rightarrow y = 1 - x + y$ if $x > y$

- negation : $n_L(x) = 1 - x$

3.1.4.1.1 Axioms

Proposition 3 An alternative axiomatization of $Ł$ is:

(Ł1) $\alpha \to (\beta \to \alpha)$

(Ł2) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

(Ł3) $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$

(Ł4) $((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$

Proof.
(Ł1) and (Ł2) are respectively (1_{BL}) and axiom $(A1)$.
(Ł3) follows from $Ł \vdash (\alpha \to \beta) \equiv (\neg\beta \to \neg\alpha)$, which is proved because $BL \vdash (\alpha \to (\beta \otimes \neg\beta)) \to \neg\alpha$,
and this is proved because
$BL \vdash (\beta \otimes (\beta \to \bar{0})) \to \bar{0}$.
Thus $BL \vdash (\alpha \to (\beta \otimes \neg\beta)) \to (\alpha \to \bar{0})$
(Ł4) follows from
$BL \vdash (\neg\alpha \otimes (\neg\alpha \to \neg\beta)) \to (\neg\beta \otimes (\neg\beta \to \neg\alpha))$(by A4)
$Ł \vdash (\neg\alpha \otimes (\beta \to \alpha)) \to (\neg\beta \otimes (\alpha \to \beta))$
$Ł \vdash \neg((\alpha \to \beta) \otimes \neg\beta) \to \neg((\beta \to \alpha) \otimes \neg\alpha)$
and so $Ł \vdash ((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$.

Definition 6 In $Ł'$ is introduced a new connective called *strong disjunction*:

$$\alpha \underline{\vee} \beta \text{ stands for } \neg\alpha \to \beta$$

Semantically, note that the truth function $\oplus$, corresponding to the connective $\underline{\vee}$, satisfies :

$$x \oplus y = \min(x + y, 1)$$

Proof.
Because of $x \oplus y = [(x \Rightarrow 0) \Rightarrow y] = [(1 - x) \Rightarrow y]$;
thus, if $x + y \leq 1$, then $1 - x \geq y$

and $x \oplus y = 1 - (1-x) + y = x + y$; if $x + y \geq 1$.
Then $1 - x \leq y$ and $x \oplus y = 1$.

Proposition 4 Let us denote (Ł1)-(Ł4) by $Ł'$, The following formulas are provable in $Ł'$[7]:

$(1_{Ł'})$ $\alpha \to ((\alpha \to \beta) \to \beta)$

$(2_{Ł'})$ $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$

$(3_{Ł'})$ $\alpha \to \alpha$

$(4_{Ł'})$ $\bar{0} \to \alpha$

$(5_{Ł'})$ $\neg\neg\alpha \to \alpha$

$(6_{Ł'})$ $(\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$

$(7_{Ł'})$ $\alpha \to \neg\neg\alpha$

$(8_{Ł'})$ $\neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$

$(9_{Ł'})$ $\neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$

$(10_{Ł'})$ $\neg(\alpha \otimes \beta) \leftrightarrow (\neg\alpha \underline{\vee} \neg\beta)$

$(11_{Ł'})$ $\neg(\alpha \underline{\vee} \beta) \leftrightarrow (\neg\alpha \otimes \neg\beta)$

$(12_{Ł'})$ $\beta \to (\alpha \underline{\vee} \beta)$

$(13_{Ł'})$ $(\alpha \underline{\vee} \beta) \to (\beta \underline{\vee} \alpha)$

$(14_{Ł'})$ $(\alpha \underline{\vee} (\beta \underline{\vee} \gamma)) \leftrightarrow ((\alpha \underline{\vee} \beta) \underline{\vee} \gamma)$

$(15_{Ł'})$ $(\alpha \wedge \beta) \leftrightarrow (\alpha \underline{\vee} \neg\alpha) \otimes \beta$

$(16_{Ł'})$ $(\alpha \vee \beta) \leftrightarrow (\alpha \otimes \neg\beta) \underline{\vee} \beta$

$(17_{Ł'})$ $\alpha \underline{\vee} \neg\alpha$

$(18_{Ł'})$ $((\alpha \otimes \neg\beta) \underline{\vee} \beta) \leftrightarrow (\alpha \underline{\vee} (\beta \otimes \neg\alpha))$

$(19_{Ł'})$ $((\alpha \underline{\vee} \neg\beta) \otimes \beta) \leftrightarrow (\alpha \otimes (\beta \underline{\vee} \neg\alpha))$

$(20_{Ł'})$ $(\alpha \vee \alpha) \to \alpha$

[7]The complete proofs can be found in [9], 65-70.

$(21_{L'})$ $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

$(22_{L'})$ $(\alpha \to \beta) \vee (\beta \to \alpha)$

Now, given these formulas, we can show that (A1)-(A7) are proved in $L^{'}$.

Lemma 1 $L^{'}$ proves the axioms of BL as well as the axiom $(\neg\neg)$.

Proof.
(A1) is $(L2)$
(A3) Using $(6_{L'})$, we have that
$L^{'} \vdash (\alpha \otimes \alpha) \to \neg(\alpha \to \neg\beta) \to \neg(\beta \to \neg\alpha) \to (\beta \otimes \alpha)$.
(A2) In the presence of (A3) it suffices to prove $L^{'} \vdash (\alpha \otimes \beta) \to \beta$.
Now, $L^{'} \vdash \neg\beta \to (\alpha \to \neg\beta)$, thus $L^{'} \vdash \neg(\alpha \to \neg\beta) \to \neg\neg\beta$
and then $L^{'} \vdash (\alpha \otimes \beta) \to \beta$.
(A4) See $(8_{L'})$ and the definition of $\wedge$.
(A5) $L^{'} \vdash [\alpha \to (\beta \to \gamma)] \leftrightarrow [\alpha \to (\neg\gamma \to \neg\alpha)] \leftrightarrow$
$\leftrightarrow [\neg\gamma \to (\alpha \to \neg\beta)] \leftrightarrow [\neg\gamma \to \neg(\alpha \otimes \beta)] \leftrightarrow [(\alpha \otimes \beta) \to \gamma.]$
For $(\neg\neg)$ see $(5_{L'})$ to complete the proof.
(A6) We start with $(21_{L'})$ and then we have that
$L^{'} \vdash ((\alpha \to \beta) \to \gamma) \to [((\beta \to \alpha) \to \gamma) \to (((\alpha \to \beta) \vee (\beta \to \alpha)) \to \gamma)]$.
Thus, we have
$L^{'} \vdash [((\alpha \to \beta) \to \gamma) \otimes ((\beta \to \alpha) \to \gamma)] \to [((\alpha \to \beta) \vee (\beta \to \alpha)) \to \gamma]$.
Now, for $(15_{L'})$ we have obtain
$L^{'} \vdash [((\alpha \to \beta) \to \gamma) \otimes ((\beta \to \alpha) \to \gamma)] \to \gamma$.
(A7) It is proved in $(4_{L'})$.

REMARK. From now, on we know that L and L' are equivalent theories (they prove the same formulas); thus L will denote either of them.

3.1.4.2 Gödel-Dummett's system

Gödel logic is a many-valued system based on BL[8], adding the axiom of idempotence of the conjunction:

$$\alpha \to (\alpha \otimes \alpha)$$

Definition 7 We have the following operations on G:

[8]Actually, Gödel's system was formulated some decades before BL logic, as underlined in the introduction to this chapter (see p. 25-27).

- t − norm : $x * y = \min(x, y)$

REMARK: Gödel's t-norm is *non-Archimedean*, because it is idempotent on all numbers.

- t − conorm : $x \diamond y = \max(x, y)$
- residuum : $x \Rightarrow y = 1 \quad \text{if } x \leq y$

  $x \Rightarrow y = y \quad \text{if } x > y$
- negation : $n_G(x) = 1 \quad \text{if } x > 0$

  $n_G(x) = 0 \quad \text{if } x = 0$

Lemma 2 G proves $(\alpha \otimes \beta) \equiv (\alpha \wedge \beta)$

Proof.
Clearly, $BL \vdash (\alpha \otimes \beta) \to (\alpha \wedge \beta)$;
on the other hand, $BL \vdash (\alpha \wedge \beta) \to \alpha$, $BL \vdash (\alpha \wedge \beta) \to \beta$.
Hence $BL \vdash [(\alpha \wedge \beta) \otimes (\alpha \wedge \beta) \to (\alpha \otimes \beta)]$,
and finally $G \vdash (\alpha \wedge \beta) \to [(\alpha \wedge \beta) \otimes (\alpha \wedge \beta)]$.

Thus we may forbear in G of $\otimes$, and so we can present Gödel logic equivalently as a system G':

Definition 8 The axiom system G' has the primitive connectives $\wedge, \to$, propositional constant $\bar{0}$ and axioms (A1)-(A3) and (A5)-(A7) of BL, plus the axiom $\alpha \to (\alpha \wedge \alpha)$ (G4).

Other connectives are defined as follows:

$\neg\alpha$ is $\alpha \to \bar{0}$
$\alpha \equiv \beta$ is $(\alpha \to \beta) \wedge (\beta \to \alpha)$
$(\alpha \vee \beta)$ is $((\alpha \to \beta) \to \beta) \wedge ((\beta \to \alpha) \to \alpha)$

Lemma 3

(i) $$G' \vdash (\alpha \wedge (\alpha \to \beta)) \to (\beta \wedge (\beta \to \alpha))$$

(ii) $$G' \vdash (\alpha \wedge \beta) \equiv \alpha \wedge (\alpha \to \beta)$$

Proof.
(i) Obviously, if BL proves a formula α (using only connectives $\to, \otimes, \bar{0}$), then G' proves the result α' of replacing each $\otimes$ by $\wedge$.

Thus, from $(\alpha \otimes (\alpha \to \beta)) \to \beta$,
we can get $G' \vdash (\alpha \wedge (\alpha \to \beta)) \to \beta$.
Then, from $\alpha \to (\beta \to \alpha)$, (A2) and (A1)
we obtain $G' \vdash (\alpha \wedge (\alpha \to \beta)) \to (\beta \to \alpha)$,
and hence, by $((\alpha_1 \to \beta_1) \otimes (\alpha_2 \to \beta_2)) \to ((\alpha_1 \otimes \alpha_2) \to (\beta_1 \otimes \beta_2))$
and by using (G4) for the formula $\alpha \wedge (\alpha \to \beta)$.
(ii) First, we observe that $G' \vdash \beta \to (\alpha \to \beta)$
and $G' \vdash (\alpha \wedge \beta) \to (\alpha \wedge (\alpha \to \beta))$.
Conversely, $G' \vdash (\alpha \wedge (\alpha \to \beta)) \to \beta$,
and so $G' \vdash [\alpha \wedge (\alpha \to \beta)] \to [\alpha \wedge \alpha \wedge (\alpha \to \beta)] \to [\alpha \wedge \beta]$.

REMARK. G and G' are equivalent in the sense that $G \vdash \alpha$ iff $G' \vdash \alpha'$ (where α' results from α by identifying $\otimes$ and $\wedge$). Thus in the sequel we shall not distinguish between them.

It is interesting to highlight the link among G and the *Intuitionistic Logic* I .

Lemma 4 G includes all the axioms of BL, so it proves all the axioms of I:

(I1) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
(I2) $\alpha \to (\alpha \vee \beta)$
(I3) $\beta \to (\alpha \vee \beta)$
(I4) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$
(I5) $(\alpha \wedge \beta) \to \alpha$
(I6) $(\alpha \wedge \beta) \to \beta$
(I7) $(\gamma \to \alpha) \to ((\gamma \to \beta) \to (\gamma \to (\alpha \wedge \beta)))$
(I8) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma)$
(I9) $((\alpha \wedge \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma))$
(I10) $(\alpha \wedge \neg\alpha) \to \beta$
(I11) $(\alpha \to (\beta \wedge \neg\beta)) \to \neg\beta$

Proof.
BL proves (I1)-(I7), (I10) and (I11) as they stand,
and (I8)-(I9) placing $\otimes$ instead of $\wedge$.

REMARK. Alternatively, we can demonstrate that I, extended by the axiom (A6), proves all axioms of G plus $(\alpha \vee \beta) \equiv ((\alpha \to \beta) \to \beta) \wedge ((\beta \to \alpha) \to \alpha)$.

Proof.
In fact we have $I \vdash \alpha \to ((\alpha \to \beta) \to \beta)$
and $I \vdash \alpha \to (\delta \to \alpha)$ (for each δ formula)

and similarly for β.
Thus, $I \vdash (\alpha \vee \beta) \to [((\alpha \to \beta) \to \beta) \wedge ((\beta \to \alpha) \to \alpha)]$.
Conversely, $I \vdash (\alpha \to \beta) \to [((\alpha \to \beta) \to \beta) \to \beta]$,
thus $I \vdash (\alpha \to \beta) \to [((\alpha \to \beta) \to \beta) \wedge ((\beta \to \alpha) \to \alpha)] \to (\alpha \vee \beta)$.
Similarly, $I \vdash (\beta \to \alpha) \to [...] \to (\alpha \to \beta)$,
thus, by (A6) we obtain $I \vdash [...] \to (\alpha \vee \beta)$.

3.1.4.2.1. Deduction theorem for G G is the only of these three propositional systems, having a standard deduction theorem.

This result follows from the deduction theorem for BL, noting that in G for each n , α^n is equivalent to α So we have that for each theory T:

$$\mathrm{T} \cup \{\alpha\} \vdash \beta \text{ iff } \mathrm{T} \vdash (\alpha \to \beta).$$

3.1.4.3 The Product logic

Product logic Π is a formal system whose axioms are the axioms of BL plus the following:

(Π1) $\quad \neg\neg\gamma \to ((\alpha \odot \gamma \to \beta \odot \gamma) \to (\alpha \to \beta))$

(Π2) $\quad (\alpha \wedge \neg\alpha) \to \bar{0}$

Definition 9 We have the following operations on Π:

- t – norm : $\quad x * y = xy$

REMARK: The product t-norm is *Archimedean* and *strict*.

- t – conorm : $\quad x \diamond y = x + y - xy$
- residuum : $\quad x \Rightarrow y = 1 \quad$ if $x \leq y$

  $x \Rightarrow y = \frac{y}{x} \quad$ if $x > y$

  (this residuum is known as *Goguen implication*)
- negation : $\quad n_\Pi(x) = 1 \quad$ if $x > 0$

  $n_\Pi(x) = 0 \quad$ if $x = 0$

Convention. Henceforth, in this section $\to$ without any subscript will be Goguen implication, while the product conjunction will be denoted by $\odot$.

Lemma 5 The axioms are tautologies over the algebra $[0,1]_{\Pi}$ of the truth functions.

Proof.
(Π1) Let e an evaluation; if $e(\gamma)=0$, then $e(\neg\neg\gamma)=0$
and so $e(\neg\neg\gamma\to\delta)=1$ (for each δ formula).
If $e(\gamma)>0$, then $e(\neg\neg\gamma)=1$,
and either $e(\alpha\odot\gamma)\leq e(\beta\odot\gamma)$,
thus $e(\alpha)\leq e(\beta)$,
hence $e((\alpha\odot\gamma)\to(\beta\odot\gamma))=e(\alpha\to\beta)=1$.
Otherwise, $e(\alpha\odot\gamma)>e(\beta\odot\gamma)$
then $e(\alpha)>e(\beta)$
and so $e((\alpha\odot\gamma)\to(\beta\odot\gamma))=e(\alpha\to\beta)=\frac{e(\beta)}{e(\alpha)}$.
(Π2) Since in Π $\neg$ is Gödel negation, either $e(\alpha)$ or $e(\neg\alpha)$ must be 0.

Lemma 6 Π proves the following formulas:

$$(1_{\Pi})\qquad \neg(\alpha\odot\beta)\to\neg(\alpha\wedge\beta)$$

$$(2_{\Pi})\qquad (\alpha\to\neg\alpha)\to\neg\alpha$$

$$(3_{\Pi})\qquad \neg\alpha\vee\neg\neg\alpha$$

Proof.
(1_{Π}) This formula is equivalent to those:
$((\alpha\odot\beta)\to 0)\to((\alpha\wedge\beta)\to\bar{0})$,
$[(\alpha\to(\beta\to 0))\odot(\alpha\wedge\beta)\to\bar{0}]$,
$[(\alpha\to\neg\beta)\odot(\alpha\wedge\beta)]\to\bar{0}$.
Now, the following chains of implications are provable:
$[(\alpha\to\neg\beta)\odot(\alpha\wedge\beta)]\to[(\alpha\to\neg\beta)\odot\alpha]\to\neg\beta$,
$[(\alpha\to\neg\beta)\odot(\alpha\wedge\beta)]\to[(\alpha\to\neg\beta)\odot\beta]\to\beta$,
and then $[(\alpha\to\neg\beta)\odot(\alpha\wedge\beta)]\to[\beta\wedge\neg\beta]\to\bar{0}$.
(2_{Π}) From (1_{Π}) we have $\neg(\alpha\odot\alpha)\to\neg\alpha$,
thus $(\alpha\odot\alpha\to\bar{0})\to(\alpha\to\bar{0})$,
$(\alpha\to(\alpha\to\bar{0}))\to(\alpha\to\bar{0})$,
and so $(\alpha\to\neg\alpha)\to\neg\alpha$.
(3_{Π})The following implications are provable:
by (2_{Π}) $(\neg\alpha\to\neg\neg\alpha)\to\neg\neg\alpha$
and by BL $\neg\neg\alpha\to(\neg\alpha\vee\neg\neg\alpha)$

so $(\neg\alpha \to \neg\neg\alpha) \to (\neg\alpha \vee \neg\neg\alpha)$.
On the other hand we have by BL
$(\neg\neg\alpha \to \neg\alpha) \to (\neg\neg\alpha \to \neg\neg\neg\alpha)$,
thus by (1_Π) here $(\neg\neg\alpha \to \neg\neg\neg\alpha) \to \neg\neg\neg\alpha$,
then $\neg\neg\neg\alpha \to \neg\alpha$ by BL,
and $(\neg\neg\alpha \to \neg\alpha) \to \neg\alpha$
$(\neg\neg\alpha \to \neg\alpha) \to (\neg\alpha \vee \neg\neg\alpha)$.
Now, we get $(\neg\alpha \vee \neg\neg\alpha)$,
applying axiom (A6) to $\neg\alpha$, $\neg\neg\alpha$ and $\neg\alpha \vee \neg\neg\alpha$.

Lemma 7 The axiom $(\Pi 2)$ can be equivalently replaced by each of the formulas (1_Π), (2_Π) and (3_Π) together with BL+$(\Pi 1)$:

Proof.
(i) Take $(\alpha \to \neg\alpha) \to \neg\alpha$ together with $BL + \Pi 1$.
We have the following chain of provable implications:
$(\alpha \wedge \neg\alpha) \to [\alpha \odot (\alpha \to \neg\alpha)] \to [\alpha \odot \neg\alpha] \to \bar{0}$.
(ii) Now take $\neg(\alpha \odot \alpha) \to \neg\alpha$;
we get $(\alpha \odot \alpha \to \bar{0}) \to (\alpha \to \bar{0})$,
and hence $(\alpha \to (\alpha \to \bar{0})) \to \alpha \to \bar{0}$
which is (1_Π).
(iii) Let consider $\neg\alpha \vee \neg\neg\alpha$.
Then the following are provable:
$\neg\neg\alpha \to (((\alpha \odot \alpha) \to (\alpha \to \bar{0})) \to (\alpha \to \bar{0}))$
which is axiom $(\Pi 1)$.
Then $\neg\alpha \to (\delta \to (\alpha \to \bar{0}))$ (for each formula δ)
thus $(\neg\neg\alpha \vee \neg\alpha) \to ((\alpha \odot \alpha \to \bar{0}) \to (\alpha \to \bar{0}))$.
Observing that $\bar{0}$ is equivalent to $(\alpha \odot \bar{0})$in BL,
hence we get $(\alpha \odot \alpha \to \bar{0}) \to (\alpha \to \bar{0})$,
i.e $\neg(\alpha \odot \alpha) \to \neg\alpha$ and it was (2_Π).

3.1.5 The t-norms fundamental theorem.

The fundamental theorem proves that all possible t-norms are refered to $*_L$, $*_G$ and $*_\Pi$.

For each *continuous* t-norm the set E of all its idempotents is a closed subset of $[0,1]$, and hence its complement is a union of a set $\mathcal{I}_{open}(E)$ of countably many non-overlapping open intervals. In other words, the set of the idempotents is open.

Now, let define $\mathcal{I}(E) = \{[a,b] \mid (a,b) \in \mathcal{A}\}$ where $\mathcal{A}$ is the set of open intervals.

For each $[a,b] \in \mathcal{I}(E)$ let $(* \restriction [a,b])$be the restriction of $*$ to $[a,b]^2$.

Proposition 4 If $x, y \in [0,1]$ are such that there is no $I \in \mathcal{I}(E)$ with $x, y \in I$, then $x * y = \min(x, y)$.

Proof.
If $x < y$ and x, y do not belong to the same interval $I \in \mathcal{I}(E)$,
then there is an idempotent a, $x \leq a \leq y$,
and we can observe that $x{*}y = x$.

Proposition 5 For each $I \in \mathcal{I}(E)$, where $I = [a,b]$, does exist a bijective linear map $f : [a,b] \to [0,1]$ which transforms the restriction of $*$ to $[a,b]$ into a continuous t-norm.

Proof.
Let $f(x) = \frac{1}{b-a}x - \frac{a}{b-a}$ the function mentioned above.
$f(x)$ transforms the restriction of $*$ to $[a,b]$
in a continuous t-norm, because from $a \leq x \leq b$
we obtain that $a * x = a$ e $b * x = b$
(in fact, from $a < x$ we have that $a = a * a \leq a * x \leq a$).
In other words, $* \mid [a,b]$ is closed which respect to $[a,b]$ and,
because there are not idempotents in $*$ which respect to I,
then there are not idempotents which respect to $* \mid [a,b]$.
It means that it preservs the property to be Archimedean.

After there preliminaries, we can suppose to work with a continuous and Archimedean t-norm, and we can prove two lemmas.

Lemma 8

(1) For each $x < 1$, $\lim\limits_{n\to\infty} x^{*n} = 0$;

(2) if $*$ is nilpotent, then each $x < 1$ is nilpotent;

(3) if $0 < x^{*n} < 1$, then $(m > n) \to (x^{*m} < x^{*n})$.

Proof.
(1) Since the sequence x^{*n} is non-increasing and bounded,
then $\lim\limits_{n\to\infty} x^{*n}$ exists.
Let define $\bar{x}$ this limit.
First of all $\bar{x}$ is idempotent, due to
$\bar{x} * \bar{x} = \lim\limits_{n} x^{*n} * \lim\limits_{m} x^{*m} = \lim\limits_{m,n\to\infty} x^{*(m+n)} = \bar{x}$.

Furthermore, $*$ is Archimedean and $x < 1$, then $\bar{x} = 0$.
(2) If $z > 0$ is nilpotent, then each $x < z$ is idempotent too.
In fact, if $z \leq x \leq 1$, then for some m we have that $x^{*m} < z$,
so x is thus nilpotent too.
(3) Now, let suppose $m > n$ and let $x^{*m} = x^{*n}$.
Put $y = x^{*n}$ and $z = x^{*(m-n)}$.
By this way, we obtain $y = y * z = y * z^{*k}$(for each k),
and then $y = y * 0 = 0$, (because $\lim_{n\to\infty} z^{*n} = 0$).
It is in conflict with $0 < x^{*n} < 1$.

Lemma 9 For each positive $x < 1$ and for each positive n , there is a unique y such that $y^{*n} = x$.

Proof.
Assume $n > 1$.
The existence of n follows from continuity of the function $f(y) = y^{*n}$,
where $f(0) = 0$ and $f(1) = 1$;
It is evident that if $y^{*n} = x$, then $0 < x < y < 1$.
Now, let $x < z < y$ and $z^{*n} = y^{*n}$.
In this case we have $x \leq u \leq y$ where u is idempotent,
and then $x * y = x\ z = y * t$ (for each t).
So $y^{*n} = x^{*n} = y^{*n} * t^{*n} = y^{*n} * t^{*(kn)}$(for each $k > 0$)[9].
Now, from **lemma 7** follows that $\lim_k t^{*(kn)} = 0$
and by continuity, $x = y^{*n} = y^{*n} * 0 = 0$.
The problem is that it is a contradiction
with the initial hypothesis $0 < x < y < 1$.

Definition 10 For each $x \in [0,1]$, $x^{*\frac{1}{n}}$ is the unique $y \in [0,1]$ such that $y^n = x$. For a rational number $r = \frac{m}{n}$, we have that $x^{*r} = \left(x^{*\frac{1}{n}}\right)^{*m}$.

REMARK: Actually, it is not a good definition, beacause we have to prove a Lemma which extend the validity of this definition for all the rationals.

Lemma 10

(4) If $\frac{m}{n} = \frac{m'}{n'}$, then $x^{*\frac{m}{n}} = x^{*\frac{m'}{n'}}$;

(5) $x^{*r} * x^{*s} = x^{*(r+s)}$ for each $x \in [0,1]$ and r, s positive rational;

(6) if $x > 0$, then $\lim_{n\to\infty} x^{*\frac{1}{n}} = 1$.

[9]Indeed, if $x \leq u \leq y$ and u is idempotent, then $x * y = x$

Proof.
(4) We may suppose $m' = km$ e $n' = kn$.
Then $x * \frac{m'}{n'} = (x^{*\frac{1}{kn}})^{*km} = \left(\left(x^{*\frac{1}{kn}}\right)^{k}\right)^{m} = \left(x^{*\frac{1}{n}}\right)^{*m} = x^{*\frac{m}{n}}$.
(5) Let $r = \frac{m}{n}$ e $s = \frac{k}{n}$;
then $x^{*r} * x^{*s} = \left(x^{*\frac{1}{n}}\right)^{*m} * \left(x^{*\frac{1}{n}}\right)^{*k} = \left(x^{*\frac{1}{n}}\right)^{*(m+k)} = x^{*(r+s)}$.
(6) If $x > 0$, then the sequence $\left\{x^{*\frac{1}{n}} \mid n\right\}$ is increasing
and its limit is an idempotent;
thus the limit is 1.

Let $*$ an Archimedean t-norm. Due to the complexity to consider all the real interval $[0, 1]$, we prefer to work on a dense set, where each element of $[0, 1]$ is a limit of this set. Consider for instance $\frac{1}{2}$ and all its exponentiation $c_r = \left(\frac{1}{2}\right)^r = \frac{1}{2^r}$, for each rational $r \geq 0$.

The set $\{c_r\}$ is a dense subset in $[0, 1]$ and we can write each element of $[0, 1]$as a limit of a sequence $\left(\frac{1}{2}\right)^r$.

Therefore, there are two relations:

(*) $\quad c_r + c_s = c_{r+s}$;

(**) $\quad$ if $r > s$, then $c_r < c_s$.

Lemma 11

(7) If the t-norm $*$ is strict, then $\langle [0, 1] , * \rangle$ is isomorphic to $\langle [0, 1] , *_{\Pi} \rangle$, where $x *_{\Pi} y = xy$ (the Product t-norm).

(8) If $*$ is nilpotent, then $\langle [0, 1] , * \rangle$ is isomorphic to $\left\langle \left[\frac{1}{4}, 1\right] , \right\rangle *_{CP}$, where $x *_{CP} y = \max\left(\frac{1}{4}, xy\right)$.

