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A categorical equivalence for product algebras

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To Franco Montagna,
who let me into the world of research, allowing me to feel and fall for
the moments of wonder when everything finally works.

To everyone who saw me look for
and then choose with conviction my own debatable way,
to the ones who held a candle to enlighten it,
and to those who just stand by me and trustfully smile.

Contents

Introduction **4**

1 Fuzzy Logic **9**

1.1 Vagueness: why fuzzy logic? 9

1.2 What does fuzzy logic mean? 10

1.3 Many-valued propositional calculi 14

1.3.1 Basic Logic 15

1.3.2 Łukasiewicz logic and Gödel logic 16

1.3.3 Product logic 17

2 Algebraic semantics of fuzzy logics **18**

2.1 CIPRLs 18

2.1.1 Varieties 20

2.2 Hoops 21

2.2.1 Filters 24

3 Some categorical equivalences in algebraic logic **26**

3.1 Stone's theorem 26

3.2 Cancellative hoops and l-groups 28

3.3 Cancellative hoops with strong unit and MV algebras 32

3.3.1 From $\mathcal{C}$ to $\mathcal{MV}$ 32

3.3.2 From $\mathcal{MV}$ to $\mathcal{C}$: settling the equivalence 35

4 A categorical equivalence for product algebras 42

4.1 The greatest boolean subalgebra and the greatest cancellative subhoop of a product algebra 42

4.2 External join, the category of product triplets and the functor Φ 49

4.3 Inverting the functor Φ 53

4.4 Concluding the proof of the categorical equivalence 63

4.5 Additional results 65

4.5.1 Special cases of the equivalence theorem 65

4.5.2 Filters 65

Conclusions and open problems 68

Introduction

Fuzzy logic is the setting where the algebraic research presented in this thesis takes place. It comes out from the need of dealing not only with absolute concepts of truth and falsity, but also with their shades of meaning. In a Platonic world of pure ideas, perfect and unchangeable, we certainly agree that any sentence, any statement, either is completely true or it is completely false. But when it comes to deal with reality, sometimes we may need to be not that categorical.

For instance, if asked, we all surely consider ourselves able to distinguish between a lake and a puddle. Now suppose that on the shore of our lake stands a man, with a red cup in his hands. He fills his cup of water from the lake, then he pours it away, over and over again. Presuming that he has enough time to spend that way, at a certain point he will have turned the lake into a puddle. But does this mean there is a cup of water whose single spilling makes a lake become a puddle? Obviously not, thus there is a time frame during which we may not be able to tell if we are staring at a lake or at a puddle, and such sentences like “this is a lake” cannot be considered neither true nor false, but possibly true “at some degree”.

Hence, at an accurate analysis, reality seems vague, reality seems fuzzy, or better as we are going to discuss later on, even though reality may be considered precise, our language and thus the way we relate to reality is certainly vague. This kind of reasoning will result to affect logic as well, since it affects concepts of truth and falseness, around which any logic is built.

The most well-known fuzzy logics result to be algebraizable in the sense of Blok and Pigozzi, hence the study of their equivalent algebraic semantics is extremely interesting even from a logical point of view. In this thesis we will concentrate our research on categorical equivalences, that are very important in algebra and in algebraic logic in particular. The main achievement is a categorical equivalence for product algebras. As far as we know, this is the first equivalence involving the whole variety of product algebras.

The most important many-valued logics are Łukasiewicz logic, product logic, Gödel logic and Hájek's logic BL. Their equivalent algebraic semantics are constituted by the varieties of MV-algebras, of product algebras, of Gödel algebras and of BL-algebras, respectively. Moreover, the varieties of MV-algebras, product algebras and Gödel algebras are generated by the standard algebras on $[0,1]$, that is to say respectively from the algebra $[0,1]_{MV} = ([0,1], \cdot_{MV}, \rightarrow_{MV}, \max, \min, 0, 1)$, by the algebra $[0,1]_{\Pi} = ([0,1], \cdot_{\Pi}, \rightarrow_{\Pi}, \max, \min, 0, 1)$ and, finally, by the algebra $[0,1]_G = ([0,1], \cdot_G, \rightarrow_G, \max, \min, 0, 1)$.