Proof.
Let consider an isomorphism between two dense sets of their
algebras and define d in two different ways:
(7) $d = \frac{1}{2}$ if the t-norm i strict;
(8) $d = \max\{x \mid x * x = 0\}$ if the t-norm is nilpotent.
In the last of these cases, $0 < d < 1$.

On d we build the set $d^{*r} = dr$,
in order to have an isomorphism between c_r starting with d_r.
Moreover, for each r, s , $d_r * d_s = d_{r+s}$, and

$(7')$ if the t-norm is strict, then for each r, s if $r > s$ then $d_r \to d_s$.
$(8')$ if the t-norm is nilpotent, the implication remains
valid only for $r, s \leq 2$ $(r < s \leq 2)$,otherwise $d_r = 0$.

Assume $0 < r < s$ and let m_1, m_2, n be such that $r = \frac{m_1}{n}$, $s = \frac{m_2}{n}$.
If we pose $x^{*\frac{1}{n}}$, then $d_r = x^{*m_1}$ and $d_s = x^{*m_2}$.
Hence $m_1 > m_2$ e $0 < x < 1$by 1.4.3 ,
if $x^{*m_2} > 0$, then $x^{*m_1} < x^{*m_2}$.
Particularly, if the t-norm is nilpotent, $d_r = 0$ iff $r \geq 2$,
i.e. iff $c_r \leq \frac{1}{4}$.

The aim is now to prove that $D = \{d_r\}$ is dense in $[0, 1]$.
Let $0 < x < 1$. We shall approximate x from aboveby elements d_r
(where $r = \frac{m}{2^n}$).
Now let n_0 such that $d_{\frac{1}{2^{n_0}}} \geq x$ [10] and let for each $n \geq n_0$, $r_n = \frac{m}{2^n}$
(for the largest m such that $d_{\frac{m}{2^n}} \geq x$).[11]
Finally, observe $d_{\frac{m+1}{2^n}} = d_{\frac{m}{2^n}} * d_{\frac{1}{2^n}} < x \leq d_{r_n}$,
and if we define these relations in terms of limit:
$\lim_{n\to\infty} \left(d_{r_n} * d_{r(1,n)}\right) = \lim_{n\to\infty} d_{r_n} = x$[12].
By this way, we have proved that for $n \to \infty$ the extremes coincide
(i.e. their distance goes to 0), so all $d_r \in D$ are dense in the our set.

Now we have to clarify why $x *_{CP} y = \max\left(\frac{1}{4}, xy\right)$.

Lemma 12 The ordered semigroup[13] $\left\langle \left[\frac{1}{4}, 1\right], *_{CP} \right\rangle$ is isomorphic to $\langle [0, 1], *_L \rangle$, where $*_L$ is Łukasiewicz t-norm.

Proof.
Let consider the function biunivocal $f : \left[\frac{1}{4}, 1\right] \to [0, 1]$,
which is defined as $f(x) = \frac{\log_2 x + 2}{2}$.
It is an increasing function and induces on $[0, 1]$ the Łukasiewicz t-norm:
$f(x * y) = \frac{\log_2(xy)+2}{2} = \frac{\log_2(x)+\log_2(y)+2}{2} = \frac{\log_2(x)+\log_2(y)+2+2-2}{2} =$
$= \frac{(\log_2(x)+2)+(\log_2(y)+2)-2}{2} = f(x) + f(y) - 1$.

T − norms fundamental theorem
If $*$, E , $\mathcal{I}(E)$ are as above, then:

[10]In fact, by lemma 11 $\lim d_{\frac{1}{2^n}} = 1$.
[11]Remembering that for lemma 8, for each fixed n , $\lim_{m\to\infty} d_{\frac{m}{2^n}} = 0$
[12]Indeed, $\lim_{n\to\infty} d_{r(1,n)} = 1$.
[13]An ordered semigroup is a pair $(A, \cdot)$, where A is a set and $\cdot$ is associative operation on A.

(i) for each $I \in \mathcal{I}(E)$, $(* \upharpoonright I)$ is isomorphic either to the product t-norm or to Łukasiewicz t-norm (on $[0,1]$).

(ii) If $x, y \in [0,1]$ are such that there is no $I \in \mathcal{I}(E)$ (with $x, y \in I$) , then $x * y = \min(x, y)$.

Remark Observe that Łukasiewicz implication is continuous but Gödel and Goguen are not; However, it is easy to show that the residuum of each continuous t-norms is left continuous in the first (antecedent) variable and right continuous in the second (succedent) variable.

3.2 Predicate calculi

3.2.1 Preliminaries

In this section, we will discuss the many-valued predicate logics (or first order logics). In other words, we will pass from declarative language to a *predicative* one.We shall develop these logical systems similarly to the classical predicate logic: particularly, we shall deal only with two quantifiers: $\forall$ (universal) and $\exists$ (existential).

First we will develop the predicate counterpart of our propositional logic BL (henceforth $BL\forall$), and then we will study respectively Łukasiewicz, Gödel and Product predicate logics.

Language. A predicate language consists of:

- a non-empty set of *predicates:* $P, Q, R, ...$;
- a set of *object constants:* $c, d, ...$;
- *object variables:* $x, y, ...$;
- *connectives* and *truth constants* defined as in the previous section;
- *quantifiers:* $\forall$, $\exists$;
- *terms*: object variables and object constants;
- *atomic formulas:* they have the form $P(t_1, ..., t_n)$ where P is a predicate of arity n and $t_1, ..., t_n$ are terms.

If α, β are formulas and x is an object variable, then $\alpha \to \beta$, $\alpha \otimes \beta$, $(\forall x)\,\alpha$, $(\exists x)\,\alpha$, $\bar{0}$ and $\bar{1}$ are formulas; each formula results from atomic formulas by iterated use of this rule.

3.2.2 The predicative counterpart of BL

$BL\forall$ has the same language and the same rules of BL plus the quantifiers $\forall$ and $\exists$, so the axiom schemata of BL$\forall$ are these of BL plus the following:

($\forall$1)$(\forall x)\,\alpha\,(x) \rightarrow \alpha\,(t)$ with t substitutable for x in $\alpha\,(x)$;
($\forall$2) $\forall x(\vartheta \rightarrow \alpha) \rightarrow (\vartheta \rightarrow (\forall x)\,\alpha)$ where x is not free in ϑ;
($\forall$3) $\forall x(\alpha \vee \vartheta) \rightarrow ((\forall x)\,\alpha \vee \vartheta)$ where x is not free in ϑ;
($\exists$1) $\alpha\,(t) \rightarrow (\exists x)\,\alpha\,(x)$ with t substitutable for x in $\alpha\,(x)$;
($\exists$2) $\forall x(\alpha \rightarrow \vartheta) \rightarrow ((\exists x)\,\alpha \rightarrow \vartheta)$ where x is not free in ϑ

As far as the deduction rules, we have

- modus ponens (from $\alpha, \alpha \rightarrow \beta$ infer β)
- generalization (from α, infer $(\forall x)\,\alpha$)

3.2.2.1 Theorems of $BL\forall$

Theorem 2 Let α, β be arbitrary formulas and ϑ a formula non containing x freely. Then $BL\forall$ proves the following:

($1_{BL\forall}$) $\forall x\,(\vartheta \rightarrow \alpha) \equiv (\vartheta \rightarrow (\forall x)\,\alpha)$;

($2_{BL\forall}$) $\forall x\,(\alpha \rightarrow \vartheta) \equiv ((\exists x)\,\alpha \rightarrow \vartheta)$;

($3_{BL\forall}$) $\exists x\,(\vartheta \rightarrow \alpha) \rightarrow (\vartheta \rightarrow (\exists x)\,\alpha)$;

($4_{BL\forall}$) $\exists x\,(\alpha \rightarrow \vartheta) \rightarrow ((\forall x)\,\alpha \rightarrow \vartheta)$;

($5_{BL\forall}$) $\forall x\,(\alpha \rightarrow \beta) \rightarrow ((\forall x)\,\alpha \rightarrow (\forall x)\,\beta)$;

($6_{BL\forall}$) $\forall x\,(\alpha \rightarrow \beta) \rightarrow ((\exists x)\,\alpha \rightarrow (\exists x)\,\beta)$;

($7_{BL\forall}$) $((\forall x)\,\alpha \otimes (\exists x)\,\beta) \rightarrow (\exists x)\,(\alpha \otimes \beta)$.

If y is substitutable for x in $\alpha\,(x)$, then $BL\forall$ proves:

($8_{BL\forall}$) $(\forall x)\,\alpha\,(x) \equiv (\forall y)\,\alpha\,(y)$ and $(\exists x)\,\alpha\,(x) \equiv (\exists y)\,\alpha\,(y)$.

For arbitrary ϑ and ν not containing x freely, $BL\forall$ proves:

($9_{BL\forall}$) $\exists x\,(\vartheta \otimes \nu) \equiv ((\exists x)\,\vartheta \otimes \nu)$;

($10_{BL\forall}$) $\exists x\,(\vartheta \otimes \vartheta) \equiv ((\exists x)\,\vartheta \otimes (\exists x)\,\vartheta)$;

($11_{BL\forall}$) $\exists x\alpha \rightarrow \neg\,(\forall x)\,\neg\alpha$;

($12_{BL\forall}$) $\neg\,(\exists x)\,\alpha \equiv (\forall x)\,\neg\alpha$;

$(13_{BL\forall})\quad \exists x\,(\vartheta \wedge \alpha) \equiv (\vartheta \wedge (\exists x)\,\alpha);$

$(14_{BL\forall})\quad \exists x\,(\vartheta \vee \alpha) \equiv (\vartheta \vee (\exists x)\,\alpha);$

$(15_{BL\forall})\quad \forall x\,(\vartheta \wedge \alpha) \equiv (\vartheta \wedge (\forall x)\,\alpha);$

$(16_{BL\forall})\quad \exists x\,(\alpha \vee \beta) \equiv ((\exists x)\,\alpha \vee (\exists x)\,\beta);$

$(17_{BL\forall})\quad \forall x(\alpha \wedge \beta) \equiv ((\forall x)\,\alpha \wedge (\forall x)\,\beta).$

Proof.
$(1_{BL\forall})$ $\vdash (\forall x)\,\alpha \to \alpha$ by $(\forall 1)$, thus
$\vdash (\vartheta \to (\forall x)\,\alpha) \to (\vartheta \to \alpha)$by transitivity.
Generalize $\vdash \forall x\,[(\vartheta \to (\forall x)\,\alpha) \to (\vartheta \to \alpha)]$, hence by $(\forall 2)$
$\vdash (\vartheta \to (\forall x)\,\alpha) \to \forall x\,(\vartheta \to \alpha)$.

$(2_{BL\forall})$ $\vdash \alpha \to (\exists x)\,\alpha$,
$\vdash ((\exists x)\,\alpha \to \vartheta) \to (\alpha \to \vartheta)$,
generalizing and applying $(\forall 2)$ we get
$\vdash ((\exists x)\,\alpha \to \vartheta) \to \forall x\,(\alpha \to \vartheta)$.

$(3_{BL\forall})$ $\vdash (\vartheta \to \alpha) \to (\vartheta \to (\exists x)\,\alpha)$,
generalize and apply $(\exists 2)$:
$\vdash \exists x(\vartheta \to \alpha) \to (\vartheta \to (\exists x)\,\alpha)$.

$(4_{BL\forall})$ $\vdash (\alpha \to \vartheta) \to ((\forall x)\,\alpha \to \vartheta)$ thus by $(\exists 2)$:
$\vdash \exists x(\alpha \to \vartheta) \to ((\forall x)\,\alpha \to \vartheta)$.

$(5_{BL\forall})$ From $\vdash \forall x\,(\alpha \to \beta) \to (\alpha \to \beta)$ and $\vdash (\forall x)\,\alpha \to \alpha$
we get using transitivity that$\vdash \forall x\,(\alpha \to \beta) \to ((\forall x)\,\alpha \to \beta)$.
Generalizing and applying $(\forall 2)$ twice we get
$\vdash \forall x\,(\alpha \to \beta) \to ((\forall x)\,\alpha \to (\forall x)\,\beta)$.

$(6_{BL\forall})$ Analogously, we get
$\vdash \forall x\,(\alpha \to \beta) \to (\alpha \to (\exists x)\,\beta)$,
from which we obtain, using $(\forall 2)$ and $(\exists 2)$, that
$\vdash \forall x\,(\alpha \to \beta) \to ((\exists x)\,\alpha \to (\exists x)\,\beta)$.

$(7_{BL\forall})$ We can generalize it in
$\vdash \alpha \to (\beta \to (\alpha \otimes \beta))$
and then apply $(5_{BL\forall})$; so we get
$\vdash (\forall x)\,\alpha \to \forall x\,(\beta \to (\alpha \otimes \beta))$
and using $(6_{BL\forall})$
$\vdash ((\forall x)\,\alpha \to ((\exists x)\,\beta) \to \exists x\,(\alpha \otimes \beta))$.

($8_{BL\forall}$) From $\vdash (\forall x)\,\alpha(x) \to \alpha(y)$ we get $(\forall x)\,\alpha(x) \to (\forall y)\,\alpha(y)$ by generalization and $(\forall 2)$.
We get $(\forall y)\,\alpha(y) \to (\forall x)\,\alpha(x)$ in the same way.
The proof of $\vdash (\exists x)\,\alpha(x) \equiv (\exists y)\,\alpha(y)$is analogous.

($9_{BL\forall}$) Using $(\exists 1)$ we obtain $\vdash (\vartheta \otimes \nu) \to ((\exists x)\,\vartheta \otimes \nu)$.
Generalize and use $(\exists 2)$ to get $\vdash \exists x\,(\vartheta \otimes \nu) \to ((\exists x)\,\vartheta \otimes \nu)$.
Conversely observe that by $(\forall 1)$, $\vdash \forall x(\nu \to \nu)$.
Thus $\vdash ((\exists x)\,\vartheta \otimes \nu) \to ((\exists x)\,\vartheta \otimes (\forall x)\,\nu)$,
which gives by ($7_{BL\forall}$) that $\vdash ((\exists x)\,\vartheta \otimes \nu) \to \exists x\,(\vartheta \otimes \nu)$.

($10_{BL\forall}$) Write Φ for $[\vartheta(x) \to (\exists x)\,\vartheta(x)]$ (which is an istance of $(\exists 1)$;
then $\vdash (\Phi \otimes \Phi) \to (\vartheta(x) \otimes \vartheta(x)) \to (\exists x)\,\vartheta(x) \otimes (\exists x)\,\vartheta(x))$.
Eliminate $(\Phi \otimes \Phi)$by modus ponens, generalize and apply $(\exists 2)$; you get
$\vdash (\exists x)\,(\vartheta(x) \otimes \vartheta(x)) \to ((\exists x)\,\vartheta(x) \otimes (\exists x)\,\vartheta(x))$.
It can be written as
$\vdash (\exists x)\,\vartheta^2(x) \to ((\exists x)\,\vartheta(x))^2$.
Conversely, observing that in the propositional calculus $(p \otimes q) \to \left(p^2 \vee q^2\right)$ for each p and q.
So, we have
$\vdash (\vartheta(x) \otimes \vartheta(y)) \to \left(\vartheta^2(x) \vee \vartheta^2(y)\right)$,
by $(\exists 1)$ we obtain $\vdash (\vartheta(x) \otimes \vartheta(y)) \to \left((\exists x)\,\vartheta^2(x) \vee (\exists y)\,\vartheta^2(y)\right)$
and by ($8_{BL\forall}$) $\vdash (\vartheta(x) \otimes \vartheta(y)) \to \left((\exists z)\,\vartheta^2(z) \vee (\exists z)\,\vartheta^2(z)\right)$.
Then $\vdash (\forall x \forall y)\left[(\vartheta(x) \otimes \vartheta(y)) \to (\exists z)\,\vartheta^2(z)\right]$
and so $\vdash (\forall x \forall y)\left[\vartheta(x) \to \left(\vartheta(y) \to (\exists z)\,\vartheta^2(z)\right)\right]$.
from which $\vdash (\exists x)\,\vartheta(x) \to (\forall y)\left(\vartheta(y) \to (\exists z)\,\vartheta^2(z)\right)$
and $\vdash (\exists x)\,\vartheta(x) \to \left((\exists y)\,\vartheta(y) \to (\exists z)\,\vartheta^2(z)\right)$.
Thus $\vdash (\exists x)\,\vartheta(x) \to \left((\exists x)\,\vartheta(x) \to (\exists x)\,\vartheta^2(x)\right)$
and finally $\vdash ((\exists x)\,\vartheta(x))^2 \to (\exists x)\,\vartheta^2(x)$.
This is just another way of writing
$\vdash (\exists x)\,(\vartheta(x) \otimes \vartheta(x)) \to ((\exists x)\,\vartheta(x) \otimes (\exists x)\,\vartheta(x))$.

($11_{BL\forall}$) By ($7_{BL\forall}$) we have that $\vdash (\exists x)\,\alpha \to ((\forall x)\,\neg\alpha \to (\exists x)\,(\alpha \otimes \neg\alpha))$;
but$\vdash (\alpha \otimes \neg\alpha) \to \bar{0}$,
thus $\vdash \forall x\,((\alpha \otimes \neg\alpha) \to \bar{0})$
and $\vdash \exists x\,(\alpha \otimes \neg\alpha) \to \bar{0}$;
hence $\vdash (\exists x)\,\alpha \to ((\forall x)\,\neg\alpha) \to \bar{0}$.

($12_{BL\forall}$) $\vdash (\neg\exists x)\,\alpha(x) \otimes \alpha(x) \to \neg\,(\exists x)\,\alpha(x) \otimes (\exists x)\,\alpha(x)$
thus $\vdash (\neg\,(\exists x)\,\alpha(x) \otimes \alpha(x)) \to \bar{0}$
and $\vdash \neg\,(\exists x)\,\alpha(x) \to (\alpha(x) \to \bar{0})$.
Generalize and apply $(\forall 2)$to get

$\vdash \neg (\exists x)\, \alpha(x) \to (\forall x)\,(\alpha(x) \to \bar{0})$.
The converse implication follows from $(11_{BL\forall})$ by BL.

$(13_{BL\forall})$ $\vdash (\exists x)\,(\vartheta \wedge \alpha) \to (\exists x)\,\vartheta \to \vartheta$ and $\vdash (\exists x)\,(\vartheta \wedge \alpha) \to (\exists x)\,\alpha$;
thus $\vdash (\exists x)\,(\vartheta \wedge \alpha) \to (\vartheta \wedge (\exists x)\,\alpha)$.
Conversely,$\vdash (\vartheta \to \alpha(x)) \to (\vartheta \to (\vartheta \wedge \alpha(x))) \to$
$\to (\vartheta \to (\exists x)\,(\vartheta \wedge \alpha(x)) \to ((\vartheta \wedge (\exists x)\,\alpha(x)) \to (\exists x)\,(\vartheta \wedge \alpha(x))))$.
Then $(\alpha(x) \to \vartheta) \to (\alpha(x) \to (\vartheta \wedge \alpha(x))) \to (\alpha(x)\,(\exists x))\,(\vartheta \wedge \alpha(x))$
$\to ((\vartheta \wedge (\exists x)\,\alpha(x)) \to (\exists x)\,(\vartheta \wedge \alpha(x)))$.
Thus we get $\vdash (\vartheta \wedge (\exists x)\,\alpha(x)) \to (\exists x)\,(\vartheta \wedge \alpha(x))$.

$(14_{BL\forall})$ $\vdash \vartheta \to (\exists x)\,(\vartheta \vee \alpha(x))$ and $\vdash (\exists x)\,\alpha(x) \to (\exists x)\,(\vartheta \vee \alpha(x))$,
thus $\vdash (\vartheta \vee (\exists x)\,\alpha(x)) \to (\exists x)\,(\vartheta \vee \alpha(x))$.
Conversely $\vdash (\exists x)\,(\vartheta \vee \alpha(x)) \to (\exists x)\,(\vartheta \vee (\exists x)\,\alpha(x)) \to (\vartheta \vee (\exists x)\,\alpha(x))$.

$(15_{BL\forall})$ $\vdash (\forall x)\,(\vartheta \wedge \alpha) \to \vartheta$ and $\vdash (\forall x)\,(\vartheta \wedge \alpha) \to (\forall x)\,\alpha$.
Thus $\vdash (\forall x)\,((\vartheta \wedge ((\forall x)\,\alpha) \to (\vartheta \wedge (\forall x)\,\alpha)))$.
Conversely, $\vdash [\vartheta \wedge (\forall x)\,\alpha] \to [(\forall x)\,(\vartheta \wedge (\forall x)\,\alpha)] \to [(\forall x)\,(\vartheta \wedge \alpha)]$.

$(16_{BL\forall})$ We have that $\vdash (\exists x)\,\alpha \to (\exists x)\,(\alpha \vee \beta)$ and $\vdash (\exists x)\,\beta \to (\exists x)\,(\alpha \vee \beta)$
which gives the implication $\leftarrow$.
Conversely, by $(14_{BL\forall})$ we obtain
$\vdash [(\exists x)\,(\alpha \vee \beta)] \to [(\exists x)\,(\alpha \vee (\exists x)\,\beta)] \to [(\exists x)\,\alpha \vee (\exists x)\,\beta]$.

$(17_{BL\forall})$ The direction $\to$ is obvious.
Conversely, by $(15_{BL\forall})$,
$\vdash [((\forall x)\,\alpha \wedge (\forall x)\,\beta)] \to [(\forall x)\,(\alpha \wedge (\forall x)\,\beta)] \to [(\forall x)\,(\alpha \wedge \beta)]$.

3.2.2.1.1 Deduction theorem for $BL\forall$. Let T be a theory over $BL\forall$ and let α, β be closed formulas of the language of T.

Then $(\mathrm{T} \cup \{\alpha\}) \vdash \beta$ iff there is an n such that T$\vdash \alpha^n \to \beta$

Proof.
The proof is an extension of the proof of the deduction theorem for BL; we have to discuss the case of generalization.
Thus let assume $T\vdash \alpha^n \to \gamma_j$where γ_j is $(\forall x)\,\gamma_j$;
then $T\vdash (\forall x)\,(\alpha^n \to \gamma_j)$ and since α is closed it follows that
$T\vdash \alpha^n \to (\forall x)\,\gamma_j$.
Thus, by $(\forall 2)$ we obtain that $T\vdash \alpha^n \to \gamma_j$.

3.2.3 Extensions of $BL\forall$.

Let $\mathcal{C}$ be a schematic extension of the basic propositional logic BL. We associate with $\mathcal{C}$ the corresponding predicate calculus $\mathcal{C}\forall$ (over a given predicate language $\mathcal{J}$) by taking as logical axioms alla formulas resulting from the axioms of $\mathcal{C}$ by substituting arbitrary formulas of $\mathcal{J}$ for propositional variables, and the axioms $(\forall 1)$, $(\forall 2)$,$(\forall 3)$,$(\exists 1)$, $(\exists 2)$ for quantifiers. Moreover, we take as deduction rules the modus ponens and the generalization rule (from α infer $(\forall x)\,\alpha$).

In particular, we are interested in three stronger logics of $BL\forall$: $L\forall$, $G\forall$ and $\Pi\forall$.

3.2.3.1 Łukasiewicz predicate logic.

$L\forall$ is an extension of L in which are proved the following formulas:

$(1_{L\forall})$ $(\exists x)\,\alpha \equiv \neg\,(\forall x)\,\neg\alpha$;
$(2_{L\forall})$ $(\forall x)\,(\alpha\,(x) \vee \vartheta) \equiv (((\forall x)\,\alpha\,(x)) \vee \vartheta)$
where ϑ does not contain x freely;
$(3_{L\forall})$ $(\forall x)\,(\alpha\,(x) \otimes \vartheta) \equiv ((\forall x)\,\alpha\,(x) \otimes \vartheta)$
where ϑ does not contain x freely;
$(4_{L\forall})$ $(\vartheta \to (\exists x)\,\alpha) \to (\exists x)\,(\vartheta \to \alpha)$;
$(5_{L\forall})$ $((\forall x)\,\alpha \to \vartheta) \to (\exists x)\,(\alpha \to \vartheta)$;
$(6_{L\forall})$ $(\exists x)\,\alpha^n \equiv ((\exists x)\,\alpha)^n$ (for each natural $n \geq 1$);
$(7_{L\forall})$ $(\exists x)\,n\alpha \equiv n\,((\exists x)\,\alpha)$(for each natural $n \geq 1$).

Proof.
$(1_{L\forall})$ By $(11_{BL\forall})$ and $(12_{BL\forall})$ we have that $\vdash (\exists x)\,\alpha \to \neg\,(\forall x)\,\neg\alpha$
and $\vdash \neg\,(\exists x)\,\alpha \to (\forall x)\,\neg\alpha$.
Now we know that $L \vdash (\neg\alpha \to \beta) \to (\neg\beta \to \alpha)$,
thus $L\forall \vdash \neg\,(\forall x)\,\neg\alpha \to (\exists x)\,\alpha$.

REMARK. From this proof it follows that $\exists$ is definable in $L\forall$ from $\forall$. Thus an alternative presentation of $L\forall$ is to allow only connectives $\neg, \to$and the quantifier $\forall$ (i.e. taking $\bar{0}$,$\otimes$,$\underline{\vee}$,$\wedge$,$\vee$,$\exists$ as defined symbols), take axioms $(L1) - (L4)$ for the propositional calculus and $(\forall 1)$, $(\forall 2)$ for predicate calculus. By this way, $(\exists 1)$, $(\exists 2)$ and $(\forall 3)$ become provable as the following chains of equivalences and implications using theorems $(1_{BL\forall})$- $(4_{BL\forall})$ (where ϑ does not contain x freely):

(i) $(\alpha\,(t) \to (\exists x)\,\alpha\,(x)) \equiv (\alpha\,(t) \to \neg\,(\forall x)\,\neg\alpha\,(x)) \equiv ((\forall x)\,\neg\alpha\,(x) \to \neg\alpha\,(t))$;
(ii) $(\forall x)\,(\alpha \to \vartheta) \equiv (\forall x)\,(\neg\vartheta \to \neg\alpha) \equiv (\neg\vartheta \to (\forall x)\,\neg\alpha) \equiv$
$\equiv (\neg\,(\forall x)\,\neg\alpha \to \vartheta) \equiv ((\exists x)\,\alpha \to \vartheta)$.

(iii) $(\forall x)(\alpha(x) \vee \vartheta) \to (\forall x)((\vartheta \to \alpha(x)) \to \alpha(x)) \to$
$\to [(\forall x)(\vartheta \to \alpha(x)) \to (\forall x)\alpha(x)] \to$
$\to [(\vartheta \to (\forall x)\alpha(x)) \to (\forall x)\alpha(x)] \to$
$\to [((\forall x)\alpha(x)) \vee \vartheta]$.

($2_{L\forall}$) Starting with the evidence that $\vdash (\forall x)\alpha(x) \to (\forall x)(\alpha(x) \vee \vartheta)$
and $\vdash \vartheta \to (\forall x)(\alpha(x) \vee \vartheta)$;
thus $\vdash ((\forall x)\alpha(x) \vee \vartheta) \to (\forall x)(\alpha(x) \vee \vartheta)$.
The converse implication is the axiom ($\forall 3$),
and it has been proved just above.

($3_{L\forall}$) The direction $\leftarrow$is trivial,
because we have that $\vdash ((\forall x)\alpha \otimes \vartheta) \to (\alpha \otimes \vartheta)$
and we can generalize and shift $\forall$.
Conversely, let $\forall x(\alpha(x) \otimes \vartheta)$ be γ;
we prove $\gamma \to ((\forall x)\alpha \otimes \vartheta)$.
We have $\vdash (\forall x)(\gamma \to (\alpha(x) \otimes \vartheta))$.
Now, by the fact that
$\vdash_L (\neg q \to p) \to [((p \otimes q) \underline{\vee} \neg q) \equiv p]$ (for each p and q)
and for ($1_{BL\forall}$),
we obtain that $\vdash (\neg\vartheta \to \alpha(x)) \to [(\gamma \underline{\vee} \neg\vartheta) \to \alpha(x)]$
and then $\vdash (\alpha(x) \to \neg\vartheta) \to ((\alpha(x) \otimes \vartheta) \to \bar{0})$.
Thus $\vdash (\alpha(x) \to \neg\vartheta) \to (\gamma \to \bar{0})$
and by ($2_{BL\forall}$) and ($4_{BL\forall}$):
$\vdash [(\gamma \underline{\vee} \neg\vartheta) \to \alpha(x)] \vee (\gamma \equiv \bar{0})$.
Now, let generalize and use (2_L) to obtain
$\vdash (\forall x)[(\gamma \underline{\vee} \neg\vartheta) \to \alpha(x)] \vee (\gamma \equiv \bar{0})$,
thus $\vdash [(\gamma \underline{\vee} \neg\vartheta) \to (\forall x)\alpha(x)] \vee (\gamma \equiv \bar{0})$.
But since $\vdash (\gamma \to \vartheta)$ by ($1_{BL\forall}$),
we can use the fact that $\vdash_L (p \to q) \to ((p \wedge q) \equiv p)$
and get $\vdash [\gamma \to ((\forall x)\alpha(x) \otimes \vartheta)] \vee \gamma \equiv \bar{0}$.
Finally, we get $\vdash \gamma \to ((\forall x)\alpha(x) \otimes \vartheta)$ as desired.

($4_{L\forall}$) In Ł$\forall$ we can prove these chains of implications:
$\vdash \neg(\exists x)(\vartheta \to \alpha) \to (\forall x)(\vartheta \otimes \neg\alpha) \to [\vartheta \otimes (\forall x)\neg\alpha] \to$
$\to \neg[\vartheta \to \neg(\forall x)\neg\alpha] \to \neg[\vartheta \to (\exists x)\alpha]$.
Thus $\vdash \neg(\exists x)(\vartheta \to \alpha) \to \neg[\vartheta \to (\exists x)\alpha]$
and hence $\vdash [\vartheta \to (\exists x)\alpha] \to (\exists x)(\vartheta \to \alpha)$.

($5_{L\forall}$) Using (4_L) we obtain that
$\vdash ((\forall x)\alpha \to \vartheta) \to (\neg\vartheta \to \neg(\forall x)\alpha) \to (\neg\vartheta \to (\exists x)\neg\alpha) \to$
$\to (\exists x)(\neg\vartheta \to \neg\alpha) \to (\exists x)(\alpha \to \vartheta)$.

$(6_{L\forall})$ We can proceed as in $(10_{BL\forall})$.

$(7_{L\forall})$ We will prove this case for $n = 2$,
but it is easy to generalize it for arbitrary n.
From $\vdash \alpha \to (\exists x)\, \alpha$ we get
$\vdash (\alpha \underline{\vee} \alpha) \to ((\exists x)\, \alpha \underline{\vee} (\exists x)\, \alpha)$ (thus $\vdash 2\alpha \to 2\, (\exists x)\, \alpha$),
and by generalization we get
$\vdash (\exists x)\, 2\alpha \to 2\, (\exists x)\, \alpha$.
Conversely, $\vdash (\exists x)\, \alpha \to (\exists x)\, \alpha$ gives by (5_L)
that $\vdash (\exists x)\, ((\exists x)\, \alpha \to \alpha)$.
Moreover, by L we obtain that
$\vdash ((\exists x)\, \alpha \to \alpha)^2 \to (2\, (\exists x)\, \alpha \to 2\alpha)$,
and generalizing and applying $(6_{BL\forall})$ we get
$\vdash (\exists x)\, ((\exists x)\, \alpha \to \alpha)^2 \to (\exists x)\, (2\, (\exists x)\, \alpha \to 2\alpha)$,
and by modus ponens, $(9_{BL\forall})$, $(10_{BL\forall})$ and $(3_{BL\forall})$
we obtain $\vdash 2\, (\exists x)\, \alpha \to (\exists x)\, 2\alpha$.

3.2.3.2 Gödel and Product predicate logics

Regarding these two extended systems, in this part of our work we can only say that $G\forall$ and $\Pi\forall$ are respectively extensions of G and Π for the predicative calculus, but in order analyze adequately their features, we would first introduce some fundamental semantic notions, which would be out of our aim. [14] However, we just know that Gödel predicate logic has a recursive axiomatization that is complete with respect to the semantics over $[0, 1]$, whereas for Łukasiewicz logic and Product logic we do not have a recursive complete axiomatization, that is there is no a recursive system of axioms and deduction rules for which provability would equal 1-tautologicity over $[0, 1]$.

[14] For a more detailed explanation see [9].

Chapter 4

Fuzzy systems: the semantic framework

In this chapter we will explore some significant results concerning the logical systems we have analyzed in the previous chapter, from a semantic point of view. In the first part, once again, we will focus on the propositional calculus, in order to show some results concerning its axiomatization.

In particular, first, we will present some general semantic notions, which will be useful in order to prove these results.

4.1 Preliminaries

In this section we will present some general notions, coming from *Universal Algebra*. The aim of the Universal Algebra is to provide some definitions and results that are common to different algebraic structures, through a higher level of abstraction.

Let $\mathcal{L}$ be a language such that:

- $Rel(\mathcal{L})$ is the set whose elements are the *predicates*;
- $Fun(\mathcal{L})$ is the set whose elements are called *functors;*
- $Cost(\mathcal{L})$is the set whose elements are called individual constants;
- there is a function $ar : (Rel(\mathcal{L}) \cup Fun(\mathcal{L})) \to \omega$ such that, if $R \in Rel(\mathcal{L})$,$F \in Fun(\mathcal{L})$, then $ar(R)(ar(F))$ is called ariety of $R(F)$. Moreover, $ar(R), ar(F) \geq 1$.

Definition 11 The quadruple $\tau := \langle Rel(\mathcal{L}), Fun(\mathcal{L}), Cost(\mathcal{L}), ar\rangle$ is called *type* of $\mathcal{L}$.

Definition 12 An Algebra $\mathcal{M} := \langle M, \rho_1, ..., \rho_n\rangle$ is called *structure* of type τ, or realization of $\mathcal{L}$, iff:

(i) M is a non empty set, called *domain* of the structure $\mathcal{M}$,

(ii) ρ_i is a function defined on $Rel(\mathcal{L}) \cup Fun(\mathcal{L}) \cup Cost(\mathcal{L})$ such that:

$\rho(c) \in M$ if $c \in Cost(\mathcal{L})$;
$\rho(R) \subseteq M^n$ if $ar(R) = n$, $R \in Rel(\mathcal{L})$;
$\rho(F) : M^n \to M$ if $ar(F) = n$, $F \in Fun(\mathcal{L})$.

Convention If $\mathcal{M}$ is fixed, sometimes we prefer to write $R^{\mathcal{M}}$, $F^{\mathcal{M}}$ and $c^{\mathcal{M}}$, respectively instead of $\rho(R)$, $\rho(F)$ and $\rho(c)$.

Definition 13 Let σ an interpretation on M, that is a function $\sigma : Var \to M$.

So $M^{\omega} := \{\sigma \mid \sigma : \omega \to M\}$ is the set of the possible interpretations on M.

Definition 14 If $\mathcal{M} = \langle \mathrm{M}, \rho_1, ..., \rho_n\rangle$ and $\mathcal{N} = \langle \mathrm{N}, \eta_1, ..., \eta_n\rangle$ are structures of the same type, $\mathcal{M}$ is a *subalgebra* of $\mathcal{N}$ iff

(iii) $M \subseteq N$

(iv) *a*) if $c \in Cost(\mathcal{L})$, so $c^{\mathcal{M}} = c^{\mathcal{N}}$

\- *b*) if $R \in Rel(\mathcal{L})$, $ar(R) = n$ so $R^{\mathcal{M}} = R^{\mathcal{N}} \cap M$

\- *c*) if $F \in Fun(\mathcal{L})$, $ar(F) = n$ and $a_1, ..., a_n \in M$ so $F^{\mathcal{M}}(a_1, ..., a_n) = F^{\mathcal{N}}(a_1, ..., a_n)$.

Definition 15 If $\mathcal{M} = \langle \mathrm{M}, \rho_1, ..., \rho_n\rangle$ and $\mathcal{N} = \langle \mathrm{N}, \eta_1, ..., \eta_n\rangle$ are structures of the same type, $\varphi : M \to N$ is a *morphism* between $\mathcal{M}$ and $\mathcal{N}$, iff:

(v) if $c \in Cost(L)$ then $\varphi\left(c^{\mathcal{M}}\right) = c^{\mathcal{N}}$;

(vi) if $F \in Fun(L)$, $ar(F) = n$, $a_1, ..., a_n \in M$ then $\varphi\left(F^{\mathcal{M}}(a_1, ..., a_n)\right) = F^{\mathcal{N}}(\varphi(a_1), ..., \varphi(a_n))$.

Definition 16 $\varphi : \mathcal{M} \to \mathcal{N}$ is an *homomorphism* of $\mathcal{M}$ in $\mathcal{N}$ iff:

(vii) φ is a morphism;

(viii) φ preserves the relations of $\mathcal{M}$, i.e:

if $R \in Rel(\mathcal{L})$, $ar(R) = n$, $a_1, ..., a_n \in M$, $\langle a_1, ..., a_n \rangle \in R^{\mathcal{M}}$, then $\langle \varphi(a_1), ..., \varphi(a_n) \rangle \in R^{\mathcal{N}}$.

Definition 17 If $\mathcal{N} = \rho[\mathcal{M}] = \{\rho(a) : a \in M\}$ (where φ is an homomorphism of $\mathcal{M}$ in $\mathcal{N}$), then $\mathcal{N}$ is called *homomorphic image* of $\mathcal{M}$.

4.2 BL Algebras

After some preliminary notions, it is time to start an *algebraization of BL*. In particular, we will introduce a variety of algebras, called *BL-Algebras*.

But, to define BL algebras, we need other two preliminary notions:

Definition 18 A *partially ordered set (poset)* is a couple (A, ρ) where A is a set, and ρ an order relation over A[1].

Definition 19 Let $(A, \leq)$ a poset and $x \in A$:

(i) x is called *greatest element* of A if for each $y \in A$, $y \leq x$;

(ii) x is called *least element* of A if for each $y \in A$, $x \leq y$;

(iii) x is called *maximal element* of A if for each $y \in A$, $x \leq y$, then $x = y$;

(iv) x is called *minimal element* of A if for each $y \in A$, $y \leq x$, then $x = y$.

Proposition 6 Let $(A, \leq)$ a poset and let $B \subseteq A$ and $x \in A$:

(v) x is called *upper bound* of B if for each $y \in B$, $y \leq x$;

(vi) x is called *lower bound* of B if for each $y \in B$, $x \leq y$.

Definition 20 Let a *poset* $(A, \leq)$, and let $B \subseteq A$.

(vii) we call *supremum* $(\sup_A(B))$ of B in A, if it does exist, the least of the upper bound of B;

(viii) we call *infimum* $(\inf_A(B))$ of B in A, if it does exist, the greatest lower bound of B.

[1] A partial order is a binary relation $\leq$ over a set A which is *reflexive*, *antisymmetric*, and *transitive*. In other words, a partial order is an antisymmetric preorder. A set with a partial order is called a partially ordered set (also called a *poset*).

Definition 21 A *lattice* is a *poset* $(A,\leq)$ such that, for each $x, y \in A$, a $\sup(\{x,y\})$ and $\inf(\{x,y\})$ exist.

If A is a lattice, and $x, y \in A$ we can write:

$$x \wedge y = \inf(\{x,y\})$$

$$x \vee y = \sup(\{x,y\})$$

Definition 22 On a lattice $\mathcal{A}$ the following identities are true:

-(idempotence) $x \cap x = x \qquad x \cup x = x$

-(commutativity) $x \cap y = y \cap x$

-(associativity) $x \cap (y \cap z) = (x \cap y) \cap z \qquad x \cup (y \cup z) = (x \cup y) \cup z$

-(absorption) $x \cap (x \cup y) = x \qquad x \cup (x \cap y) = x.$

Actually, in infinite-valued systems we deal with *residuated lattices.*

Definition 23 A *residuated lattice* is an algebraic structure

$$\mathcal{R} = (A, \leq, *, \Rightarrow, 0, 1)$$

with four binary operations and two constants such that:

(i) $(A, \leq, 0, 1)$ is a lattice with the greatest element 1 and the least element 0;

(ii) $(A, *, 1)$ is a commutative semigroup[2] with the unit element 1 (i.e. $*$ is commutative, associative, $1 * x = x$ for all x);

(iii) $*$ and $\Rightarrow$ form an adjoint pair, which means

(1*) for all x, y, z $z \leq (x \Rightarrow y)$ iff $x * z \leq y$

Convention Henceforth, we will use $\mathcal{A}$ to indicate residuated lattices.

Definition 24 A residuated lattice is a BL-algebra iff the following two identities hold for all $x, y \in A$:

[2]A semigroup is a set together with a binary operation $\cdot$ (that is, a function $\cdot : S \times S \to S$) that satisfies the associative property. A semigroup is commutative if $\cdot$ is commutative.

(2*) $x \cap y = x * (x \Rightarrow y)$;

(3*) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

Definition 25 A residuated lattice is *linearly ordered* if its lattice ordering is linear, which means that, for each pair x, y:

$$x \cap y = x \text{ or } x \cap y = y$$

$$x \cup y = x \text{ or } x \cup y = y.$$

REMARKS.

- Note that the class of linearly ordered residuated lattices is not a variety because it is not closed under direct products.
- Each continuous t-norm on $[0, 1]$ determines a BL- algebra, with its standard linear ordering.

Lemma 13 In each BL-algebra, the following hold for each x, y, z :

(4*) $x * (x \Rightarrow y) \leq y$ and $x \leq (y \Rightarrow (x * y))$;

(5*) $x \leq y$ implies $(x * z) \leq (y * z)$, $(z \Rightarrow x) \leq (z \Rightarrow y)$, $(y \Rightarrow z) \leq (x \Rightarrow z)$;

(6*) $x \leq y$ iff $x \Rightarrow y = 1$

(7*) $(x \cup y) * z = (x * z) \cup (y * z)$;

(8*) $x \cup y = ((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)$.

Theorem 3 The logic BL is sound with respect to BL-tautologies: if a formula α is provable in BL, then α is an $\mathcal{A}-$tautology for each BL-algebra $\mathcal{A}$. More generally, if T is a theory over BL and T proves α, then, for each BL-algebra $\mathcal{A}$ and each $\mathcal{A}$-evaluation e of propositional variables assigning the value 1 to all the axioms of T, we have $e(\alpha) = 1$[3].

[3] See 3.1.2

Proof.
We have to prove that all axioms of BL are $\mathcal{A}-$tautologies and also that the definition of $x \cup y$ from $\Rightarrow$ is a tautology.
As we have proved in **3.1.3.1.1**,
all axioms of BL are tautologies
and now we can prove our statement in the same way.
The only exception is $(A6)$.
In particular, :
$((x \Rightarrow y) \Rightarrow z) * (((y \Rightarrow x) \Rightarrow z) \Rightarrow z) = 1$
for $x \leq y$ iff $x \Rightarrow y = 1$
then $\underbrace{(x \Rightarrow y) \Rightarrow z}_{X} \leq \left(\left(\underbrace{(y \Rightarrow x) \Rightarrow z}_{Y}\right) \Rightarrow z\right)$
but for (1^*) iff $X * Y \leq z$.
But, $X * Y = (X * Y) * \underbrace{(x \Rightarrow y) \cup (y \Rightarrow x)}_{=1}$
that is $= ((X * Y) * (x \Rightarrow y)) \cup ((X * Y) * (y \Rightarrow x))$
which is $\leq (((x \Rightarrow y) \Rightarrow z) * (x \Rightarrow y)) \cup (((y \Rightarrow x) \Rightarrow z) * (y \Rightarrow x)) \leq$
$\leq z \cup z = z$.

Definition 26 Let T be a fixed theory over BL. For each formula α let $[\alpha]_{\mathrm{T}}$ be the set of all formulas β such that $\mathrm{T} \vdash \alpha \equiv \beta$.
Then, A_{T} is the set of all the classes $[\alpha]_{\mathrm{T}}$.
We define:
$0 = [0]_{\mathrm{T}}$;
$1 = [1]_{\mathrm{T}}$;
$[\alpha]_{\mathrm{T}} * [\beta]_{\mathrm{T}} = [\alpha \otimes \beta]_{\mathrm{T}}$;
$[\alpha]_{\mathrm{T}} \Rightarrow [\beta]_{\mathrm{T}} = [\alpha \to \beta]_{\mathrm{T}}$;
$[\alpha]_{\mathrm{T}} \cap [\beta]_{\mathrm{T}} = [\alpha \wedge \beta]_{\mathrm{T}}$;
$[\alpha]_{\mathrm{T}} \cup [\beta]_{\mathrm{T}} = [\alpha \vee \beta]_{\mathrm{T}}$.

Lemma 14 $\mathcal{A}_{\mathrm{T}}$ is a BL-algebra.
Proof.
Let $\mathcal{A}_{\mathrm{T}}$ be a *residuated lattice* (with all properties specified above), and let $(A_{\mathrm{T}}, *, 1)$ be a commutative semigroup.
First, observe that the lattice ordering $\leq$ satisfies the following:

$$[\alpha]_{\mathrm{T}} \leq [\beta]_{\mathrm{T}} \ iff \ \mathrm{T} \vdash \alpha \to \beta.$$

In fact, if $\mathrm{T} \vdash \alpha \to \beta$ then $\mathrm{T} \vdash \alpha \equiv (\alpha \wedge \beta)$.
So we have the consequences $[\alpha]_{\mathrm{T}} = [\alpha]_{\mathrm{T}} \cap [\beta]_{\mathrm{T}}$ and $[\alpha]_{\mathrm{T}} \leq [\beta]_{\mathrm{T}}$.

Conversely, if $[\alpha]_{\mathrm{T}} \leq [\beta]_{\mathrm{T}}$ then $\mathrm{T} \vdash \alpha \equiv (\alpha \wedge \beta)$ and since $(\alpha \wedge \beta) \to \beta$ then $\mathrm{T} \vdash \alpha \to \beta$.

Thus, $[\gamma]_{\mathrm{T}} \leq [\alpha]_{\mathrm{T}} \Rightarrow [\beta]_{\mathrm{T}}$ iff $\mathrm{T} \vdash \gamma \to (\alpha \to \beta)$
iff $\mathrm{T} \vdash (\gamma \otimes \alpha) \to \beta$ (for (A5a))
iff $\mathrm{T} \vdash [\gamma \otimes \alpha]_{\mathrm{T}} \leq [\beta]_{\mathrm{T}}$.

But it is just the definition: $\forall x, y, z\ z \leq (x \Rightarrow y)$ iff $x * z \leq y$, thus $\mathcal{A}_{\mathrm{T}}$ is a residuated lattice.

Now, previously we have defined that an algebra is a BL-algebra iff, for all $x, y \in A$:

(2*) $x \cap y = x * (x \Rightarrow y)$;

(3*) $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

So, in this case, (2*) follows from the definition of $\wedge$ (which is $\alpha \wedge \beta \equiv \alpha \otimes (\alpha \to \beta)$), and so $[\alpha]_{\mathrm{T}} \cap [\beta]_{\mathrm{T}} = [\alpha]_{\mathrm{T}} * [\alpha \Rightarrow \beta]_{\mathrm{T}}$; and (3*) follows from $(\alpha \to \beta) \vee (\beta \to \alpha)$, that is $([\alpha]_{\mathrm{T}} \Rightarrow [\beta]_{\mathrm{T}}) \cup ([\beta]_{\mathrm{T}} \Rightarrow [\alpha]_{\mathrm{T}}) = 1$.

Definition 27 Let $\mathcal{A} = (A, \cup, \cap, *, \Rightarrow, 0, 1)$ be a residuated lattice. A *filter* on $\mathcal{A}$ is a non empty set $F \subseteq A$ such that for each $x, y \in A$:

- $x \in F$ and $y \in F$ implies that $x * y \in F$;
- $x \in F$ and $x \leq y$ implies that $y \in F$.

F is a *prime filter* iff for each $x, y \in A$:

- $(x \Rightarrow y) \in F$ or $(y \Rightarrow x) \in F$.

Lemma 15 Let $\mathcal{A}$ be a BL-algebra and let F be a filter. Put:

- $x \sim_F y$ iff $(x \Rightarrow y) \in F$ and $(y \Rightarrow x) \in F$.

Then:

(i) $\sim_F$ is a congruence and the corresponding quotient algebra $\mathcal{A}/\sim_F$ is a BL-algebra;

(ii) $\mathcal{A}/\sim_F$ is linearly ordered iff F is a prime filter.

Lemma 16 Let $\mathcal{A}$ be a BL-algebra and let $a \in A$, $(a \neq 1)$. Then there is a prime filter F on $\mathcal{A}$ not containing a.

Lemma 17 Each BL-algebra is a subalgebra of the direct product of a system of linearly ordered BL-algebras.

Proof.

Let $\mathcal{U}$ be the system of all prime filters on $\mathcal{A}$. For $F \in \mathcal{U}$ let $\mathcal{A}_F = \mathcal{A}/F$ and let

$$\mathcal{A}^* = \prod_{F \in \mathcal{U}} \mathcal{A}_F.$$

So, $\mathcal{A}^*$ is the direct product of linearly ordered residuated lattices $\{\mathcal{A}_F \mid F \in \mathcal{U}\}$of $\mathcal{A}^*$(this follows from the fact that $\mathcal{A}/\sim_F$ is linearly ordered iff F is a prime filter.) For $x \in \mathcal{A}$ let $i(x) = \{[x]_F \mid F \in \mathcal{U}\}$ be the element of $\mathcal{A}^*$. Clearly, this embedding preserves operations.

It remains to show that i is injective, which means that if $x \neq y$, then $i(x) \neq i(y)$ for each $x, y \in F$). Let us assume $x \neq y$. Then $x \nleq y$ or $y \nleq x$. Now, let suppose $x \nleq y$, then $(x \Rightarrow y) \neq 1$ in $\mathcal{A}$, and for the previous lemma, there is a prime filter F on $\mathcal{A}$ which does not containing $(x \Rightarrow y)$. Then, in $\mathcal{A}/F$, $[x]_F \nleq [y]_F$, so $[x]_F \neq [y]_F$ and consequently $i(x) \neq i(y)$. Analogously, if $y \nleq x$.

Corollary 1 Each formula which is an $\mathcal{A}$-tautology for all linearly ordered BL-algebras, is an $\mathcal{A}$-tautology for all BL-algebras.

4.2.1 A Completeness Theorem for BL-algebras

That BL is complete means that for each formula α the following three things are equivalent:

(i) α is provable in BL;

(ii) for each linearly ordered BL-algebra $\mathcal{A}$, α is an $\mathcal{A}$-tautology;

(iii) for each BL-algebra $\mathcal{A}$, α is an $\mathcal{A}$-tautology.

Theorem 4 BL is complete.

Proof.

(i)$\rightarrow$(ii): it follows from the fact that if $BL \vdash \alpha$ then α is a $\mathcal{A}$-tautology for each BL-algebra $\mathcal{A}$, which is proved by the proof that all axioms of BL are $\mathcal{A}$-tautologiesand that the rule of modus ponens preserves tautologicity.

(ii)$\rightarrow$(iii) : it follows from the corollary just presented.

(iii) $\rightarrow$ (i) : Let suppose (iii) and prove (i). To this end recall the fact that $\mathcal{A}_T$ is a BL-algebra, which means that in this case the algebra $\mathcal{A}_{BL}$ of classes of equivalent formulas of BL is a BL-algebra; thus, an α satisfying (iii) is an $\mathcal{A}_{BL}$-tautology. In particular, let

$$v(\alpha) = [\alpha]_{BL} = [1]_{BL}$$

for each $\mathcal{A}_{BL}$-evaluation v.

Thus, $BL \vdash \alpha \equiv 1$ and so $BL \vdash \alpha$.

Definition 28 Let $\mathcal{C}$ be a schematic extension of BL and let $\mathcal{A}$ be a BL-algebra. $\mathcal{A}$ is a $\mathcal{C}$-algebra iff all axioms of $\mathcal{C}$ are $\mathcal{A}$-tautologies.

Now, we shall generalize the Completeness Theorem to obtain a Strong Completeness theorem:

Theorem 5 Let $\mathcal{C}$ be a schematic extension of BL and let α be a formula. The following are equivalent:

(i') $\mathcal{C}$ proves α;

(ii') α is an $\mathcal{A}$-tautology for each linearly ordered $\mathcal{C}$-algebra $\mathcal{A}$;

(iii') α is an $\mathcal{A}$-tautology for each $\mathcal{C}$-algebra $\mathcal{A}$.

Proof.

(i')$\equiv$(ii'): it is evident because it is soundness.

(ii')$\equiv$(iii'): it follows from the fact that each BL-algebra is a subalgebra of the direct product of a system of linearly ordered BL-algebras, which means that an arbitrary $\mathcal{C}$-algebra is embedded into a direct product of its linearly ordered factor algebras, and these factor algebras are $\mathcal{C}$-algebras too.

(i')$\equiv$(iii'): it is proved because the algebra $\mathcal{A}_C$ of classes of mutually $\mathcal{C}$-equivalent formulas is itself a $\mathcal{C}$-algebra. In fact, if $\Phi(\alpha_1, ..., \alpha_n)$ is an instance of he axiom schema $\Phi(p_1, ..., p_n)$ and $e(p_i) = [\beta_i]_C$ then $e(\Phi(\alpha_1, ..., \alpha_n)) = [\Phi(\alpha'_1, ..., \alpha'_n)]_C$ (where α_i results by substituting β_i for p_i), thus $\Phi(\alpha'_1, ..., \alpha' n)$ is also an instence of the schema and therefore $[\Phi(\alpha'_1, ..., \alpha'_n)]_C = [1]_C$.

Finally, we can prove a general *Strong Completeness Theorem*:

Theorem 6 Let T be a theory over $\mathcal{C}$ and let α be a formula.. Then the following are equivalent:

(i') $\mathrm{T} \vdash_{\mathcal{C}} \alpha$;

(ii') for each linearly ordered $\mathcal{C}$-algebra $\mathcal{A}$ and each $\mathcal{A}$-model e of T, $e_A(\alpha) = 1_A$;

(iii') for each $\mathcal{C}$-algebra $\mathcal{A}$ and each $\mathcal{A}$-model e of T, $e_A(\alpha) = 1_A$.

Proof.

Soundeness follows from the fact that axioms of BL are given by axiom schemata and if α is an axiom and β results from α by substitution, then also β is an axiom. So here, all axioms of $\mathcal{C}$ are true in all $\mathcal{C}$-models of T by the definition of a model; and formulas true in a model are closed under MP.

Conversely, assume $\mathrm{T} \nvdash \alpha$ and let $\mathrm{T}' \supseteq \mathrm{T}$ be complete and $\mathrm{T}' \nvdash \alpha$. Thus, setting, for each β, $e(\beta) = [\beta]_{\mathrm{T}}$ we get an $\mathcal{A}_{\mathrm{T}}$-model of T in which $e(\alpha) < 1_{\mathcal{A}\mathrm{T}}$. So, $\mathcal{A}_{\mathrm{T}}$ is a linearly ordered $\mathcal{C}$-algebra.

Now we can consider some schematic extensions of BL-algebras. We will call these algebras for the three *propositional calculi* examined above (L, G and Π), respectively *MV-algebras*, *G-algebras* and $\Pi-$*algebras*, and our final aim will be to prove a completeness theorem for all these structures.

4.3 MV-algebras

Definition 29 In general, a MV-algebra is an algebraic structure $\langle A, \oplus, \neg, 0\rangle$ consisting of

- a non empty set A;
- a binary operation $\oplus$ on A;
- a unary operation $\neg$ on A;
- a constant 0 denoting a fixed element of A;

and satisfying the following identities:

- $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
- $x \oplus 0 = x$
- $x \oplus y = y \oplus x$
- $\neg\neg x = x$
- $x \oplus \neg 0 = \neg 0$
- $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$

Acually, a MV-algebra can equivalently be defined ([9]) as a BL-algebras, satisfying the additional identity

$$x = ((x \Rightarrow 0) \Rightarrow 0).$$

REMARK:

Each formula α determines the corresponding term $\alpha^{\bullet}$ of the language of residuated lattices, and the completeness proved above, implies that:

- $L \vdash \alpha$ iff the identity $\alpha^\bullet = 1$ is valid in each linearly ordered MV-algebra.
- $L \vdash \alpha \equiv \beta$ iff the identity $\alpha^\bullet = \beta^\bullet$ is valid in each linearly ordered MV-algebra.

Our definition of a MV-algebra is natural in the context of residuated lattices; but there are equivalent simpler definitions, for example we can elaborate one such definition based on Łukasiewicz's original axioms (Ł1-Ł4). In particular, we shall call these algebras satisfying the new definition *Wajsberg Algebras*, and we will show their relation to MV-algebras.

Definition 30 A Wajsberg Algebra is an algebra $\mathcal{W} = \langle W, \Rightarrow, 0\rangle$ in which the following identities are valid (put $\neg x = x \Rightarrow 0$, $1 = (0 \Rightarrow 0)$):

(W1) $(1 \Rightarrow y) \Rightarrow y$

(W2) $(x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z)) = 1$

(W3) $((\neg x \Rightarrow \neg y) \Rightarrow (y \Rightarrow x)) = 1$

(W4) $((x \Rightarrow y) \Rightarrow y) = ((y \Rightarrow x) \Rightarrow x)$.

Lemma 18 The following identities are true in each Wajsberg algebra:

(i) $(x \Rightarrow x) = 1$

(ii) $x = y$ iff $(x \Rightarrow y) = 1$ and $(y \Rightarrow x) = 1$

(iii) $(x \Rightarrow 1) = 1$

(iv) $(x \Rightarrow (y \Rightarrow x)) = 1$

(v) $((x \Rightarrow y) \Rightarrow y) \Rightarrow ((y \Rightarrow x) \Rightarrow x) = 1$

Lemma 19 For arbitrary formulas α, β and each Wajsberg algebra $\mathcal{W}$:

(vi) $L \vdash \alpha$ implies that the identity $\alpha^\bullet = 1$ is valid in $\mathcal{W}$;

(vii) $L \vdash \alpha \equiv \beta$ implies that the identity $\alpha^\bullet = \beta^\bullet$ is valid in $\mathcal{W}$.

Proof.
(vi) If $x = 1$ and $x \Rightarrow y = 1$, then $(1 \Rightarrow y) = 1$ and finally $y = 1$.
(vii) It follows by (ii) that is $x = y$ iff $(x \Rightarrow y) = 1$ and $(y \Rightarrow x) = 1$.

Theorem 7 The restriction of an MV-algebra to $\Rightarrow, 0$ is a Wajsberg algebra, and each Wajsberg algebra expands to a MV-algebra. In more detail:

(1) if $\mathcal{A}$ is a MV-algebra, then $\mathcal{W} = \langle W, \Rightarrow, 0\rangle$ is a Wajsberg algebra;

(2) if $\mathcal{W} = \langle W, \Rightarrow, 0\rangle$ is a Wajsberg algebra, and if $*, \cap, \cup, 1$ are defined in the obvious way ($x * y = \neg x \Rightarrow \neg y$, $x \cap y = x * (x \Rightarrow y)$, $x \cup y = (x \Rightarrow y) \Rightarrow y$), then $\mathcal{A} = \{A, \cap, \cup, *, \Rightarrow, 0, 1\}$ is a MV-algebra.

Proof.

(1) It follows from the fact that the counterparts of the axioms (Ł1-Ł4) are provable and so are $(1 \Rightarrow x) = x$ and $((x \Rightarrow y) \Rightarrow x) = ((y \Rightarrow x) \Rightarrow x)$.

(2) It follows by observing that MV-algebras are characterized by finitely many identities and that the corresponding formulas are Ł-provable; thus, if $\mathcal{W}$ is a Wajsberg algebra, and $\mathcal{A}$ its expansion by the obvious definitions of $*, \cap, \cup$, then the identities in question are valid in $\mathcal{A}$. Thus from now on we may identify Wajsberg algebras with MV-algebras.

4.3.1 A completeness theorem for MV-algebras

Definitions 31 A linearly ordered *Abelian semigroup* is a structure $\langle G, +, \leq\rangle$ such that $\langle G, \leq\rangle$ is a linearly ordered set and the following monotonicity axiom[4] is true in $\langle G, +, \leq\rangle$.

Definition 32 An *abelian Group* is a structure $\langle G, +, 0, -\rangle$such that $\langle G, +\rangle$ is an abelian semigroup, 0 is its zero element and - is the operation of inverse, i.e.$x + -x = 0$ for each x.

Definition 33 A *linearly ordered Abelian group is a structure* $\langle G, +, 0, -, \leq\rangle$ such that $\langle G, +, 0, -\rangle$ is an abelian group and $\langle G, +, \leq\rangle$ is a linearly ordered Abelian semigroup.

Definition 34 Let $\mathcal{G} = \langle G, +_G, \leq_G\rangle$ be a linearly ordered abelian group and let $e \in G$, $0 <_G e$ be a positive element. $MV(\mathcal{G}, e)$ is the algebra $\mathcal{A} = \langle A, \Rightarrow, 0_G\rangle$ whose domain A is the interval $[0, e]_G = \{g \in G \mid 0 \leq_G g \leq_G e\}$, $x \Rightarrow y = e$ and $x \Rightarrow y = e - x + y$ otherwise.

Theorem 8 Let $\alpha(x, ..., y)$ be a propositional formula in the language of abelian groups and let $\mathbf{Re} = (Re, +, \leq)$ where Re is the set of all real numbers, $+$ the addition of reals, $*$ the multiplication of reals . If the formula $(\forall x, ..., y)\, \alpha(x, ..., y)$ is true in the abelian group $\mathbf{Re}$, then is true in all the abelian groups.

[4] $x \leq y \to (x + z \leq y + z)$

Theorem 9 Each abelian group is partially embeddable into **Re**.

Remark We may assume that $\mathcal{G}$ is an abelian group with some additional operations $F_1, ..., F_k$ definable by open formulas from the group operations and ordering. That is, there are open formulas α_i such that $z = F_i(x, ..., y) \equiv \alpha_i(x, ..., y, z)$ is true in G and in **Re**.

Lemma 20 $MV(\mathcal{G}, e)$ is a linearly ordered MV-algebra.

Theorem 10 For each linearly ordered MV-Algebra $\mathcal{A}$ there is a linearly ordered Abelian Group $\mathcal{G}$ and a positive element $e \in \mathcal{G}$ suc that $\mathcal{A} = MV(\mathcal{G}, e)$.

Lemma 21

(1) If an identity $\sigma = \tau$ in the language of MV-algebras is valid in the standard MV-algebra $[0,1]$ with truth functions, then it is valid in each linearly ordered MV-algebra.

(2) Consequently, if a formula α is a tautology over the standard MV-algebra, then α is an $\mathcal{A}$-tautology for each linearly ordered MV-algebra $\mathcal{A}$.

(3) More generally, if T is a finite theory and α is true in each $[0,1]_L-$ *model* of T, then for each linearly ordered MV-algebra $\mathcal{A}$, α is true in each $\mathcal{A} - model$ of T.

Proof.

Recall theorem **8** and the previous remark we know that the same is true if we introduce new operations by open definitions. In particular, here we will expand the theory of abelian groups by the ternary operiation $x \Rightarrow_e y$ defined as follows:

$x \Rightarrow_e y = e$ if $x \leq y$, otherwise $x \Rightarrow_e y = e - x + y$.

Now, each term σ of MV-algebras (assume it is constructed from variables using only 0 and $\Rightarrow$) we associate a term σ_e^* of abelian groups putting $x_{i_e}^* = x_i$ and $0_e^* = 0$, $(\sigma_1 \Rightarrow \sigma_2)_e^* = (\sigma_1)_e^* \Rightarrow (\sigma_2)_e^*$.

Now, let $\mathcal{A}$ be a MV-algebra and σ, τ terms such that the identity $\sigma = \tau$ is not valid in $\mathcal{A}$ (which means that for some $\mathbf{a} = a_1, ..., a_n \in \mathcal{A}$, $\mathcal{A} \models \sigma(a) \neq \tau(a)$).

Let $\mathcal{G}$ be an abelain group such that $\mathcal{A} = MV(\mathcal{G}, e)$ for an appropriate $e \in G$; thus $\mathcal{G} \models \sigma_e^*(\mathbf{a}) \neq \tau_e^*(\mathbf{a})$ and $\mathcal{G} \models 0 \leq \mathbf{a} \leq e$.

By the theorem **8** , there are reals $e > 0$, $0 < a_1, ..., a_n < e$ such that $Re \models \sigma_e^* (\mathbf{a}) \neq \tau_e^* (\mathbf{a})$. By dividing by e we get $b_1, ..., b_n$ such that $Re \models \sigma_e^* (\mathbf{b}) \neq \tau_e^* (\mathbf{b})$ where $\mathbf{b}$ is a tuple of elements of $[0, 1]$. Hence, the standard MV-algebra over $[0, 1]$ satisfies $\sigma (\mathbf{b}) \neq \tau (\mathbf{b})$.

The proof of (3) is obtained in a similar way, just observing that $\mathcal{A}$ is a linearly ordered MV-algebra and $\mathbf{a}$ is a tuple of its elements such that $\mathcal{A} \models \sigma_1 (\mathbf{a}) = \tau_1 (\mathbf{a})$,...,$\mathcal{A} \models \sigma_n (\mathbf{a}) = \tau_n (\mathbf{a})$ but $\mathcal{A} \models \sigma (\mathbf{a}) \neq \tau (\mathbf{a})$, then the theorem **8** gives us a tuple $\mathbf{b}$ of elements of $[0, 1]$ such that

$[0, 1]_L \models \sigma_1 (\mathbf{b}) = \tau_1 (\mathbf{b}), ..., \sigma_n (\mathbf{b}) = \tau_n (\mathbf{b})$ where $\sigma (\mathbf{b}) \neq \tau (\mathbf{b})$.

Thus, if $\mathrm{T} = \{\alpha_1, ..., \alpha_n\}$ and α is not true in an $\mathcal{A} - model$ μ of T (where μ is an $\mathcal{A}$-evaluation α containing propositional variables $p_1, ..., p_n$) then for $a_i = e (p_1)$, $\mathbf{a} = (a_1, ..., a_n)$ (where $i = 1, ..., n$) we get $\alpha_1^* (\mathbf{a}) = 1_A, ..., \alpha_n^* (\mathbf{a}) = 1_A$ and $\alpha^* (\mathbf{a}) \neq 1_A$. The above gives $\mathbf{b} = (b_1, ..., b_n)$ such that $\alpha_i^* (\mathbf{b}) = 1$ and $\alpha^* (\mathbf{b}) \neq 1$, thus any $[0, 1]$ evaluation μ' such that $\mu' (p_i) = b_i$ is a $[0, 1]_L - model$ of T in which α is not 1-true.

Corollary 2 Let $[0, 1]_L$ denote the standard MV-algebra on $[0, 1]$ with truth functions on Łukasiewicz logic.

(1) A formula α is an 1-tautology of Ł iff it is an $\mathcal{A}$-tautology for each linearly ordered MV-algebra $\mathcal{A}$.

(2) Let T be a finite theory over Ł. The following are equivalent.

- $\alpha$is true in each $[0, 1]_L - model$ of T,

- for each linearly ordered MV- algebra $\mathcal{A}$, α is true in each $\mathcal{A}$-model of T.

Theorem 11

(1) A formula α is provable in L iff it is a 1-tautology of Ł.

(2) Let T be a finite theory over L, α a formula. T proves α on L iff α is true in each model of T.

Proof.

This follows by the strong completeness theorem for BL-algebras and the preceding corollary.

4.4 Π algebras

Definition 35 A $\Pi - algebra$ is a BL-algebra satisfying the following conditions:

- $\neg\neg z \leq ((x * z \Rightarrow y * z) \Rightarrow (x \Rightarrow y));$

- $x \cap \neg x = 0.$

Lemma 22 The following holds in each linearly ordered product algebra:

(i) if $x > 0$ then $\neg x = 0$;

(ii) if $x > 0$ then $x * z = y * z$ implies $x = y$;

(iii) if $z > 0$ then $x * z < y * z$ implies $x < y$.

Proof.

(i) $0 = x \cap \neg x = \min(x, \neg x)$, hence if $x > 0$ then $\neg x = 0$.

(ii) If $z > 0$ then $\neg\neg z = 1$ thus if $x * z \leq y * z$, then $(x * z) \Rightarrow (y * z) = 1$. And $x \Rightarrow y = 1$, hence $x \leq y$. Thus $x * z = y * z$ implies $x = y$. On the other hand, evidently $x \leq y$ implies $x * z \leq y * z$, thus if $x * z < y * z$, which means $x * z \leq y * z$ and not $x * z \leq y * z$, then $x < y$.

(iii) It is implied by (ii).

Theorem 12 Let $\mathcal{A} = \langle A, *, \Rightarrow, \cap, \cup, 0_A, 1_A \rangle$ be a linearly ordered product algebra. Then there is a linearly ordered Abelian Group $\mathcal{G} = \langle G, +_G, 0_G, \leq_G \rangle$ such that $A - \{0\} = Neg_G = \{g \in G \mid g \leq_G 0_G\}$ such that, for all $g, h \in A - \{0\}$:

(1) $0_G = 1_A$;

(2) $g +_G h = g * h$;

(3) $g \leq_G h$ iff $g \leq h$;

(4) for $g \geq h$, $g \Rightarrow h = h -_G g$.

Proof.

Observe that $A - \{0_A\}$ is linearly ordered commutative semigroup and 1_A is its greatest and neutral element. It is closed under $*$ due to $\neg(\alpha \odot \beta) \rightarrow \neg(\alpha \wedge \beta)$ which gives: $x * y = 0$ implies $\min(x, y) = 0$. Moreover, observe that $A - \{0_A\}$ satisfyies that fact that for each $0_A < x \leq y$ the equation $x * z = x$ has a solution (namely $y \Rightarrow x$ ad recall that $y * (y \Rightarrow x) = \min(x, y)$)

and this solution is unique due to the fact that if $z > 0$ then $x * z = y * z$ implies $x = z$.

Definition 36 For each linearly ordered abelian group $\mathcal{G}$ let $\Pi(\mathcal{G})$ be the algebra $\mathcal{A} = \langle A, *, \Rightarrow, \cap, \cup, 0_A, 1_A \rangle$ where $A = Neg_G \cup \{-\infty\}$ where $-\infty$ is a new element, less than all $x \in Neg_\infty$ and such that:

- $x * y = x +_G y$ for $x, y \in Neg_G$;
- $(-\infty) * x = x * (-\infty) = (-\infty)$ for all $x \in A$;
- $x \Rightarrow y = 1$ for $x \leq y$ $x, y \in A$;
- $x \Rightarrow y = y -_G x$ for $x > y$ $x, y \in A - \{-\infty\}$;
- $x \Rightarrow -\infty = -\infty$ for $x > -\infty$;
- $x \cap y = \min(x, y)$ $x \cup y = \max(x, y)$;
- $0_A = -\infty$, $1_A = 0_G$.

4.4.1 A Completeness theorem for $\Pi-$algebras

(1) A formula α is provable in the product logic iff it is a 1-tautology of the product logic.

(2) Let T be a finite theory over Π, and α a formula. T proves α over the product logic iff it is true in each model of T.

4.5 G-algebras

Definition 37 A $G - algebra$ is a BL-algebra satisfying the identity

$$x * x = x.$$

Lemma 23 Let $\mathcal{H}$ be a linearly ordered G-algebra.

(1) For each x, y $x > y$ implies $(x \Rightarrow y) = y$;

(2) each subset of $\mathcal{H}$ containing 0_H and 1_H is a subalgebra.

Proof.

(1) $z \leq (x \Rightarrow y)$ implies $x \cap z \leq y$; thus if $x > y$ we get $x \cap z < x$, and $x \cap z = x$. Thus $z \leq y$. Hence, $(x \Rightarrow y) \leq y$; conversely $y \leq (x \Rightarrow y)$ in each residuated lattice.

(2) In linearly ordered algebra $\mathcal{H}$ we have that $x \cap y = x$ or $x \cap y = y$, (and similarly for $\cup$), then $(x \Rightarrow y) = 1$ or $(x \Rightarrow y) = y$.

Corollary 3

(3) If $\mathcal{H}_1, \mathcal{H}_2$ are two finite linearly ordered G-algebras of the same cardinality, then they are isomorphic.

(4) Each at most countable linearly ordered G-algebra is isomorphic to a subalgebra of the standard linearly ordered G-algebra $[0,1]_G$; moreover, it is isomorphic to a subalgebra of the G-algebra of rational elements of $[0,1]_G$.

Proof.

(3) They are isomorphic as linearly ordered sets; but the linear order determines all the operations.

(4) This follows from the fact that each countable linear order can be isomorphically embedded to rationals from $[0,1]$.

Lemma 24 If an identity $\tau = \sigma$ in the language of G-algebras is valid in the standard Heyting algebra[5] $[0,1]_G$ of truth functions then it is valid in all linearly ordered G-algebras.

Proof.

Let $\tau = \sigma$ be violated by $\alpha_1, ..., \alpha_n \in \mathcal{H}$; thus it is violated in $\mathcal{H}_1 = \{0, \alpha_1, ..., \alpha_n, 1\}$ as a subalgebra of $\mathcal{H}$. Take an isomorphic copy $\mathcal{H}_2$ which is a subalgebra of $[0,1]_G$; so $\tau = \sigma$ is violated in $\mathcal{H}_2$ and hence in $[0,1]_G$.

4.5.1 A Completeness Theorem for G-algebras

(1) A formula α is provable in the Gödel logic iff it is a 1-tautology of the Gödel logic.

(2) For *each theory* T over G, and α a formula. T proves α (over the Gödel logic) iff it is true in each model of T(over G).

[5]A *Heyting algebra* is a bounded lattice equipped with a binary operation $x \rightarrow y$ of implication such that $(x \rightarrow y) \wedge x \leq y$, and moreover $x \rightarrow y$ is the greatest such in the sense that if $z \wedge x \leq y$ then $z \leq x \rightarrow y$.

REMARK.

We can notice that in this case we get a *strong completeness* for *arbitrary,* not necessarily finite, theories.

Proof.

To prove (1) and (2): if $T \nvdash \alpha$ then there is a model e of T over the rationals from $[0,1]$ such that $e(\alpha) < 1$. By the strong completeness for BL we have just proved, T has a model e over $\mathcal{A}_{\hat{T}}$ (where $\hat{T}$ is a completion of T) such that $e(\alpha) < 1_{\hat{T}}$; by the corollary just presented, $\mathcal{A}_{\hat{T}}$ can be isomorphically embedded into the rationals from $[0,1]$(as a Heyting algebra) which gives the result.

Just a final gloss about the behaviour of Gödel logic with respect to *partial truth.*

Theorem 13 For each theory T over G, each formula α and each rational r such that $0 < r \leq 1$, $T \vdash \alpha$ iff each evaluation e such that $e(\gamma) \geq r$ for each axiom $\gamma \in T$ satisfies $e(\alpha) \geq r$, which means that e makes all axioms $r - true$ then it makes $\alpha\ r - true$.

Proof.

Let assume $T \vdash \alpha$ and $e(\gamma) \geq r$ for each $\gamma \in T$; if $\alpha_1, ..., \alpha_n$ is a proof of α then show $e(\alpha_i) \geq \gamma$ by induction, observing that if $e(\alpha_i) \geq \gamma$ and $e(\alpha_i \to \alpha_j) \geq \gamma$ then $e(\alpha_j) \geq \gamma \wedge \gamma = \gamma$.

Conversely, if r_0 is such that $0 < r_0 < 1$ and each e making T $r_0 - true$ makes $\alpha\ r_0 - true$ then for each $0 < r < 1$, each making T $r - true$ makes $\alpha\ r - true$. In other words, let take any monotone one-one mapping of $[0,1]$, onto itself such that $i(r) = r_0$ and observe that assigning to each e an evaluation e' such that $e'(p) = i(e(p))$ we map the set of all evaluations onto itself and for each formula β, $e(\beta) \geq r$ iff $e'(\beta) \geq r_0$. Thus, for each $r < 1$ we get: if e makes T r-true then it makes α $r-$true. It follows easily that this must also hold for $r = 1$ and hence $T \vdash \alpha$ by the above completeness theorem.

Chapter 5

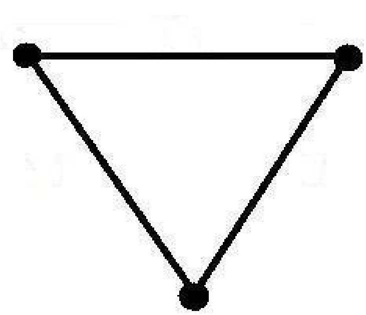
A many-valued approach to Vagueness

In this chapter, we will explore two aspects of the issue of vagueness: in particular, we will return to the sorites paradox, in order to highlight why and how we can try a many-valued approach to vagueness, if it is conceived in terms of closeness.

Therefore, before arguing in favor of a link between vagueness-as-closeness and the many-valued approach, we need to retrieve the definition of *sorites-susceptibility* given in the first chapter[1].

In fact, we can see the situation like a triangle, whose vertices are represented by the three main concepts we have hitherto identified: Sorites susceptibility, vagueness-as-closeness and the many-valued approach:

Many-valued approach   Vagueness-as-Closeness



Sorites susceptibility

[1]See 2.1.2.1.

5.1 About Sorites susceptibility

Let start with (1): the relationship between the *Vagueness-as-Closeness* definition and the *sorites susceptibility.* As we anticipated in the first chapter, the main point is that if we suppose that a predicate conforms to Closeness, we can see both why a Sorites paradox for this predicate is compelling, and also how the paradox is mistaken.

But first of all, let's remember one of the possible versions of the paradox (where P is a predicate):

1. The first object in the series is P.
2. For any object in the series (except the last), if it is P, then the next object is P too.
3. Therefore, the last object in the series is P.

A Sorite series for a predicate P is a series of objecs, which begins with an object which is P, and ends with an object which is not P, and where the adjacent items are very close in $P-$relevant respects.

The *busillis* is therefore the second premiss:

2. For any object in the series (except the last), if it is P, then the next object is P too.

and in particular, the question is based on the choice between - again - a *Tolerance reading* and a *Closeness reading.*

Indeed, in the Tolerance reading, the conclusion follows from the premisses, whereas on the Closeness reading it does not, because each following statement "x is P" must be very similar in respect of truth to the previous, but not exactly the same in respect of truth. And if we say that in the second premiss of the argument P conforms to Closeness, the whole argument is invalid: this is the reason why the paradox is mistaken.

The reader must note that this fact mirrors perfectly the reactions of ordinary speakers to Sorites paradoxes, involving vague predicates, and gives us reason to believe that ordinary speakers do accept that vague predicates satisfy Closeness (and not Tolerance).

It means that the vagueness-as-Closeness definition allows the possibility of a predicate which is vague, but not sorites-susceptible. In fact, if we have shown that if S believes that P conforms to Closeness, then giving a Sorites series for P, means that we can build a sorites paradox for P which S is complelling (and mistaken).

The problem is, therefore, that - as Scott Soames[2] suggests - not all predicates are sorites predicates, that is, sometimes it is difficult to imagine a Sorites series for P: a series of possible things ranging from one which is P to one which is $non - P$, with adjacent items in the series being very close in $P-$relevant respect.

By this way, we have proved why sorites-susceptibility does not belong to the definition of vagueness, rather it is only a mark of it: if a Sorites series does exist for a predicate P, then supposing that P conforms to Closeness, explains why the associated Sorites argument is compelling (and mistaken).

In other words, if P is not Sorites- susceptible, it does not mean that P is not vague. It might be that P is vague, but there is no readily imaginable Sorites series for P.

It is significant to note that these last arguments give an exhaustive answer to the question about the definition of sorites susceptibility, posed in the second chapter.[3]

Now, consider (2): the relationship among the *Sorites Susceptibility* and the *many-valued approach*.

As Smith argues, it is absurd that one of the few things that most philosophers have found appealing about the standard many-valued account, is the resolution of the Sorites, when actually this resolution fails to solve the problem.

Here we shall present the standard resolution of the Paradox, and then we will show why this resolution is a flop. Then, we will suggest a way to solve the Sorites without using the Łukasiewicz conditional.

Consider a standard version of the paradox:

I.	If a pile 10000 is a heap, then pile 9999 is a heap.
2.	If a pile 9999 is a heap, then pile 9998 is a heap.
$\vdots$	
9999.	If pile 2 is a heap, then pile 1 is a heap.
10000.	Pile 10000 is a heap.
$\therefore$10001.	Pile 1 is a heap.

[2][22], 217.
[3]See 2.1.2.1.

According to the definition given in 3.1.4.1, the account "if...then..." here is read as the Łukasiewicz conditional, which has the following truth conditions:

- $\alpha \Rightarrow \beta = 1 \quad if\ \alpha \leq \beta$
- $\alpha \Rightarrow \beta = (1 - \alpha + \beta)$ otherwise

First of all, we must note that not all the premisses in a sorites paradox are 1 true.

In fact, at the beginning of the series of a Sorites Paradox, both antecedent and consequent of a conditional are 1 true, and so are the conditionals. But if we move along the series, we arrive at one point where the antecedent is so slightly more true than the consequent: in this part of the reasoning, the conditionals are ever so slightly less than 1 true.

Thus, the argument proceeds for a while until both antecedent and consequent are 0 true, and hence the conditionals are 1 true again.

So, what is the problem with the argument? The problem is that, not all premisses are fully true, but it is compelling because all the premisses are *very nearly* 1 true. Neverheless, the little amount of falsity in some of the premisses, does accumulate as we move along the series, and so - by the time we get to the end of the series - the conclusion will be 0 true.

Thus, the standard many-valued explanation about the plausibility of the conditional formulation of the Sorites paradox, does not extend to other formulations. Thus, it is obvious that, if a solution solves a problem only when it is formulated in one specific way, then the solution is not really adressing the fundamental problem at all.

However, we must also note that the equivalent formulations of the Sorites paradox are not always equally compelling. For instance, versions of the paradox with premisses of the form "it is not the case that Pa and not Pa" are generally as compelling as versions with premisses of the form "if Pa, then Pa". But versions with premisses of the form "either it is not the case that Pa or it is the case that Pa", seem not to be compelling. In particular, Sorites premisses of this last sort are not compelling because—for whatever reason—we simply do not (without a deal of difficulty) hear them as expressions of Tolerance.

The main problem that arises here, concerns the possibility of building another definition of a material conditional which does not involve the Łukasiewicz conditional, and which is better in order to solve the Sorites paradox.

Here we will present Smith's proposal, but firstly let us remember what is our claim: the sorites- susceptibility is only a *symptom* of vagueness (in our sense), it is not a part of the vagueness-as-closeness definition. Then, even if we would be able to construct a new type of fuzzy material conditional to solve the paradox, it would not be enough to become a fundamental part of our definition.

Now, it is time to enter into the heart of the alternative proposal. The issue is what could we say about sentences of the form "if α then β". The stategy here consists in to mantain the equivalence[4] of

$$\alpha \Rightarrow \beta, \qquad n(\alpha) \diamond \beta \quad \text{and} \quad n(\alpha * n(\beta))$$

and then the usual connection between consequence and the conditional: β is a consequence of α just in case $\alpha \Rightarrow \beta$ is a tautology.

In the account presented here, the tautology property is having a value of at least 0.5, and in the case of the fuzzy material conditional defined above, we observe that:

- if $\alpha \leq \beta$, then $\quad \alpha \Rightarrow \beta \geq 0,5$
(for if $\beta \geq 0,5$, then $(\alpha \Rightarrow \beta) = (n(\alpha) \diamond \beta) \geq 0,5$;

- while, if $\beta < 0,5$, then $\alpha < 0,5$,
so $n(\alpha) > 0,5$,
and then $(\alpha \Rightarrow \beta) = (n(\alpha) \diamond \beta) > 0,5$.

Let's do an example. Consider the set of italian basins, in particular Lake Garda as a borderline case of "pollute", and Iseo's Lake who has one more point in percentage than Lake Garda. Let us suppose "Lake Garda is polluted" is 0,5 true and "Lake Iseo is pollute" is 0.51 true. So, according to this proposal, "if Lake Garda is polluted, then Lake Iseo is polluted" is 0,51 true, while in the standard fuzzy semantics it would be 1 true, according to the Łukasiewicz conditional.

Now, suppose that I say "if Lake Garda is polluted, then Lake Iseo is pollute" and *I mean just that;* it is evidente that in this case it does not seem that the sentence should definitively be true. Furthermore, suppose Alice informs Bill that the Iseo's Lake has one more point in percentage than the Lake Garda, and then Bill can see these two basins in front of him. Suppose also that Bill could recognize Lake Iseo as a borderline case for "pollute". If Bill now says "if Lake Garda is polluted, then Lake Iseo is polluted" (and he means just that), then far from being clearly true, his

[4]Two wfs α and β are logically equivalent (written $\alpha \equiv \beta$) if they have the same truth value on every interpretation.

statement is odd, because it is not clearly false that Lake Garda is polluted, but only quite odd.

The main point here, is therefore that there is not a contrast among the intuitive assertibility status of the sentence, and the truth value assigned to the latter well formula by this semantic context.

But now, let us return to the question about the nature of the Sorites Paradox. We argued that if we suppose that a predicate conforms to Closeness, we can see both why a Sorites paradox for this predicate is compelling and also how the paradox is mistaken. To understand better these last statements, it would be helpful to precise again the Tolerance and Closeness readings of the second premiss of the argument, which had been mentioned above. To do this, we must firstly add some symbols to our formal language:

- if α is a closed well formula, then $[\alpha]$ is a term and we will use it as a name of the degree of truth of α

- $=$ denotes an identity predicate and we treat it in a classical way, i.e. as an item of logical vocabulary.

- $\approx_T$ denotes the relation which holds between truth values that are very close in respect of truth (which means that two sentences are close in respect of truth, if their truth values stand in this relation). [5]

Now we are ready to consider the Tolerance reading of the Sorites argument:

I. $[Hp_{10000}] = [Hp_{9999}]$
2. $[Hp_{9999}] = [Hp_{9998}]$
$\vdots$
9999. $[Hp_2] = [Hp_1]$
10000. $[Hp_{10000}] = 1$
$\therefore$10001. $[Hp_1] = 1$.

On the other hand, on the Closeness (but not Tolerance reading), the full argument is:

I. $[Hp_{10000}] \approx_T [Hp_{9999}]$
2. $[Hp_{9999}] \approx_T [Hp_{9998}]$

[5] See 2.1.2.5.

⋮
9999. $[Hp_2] \approx_T [Hp_1]$
10000. $[Hp_{10000}] = 1$
$\therefore$10001. $[Hp_1] = 1$.

The main difference is that the first of these arguments is classically valid, and hence valid in the standard version of *fuzzy-logic*. On the contrary, the latter is valid neither in classical logic, nor in the standard fuzzy approach.

However, the problem with the former is that we are inclined to consider statements of Tolerance as true, while they cannot all be true; on the other hand, read as $[Hp_n] \approx_T [Hp_{n-1}]$- that is an expression of Closeness - all the conditional premisses are 1 true. Nevertheless, on this second reading, the argument is not valid.

In sum, a final aspect to specify, deals with the speaker's approach to the sorites paradox, which explains why people find the original argument compelling and unconvincing at the same time, and why they think differently about the second. In fact, we must remember that one of our aims is to consider these issues from the speaker's point of view, in order to describe how language is *used*.

Therefore, in this particular situation, the busillis is to indentify the two different views they might take on what the same argument (i.e. the original Sorites Argument as presented at the beginning) is *saying*.

Indeed, these two arguments, expressed in our formal language - using respectively $=$ and $\approx_T$ - allow us to conclude by confirming our thesis: when we have a system of semantics that accommodates Closeness (without Tolerance), we can take the solution of the Sorites set out previously. We do not need, for example, any proprietary truth definitions for conditionals, and it means that

> the standard fuzzy response to the Sorites—employing the Łukasiewicz conditional—is thus a red herring.[6]

Actually, Smith's proposal is not the only suggestion to "solve" the sorites paradox (we can mention for instance Graham Priest's idea[7])

To conclude, the fact that the fuzzy approach overtrows the Sorites Argument, it is not a bad consequence for us, because we have proved that anyway, sorites susceptibility does not belong to a fundamental definition of vagueness; therefore, the many-valued approach does not contradict our

[6] [29], 273.
[7] See [15] .

vagueness-as-Closeness definition: rather, it allows us to draw the line representing the final side of our triangle: the link between the many valued-approach and the vagueness-as-closeness definition.

5.2 A bridge among vagueness and degrees of truth.

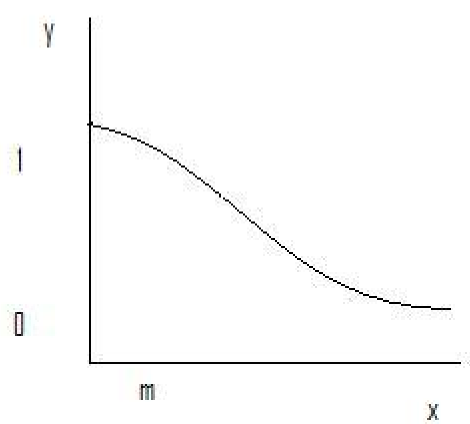
Finally, we will just deal with the third side of our triangle, which represents the main pillar of this thesis, the legitimation of our proposal.

It is interesting to underline that the elements of the legitimation we will explain in this chapter, arise from some critics which had been formulated against the degree approach to vagueness. For Smith, they are useful to emphasize more what is the strenght of his proposal.

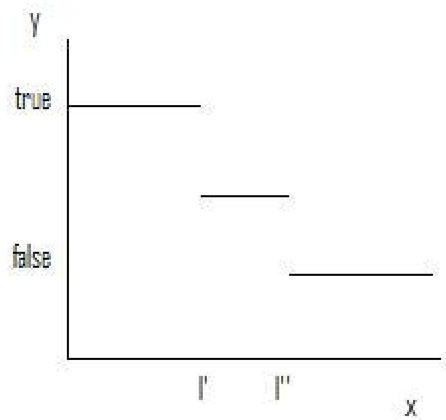
The first question we must consider is how this logical framework can accommodate Closeness. In order to answer this question, let us remind that the fuzzy systems we are considering, and let us take as their set of degrees of truth the real numbers between 0 and 1 inclusive. Now, there are certainly pairs p and q of reals in $[0, 1]$ such that sentence S's truth value is p and sentence T's truth value is q. By this way, S and T are very similar in respect of truth (which means for instance that $\mid p - q \mid = 0,0001$).

Therefore, the fuzzy framework can accommodate Closeness, because it has a sufficiently rich structure of truth values to allow arbitrarily small steps in $P-$relevant respect, to correspond to arbitrarily small steps in truth.

Particularly, the situation could be represented as follows: (in the $x-$axys we have the points on the strip, and in the $y-$axys we have the truth value of "point p is red")



Now, take for instance the point m and consider the predicate "is red". Now, this point is indeed the last point which is definitively red, but in this case it is not a problem; it would be if we considered non-fuzzy situations like for instance a three-valued view:



In fact, in this last case the points l' and l'' were jump points of the function which assigns truth values to sentences concerning points on the strip.

On the other hand, in the fuzzy framework the value of the function does change at m, and this change is *gradual*. Therefore, the existence of m - i.e. of a last red point - does not necessitate a violation of Closeness.

This last statement allows us to specify which are the differences among assuming two or three truth values, or a large set of degrees of truth. Smith's main thesis is that a large finite number of truth values would be *sufficient* for accommodate Closeness.

Let us begin specifying that, whether or not two sentences are very close in respect of truth, does not depend on what they say, but only it is meant to be a function of how true they are. So, we could say that S and T are very close in respect of truth if S's truth value is 1 and T's is $1 - \Delta$, but not if S's truth value is, say, 0,5 and T's is $0,5 - \Delta$. This means that any two sentences, whose degrees of truth are within Δ of each other, are very close in respect of truth. Therefore, even if our set of truth values is the large but finite set $\{0, \Delta, 2\Delta, ..., 1 - 2\Delta, 1 - \Delta, 1\}$, it can accommodate Closeness. However, it is not important to know exactly how many degrees of truth we need to adapt to Closeness, rather to have a significant number of them.

In fact, if we take again a Sorites series $x_1, ..., x_n$ for the predicate P , we need it to be the case that

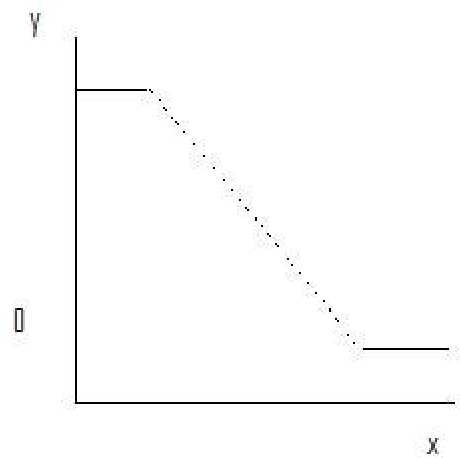
- Px_1 is true,

- Px_n is false,
- and Px_i , Px_{i+1} are always very similar in respect of truth.

It can happen only if two sentences have different gradations of truth, and yet it still be the case that the two sentences are very similar in respect of truth.

But - and it should be stressed - for Smith the *nature of vagueness* does not require that vague predicates have continuous characteristic functions[8] over continuously varying domains. Closeness requires only that the characteristic functions have no jump points, not that they have no points of discontinuity.

In detail, we can also deal with a function, whose graph can be represented as follows:



5.2.1 Continuity and Closeness

Linked to these last statements, the most important thing to emphazise at the beginning of this paragraph, is that even if we work with an extended infinite domain of truth values, it does not implies that this domain must be necessary continuous. The main thesis here is that a continuum of degrees of truth could accommodate Closeness, but it is not necessary in order to do it: a large finite number of truth values would be sufficient.

Anyway, even if assuming a continuum range of degrees of truth could appear more appealing than the position that there are only finitely many,

[8]For a definition of characteristic function, see here 2.1.2.5.

Smith's aim is just to destroy this idea.

However, although the nature of vagueness does not require that vague predicates have continuous characteristic functions, we must specify that intuitively there is an affinity between Closeness and the idea of *continuity,* because the notion of a continuous function is based on the intuitive idea that a small change in input produces at most a small change in the value of the function.

Nevertheless, it does not mean that a predicate is vague just in case its characteristic function is continuous. In particular, Smith argues that the continuity proposal is a good thing in the context of a highly general mathematical definition, but it is a bad thing in a definition of vagueness. In fact, our claim is to describe the *reality* of the use of the language, and sometimes in the reality we deal with discrete domains. Therefore, we are not arguing that a continuity proposal is a wrong idea; rather, we are saying that it is a limit to think that it may be able to make an exhaustive account of the notion of vagueness-as-closeness.

Now, let's precise the continuity position. Actually, these topics are mathematically very complex, and a proper discussion would require a closer examination, that goes beyond the scope of this work. We will only try to introduce some mentions of the vagueness-as-continuity mathematical proposal.

The main point of this idea is the notion of *topology.* First of all, some preliminary definitions:

(i) A *topology* $\mathscr{F}$ on a set S is a set of subsets of S (known as "open sets") which satisfies the conditions that $\emptyset$ and S are in $\mathscr{F}$, and $\mathscr{F}$ is closed under finite intersections and arbitrary unions.

(ii) A set for which a topology has been specified is called a "topological space".

(iii) A function from a topological space S to a topological space T is *continuous* just in case the pre-image[9] of every open set in T is open in S.

(iv) A *base* for a topology $\mathscr{F}$ on S is a set $\mathscr{B}$ of open subsets of S such that every open subset of S is a union of sets in $\mathscr{B}$.

[9] The pre-image of a subset Y of T, under the function f: $S \rightarrow T$, is the subset X of S which contains every element of S which is mapped by f to an element of Y.

But where do these topologies come from in this context? The main idea is that the topology of the codomain (that is the set of truth values of our logical framework) codifies the notion of closeness in respect of truth, while on the domain we need one topology for each predicate P, with the topology associated with P codifying the notion of closeness in $P-$relevant respect . In other words, saying that the predicate P is vague means that its chacteristic function - from the domain of discourse endowed with the $P-$topology, to the topological space of truth values - is continuous.

Moreover, it assumes that there is a three-place similarity relation in the domain for each predicate P - and this relation is a *base* for a topology, if we stipulate that:

(v) A subset S of the domain is a *base* element iff it satisfies the following condition: $\left(x \overset{P}{\leq} y \wedge y \in S \wedge z \in S\right) \rightarrow x \in S$ (which means that if x is at least as close to z as y is, and y and z are both in S, then x is in S.)

As Smith suggests, this reasoning leads to an absurd conclusion. In fact, it is easy to infer that, due to the reflexivity of the relative similarity relation [10], every singleton of a member of the domain satisfies the above condition and hence is a basic element.

Now, it is obvious that, taking arbitrary unions of singletons, gives us every subset of the domain, and so we can conclude that in the resulting topology every subset of S is open, so it will always be the *discrete* topology. This is markedly in contradiction with the statement that any function from a set endowed with the discrete topology is continuous. In fact even when we have discrete domains, we still want to distinguish between vague and precise predicates defined over these domains. However, every function from a discrete domain is continuous, and so if "vague" means "has a continuous characteristic function", then *each* predicate is vague relative to a discrete domain. Therefore, this proposal leads us the absurd conclusion is that, on the present proposal, each predicate P will turn out to be vague!

What could be a solution for this problem? Once again, Smith advises us to anchor our arguments to the reality of the natural language, and in this sense these stipulated topologies are less grounded in ordinary experience and practice than his proposal that for each predicate P there is an associated three-placed relative similarity relation on the domain and an associated two-place absolute similarity relation[11].

[10] See 2.1.2.2.

[11] See again 2.1.2.2.

Furthermore, sometimes domains really are discrete, for instance, in most of the standard examples of sorites arguments. Consider the example given by Smith itself:

> Suppose our domain consists of a line of men, from one with no hair up to one with a full head of hair, each differing from the next by just a hair. Consider the precise predicate 'has 100 or less hairs on his head' (as opposed to the vague predicate 'is bald'). Its characteristic function assigns True to the first 101 men, and then jumps to False for the rest of the men. But this jump is not enough to make this function discontinuous—so on the proposal in question, this predicate comes out as being vague. [12]

Conversely, we must specify that it is false that each predicate P automatically satisfies Closeness relative to a discrete domain of discourse. The difference here between the continuity proposal and the Closeness proposal is that the latter in based on the notion of absolute similarity between the elements of the domain. The crux of the matter is that - as underlined at the beginning of the paragraph - this notion of absolute similarity is linked to an intuitive notion of continuity (a small change in input produces at most a small shange in the value of the function), but not in the final mathematical definition of continuity. Thus, again, this is a good thing in the context of a highly general mathematical definition, but not in a definition of vagueness in the context of ordinary language.

To sum up, the most significant conclusion we can get is that following the continuity proposal, we are not able to distinguish adequately the predicates which are vague from which are not vague, while from a closeness viewpoint it is possible.

An example of this final statement is given by reconsidering the discrete domain of men and the predicate "has 100 or less hairs on his head". This predicate comes out as precise by the Closeness definition because its characteristic function assigns True to man 100, and False to man 101. In fact, these two men are very close, in absolute terms, in hair count, and so in respects relevant to whether someone has n or less hairs on his head (for each n). Therefore, from this view, the predicate is precise, in contrast to the continuity proposal, which concludes that it is vague.

In conclusion, it would be clear why we have explained in the previous chapter some extended many-valued systems whose sets of degrees of truth were so large or infinite: because Smith proved that the winning scenario to think about vagueness-as-closeness is the notion of *absolute similarity,* in

[12] [29], 154.

which the intuitive idea of continuity could be saved, but it is not necessary in order to deal with a universe of discourse. Indeed, this universe should be treated through a large or an infinite domain of degrees of truth, without the mathematical characterization of continuity.

5.2.2 Two sorts of degrees of truth

Another point for the legitimation of a degree approach to vagueness, arises if we underline that the Closeness definition provides the link between the two sorts of degrees, expressed in the definition itself: the degrees of truth of predication P, and the degrees$_m$ of P. In other words, vagueness is correctly understood in terms of Closeness, even because it provides the link between two sorts od degrees.

For instance, let us consider the two predicates "polluted" and "more polluted". It is evident that there are some connections among these two predicates, but it is not simple to identify them. In particular, if a is more pollute than b, then the fact that "a is polluted" is truer than "b is polluted" is not a good implication; but at the same time, we cannot deny that sentences of the form "a is polluted" is true to intermediate degrees.

Smith suggests that the key to understanding this question, lies in the statement that we have two "domains" involved:

- first, there are *objects* that have a certain degree of pollution: basins, soils and so on (they form a set we can call O);
- then, there are *degrees of pollution* that these things have: these degrees are *objects* too (and they form a set we can call H).

Furthermore, we have:

- an ordering relation $\leq$;
- a mapping $h : O \to H$;
- the set of real numbers $\mathbb{R}$.

and various mappings from the set of degrees of pollution to the set of real numbers and each of them may be thought of as giving a name to each degree of polluteness.

For example, suppose that the concentration of CO_2 in the air in the city center of Milan (CM) is x (which means that $h(CM) = x$). Now, imagine that a map f , from the set of the degrees of polluteness to the set of reals, assigns x the number 5; intuitively, $f(h(CM))$ is the concentration of CO_2 in the air in the city center of Milan, as measured in *ppm*[13].

Conversely, consider another map m , from the set of degrees of concentration of CO_2 in the air to the set of reals, which assigns x the number 1,8; intuitively, $m(h(CM))$ is degrees of concentration of CO_2 in the air in the city center of Milan, expressed in *Decipol*[14].

Now, the situation with regard to "more polluted" seems to be clear: for any objects x and y in O, x is more pollute than y just in case $h(y) < h(x)$, (that is $h(y) \leq h(x)$ and $h(x) \nleq h(y)$).

Conversely, we cannot say the same for "polluted"; in fact, we could state that there is a distinguished subset T of H, such that for any object x in O, x is polluted just in case $h(x) \in T$[15]. In other words, x is polluted just in case x is *sufficient* polluted, and just this word - sufficient - express the idea that T should be closed upwards, which means that for any x and y in H, if $x \leq y$ and $x \in T$, then $y \in T$.

This fact gives us a fundamental relation among "polluted" and "more polluted":

- for any x and y in O, if x is more polluted than y, and y is pollute, then x is polluted.

Nevertheless, this definition has a weakness: it ignores the vagueness of "polluted". For this reason, we prefer to consider a fuzzy subset T instead of the classical subset T and modify the requirement that T be closed upwards to the requirement that for any x and y in H, if $x \leq y$ then x's degree of membership in T is less than or equal to y's degree of membership in T.

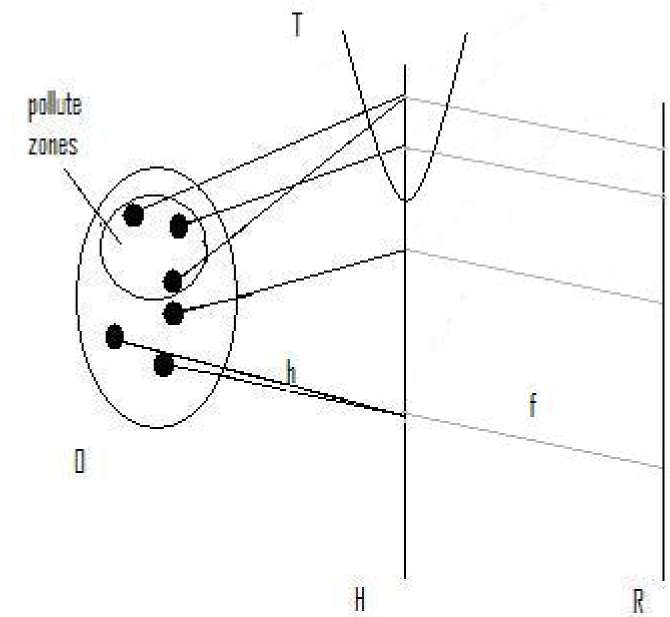
These patterns might clarify the two different situations[16]:

[13] Parts-per-million, (10^{-6}).

[14] The Decipol is a unit used to measure the perceived air quality and it was introduced by Danish professor P. Ole Fanger. One decipol (dp) is the perceived air quality (PAQ) in a space with a sensory load of one olf (one standard person) ventilated by 10 L/s. It was developed to quantify how the strength of indoor pollution sources indoors influence air quality as it is perceived by humans. Unit of perceived indoor air quality, measured indirectly from the concentration of carbon dioxide and the amount of fresh air supplied. Higher the decipol number, more polluted the air.

[15] Actually, according to the given definition of Closeness, instead of a single distinguished subset T of H, we would need different subsets T_F for different kind F of things.

[16] These figures are taken from [29], 217-218. The first figure represents the standard viewpoint, and the latter Smith's proposal.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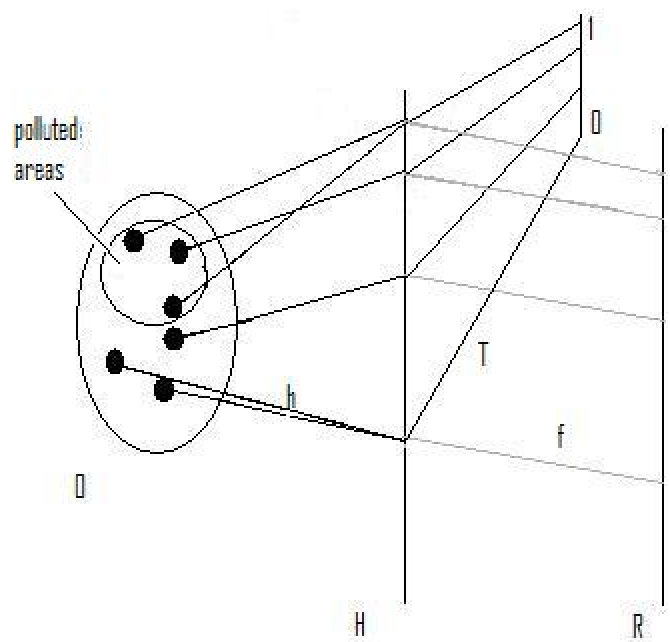
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h
f
D
H
R

About this last figure, observe that for clarity, $[0,1]$ is drawn separately from $\mathbb{R}$.

Therefore, "a is polluted" will be true to whatever degree $h(a)$ is in T, and also we can rewrite the definition given above:

- for any x and y in O, if x is more polluted than y, then the degree of truth of "x is polluted" is at least as great as the degree of truth of "y is polluted".

This definition accommodates the vagueness of "polluted", because if a and b in O are very close in respect of polluteness, then it can now be the case that "a is polluted" and "b is polluted" are very close in respect of truth, so, we are not committed to the idea that if a is more polluted than b, then 'a is polluted' is truer than "b is polluted".

Furthermore, in the second figure, we have degrees$_m$ of the concentration of CO_2, degrees of truth of sentences of the form "a is polluted", a map f from H to $\mathbb{R}$ and, above all, the composite map[17] $f \circ h$ from O to $\mathbb{R}$.

Moreover, we have other two distinct functions: a map T from H to $[0,1]$ and then a composite map $T \circ h$ from O to $[0,1]$. It is just this last composite function, which finally captures the vagueness of "polluted".

* * *

In conclusion, in these chapters we have considered the problem of the nature of vagueness in ordinary language. We have given a definition of vagueness in order to explain our aim, which is an attempt of approaching this philosophical problem through a many-valued logical system. We have found a definition able to show this link, in some papers of Nicholas Smith, whose position is known as the *vagueness-as-closeness* proposal.

During this analysis, also emerged the question about the location of vagueness, that is the question - we will reconsider further - of recognizing vagueness in the relationship between the language and the world, or in the world itself.

Then, we have presented the main many-valued logical systems based on t-norms, in order to understand better the "bridge" that Smith has identified being the legitimation of a fuzzy approach to the linguistic feature.

In the fifth chapter, we have in fact presented the main characteristics of this "bridge" and some consequences that arise if we accept it.

[17] Function composition is the application of one function to the results of the other.

Indeed, it has seemed significant to me, to present in detail both the "poles of the battery" - which are the definition of vagueness as closeness, and the fuzzy systems - above all, due to the fact that, as underlined in the introduction, our interest is *inherently bidirectional*, which means that we are not looking for a philosophical annswer to the question, concerning the issue of vagueness related to the use of the language, if it is not coherently supported by some logical systems we have built to treat it.

Conversely, also the logical systems cannot give an answer to any philosophical problems in general, if we don't define precisely what are the boundaries of the conceptual backdrop we want to consider for a close examination, and these limits are just provided by the vagueness-as-closeness definition.

Actually, the arguments in favor of this relationships are not finished with the end of this part: we have presented only those are related with the nature of the definition of vagueness. In the following chapter, we will study the remaining part of Smith's argumentation, through an examination of the possible interpretations of some semantic aspects, concerning the use of the natural language. In this way, will surface some other facets just of the "bridge" mentioned above.

Chapter 6

Fuzzy Plurivaluationism

In this second part we will focus on the interpretation of the fuzzy framework, in order to describe formally and philosophically the features of our linguistic usage, emphasizing on the *number* of the possible interpretations of the models we could consider in a fuzzy system.

Smith's viewpoint is called *plurivaluationism,* and the main characteristic of this position is that it does not consider only a unique interpretation of the semantics, but more. Thus, we will present here a possible interesting interpretation of the semantics built on fuzzy logic.

In the first part of this chapter, we will introduce classical *Plurivaluationism,* and we will discuss some strenghts and weaknesses of considering it as an interpretation about the semantical structure of many-valued propositional systems.

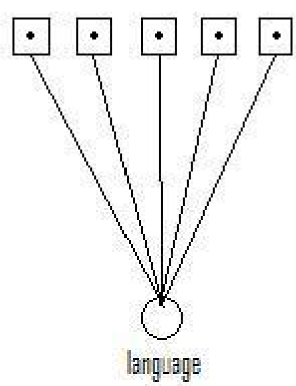
The second part is devoted to a presentation of the picture in which Smith intends to move, in the sea of the considerations about linguistic usage. We will specify his philosophical scenario, to understand better what should be the role of considering degrees of truth, and some critics against this position, for instance about the role of truth-functionality. At the same time - in order to mantain the bidirectionality mentioned above - in this way we may find an answer about the legitimacy of a logical approach to the philosophical problems of vagueness.

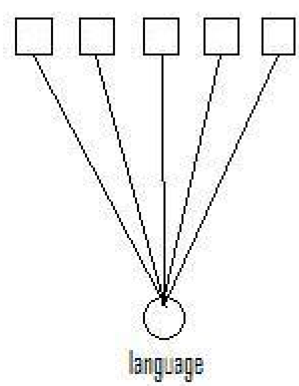
6.1 Fuzzy Plurivaluationism

Standard Plurivaluationism is an interpretation of the language, which denies the idea that each discourse has a unique intended interpretation. However, it accepts all the others parts of the classical picture, which are: the employment of a two-element Boolean algebra of truth values, a total interpretation function and total characteristic functions of sets, and the fact that the truth values of compound wfs are determined in a recursive fashion, from the truth values of their components. Therefore, there is a distinction between sentences which are true on every interpretation (i.e. logically true), and those which are not, but nevertheless, we are not able to distinguish amongst the latter between those sentences which are *actually* true and which are *actually* false.

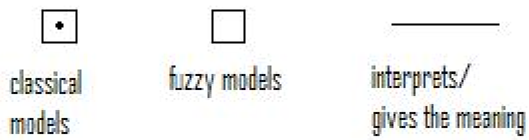
Fuzzy Plurivaluationism arises from the observation that both standard plurivaluationism and standard fuzzy framework are overall interesting ideas, but, at the same time, they contain weakenesses, which can be deleted if these two theories are combined together, by a coherent selection of the convincing features of both the proposals. In more detail, Fuzzy Plurivaluationism combines fuzzy models with semantic indeterminacy, of the sort involved in plurivaluationism.

If we want to use an image, we could state that if classic plurivaluationism could be represented as in the first of the following figures, fuzzy plurivaluationism could be described by the second one:





key:



In the following paragraphs we will try to justify the modification of the standard plurivaluationism, in order to take into account the power of fuzzy machinery; at the same time, we will use the overal idea of plurivaluationism, to give a more adequate interpretation of the fuzzy systems when they describe linguistic facts.

6.1.1 The problem of the intended interpretation

The first question is: why considering only one fuzzy interpretation is not enough? Actually, there is something implausible about assigning a unique fuzzy degree of truth to a vague sentence - but this sort of implausibility is not against the very nature of vagueness. In fact, if a vague discourse is assigned a unique intended fuzzy interpretation, this would not compromise the vagueness of that discourse. Rather, in Smith's opinion the idea that each vague discourse is assigned a unique intended fuzzy interpretation, does offend *intuition*.

Intuitively, it is not correct saying that there is one unique element of $[0, 1]$ that represents the authentic degree of truth of the sentence "Lake Garda is polluted", whereas all other choices are incorrect. First of all, the problem now is not *how* we could assert that "Lake Garda's degree of pollution is 0.8". In fact, Lake Garda's pollution is not within our control, because it is determined by natural and artificial phaenomena, and not by our usage of the word "polluted" in the past.

Rather, the problem is most clearly set up as follows. On the one side, we have our uninterpreted language, and on the other we have its fuzzy interpretations $\mathcal{I}_1, ..., \mathcal{I}_4$. All these interpretations have the same domain, and assign Lake Garda as the denotation of the name "Lake Garda". Conversely, they assign four different functions $f_1, ..., f_4$, respectively, to the predicate "is pollute". We can consider the following examples:

f_1: Lake Garda $\mapsto$ 0.8, Lake Iseo $\mapsto$ 1, Lake Como $\mapsto$0
f_2: Lake Garda $\mapsto$0.8, Lake Iseo $\mapsto$ 0, Lake Como $\mapsto$1
f_3: Lake Garda $\mapsto$0.806, Lake Iseo $\mapsto$ 1, Lake Como $\mapsto$ 0
f_4: Lake Garda $\mapsto$0.799, Lake Iseo $\mapsto$ 1, Lake Como $\mapsto$ 0.

Now consider the sentence “Lake Garda is polluted”: it is 0.8 true on $\mathcal{I}_1$, 0.8 true on $\mathcal{I}_2$, 0.806 true on $\mathcal{I}_3$, and 0.799 true on $\mathcal{I}_4$. The question is: how true is it *simpliciter*? The answer depends on which interpretation is the intended one. The problem is that some interpretations are clearly incorrect: for instance, ones which assign the number 5 as the denotation of “Lake Garda”, ones whose domain contains only the real numbers, ones which assign to “is polluted” a function which maps all prime numbers to 1 and $\mathcal{I}_2$ above, which assigns to “is polluted” a function which maps something which is clearly polluted to 0, and which maps something which is clearly not polluted to 1.

But - following this example - what about $\mathcal{I}_1$, $\mathcal{I}_3$ and $\mathcal{I}_4$? What could single out one of these as the intended one, and render the others incorrect? Rather, it seems that the picture on which sentences have a unique degree of truth - their degree of truth on the intended fuzzy interpretation - is not correct.

However, there is no fact of the matter concerning what is the authentic interpretation, we can now extract the abstract form of Smith’s argument:

(1) Facts of type T do not determine a unique intended interpretation of discourse D.

(2) No facts of any type other than T are relevant to determining the intended interpretation of D.

(3) From (1) and (2): all the facts together do not determine a unique intended interpretation of D.

(4) It cannot be a primitive - that is a fact not determined by other facts - that some interpretation $\mathcal{I}$ is the unique intended interpretation of D.

(5) From (3) and (4): it is not a fact at all that D has a unique intended interpretation.

Now, let us consider a discourse D involving vague predicates. As for type T, there is widespread agreement concerning the sorts of facts it should contain:

- all the facts as to what speakers of D actually say and write, including the circumstances in which these things are said and written, and any causal relations obtaining between speakers and their environment.

- All the facts as to what speakers of D are disposed to say and write in all kinds of possible circumstances.

- All the facts concerning the *eligibility as referents* of objects and sets.
- All the facts concerning the simplicity or complexity of the candidate interpretations.

In other words, the premiss (2) says that, if anything determines that some interpretation is the intended interpretation of discourse D, it is facts regarding the usage of speakers of D, together with facts about the intrinsic eligibility as referents of the objects and sets, assigned as referents in that interpretation, and together with the facts about the intrinsic simplicity of the interpretation. Smith definitively argues that if these things do not determine the meanings of parts of D uniquely, then nothing does. The facts of type T do not determine neither that any $\mathcal{I}_1, \mathcal{I}_3$ and $\mathcal{I}_4$ is the unique intended interpretation of our vague language, nor that any of them is incorrect.

6.1.2 About linear ordering

Considering a fuzzy plurivaluationist scenario forces us to review our ideas about linear ordering. In fact, in a standard fuzzy account for any two sentences whatsoever, either they are precisely as true as one another, or one is strictly more true than the other[1]: there are not two incomparable sentences in respect of truth.

In Smith's opinion it is not that a linear ordering of sentences would violate the nature of vagueness.

Suppose that our practice does not determine that the sentence "Lake Garda is polluted" should be 0.2 true rather that 0.3 true, nor does it determine that the sentence "Bob's house is nice" should be 0.2 true than 0.3 true. This leaves an open question: what does our practice determine about the relative degrees of truth of "Lake Garda is polluted" and "Bob's house is nice"? In other words, does our practice determine a relative ordering in respect of truth of *all* sentences - that is does it determine for any two sentences either that they are exactly the same in respect of truth, or that one is strictly more true than the other? The position currently under consideration says Yes. In fact, let us suppose that our practice does determine something about the relative truth of "Lake Garda is polluted" and "Bob's house is nice", in particular that the former is truer than the latter. Thus, on every acceptable interpretation, the sentence "Lake Garda is polluted" is truer than "Bob's house is nice". Therefore, we can conclude that - as a matter of facts - one sentence is truer than the other because it holds on

[1] For a logical definition and some observations about linear ordering, see 3.1.

every acceptable interpretation. On this point the fuzzy plurivaluationist view agrees with the original fuzzy account.

However, it does remain a huge philosophical question to clarify: what do we mean by *measuring truth*? That is, are we in essence measuring the degree of truth of sentences, where we have assignments of degrees of truth to sentences which are not unique?

Now, let us suppose we are measuring height. We start with two primitive operations: comparison and concatenation. We combine these two elements and we obtain an objective basis from which starting our measurement of height. Conversely, in the case of assigning truth values to sentences, someone might think that an analogous thing is going on, but actually, while there are some analogies between what we are doing and the process of measurement, the overall idea is quite different. In fact, the piece of wood whose we are measuring height has one unique height, and we could simply naming this height in different ways (using centimeters, inches and so on). In the present case, on the other hand, where we are assigning degrees of truth to vague sentences , and these assigments are not unique, the whole point is precisely that there is indeterminacy as to *how true* the sentence is, not simply as to *how to name* it. We do not want a picture in which the real truth values are simply *named* by real numbers: our acceptable interpretations are not different acceptable descriptions of one unique semantic reality. Rather, these acceptable interpretations are different semantic realities, each equally real, and it means that there is genuine indeterminacy here, not a choice as to how to describe one determinate event. In other words, in our case the indeterminacy of assigments of truth value is analogous not to a multiplicity of acceptable measuring systems, but to *indeterminacy* of the correct height assignment within one such system and it is not best ruled by the machinery of standard measurement theory.

I want to conclude this paragraph focusing on the question about linear ordering from the point of view of *our practice*. Particularly, a final observation which seems to be significant in our overview, is the distinction between what is *acceptable* and what is *mandatory*. Our practice imposes some constraints or correct interpretations of any discourse. However, anything directly required by these constraints is mandatory, and anything not ruled out by these constraints is acceptable. This means that nothing about our practice requires a particular ordering of “Lake Garda is polluted” and “Bob’s house is nice” in respect of truth. In fact, when we first consider the linear ordering worry for the standard fuzzy framework, we think that we do not mandate this ordering. Moreover, what we have fixed leaves it open which sentence is truer. But what is wrong with this other interpretation in which “Bob’s house is nice” is strictly truer than “Lake Garda is polluted”? The point is that *all* the interpretations are equally compatible with our constraints on correct interpretations, thus the fuzzy framework has not an legitimation of saying that only one of them is the correct one.

To sum up, Smith’s relevant point is that our practice is *silent* on the matter, about the ordering of two given sentences: nothing about it mandates anything about the ordering. As Smith suggests, “precisely this sort of silence that set the artificial precision and linear ordering problems for the fuzzy view in the first place.”[2]

To conclude, in this section we have considered the problems that arise from a uinque intended interpretation of each speech act, and consequently we have presented fuzzy plurivaluationism as the correct answer to these issues. Thus, the “plurivaluations” leads us to question the linear ordering, which is usually implicit in the fuzzy logical systems, and also in our common intuitive representation of natural language. However, even if fuzzy plurivaluationism forces us to reconsider some of our usual beliefs, it does not mean that all of them are incorrect representations. Rather, they must be conceived as refinements of the “glasses” with which we interpret speakers’ linguistic usage.

Therefore, after having justified the necessity of a fuzzy plurivaluationist view, it is time to include it into a more general philosophical context.

6.2 Truth and assertibility

The idea that truth comes in degrees allows some important developments concerning some considerations of the philosophy of language. At the same time, the following philosophical considerations could reveal an interesting interpretation of the logical framework used. In this part we will analyze the consequence that this theoretical viewpoint implies.

At the beginning of chapter 2 we have said that our background is represented by a research about the use of the language, therefore from a descriptive conception of the language. Here, we will specify this statement, emphazysing on a particular point of view, called *conversational pragmatics,* proposed by Robert Stalkaner, which mirrors Smith’s opinion.

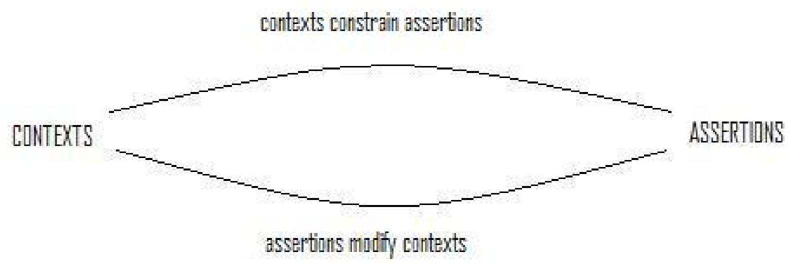
On this view, any conversation takes place in a *context,* which means that a conversation is taken to consist in a series of assertions by the conversationalists, who have the purpose of modify the context by adding the content

[2] [29], 304.

of what is asserted to the set of presupositions, which are the information that speakers take for granted.

Moreover, assertions may be accepted or not, by other conversationalists, and in this second case, the context remains the same.

In other words, assertions are related to the context by a two way- interaction



where the assumption that contexts constraint assertions opens the way to Gricean opinion about how assertibility can diverge from truth, while the converse direction opens the way to conversational dynamics.

The main difference with Smith's viewpoint, is that in Stalkaner's framework a *proposition* is a function from possible worlds to a *crisp* set of truth values, and he calls this set of possible worlds the *context set.* So, we can state that a proposition is *presupposed* if and only if it is true in all of these possible worlds. Furthermore, assertions work by narrowing the context set: in fact, if the assertion is accepted, worlds in which the proposition asserted is false are struck out the context set. In other words, the new context set (obtained after the assertion has been accepted) is the intersection of the old context set with the proposition asserted. In this way, it is evident that each person has her own context set.

This is Stalkaner's framework; nevertheless, the notion of assertion poses some problems, because we can - as Smith does - consider degrees of truth. If we have degrees of truth, we should also countenance degrees of assertions and corresponding degrees of belief. Furthermore, to accomodate degrees of truth, we must simply suppose that the function assigns to each possible world a fuzzy subset of the domain of that world.

In this sense, the most important thing is that Smith's opinion is that degrees of truth are essentially *degrees of belief*, which means that we have degrees of truth in the picture and therefore also countenance degrees of assertion, and corresponding degrees of belief.

But, before going into the question of the Smith's proposal to integrate degrees of truth with Stalkanerian pragmatics, we must specify some characteristics of considering degrees of truth as degrees of belief.

First of all, let consider the experiment proposed originally by Stephen Schiffer[19] , and then cited by Smith in [24] :

> Sally is a rational speaker of English, and we're going to monitor her belief states throughout the following experiment. Tom Cruise, a paradigmatically non-bald person, has consented, for the sake of philosophy, to have his hairs plucked from his scalp one by one until none are left. Sally is to witness this, and will judge Tom's baldness after each plucking. The conditions for making baldness judgments—lighting conditions, exposure to the hair situation on Tom's scalp, Sally's sobriety and perceptual faculties, etc.—are ideal and known by Sally to be such. . . . Let the plucking begin. Sally starts out judging with absolute certainty that Tom is not bald; that is, she believes to degree 1 that Tom is not bald and to degree 0 that he is bald. This state of affairs persists through quite a few pluckings. At some point, however, Sally's judgment that Tom isn't bald will have an ever-so-slightly-diminished confidence, reflecting that she believes Tom not to be bald to some degree barely less than 1. The plucking continues and as it does the degree to which she believes Tom not to be bald diminishes while the degree to which she believes him to be bald increases. . . . Sally's degrees of belief that Tom is bald will gradually increase as the plucking continues, until she believes to degree 1 that he is bald. Although I'll have a little more to say about this later, for now I'm going to assume that the qualified judgments about Tom's baldness that Sally would make throughout the plucking express partial beliefs. After all, the hallmark of partial belief is qualified assertion, and, once she was removed from her ability to make unqualified assertions, Sally would make qualified assertions in response to queries about Tom's baldness.[3]

Now, we can do - with Smith - some comments about the fact that Sally has degrees of belief. First, these three options seem to emerge:

(i) Sally fully believes that Tom is not bald until a particular hair is removed, from which point on she fully believes he is bald;

(ii) Sally fully believes that Tom is not bald until a particular hair is removed, at which point she enters an indeterminate state in which she does not believe (i.e with a degree = 0) that Tom is bald, and then when another particular hair is removed, Sally comes to fully believe that Tom is bald;

[3] [24] ,227-228.

(iii) Sally does not have attitudes towards propositions such as "Tom is bald", but only towards propositions such as "Tom is bald to degree x" each of which she either fully believes or fully rejects.

Nevertheless, Smith suggests that these approaches do not fit the phenomena. In particular, Sally certainly seems to be unsure as to what to believe and say about Tom's baldness (and this makes (i) and (iii) weaker); moreover, she does not have one catch-all confused state which she enters, remains in, then leaves. Rather, she seems to become less and less sure that Tom is bald and then later, more and more sure that he is (and this is contra (ii)).

Furthermore, (iii) underlines other problems too, expecially that there is a separation between truth on one hand, and belief and assertions on the other. In fact, it seems that we have a semantics which assigns degrees of truth to atomic propositions such as "Tom is bald", but we are then told we cannot believe or assert such propositions. We must believe and assert meta-level propositions of the form "'Tom is bald' is true to degree x" or, equivalently, propositions about degrees, such as "Tom degree of baldness is x".

However, the main problem that emerges in this example is that partial beliefs arising from *vagueness* do not behave in the same ways as partial beliefs of the familiar kind arising from *uncertainty*. To understand better this last statement, let consider a further example, given by Stephen Shiffer[4] and John MacFarlane[5]:

> [...] suppose that Sally is about to meet her long-lost brother Sali. She has been told that he is either very tall or very short, but she has no idea which (so she does know that he is not a borderline case), and she has been told that he is either hirsute or totally bald, but she has no idea which (so she does know that he is not a borderline case). As a result of her uncertainty, she believes both of the propositions 'Sali is tall' and 'Sali is bald' to degree 0.5. Suppose also that Sally regards these two propositions as independent: supposing one to be true would have no bearing on her beliefs about the other. Then, for familiar reasons, she should believe 'Sali is tall and bald' to degree 0.25. Now suppose that midway through Schiffer's experiment, when Sally's degree of belief that Tom is bald is 0.5, she also believes to degree 0.5 that Tom is tall—on the basis of looking at him and seeing that he is a classic borderline case of tallness. Then what should be her degree of belief that Tom is tall and bald? The answer 0.5 suggests itself very strongly: certainly the answer 0.25 seems wrong. If you don't

[4]See [19]

[5]See [12].

> think so, then just add more conjuncts (e.g. funny, nice, intelligent, cool, old—where Sally knows of Sali only that he is not a borderline case of any of them, and of Tom that he is a classic borderline case of all of them): the more independent conjuncts you add, the lower the uncertainty-based degree of belief should go, but this is clearly not the case for the vagueness-based degree of belief.[6]

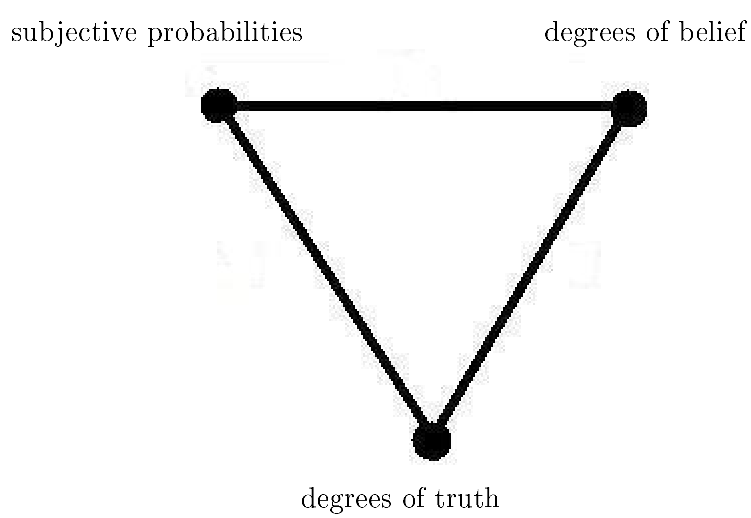
In Shiffer's opinion there are *two kinds* of degree of belief:

- uncertainty-based degrees of belief, or *SPB*'s (stands for "standard partial belief")
- vagueness-based degrees of belief, or *VPB*'s (stands for "vagueness-related partial belief"').

The difference between them is that an assigment of SPB's to propostitions obeys the laws of probability, whereas an assigment of VPB's to propositions obeys the laws of *standard fuzzy propositional logic.*

6.2.1 Expected truth values as degrees of belief

In order to clarify Smith's argument, we must say that the main point for Smith is to individuate three main concepts which play a significant role in our investigation, and explore their relationships. In order to do that, we can represent the situation with a new triangle:



[6] [29] , 229-230.

The essential problem is to give a clear account of the relationship between degrees of belief and subjective probabilities. Also the solution presented here involves degrees of truth: the proposal is that one's degree of belief in a proposition S is one's expectation of S's degree of truth.

In other words, the picture proposed by Smith has essentially three components, all tied together:

(1) an agent's epistemic state as a subjective matter;

(2) the degrees of truth of propositions;

(3) an agent's degrees of belief in propositions.

Let us specify each of these points:

(1) An agent's epistemic state represents a probability measure over the space of *possible worlds,* which means that, if W is the set of possible worlds, X a subset of W, the agent's epistemic state P is a function which assigns a real number between 0 and 1 inclusive to each subset of W. In other words:

$$P : X \to [0, 1] \, .$$

Intuitively, the measure assigned to a set S of worlds indicates how likely the agent thinks it is that the actual world is one of the worlds in S.

Given these definitions, the three probability axioms are the following:

(P_1) for every set $X \subseteq W$, $P(X) \geq 0$;

(P_2) $P(X \cup Y) = P(X) + P(Y)$ provided $X \cap Y = \emptyset$;

(P_3) $P(W) = 1$.

(2) At each possible world, each proposition has a particular degree of truth. Particularly, each proposition S determines a function S' such that:

$$S' : W \to [0, 1]$$

which is the function that assigns to each world $w \in W$ the degree of truth of S at w.

The relationships between the functions associated with various propositions will be constrained in familiar ways by the logical relationships between these propositions, for instance:

$(S \vee T)'(w) = \max \{S'(w)\}, T'(w)$;
$(S \wedge T)'(w) = \min \{S'(w), T'(w)\}$;

$(\neg S)'(w) = 1 - S'(w)$.

(3) The agent's degree of belief in S with her *expected truth value* of S. We must consider two cases: the case where there are finitely many possible worlds, and the case where the possible worlds are uncountably many. Summing up the proposal, an agent's degree of belief are the resultant of two things: the agent's subjective uncertainty about which way the actual world is (represented by a probability measure over the space of all possible worlds - or at least over a σ-field of subsets of this space - with the measure assigned to a set of worlds specifying how likely the agent thinks it is that the actual world is in that set), and the objective facts about how true each propositionis in each world.

Therefore, Smith's view countenances the subjective probability measure - it models the agent's epistemic state - but regards degrees of belief as resultants of this state and degrees of truth.

To be more precise, let make the following definitions:

Definition (vagueness-free situation). An agent is in a vagueness-free situation (VFS) with respect to a proposition S if and only if there is a measure 1, a set T of worlds (that is a set T such that $P(T) = 1$) such that $S(w) = 1$ or $S(w) = 0$ for every $w \in T$.

An agent is in a VFS with respect to a set Γ of propositions if she is in a VFS with respect to each of the propositions in Γ.

Definition (uncertainty-free situation). An agent is in an uncertainty-free situation (UFS) with respect to a proposition S if and only if there is a measure 1 set T of worlds and a $k \in [0,1]$ such that $S(w) = k$ for every $w \in T$.

An agent is in an UFS respect to a set Γ of propositions if she is in a UFS with respect to each of the propositions in Γ.

Now, given these definitions, we can formulate four propositions which explain when degrees of belief bekave like probability assignments and when they do not.

Proposition (degrees of belief behave like probabilities in VFSs). So, let Γ be a class of well formed formulas, closed under the operations of forming formulas, using propositional connectives, such that each formula is in a VFS with respect to Γ. Thus, we have these three conditions:

(1) for all wfs $\alpha \in \Gamma$, $0 \leq E(\alpha) \leq 1$;

(2) for all tautologies $\alpha \in \Gamma$, $E(\alpha) = 1$;

(3) if $\alpha_1 \in \Gamma$ and $\alpha_2 \in \Gamma$ are mutually exclusive, then $E(\alpha_1 \vee \alpha_2) = E(\alpha_1) + E(\alpha_2)$.

Proposition (degrees of belief behave like degrees of truth in UFSs). Let Γ the class of formulas mentioned above, then one's degrees of belief of wfs in Γ behave like degrees of truth, in the sense that these conditions are satisfied:

(4) $E(\neg\alpha) = 1 - E(\alpha)\,;$
(5) $E(\alpha_1 \vee \alpha_2) = \max\{E(\alpha_1), E(\alpha_2)\}\,;$
(6) $E(\alpha_1 \wedge \alpha_2) = \min\{E(\alpha_1), E(\alpha_2)\}.$

After having explained in detail the three vertices of the triangle, the main character that emerges is the *subjectivity* of the language. It reminds us to consider degrees of truth in a relative way, that introduces the necessity of a plurivaluationist view, which represents the "anthropological" aspect of the matter.

6.2.2 The question of assertibility

Now, it is time to return to the Stalkanerian pragmatic, to discuss how it might work when we admit degrees of truth. Particularly, in this picture each conversationalist is in a epistemic state given by a probability measure over possible worlds. Moreover, he has a set of presuppositions, and - this is Smith's key - presuppositions are considered to be matter of degree.

He also states that the way in which one expresses an intermediate degree of belief is via an unconfident or hesitant *utterance:* we can suppose that there are degrees of confidence of utterances, distinguished by response time, tone of voice, and so on, corresponding to the degrees of truth.

In more detail, the agent's set of presuppositions determines a set of worlds, that is, a context set:

Context set the set of all worlds in which each proposition is true to the degree to which it is presupposed.

Furthermore, unconfident utterance of S counts as an assertion only if the utterer is in a UFS with respect to S, and the actual world is one in which S is $n-$true. In other words, each conversationalist will conditionalize his probability measure on the set S_n, of worlds in which S is n-true.

There still remains the question of *assertibility.*

Assertibility If we are in the case that we have only one speech act of assertion, a sentence is assertible in a given context if one can assert it without promting a legitimate challenge from one's conversational partners, that is, if one's assertion passed or is accepted.

It is evident that there is a significant difference between *truth*, *assertion* and *assertibility*: in fact, while truth and assertion are (in this picture) matters of degree, the notion of assertibility or acceptability are not: they are pass/fail notions, not graded ones. The essential point remains that introducing degrees of assertion does not in itself provide a reason for thinking of assertibility as a graded notion.

But what could be the relationship between truth and assertibility? In Smith's opinion, it consists in a generalization of the classical idea that S is assertible when true to the context of degrees of truth and degrees of assertion. A sentence S is $n-$assertible when true to degree n. That is, the degree of confidence which is appropriate in an assertion of S is the one that corresponds to S's degree of truth. In other words, if S is 0.5 true, so it will be 0.5 assertible.

Furthermore, we can apply the predicate with varying degrees of confidence or hesitation, which means that if we have a predicate P and an object x, we can assert Px with varying degrees of confidence.

To conclude, it seems that Smith proposes an empirical approach, in which everything revolves around the speech acts, i.e. around the *use* of the language.

The fact that Smith's aim is essentially empirical, is evident if we consider the question of *truth-functionality*.

One of the main objections to the degree approach to vagueness is that a recursive many valued views does not cohere with ordinary usage of compound sentences about borderline cases. Smith answers to this question with an empirical linguistic work, in order to underline that there is no basis for thinking that the truth-functional degree theorists has a problem in this area.

To study the attitude of speakers to certain sentences involving vague predicates, he has put informal questionnaries to undergraduate students and non-philosophers.[7] We show below some examples of the sentences Smith was interested in, where x is a colour sample midway between clear red and clear orange:

1. x is red.

2. x is not red.

3. x is orange

[7]This experiment is presented in [29] ,but actually Smith takes this experiment from Bonini, Nicolao Osherson, Viale,Williamson, *On the psychology of vague predicates.* Mind and Language, (1999), 14: 377–93.

4. (a) x is red or x is not red.
   (b) x is or ins't red.
   (c) x is either red, or it isn't.
   (d) either x is red, or it is not red.

⋮

5. (a) x is red or x is orange.
   (b) x is red or orange.
   (c) x is either red or orange.
   (d) either x is red, or it's orange.

⋮

6. (a) it is not the case that x is red or that x is not red.
   (b) x is neither red, nor not red.
   (c) x's neither red, nor not.

⋮

7. (a) x is red and x is not red.
   (b) x is and isn't red.
   (c) x is red, and not red.
   (d) x is both red and not red.

⋮

8. (a) x is red and x is orange.
   (b) x is red and orange.
   (c) x is both red and orange.

⋮

After we have done this example of the Smith's empirical approach, we shall resume our statements:

(i) by uttering a sentence, we say or state its content: to say that a sentence expresses a content in a context is to say that to utter the sentence in that context would be to state that content.

(ii) Sentences, not contents, that are said to be assertible to such-and-such degree in context.

(iii) A sentence is assertible in a context to a degree corresponding to the degree of truth of the content that it expresses in that context.

Finally, a challenge to a truth-functional degree theory might ultimately be based on considerations of *assertibility*, or on considerations of truth. In other words, the assertibility challenge and the truth challenge take respectively this form (we denote assertibility challenge by A, and Truth challenge by T):

(iv.1_A) Sentence S is $n-$assertible in context C.

(iv.2_A) Given the fuzzy theory, S expresses the wf α in C.

(iv.3_A) α is not $n-$true.

(iv.4_A) S is assertible in C to the degree that the content it espresses in C is true.

(iv.5_A) So, given the fuzzy theory, S is not $n-$asserible in C.

(iv.1_T) In C, sentence S is $n-$true.

(iv.2_T) Given the fuzzy theory, S expresses the wf α in C.

(iv.3_T) α is not $n-$true.

(iv.4_T) So, given the fuzzy theory, S expresses a content in C which is not $n-$true.

It is evident that there is a big problem. In this sense, Smith analyzes two different answers available: contextualism and *warranted assertibility* *manueuvre (WAM).* The differences between them are the following: a contextualist denies the second premiss, and the WAM consists in denying the first step. But, let specify why. The former thinks that α is the obvious reading of S, and in many contexts S does express α. Nevertheless, in context C, S expresses β and β (unlike α) is $n-$true. Conversely, the WAM is this: S does express α in C, but in C the truth norm governing assertion is overridden by other norms of assertion, which means that in C, S's assertibility does not go simply by the truth of its content. In other words, the WAM consists in denying the first step, via the claim that the objector is confusing warranted assertibility with truth.

But what is Smith's opinion? He underlines that our readings of sentences colud be of three sorts:

(i) the surface reading;

(ii) readings involving the predicate "is true to degree 1";

(iii) readings invoked in particular sorts of context: for instance, hearing a sentence as saying that all samples have to be classified in one of two ways. In other words, hearing a sentence could be an expression of Tolerance in the context of reasoning about a Sorites series.

Let reconsider for instance

8. (a) x is red and x is orange.

Again, given these conflicting viewpoints, he proposes to accept that proper empirical studies might show that almost all speakers find $(8a)$ "x is red and x is orange" to be 0-assertible, or that almost alla speakers find $(8a)$ to be 0.5 assertible, or that significant numbers of speakers go each way in a given context, or finally, that all speakers go the same way in each context, but different ways in different contexts.

More formally, in Smith's opinion there is no problem here for the truth-functional degree theorist - for there is a plausible reading of $(8a)$ which is 0.5 true $(Rx \wedge Ox)$, and another plausible reading which is 0 true. Particularly, in order to state the latter reading, we need to add some symbols to our formal language:

- where α is a wf, $\langle\alpha\rangle$ is a singular term, whose intended referent is α;

- T_1 is a one-place predicate which can be read as 'is true to degree 1';

- $T\langle\alpha\rangle$ is a wf which is 1-true if α is 1-true, and 0 if α is true to any degree other than 1.

Thus, the second plausible reading of (8a) could be $T_1\langle Rx\rangle \wedge T_1\langle Ox\rangle$.

It is evident that there is no prospect of a trouble for the fuzzy theorists, because they could state three things:

(1) if the fact is that almost alla speakers find this sentence to be 0-assertible or 0 true (when x is borderline), then the explanation of this fact is that in these contexts almost alla speakers hear this sentence as $T_1\langle Rx\rangle \wedge T_1\langle Ox\rangle$;

(2) if the fact is that almost alla speakers find this sentence to be 0.5 assertible or 0.5 true, then the explanation of this fact is that almost all speakers hear this sentence $Rx \wedge Ox$;

(3) finally, if the fact is that significant numbers of speakers go each way, then the explanation of this fact is that significant numbers of speakers hear this sentence each way.

It is evident that the first reading (the surface reading) is a plausible reading of (8a) . Conversely, about the second reading, Smith argues that it is plausible to say that someone can, given the right emphasis, tone of voice or other contextual features, hear (8a) as $T_1\langle Rx\rangle \wedge T_1\langle Ox\rangle$ that is as making a claim which is true to degree 1 just in case both Rx and Ox are true to degree 1.

Anyway, the most important thing to underline is that it is an attempt to dispel the worry that truth-functionality degree theories cannot account for ordinary usage or intuitions about the truth and/or asssertibility of compound sentences about borderline cases. However, Smith believes that painting a systematic picture of how particular features of context influence that content of sentences with a particular syntactic form would be untimely, due to the fact that we have not adequate data on which to base this kind of theories.

6.3 About *acceptable* interpretations

Smith's main thesis is that having proved that each discourse has a unique intended interpretation is not enough: we have to add the idea that each discourse has some *acceptable* interpretations, maybe many, maybe in some cases only one.

An acceptable interpretation of a discourse is simply one that is not ruled out as incorrect by the type-T facts, that is one that meets all the following constraints on correct interpretations imposed by the type-T facts:

- paradigm constraints: if speakers would all apply the predicate P to the object x in a normal conditions, then any candidate correct interpretation must assign P a function which maps x to 1. Conversely, if speakers would all withold the predicate P from the object y in normal conditions, then any candidate correct interpretation must assign P a function which maps y to 0.

- Ordering constraints: if person x and person y are of the same sex and roughly of the same height, and x's age is greater than y's, then any candidate correct interpretation must assign to the predicate 'is young' a function which maps x to a value greater than equal to the value to which it maps y.

- Exclusion constraints: any candidate correct interpretation that assigns the predicate "is green" a function which maps x to a value near 1 , and which must assign the predicate "is blue" a function which maps x to a value near or equal to 0.

But when is a sentence assertible? Following on the definition given above, a sentence is assertible to the degree that its content is true, where by "content" we mean a wf plus an interpretation, which is the interpretation relative to the context of utterance. Thus, our sentence expresses multiple contents, because it could express only one wf, but this wf has many equally acceptable interpretations. And, how confident should our utterance of the sentence be? As Smith suggests, we can try an answer: if the wf expressed has the same degree of truth - for istance 0.3 - on every acceptable interpretation, then the answer is obvious; otherwise, if the wf is true to a low degree on one acceptable interpretation and a high degree on another, then there is no degree of confidence such that an assertion of that degree would be appropriate. Here the issue is purely *pragmatic*: we cannot assert such sentences, with any degree of confidence.

But a reader could ask what is the sense of dealing with these problems since the declared argument of this work is an analysis concerning the Vagueness-as-Closeness definition? Well, on Smith's view a vague predicate like ‘is polluted’ satisfies Closeness on every *acceptable* interpretation, so hence we can state overall that it satisfy Closeness.

But what about the “last bald man” in the Sorites series? On the fuzzy plurivaluationist view there is *no overall* last bald man. On any acceptable interpretation there is a last person in the series who is mapped to 1 by the function assigned to “is bald” on that interpretation. So, there is no particular man of whom it might be said that he is the last bald man. Nevertheless, there is a last man x in the series such that the sentence “x is bald” is 1-true on every acceptable interpretation. Therefore, there is no individual such that we can talk as though he is the last bald man, but there is a last man such that we can talk as though he is bald. This is the sense to assert that fuzzy plurivaluationism admits still a last bald man.

* * *

To conclude, let us return to the original question: what is it about our practice that makes an interpretation which assigns “is polluted” a function which maps Lake Garda to 0.99 acceptable, but an interpretation which assigns “is polluted” a function which maps Lake Iseo to 0.99 unacceptable? It might simply be that saying makes it so: that a given interpretation is acceptable because it makes true what some speaker says. If so, it may be that the relationship between meaning and use is more complicated than we had though until now. However, Smith notes that there is a parameter in this proposal: our practice, that is a *group of speakers*. Given a set of speakers it is plausible to think that their practice can determine a unique set of acceptable interpretations of their words, but it does not mean - of course - that a unique set of acceptable interpretations can be fixed on once and for all.

We have to read these results as a corroboration of the guiding idea behind our dissertation, which is that all the issues in the philosophy of language are problems of determination of meaning and that language is a human artefact, where the meaning of the words depends essentially on how speakers use them.

Part II

SOME OBSERVATIONS

Chapter 7

Some observations on Smith's view

In this chapter we will present some observations about Smith's viewpoint. We will split them in three groups: those regarding the Closeness definition, those related the fuzzy approach to vagueness, and others involving Fuzzy Plurivaluationism.

As underlined in the introduction, we will take into account what has been said in literature about Smith's proposal, particularly in [1] , [14] , [26] , [27] and [31]. These papers have provided me some interesting and witty observations on Smith's work and they have enabled me to develop and to articulate better some of my objections to this theory.

The main point I want to stress is that in Smith's theory vagueness is *never* conceived as a *problem*. I think this is the crux of the matter, because it allows us to highlight from the beginning that the goal of Smith's investigation - and of course of my dissertation - is not an attempt to eliminate vagueness, rather it must be seen as a endeavor to describe formally - that is in the rigorous way provided by logic - speakers' linguistic usage.

Now that I have specified these things, we can begin to present the objections.

7.1 On the Vagueness-as-Closeness definition

7.1.1 Some issues about the Closeness definition

The first observation on which we must dwell, is concerning the vagueness of the term Closeness itself. In fact, I agree with Libor Běhounek, who states in his comment [1] that the Closeness definition is itself based on the vague terms *very close/similar*.

Let us remind the Closeness definition:

> If x and y are very close in $P-$relevant respects, then "Px" and "Py" are VERY CLOSE in respect of truth.

Smith says that the relation of absolute closeness is to be regarded as precise because he takes as a datum that the term "vague" could be applied to predicates which exhibit three characteristics: they admit of borderline cases, they generate Sorites paradoxes and they have blurred boundaries. In other words, if a predicate satisfies Closeness, then it must have these three features. Nevertheless, I think that a deep question does remain: how this predicate ("be close/similar") could be determined in *fixed* relations and fixed proportions? Which means which is the scale with which we measure closeness?

First of all, I think that we must distinguish between "is close to" and "is similar to" because the former implies the notion of measurement on a scale, whereas the latter makes use rather of the concept of analogy. Saying that "x is similar to y" does not imply in my opinion that we must misure x and y in order to put them in a unique units. Conversely, saying that "x is close to y" justifies the use of a unique scale of measurement, a unique dimension in the way of considering these two elements x and y. Therefore, I think that in our discourse we have to talk only about closeness relations, and not about similarity relations.

This differentiation amongst them may give us some instruments to analyse the position (as Běhounek's view) which argues that in general rather than graduality, it is *semantic indeterminacy* which is essential for vagueness. On the contrary, Smith thinks that if a predicate is semantically indeterminate (that is it does not make use of the Closeness notion), then it need not exhibit the three features mentioned above, so it is not vague.

To investigate this issue let us consider - with Francesco Paoli - *evaluative* predicates, which are predicates that are typically multidimensional, like "clever", "beautiful" and so on. As Pauli suggests - and I agree with him - it remains to be considered whether the Closeness principle makes sense when P is an evaluative predicate. It means that:

> what does it mean for x and y to be very close/similar in respect relevant to the application of clever?[1]

[1] [14] , 34.

First of all, here we can ascertain the validity of the differentiation I have suggested, amongst "being close to" and "being similar to": in fact, I think that evaluative predicates are considerable only from the viewpoint of the similarity relation, because their multidimensionality could not be captured by a concept which involves a measurement scale. This is the reason why I have proposed, in the explanation of Smith's view, some examples with different kinds of predicates, in particular I focused on predicates about colors ("is red", "is green") and about environmental pollution ("is polluted"). It seems to me that this differentiation is fundamental for a precise analysis of the situation.

For example, let us consider again the two predicates "'is red" and "is polluted" and respectively the sentences "x is red" and "x is polluted". Now, let us remind the notion of Closeness given in **2.2.2**:

> To sum up, if we aim to give a general theory which codifies our intuition, about closeness of objects in respect relevant to whether something is P - for a given predicate P - *we have first to determine the relevant respects. Then, we have to associate each respect to a numerical scale*, giving rise to a vector space, where each object corresponds to a vector whose coordinates are the numbers to which the object is associated on each numerical scale. Now, relative closeness could be extracted via the idea that x is at least as close to z as y is, just in case the distance between x and z is less than, or equal to the distance between y and z. On the other hand, absolute closeness may be extracted via the selection of a particular number d , and here the idea is that x and y are very close just in case the distance between them is less than d.

The first thing to underline is the following:

> we have first to determine the relevant respects. Then, we have to associate each respect to a numerical scale

and it is just the decisive element from which we must distinguish the evaluative predicates like "is red", from the non-evaluative predicates like "is polluted", as relevant respects. In particular the question is the following: how can we associate each respect to a numerical scale, if the respect is an evaluative predicate? If we consider on the one hand the sentences "x is polluted" and "y is polluted", and on the other hand "x is red" and "y is red", we are justified to assert that the in first two sentences x and y are closed in $P-$relevant respect (where P is the predicate "is polluted"). In fact, we can easily fix a sort of *unit of measurement* (which could be for instance the level of CO_2 in the air) and equally easily we can use a scale based on this unit of measurement. Conversely, I think that it is not equally easy - or perhaps even impossible - doing the same for the last two sentences. It is not simple to determine univocally which are the key features that define the predicate "is clever", because it is a matter of speaker's *linguistic sensibility*. For the individual A - for example - x and y are close in respect relevant to the predicate "is clever" if both x and y are able to solve complex quadratic equations in less than two minutes. Instead, an individual B could consider x and y as close in respect relevant to the predicate "is clever", only if they belong to the species *homo sapiens sapiens*. It seems evident to me that A and B can disagree about which are the key features of the predicate "is clever", whereas any person (if reasonable) must recognize that the level of CO_2 in the air as a possible key feature of the predicate "is polluted".

To sum up, I think that this preliminary distinction between evaluative predicates and non-evaluative predicates is fundamental to proceed in our overview.

7.1.2 On Sorites Susceptibility

As underlined in **2.2.1**, the main question about the link between vagueness and Sorites Susceptibility is the following:

> although it is easy to see intuitively that soritical predicates are vague, can we automatically conclude that all vague predicates are soritical?

In other words, the sorites susceptibility is only a symptom of vagueness but it must not be considered a part of the vagueness-as-closeness definition. Even if we have already said that for Smith the answer is negative, nevertheless some clarifications are needed.

First of all, let us concentrate on the alternative proposal of modifying the definition of the standard Łukasiewicz conditional as follows: in this new account, the tautology property is having a value of at least 0.5 instead of 1. It seems evident that by this way there is not a contrast between the intuitive assertibility status of the sentence and the truth value assigned to the latter wf by this semantic context. However, it does remain an intuitive assertibility status, and it is unclear to me how it could provide a convincing solution of the paradox: a group of speakers might disagree on which is the value in the interval $[0, 1]$ that must be assigned to the notion of tautology.

Therefore, even I am persuaded that sorites susceptibility is only a mark of vagueness, at the same time I consider Smith's idea still too factitious in order to be considered as a complete solution of the sorites paradox.

Nevertheless, I think that this attempt - that is to modify the fuzzy apparatus - should be revealed a interesting and effective way for reconsidering this paradox, because it hinges on the mathematical and logical framework, to interpret our intuition about it.

7.1.3 Is Vagueness exhausted by vague predicates?

Finally, another objection to Smith's definition, which seems to me that undermines all the pillars on which vagueness-as-closeness definition is built. I want to take a stand on Weatherson's objection, which states that Smith tries to provide a definition of vagueness, but he only tells us what is for a *predicate* to be vague. I think it is evident that if we consider respectively the Closeness definition, the legimitimation of the fuzzy approach to vagueness and fuzzy plurivaluationism, author's arguments are based on the features, on the behavior and on the interpretations of vague predicates, and it seems that for Smith semantic vagueness overlaps to the vagueness of predicates.

Actually, Weatherson focuses on non-linguistic representations (in which of course there are not predicates), in order to support the thesis that the huge theme of vagueness can not be analysed in terms of what makes a predicate vague. Instead, I think that the philosophical weakness of Smith's definition lies elsewhere.

In particular, in my opinion we must of course distinguish - as Weatherson does - between linguistic and non-linguistic representations, but - and here I disagree with Weatherson - I think we must strive to confront ourselves on a common ground, which is *linguistic* vagueness.

At the beginning of the second chapter, I stated that vagueness is a phaenomenon which interests terms belonging to different lexical cathegories, like adjectives, adverbs, nouns, predicates and so on, and that, semantically, vagueness may concern *properties* and /or *objects*. My opinion is that if we do not limit the consideration about vagueness only on predicates, we may obtain some results about the question if vagueness concerns properties and/or objects.

Actually, Smith seems to suggest himself the possibility of extending the Closeness definition to properties and relations (see **2.2.5.**), but it remains apart from the main definition of Closeness: he mentions the issue, but he does not go deeper into the question.

For example, let us consider nouns and remind the two sentences (respectively involving an evaluative and a non-evaluative predicate):

- "x is red";

- "x is polluted".

Let interrogate ouselves on what x could be. It is evident that it must be a subject, but in my opinion we must distinguish inside the set of nouns which are the *rigid designators* and treat them as a special subset which should not be described by the vagueness-as-closeness definition.

In other words, I think that for a deeper examination of the theme of linguistic vagueness, we have to consider also the complement of the set of predicates, which is composed by all nouns, adjectives, adverbs, determiners, connectives, and so on. On this set, I think we must focus on a special subset of terms which are personal nouns, and again, in this subset we must distinguish between which nouns are univocally determined, and which are vague. We can call the former group as *rigid designators*. In fact, it is evident that rigid designators like "Bob" or "Alice", are not vague terms, because all speakers could potentially agree on who they are talking about. In this sense, sentences like "Bob is bald" or "Alice is clever" mirror exactly Smith's conception of vagueness, that is represented only by the vagueness of the predicates "'is bald" and "is clever".

Conversely, it seems to me that - contra Smith's argument, explained in **6.1.1** - a common noun or a concept can not be univocally determined by a speech act. Let us consider for example "Lake Garda". If we say "Lake Garda is polluted", it is not clear for instance what part of the lake is polluted, or – more generally – what are the "borders" of this noun: one can consider for instance only the part of the earth covered by water, whereas another people may consider also the coast or the environment around the lake; in other words, if we use this noun in a sentence like "Lake Garda is polluted", we must take into account not only the vagueness of the predicate "is polluted", but also the vagueness of the subject. Actually, I think this observation is not far from Quine's argument of the inscrutability of reference, because both of them are based on the impossibility for the speaker to indicate univocally the object on which he is talking about and consequently on the impossibility for all speakers to agree on what is the object they are dealing with.

Another example of this intrinsic feature of human communication is given by general concepts, for instance a term like "democracy"[2], and the respective adjective "democratic". In a sentence like "Italy is a democratic republic based on work", is the term "democratic" fixed by a universal definition? My answer is No. Rather, I think that we can fix some semantic *conventions* (as for example the definition of "democratic" we find in a vocabulary), but I am persuaded that if some expert speakers decide to confront

[2] I am grateful to Clara Zanardi for having proposed me this significant example.

themselves on what is the right meaning of this term, they hardly agree. In fact, even if all of them may agree on a general definition of "democratic", each of them assigns to this term different facetings, and so it becomes hard to estabilish if and at what degree of confidence, a sentence like "Italy is a democratic republic based on work" is true. Furthermore – as the reader has noted – in the sentence we are considering there is another notion that could create the same problems: the term "republic".

Another element I see unclear in Smith's argument concerns what he means with *vague objects*. Actually, he does not devote too much space to clarify this theme, and I see this choice as a consequence to the opinion that vagueness consists essentially on vague predicates. Neverteless, Smith states that the Closeness definition could be applied – by extension – to vague objects[3]: their vagueness is a matter of the vagueness of certain properties and relations, and the vagueness of the latter is described by the Closeness definition. I think that this claim suffers of the following problems: - it is not clear what Smith defines as objects, and consequently what is (if it does exist) the difference between *objects* and *concept*; - it seems to me unspecified in what sense vagueness is a matter of the vagueness of certain properties and relations; - and since, how the Closeness definition could be extended to vague objects.

Actually, in a little paragraph[4] the author tries to outline a differentiation between concepts and objects, in order to show that in the standard fuzzy view - but it is not specified what happens in Smith's fuzzy view - wordly vagueness is not vagueness in concepts. However, this underscoring is not significant, because it is not taken up further, so it does not clarify Smith's final position, which involves semantic plurivaluations.

To conclude, for me the main weakness of Smith's definition is that it concerns only vague predicates, ignoring the peculiarity of the other subsets of the language, first of all of which could be considered as subjects on which the predicates are applied. And for this reason, I agree with Weatherson on the claim that "a definition of vagueness must be more general than a definition of predicate vagueness, or at least generalisable beyond this case" [5]. Therefore, I suggest two improvements:

- a generalization of the Closeness definition which involves also objects, and concepts and which takes into account the fundamental differences among them;

[3]See [29] , 158.
[4][29] ,71.
[5][31] , 6.

- what discriminates between subjects as rigid designators and not lies on the possibility - even potential - of the agreement of all speakers without objections.

A final argue concerning the lack of clarity is given by **5.2.2**, where Smith tries to justify the legitimacy of *two sorts of degrees of truth*. I see the following observations as marks of the weakenesses highlighted above. Firstly, also in this place it is not clear what does differentiate these two "domains":

- oblects that have a certain degree of pollution (named by O);
- degrees of pollution that these things have, which are objects too (named by H).

I think it is not clear - again - what does Smith intend with the term *object* and, consequently, what is the sense of a mapping h from O to H, if we interpret these two domains as objects. Lastly, it is interesting to note that the author himself has underlined the weakeness of a definition involving a term like "sufficient", due to the fact that it ignores the vagueness of "pollute" and therefore, of "sufficient".

In fact, defining it as a term expressing the idea that T (which is a subset of H) should be closed upwards. This means that for each x and y in H, if $x \leq y$ and x belongs to T, then y belongs to T too.

7.2 On the fuzzy approach to vagueness

7.2.1 Closeness and continuity

I want to start this paragraph underlying that a fuzzy approach to vagueness seems to me being very intriguing. I think that the fact that the fuzzy framework can accomodate Closeness, due to its rich structure of degrees of truth, is a convincing idea.

Nevertheless, it seems unclear what Smith observes about the relationship between Closeness and continuity. In particular, he says that, in order to describe the use of the language, we cannot accept the assertion that a predicate is vague just in case its characteristic function is continuous. My opinion is that for Smith, defining a domain is a matter of *choice*. But it is just the problem I see in his proposal: I think that if we accept a logical or a mathematical framework – as fuzzy logic – we must take into account all features and all consequences that it involves.

In particular, Smith says that there are two "places" in which continuity is not needed: (i) we do not need the continuity of the domains and (ii) we do not need the continuity of the characteristic function. Particularly, he says that we must not think that a predicate is vague if its characteristic function is continuous.

About (i), it seems to me that choosing sometimes only some values as members of the domain, it is not justified enough. Fuzzy logic – as we have seen in **3.1**, is based on the idea to generalize classical logic considering as the set of values the real range $[0,1]$, instead of the set $\{0,1\}$. Moreover, in the *t-norms* based systems used by Smith, each $n-$ary connectives has a corresponding truth function f_c such that $f_c\,[0,1]^n \rightarrow [0,1]$. Finally, we have defined when a t-norm is continuous, using the usual definition of continuous function in a interval.

My claim is that if Smith's intention is to consider the whole real interval, he also must accept the mathematical definition of its cardinality - which is $2^{\aleph_0}$ - and consequently he cannot ignore the continuum ipothesis and the deeper discourses that follows from it.

The crux is that Smith argues that a mathematical definition of continuity using the definition of topologies, has the weakness to be far away from our considerations about the *real* use of ordinary language. In this sense, I think that the mathematical definition of vagueness does not mean first of all working with topologies, rather the first essential notions we meet just when we try to study fuzzy logic – like for instance the cardinality of the real interval - are already mathematically deep and fundamental. Therefore, they form a sufficient constraint about the possible interpretations, modifications and choices we can suggest on the logical system, in order to accomodate it with our philosophical proposal.

I see it is a symptom of a huge issue that covers all attempts to treat philosophical problems with the help of the mathematical logic: we must pay attention – I think – to use the instruments given by logic without forcing them to describe or to solve our philosophical problems. We may only be led by these logical instruments and after a close examination of the results, we could state if they have helped us, and if they have not we can try to modify some aspects in a coherent mathematically way. In this sense, I think that Smith's proposal of choosing only some values from the whole real interval to form useful domains, is a forcing, because it seems to be a matter of speaker's choice, aware or not .

About (ii), instead, the problems that arise from the characteristic functions, are quite different: I think this choice is a little more justified, because the fuzzy framework does not implies that all characteristic functions must be continuous. A characteristic function f_c (as a t-norm) could be considered as continuous if and only if is continuous in the interval $[0, 1]$ which means that in each point of the interval $x_0 \in \mathbb{R}_1$ if $\lim_{x \to x_0} f(x) = f(x_0)$. At the same time, it is evident that the t-norms fundamental theorem does not hold for non-continuous t-norms, and I see it as a result that suggests us to reconsider Smith's idea more in detail.

Furthermore, Smith states that the main condition we must satisfy is not a huge amount of degrees of truth to accomodate Closeness, rather to have a significant number of them. But what does *significant* mean? How can we quantificate this vague term each time? That is, even if Smith's idea could be reformulated more clearly, how much elements can be put in the domain? And then, how can we know when the number of these elements is *sufficient* to be significant? They are all questions to be answered before going on, and for this reason I see this situation as a *vicious circle*: vagueness, through its many-sided facets, involves all the definitions we assume, so - again – I see it as a fundamental intrinsic character of the language, and last but not least, a mark of the claim I have suggeted above, about the limits of defining vagueness as the vagueness of predicates.

7.2.2 What does "fuzzy" mean?

Now, an observation on a little paragraph in [29], titled "different senses of fuzzy logic", which should be - in my opinion - more detailed than Smith does.

My claim is that it is not clear in author's explanation, what he means with "fuzzy logic" and, consequently, what is the benefit of assuming a fuzzy logical and theoretical framework. My opinion is that this unclearness allows Smith to use fuzzy logic in an arbitrary way: he seems to justify himself of considering only some aspects of the logical systems.

But let us specify the question. The author emphasizes that the sense of fuzzy logic he has accepted, is expressed by the following passage by Susan Haack about Zadeh's framework:

> Zadeh offers us not only a radically non-standard logic, but also a radically nonstandard conception of the nature of logic. It would scarcely be an exaggeration to say that fuzzy logic lacks every feature that the pioneers of modern logic wanted logic for . . . it is not just a logic of vagueness, it is—what from Frege's point of view would have been a contradiction in terms—a vague logic. (Haack 1979, 441)[6]

[6] [29], 276

In other words, Smith argues that

> Haack notes explicitly that she is concerned with fuzzy logic in the elaborate sense - not with the view discussed in this book. I think that what Haack says in these passages is right - but none of it applies to the fuzzy view as discussed here.[7]

Therefore, I think is difficult to understand exactly what is the authentic sense of fuzzy logic, in Smith's view. He always talks about fuzzy logic without other specifications, thus I see this sort of gloss as not enough to clarify the issue.

In particular, in my opinion Smith considers primarily the following *useful* aspects of this logical framework:

- the richness of degrees of truth;
- fuzzy membership functions;
- conditionals (firstly Łukasiewicz conditional);
- fuzzy algebras as possible models;
- truth-functionality of sentences.

At the same time, he seems to ignore other features of fuzzy systems, like for instance:

(i) the differences between propositional and *first order logic* about the completeness results and the axiomatisation results.

(ii) the question of *axiomatization* from the point of view of each different logical system;

Let's start with (i). I think that the crux of the matter is that Smith seems to consider expressly only *propositional calculi*, while he intentionally ignores fuzzy predicate logic.

The question I ask is: why? What is the reason of taking into account only tools provided by the propositional framework?

I suggest two possible answers:

1. Smith is persuaded that the fuzzy propositional framework gives us alone a sufficient set of logical instruments, in order to describe vagueness of the language in speakers' disposition in a exhaustive way. However, in this case it is not specified how and why first order fuzzy logic is redundant and not necessary.

[7] [29], 276

2. First order fuzzy logic has still not been developed enough, and an investigation that considers these results would be too hard. I agree on this second possible consideration, but nevertheless I do not consider it a sufficient reason to eliminate *a priori* the possibility of developing and using this logical tool.

I do not know which is the right answer in author's opinion, but I am persuaded that they are both involved. The problem they bring with themselves is that they clash with the authentic stated purpose of Smith's philosophical position, which is - again - to analyze the vagueness of all parts of the language from a descriptive viewpoint, to describe speakers' linguistic usage in a formal way.

I want to stress on the expression *all of parts,* which means that vagueness lies in all parts of our communication. Instead, it seems to me that restricting our set of logical tools to propositional logic forces us to limit our investigation and to condition our philosophical results.

Moreover, there is a more crucial argument that in my opinion would persuade Smith to involve first order fuzzy logic in his argumentation, and this argument is represented just by the Closeness definition. Since the vagueness-as-closeness definition - as we have highlighted elsewhere - is based on the vagueness of *predicates*, how does legitimate us to rule out the predicate counterpart? I think this questions is strong enough to reconsider entirely the choice of the logical tools.

Furthermore, there are two elements we must recognize in Smith's explanation. The first is that if we remind the possible variants of the definition of the sories series given in **2.2.1**, we must note that one of them uses quantifiers. The second is represented by the following definition in **2.2.4**:

> A predicate satisfies Closeness over a set S if and only if, it satisfies Closeness when the initial quantifiers "for any object a and b" in Closeness are taken as ranging only over S.

Both these two elements seems to suggest us that the author is aware of the power of first order logic, and for this reason, it is even less clear why he does not examine also the predicate fuzzy framework and its implications.

Now let us deal with (ii). It seems to me Smith has omitted in his discourse about axiomatization in fuzzy systems: the question concerning the *recursive property.* There is an interesting consequence of extending our set of tools to first order logic: the fact that Gödel predicate logic has a recursive axiomatization that is complete with respect to the semantics over

$[0,1]$, whereas for Łukasiewicz logic and Product logic[8] we do not have a recursive complete axiomatization. More precisely, tautologies of $G\forall$ (over all linearly ordered $G-$algebras) coincide with tautologies over the standard $G-$algebra $[0,1]_G$ of truth functions, therefore $G\forall$ is recursively axiomatized. Conversely, a similar result for $L\forall$ and $\Pi\forall$ is impossible, which means that there is no recursive system of axioms and deduction rules for which provability would equal 1-tautologicity over $[0,1]$.

Thus, I think that this fundamental difference must be taken into account, in a closer examination of the relationships between human language and the formal tools provided by fuzzy systems.

For these reasons, I suggest to extend our view to first order logic; otherwise Smith's proposal will remain partial and quite incoherent with the theoretical aim. The challenge is therefore being entirely led by the instruments we have choosen, and being ready to refine them, if necessary.

To conclude, I see this objections as a confirmation of the fact that, even if Smith assumes expressly a limited and arbitrary definition of "fuzzy logic", nevertheless, this stratagem does not suffices to legitimate his choices about the applications of the logical instrument of knowledge, in a close exhamination of linguistic vagueness.

7.3 On Plurivaluationism

First of all, I think that Fuzzy Plurivaluationism is a good way to explore the vagueness about the linguistic usage through a logical method, because it respects the preliminary scope expressed by Smith, of trying an analysis from the point of view of speakers' behaviour.

In this sense, it is interesting to return on the problem of axiomatization just discussed, from the semantical point of view. A first comment in fact, concerns the problem of *axiomatization* of the meaning postulates of vague predicates.

Let's start with Běhounek observation:

> Fuzzy plurivaluationistic semantics with sharp sets of fuzzy models in fact conforms better to a different conception of meaning determination, namely one which identifies the meaning of a word with the

[8] As Hajek underlines [9], the embeddability result of Łukasiewicz into Product logic extends to predicate calculus gives immediately that predicate Product logic is not axiomatizable.

> set of its meaning postulates, i.e., its semantic properties and relations that would be approved by competent speakers [...][9]

For each vague predicate, it is possible to extract at least one condition, that the intended usage of that term seems to satisfy.

As Běhounek suggests, this condition should be accepted by the majority of competent speakers, but, however, it does not mean that all these competent speakers have the same comprehension of this predicate. Thus, in the degree theoretical semantics we could reformulate the "meaning postulates" as conditions on the membership function of that predicate. In other words, the claim that Smith's constraints are not univocally determined, mirrors the fact that the fuzzy models are not sharp, and it is - for Běhounek - a mark of the impossibility of a complete and precise axiomatization.

However, although for Běhounek, Smith's plurivaluations are just the classes of models of formal fuzzy theories (and these classes can formalize the meaning postulates of vague predicates), these meaning postulates do not talk about the degrees of truth, rather, these degrees of truth remain undetermined by the theory.

Smith's answer to that objection starts from the claim that

> the main (but not the only) meaning-determining facts are speakers' usage dispositions, and the acceptable models of a discourse are those that meet all the constraints imposed by the meaning determining facts.[10]

These constraints could be - as specified elsewhere - paradigm case constraints, ordering constraints, and so on.

My opinion is that Běhounek's worry should be considered as redundant, because he disagree with Smith on a fundamental question: unlike Běhounek, Smith is persuaded that if the set of acceptable models of a discourse is axiomatizable, then an approach through its axioms should be advantageous, because this sort of approach involves useful logical instruments. Nevertheless, at the same time he thinks that it does not imply that the set of acceptable models of a discourse must be *always* axiomatizable.

Let us consider - with Běhounek - the following sentences:

- Michael J. Fox is not tall (that could be axiomatized by $\neg Ha$)
- Christopher Lee is tall (that could be axiomatized as Hb).

[9] [1] ,2.
[10] [27], 29.

Smih states that, in order to solve the issue, provided by a model that makes Hb true and in which the name b does not refer to Christopher Lee, we must be sure that (1) Christopher Lee is fully in the extension of H in the model and (2) that the formula Hb is fully true on the model. Thus, in Smith's opinion, saying (2) is a problem because we must be absolutely sure that the referent of b in each model is the extension of H on the model, and this problem cannot be solved only by adding more axioms which constraint the interpretation of the name b. In fact, as Smith says, even if the best that a set of axioms can do, is fix its models up to isomorphism, the set of acceptable models of a discourse are not be closed under isomorphism.

My opinion is that - as always - we must make some distinctions. First of all, I think Smith does not take enough into account the fuzzy systems and the differences among them. For instance, it is not clear what specific logical system does he consider. It seems to me that if we talk about $G-$algebras, - following the corollary **3**[11]- if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two finite linearly ordered $G-$algebras of the same cardinality, then they are isomorphic. I see this fact as a significant reason to accept some guidelines provided by the logical tools, because it is not *always* true that "the set of acceptable models of a discourse are not be closed under isomorphism." Moreover, I think that there is a concrete possibility that a group of speakers assume as an acceptable model of its discourses a model representable with $G-$algebras. It is sufficient that each element of this model respect the condition expressed in **4.5**, and at the same time, the constraints imposed by the type$-T$ facts, which are respectively paradigm constraints, ordering constraints and exlusion constraints.

In fact, I think that in a $G-$algebra may hold the following conditions required by Smith: if speakers would all apply a predicate P to the object x in a normal conditions, than any candidate correct interpretation must assign P a function which maps x to 1 (and conversely, if speakers would all withold the predicate P from the object y in normal conditions, then any candidate correct interpretation must assign P a function which maps y to 0). Secondly, if for instance x's age is greater than y's, then any candidate correct interpretation must assign to the predicate "is young" a function which maps x to a value greater that equal to the value to whihc it maps y. And finally, in a $G-$algebra may hold the condition that any candidate correct interpretation assigns for example the predicate "is green" a function which maps x to a value near 1, and that must assign the predicate "is blue" a function which maps x to a value near or equal to 0.

[11]See 4.5.

7.3.1 On linear ordering

I think that the question of linear ordering is a special issue on the fuzzy plurivaluatiosist proposal, that must be analysed apart. The situation seems to be the following: on the one hand we have fuzzy logic that - as shown in chapters 3 and 4 - implies the linear ordering of the system of truth degrees; on the other side, we find fuzzy plurivaluationism, which admits linear ordering in each model of the framework, but that does not allow any sort of *supertruth.*

As Běhounek has noted, even if Smith is persuaded that fuzzy plurivaluationistic semantics for gradual vague predicates solves the problem of artificial precision, nevertheless it does not imply the possibility of a supertruth due to the incommensurability of some predicates. In Běhounek's view this is a mark of the indeterminacy of the properties, a proof that the truth status of sentences is not semantically determined. Finally, these considerations led him to consider supertruth as *deducibility*, in the sense that

> the consequence relation of fuzzy logic and the corresponding deduction rules have literally been designed to determine the supertruth of sentences of fuzzy plurivaluationism.[12]

In my opinion, the answer depends on what we believe that models represents: if we interpret fuzzy models as possible worlds (of the sort of Putnam's possible worlds) I think it is impossible to accept a sentence as supertrue; instead, if these models are interpreted as *contexts of utterance*, particularly - as explained in **6.2.1.** - like agent's epistemic state as a subjective matter, and as an agent's degrees of belief in proposition, we must make some distinctions. First of all, we have to reconsider the distincion among vague predicates and non-vague predicates, which Smith has explained in **2.1.2.4.**; lastly, we must discuss separately the case of rigid designators.

Let us consider for instance the following sentences :

(1) "Bob is 185 cm tall";

(2) "Alice has ingested exactly 45mg/kg of arsenic".

I think that, by interpreting fuzzy models as contexts of utterance, these sentences could be considered supertrue. In this sense, I try to suggest a possible definition of "supertruth" as *the highest level of understanding between a group of speakers.* Nothing more. Neither a possibility of a sentence

[12] [1] , 6.

to be absolutely true (that is in all possible worlds), nor a example of an authentic and total human comprehension, nor a legitimation of the possibility to confront different predicates (as for instance "is nice" and "is bald") on a unique linear order.

Rather, only the best level of understanding on which we can accept to fix a *convention* with other speakers, about the meaning of the parts of the sentences. In fact, unlike Smith, I see in these two sentences three elements that make a difference:

- the presence in both of them, of subjects as that I have called rigid designators.
- The presence in (1) of a predicate that, even if it remains inherently vague and not measurable (because it is an evaluative predicate), it is limited by a unit of measurement, that is a convention on which all speakers may agree. And if someone prefers to disagree, anyway, the issue is a matter of unit of measurement and it is potentially possible to fix a convention which satisfies all the speakers.
- The presence in (2) of a non vague predicate (according to Smith's definition given in **2.2.4**.).

A final observation about Smith's distinction between what is *acceptable* in an interpretation, and what is *mandatory*. My claim is that Smith is right when he stresses that nothing about our practice mandates anything about the ordering of sentences which have different sorts of predicates, like these two sentences. Furthermore, I would add that nothing is mandatory, neither if we accept the definition of 'supertruth' I have suggested, due to the nature of human communication, which is at the best, in my opinion, essentially *conventional*.

7.3.2 Worldly vagueness

We have touched on this question at the beginning of this dissertation, where we have highlighted the two positions involved: the *vagueness-in language* scenario and *worldly vagueness*. I think this is the right place to reconsidering this question and giving an answer.

The crux of the question is that Smith thinks that wordly vagueness is a fundamental part of fuzzy plurivaluationism. In particular, fuzzy plurivaluationism implies on the one hand worldly vagueness (in the fuzzy models) and, on the other hand, semantic indeterminacy as the lack of a unique intended interpretation. In this way, fuzzy plurivaluatonism solves both the *location problem* and the *jolt problem* by combining worldly vagueness and semantic indeterminacy.

But let us specify the question better: what is the crucial difference between worldly vagueness and semantic indeterminacy? And then: what does Smith intend with "location problem" and "jolt problem"? About the first question, contra Weatherson, Smith thinks that semantic indeterminacy and worldly vagueness are two distinct phenomena, because they concerns different *areas*: we find semantic indeterminacy only in the relationship between language and the world - due to the lack of a unique intended interpretation of any discourse - but nevertheless it has nothing to do with with vagueness. In fact, the latter conserns the nature of the interpretations, rather than their number. This is the reason for which a classical plurivaluationist, interprets vagueness as semantic indeterminacy: he assumes only a correct intended interpretation, thus he does not question himself about the nature of the intended interpretation, whereas a fuzzy plurivaluationist focuses his attention just on the property of models of being *fuzzy*.

Instead, about the location problem and the jolt problem, we should summarize saying that, if we consider a Sorites series, the former concerns the fixing of the location of the change point, that is the point on which hold the claims "this object is P" is true and "this object is P" is false. The fact that we cannot see how our linguistic usage could fix it, to be at any particular point in the series, is called the "location problem". As far as the jolt problem, it is the difficulty of positing of a particular change point in a Sorites series for the predicate P, at which the sentence "this object is P" goes from being true to be false.

To conclude, fuzzy plurivaluationism - as a consequence of all arguments we have hitherto explained - does solve the jolt problem via its positing of degrees of truth, whereas it is a response to the location problem because of the lack of a unique intended interpretation. If we would consider only standard plurivaluationism or only the standard fuzzy theory, we never could provide an answer for both the problems.

My first claim is against the term "solve", about the location problem and the jolt problem. I think these two characteristics of fuzzy plurivaluationism do not solve the problems just mentioned, because the use of this expression may suggest a definitive elimination of the issues.

Rather, in my opinion, Smith's view should represent an interesting way to inquire where vagueness is located. I think it is fundamental to link worldly vagueness to semantic indeterminacy, but nevertheless I am not convinced that this explanation is exhaustive. I do not think that it describes sufficiently the power of assuming vagueness as an intrinsic feature of the world; it does not take into account that it is an ineradicable element of our communication, which does not concern only the impossibility for a speaker to express the authentic meaning of *concepts* and *objects*.

To use the image of an iceberg, Smith's explanation could be a complete and detailed analysis of what worldly vagueness entails, if and only if we are willing to remain "on the tip of the iceberg", that is, if we deliberately rule out the philosophical (in the sense of *anthropological*) dimension of human being, which is the communication.

Conversely, I see worldly vagueness primarily as an anthropological indication concerning one of the main intentional contents of human behavior, which must be directed to the ways of communication between people, rather than wanting to be a grasp on the world through the language.

Therefore, I agree with the author about the possible difference between semantic indeterminacy and worldly vagueness, but at the same time - and this is my proposal - a wider and deeper definition of "worldly vagueness" should incorporate also semantic indeterminacy in a broader view of the question.

To sum up, in this chapter we have suggested some observations on Smith's theory, which is essentially represented by [29] . We have followed three main "lines": objections on the vagueness-as-closeness definition, observations on the fuzzy approach to vagueness and finally some comments on fuzzy plurivaluationism.

In this sense, two junctions seem to emerge:

- the fact that the definition of vagueness as "predicate vagueness" does not represent a complete conceptualization of the matter;
- the fact that we need a complete and deep awareness of a definition of worldly vagueness, which does not remain bounded by the limits of fuzzy models, but that covers as a blanket all the aspects of the issue.

* * *

To conclude, after having explained my critics, it is time to underline the strenght of Smith's proposal, and to suggest other working hypothesis.

I think that Fuzzy Plurivaluationism is an interesting attempt to describe the intrinsic vagueness of speakers' use of the language, through a fuzzy logical tool. In particular, in my opinion the author's view could be "saved" notwithstanding all the objections, if we start focusing on his *empiricistic* *view*. I see this aspect as a wind that blows on all the facets of Smith's idea, the authentic backbone of Fuzzy Plurivaluationism.

This "wind" does emerge explicitly through the attempt to undertake a questionnaire[13] to some students, to test their opinion about the vagueness of certain sentences involving vague predicates. Actually, Smith's seems to give not too much weight to this idea, but I think it is the crucial point of his position, even if it is never explicitly mentioned. This is the reason why I see in Smith's papers a dialectical relationship between fuzzy logic and linguistic vagueness.

For Smith it is not true that fuzzy logic should solve the problem of vagueness in ordinary language, just because the latter is not conceived as a problem. Rather, it is - again - an intrinsic characteristic of human communication and, consequently, of the world as interpreted by people.

This fact allows us to take into account an empirical viewpoint as a implicit underground: the point from which Smith starts, and the point on which it ends seem to coincide in the empirical view. I see it as a *virtuous circle,* which should help us to describe the features, the borders and the consequences of vagueness in ordinary language, adding up fuzzy models to plurivaluations. In other words, into this circle there are two fundamental nodes: *fuzzy plurivaluations* - just mentioned - and the *Closeness definition*; my final claim is that Smith's idea depends essentially on the Closeness definition. If we extend the Closeness definition beyond he limits of the set of predicates, we would be clarify the issues that have been emerged on this definition, and - as a chain reaction - also the objections concerning the fuzzy approach to vagueness and aot fuzzy plurivaluationism, would be reconsidered.

Finally, just a little gloss, that must be considered as a starting point for further developments, rather than a criticism to the theory.

The observation concerns the word "wordly", which suggests me another meaning: the *geographical acceptation*. I think Smith does not tackle the question about the validity of a common fuzzy approach to the vagueness of different languages. In particular: could we be sure of the validity of a unique fuzzy logical tool to describe the lingustic usage of people speaking different languages? Could we be certain that a formal instrument that is useful for the English language, is *a priori* an effective way to describe formally another language? It is not clear if Smith is aware of this aspect, but he does not consider it as a significant thing, or if he simply ignores it.

[13]See 6.2.2.

However, my answer is that we must recognize the relativity of "applying" a logical framework to a specific language, and try to adapt the former to the latter. I am not persuaded that our efforts must be adressed to search a common logical tool, which may formalize all natural languages; rather, I think these efforts should be directed to locate the precise differentiations between the languages, which give us the suggestions for a complete formalization of all of them. In my opinion, it represents a fundamental *empirical* aspect, that must be further developed.

Chapter 8

Conclusions

I think it is interesting to conclude this dissertation with a *reverse zoom,* than the one proposed in the introduction: from an overview of the results obtained, to the overall picture that has been presented at the beginning of this survey, as a *destination.*

In this dissertation we have shown that the crux of the matter about vagueness in ordinary language, is to provide a good *definition* of vagueness. In particular, we have proved that all the objections against the theory called Fuzzy Plurivaluationism, are based on the vagueness as closeness definition, and this is the reason why I have chosen to title this thesis "Some observations on the Vagueness-as-Closeness definition".

I do not think that Smith's whole theory must be discarded; rather, that it should be ameliorated, by extending the vagueness-as-closeness definition to other parts of the language, beyond the set of predicates. By this way, I think the objections that has been emerged in the previous chapter, may be clarified.

To be more precise about my criticisms, I have split them in three main "branches": objections against the vagueness-as-closeness definition itself, those against the fuzzy approach to vagueness, and finally those regarding fuzzy plurivaluationism. In these three branches, some transversal "keystones" have been emerged, as "underground rivers".

First of all, the central role of the *speakers' linguistic usages*: I consider this element as the *context* on which the analysis takes its form. The strength of Smith's attempt is in fact, to use fuzzy logic to describe human language from a "behavioral" viewpoint, which means that the focus is on the *speech acts,* that can be formalized, and that represent the raw material from which starting an investigation on linguistic vagueness.

The second "underground river" is represented by the *bidirectional aim,* mentioned in the introduction. In this sense, we have highlight that we see the logical tools as *testers,* which can sample - as I have shown for instance in **7.2** and **7.3** - our interpretations. Fuzzy systems are, in fact, both "cause" and "effect" of this dissertation: they cause it because it was my intention to find a "practical" application of their "investigative power", and they are an effect of this analysis, due to their "methodological power", because it seems evident that a close examination of vagueness, that aimes to be a formal description of linguistic usage, lead us to the fuzzy systems and to their interpretations as plurivaluations.

Finally, *worldly vagueness*. As I have said at the end of the previous chapter, this is the primary philosophical question that we must have in mind, when we deal with the aspect of semantic vagueness.

My last claim is that linguistic vagueness is always *worldly*, and if an investigation suggests otherwise, maybe this investigation is partial. I recognize that this statement seems to be *intuitive*, and it might clash with the aim of this work, that is use a logical method. However, I think in this dissertation has been shown that a logical instrument which has not any kind of relationships with intuition, is partial. This observation is evident, for instance, in the objection concerning *Continuity and Closeness*[1], or in the necessity of a plurivaluationist view, that does not imply a unique intended fuzzy interpretation.

In other words, I see an intuitive pulse to agree to a *worldly vagueness* scenario, rather than a vagueness-in-language scenario. A belief - all intuitive - that the human grip of the world is essentially *anthropological,* in the sense that it represents the practical attitude, to mould the external (and even the internal) world, through our symbolic forms, defined by our linguistic skills.

This is the reason why Smith's approach seems to me very attractive, notwithstanding the weaknesses I have underlined. I "save" this theory due to its "empiricist *conatus*", which is powerful because it imposes ourselves a sort of worldly vagueness, although the author does still not seem to be entirely aware of its importance. For this reason, in chapter 7 I have not simply argued against Smith's theory, to destroy it. Conversely, my objections are intended to highlight some weak points of Smith's reasoning, in order to suggest a possible reassessment of them, that does not misunderstand the results provided by the logical framework, and, at the same time, which does not blight the "empirical" aim of the analysis.

[1] See 7.2.1.

To proceed with the opposite zoom, in the introduction we have been focused on the claim that the problems in the philosophy of language, concern essentially the *assignment of meanings.* This aspect takes its complete form during our analysis, particularly, it finds its formulation through the introduction of plurivaluations, that is the lack of a unique intended interpretation. The necessity of fuzzy plurivaluations, in fact, arises just from the acknowledgement that the assignment of meaning is not always - maybe never - the same for all speakers: as we have underlined in some passages, two speakers may intend different meanings for the same word. This is the "engine" of our attempt to use fuzzy models to interpret the semantic of speakers' discourses.

Now, let us return on the circular polarization of the problem of assignment of meaning: between the *nature* of vagueness and the *number* of the possible interpretations impied by a complete theory of vagueness. We have said that our analysis must be focused on these two points, and we have maintained our intention.

In fact, we have obtained a theory which answers to both these issues: about the nature of linguistic vagueness, we have shown that it is intrinsically "worldly", which means that it affects not only human capacity to describe reality, but also - and above all - the reality itself, that is conceived as composed by vague objects and vague concepts.

The proof that this is the authentic nature of linguistic vagueness, is provided by the fact that all the considerations we have done about vagueness (i.e. what should be a good definition of vagueness, or how fuzzy logic is a legitime approach to this theme, and so on) arise from the feature of vagueness to be *worldly.*

About the number of the possible interpretations, instead, we have answered by the notion of *plurality*, that is characterized by the two intrinsic "sides" of fuzzy plurivaluationism: *plurivaluations* as *fuzzy models.*

To sum up, these two points - the nature of vagueness and the number of the possible interpretations - have allowed us to deal with the characterization of human assignment of meaning in a new way, leading us to a coherent description of the consequences implied by a "worldly vagueness" idea.

To conclude, in our survey about Smith's theory, we have discovered that we have been forced to answer the question about "worldly vagueness". We are dealing with a definition of philosophical activity that finds in this examination of Smith's theory, the opportunity to express itself as a practical activity, in the sense specified at the beginning. And this definition imposes us a 360 degree turn, in the investigation about *meanings,* showing us also the most abstract methods, up to the complex - but intriguing - mathematical context, without unhinging us to our authentic anthropological dimension, which is *human communication*.

Just like the Nietzschean image of a philosopher, who, when he starts his climbing, is already on the top of the mountain.

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