The early ancestor between the theorems of categorical equivalence in algebraic logic is certainly the Stone Theorem, which links Boolean algebras to topological spaces. Exactly, Stone's theorem states that there is a duality between the category of Boolean algebras and the category of Stone spaces, i.e. every boolean algebra A is isomorphic to its Stone space S(A), and given any homomorphism from a Boolean algebra A to a Boolean algebra B it corresponds in a natural way to a continuous function from S(B) to S(A).

For MV-algebras, a famous theorem, due to Daniele Mundici [Mu], shows the existence of an equivalence between the category of MV-algebras, with morphisms the homomorphisms, and the category of lattice ordered abelian groups with a strong order unit, with morphisms the unit preserving homomorphisms. So far, as regards to algebra of fuzzy logic, only for the category of MV-algebras a satisfactory categorical equivalence has been shown. To the best of our knowledge, no equivalence is known for the whole category of BL-algebras, or even for the category of all Gödel algebras.

For product algebras, the best result has been shown in [CT], for those special product algebras which are obtained from a cancellative hoop $\mathbf{C}$ by the adding of a bottom element 0 (in the sequel, the product algebra obtained in this way will be denoted by $\mathbf{2} \oplus \mathbf{C}$). In [CT], the full subcategory of the category of product algebras whose objects are of the form $\mathbf{2} \oplus \mathbf{C}$ shown above is proved to be equivalent to the category of lattice ordered abelian groups. Note that lattice ordered abelian groups are categorically equivalent to cancellative hoops (the latter being essentially the negative cones of the former), and hence, the obvious equivalence between cancellative hoops and product algebras of the form $\mathbf{2} \oplus \mathbf{C}$, $\mathbf{C}$ a cancellative hoop, yields an equivalence with lattice ordered abelian groups as well.

But the construction of [CT] does not work for product algebras in general, as there are product algebras which are not of the form $\mathbf{2} \oplus \mathbf{C}$, for instance, any direct product of two non-trivial product algebras. Thus in order to obtain a categorical equivalence for the whole variety of product algebras, we need a more complex category.

The objects of the category $\mathcal{T}$ we are going to investigate and which will turn out to be equivalent to the one of product algebras, are triples consisting of: a Boolean algebra $\mathbf{B}$, representing the maximum Boolean subalgebra of the given product algebra; a cancellative hoop $\mathbf{C}$, representing its maximum cancellative subhoop; and an external operation $\vee_c$ from $B \times C$ into C, where for all $b \in B$ and $c \in C$, $b \vee_e c$ represents the join (in the product algebra) of b and c and satisfies some properties which will be specified later.

Moreover, the algebras $\mathbf{B}$ and $\mathbf{C}$ alone do not determine a unique product algebra, that is, in general there can be many non-isomorphic product algebras whose maximum boolean subalgebra and whose maximum cancellative subhoop are isomorphic. What makes the product algebra unique is the external join $\vee_e$.

The morphisms from an object $(\mathbf{B}, \mathbf{C}, \vee_e)$ into another object $(\mathbf{B}', \mathbf{C}', \vee_e')$, of $\mathcal{T}$, are pairs (h, k), where h is a homomorphism from $\mathbf{B}$ into $\mathbf{B}'$, k is a homomorphism from $\mathbf{C}$ into $\mathbf{C}$, and the condition $k(b \vee_e c) = h(b) \vee_e' k(c)$ is satisfied for all $b \in B$ ad $c \in C$.

We then define a functor Φ from the category $\mathcal{P}$ of product algebras into the category $\mathcal{T}$ of product triplets that to each product algebra $\mathbf{P}$ associates the triplet $(\mathbf{B}(\mathbf{P}), \mathbf{C}(\mathbf{P}), \vee_e)$, where $\mathbf{B}(\mathbf{P})$ and $\mathbf{C}(\mathbf{P})$ are the maximum boolean subalgebra and the maximum cancellative subhoop of $\mathbf{P}$, and, for $b \in B(P)$ and $c \in C(P)$, $b \vee_e c$ is the join of b and c in $\mathbf{P}$. Moreover, to each homomorphism f from a product algebra $\mathbf{P}$ into a product algebra $\mathbf{P}'$, Φ associates the pair (h, k), where h and k are the restrictions of f to $\mathbf{B}(\mathbf{P})$ and to $\mathbf{C}(\mathbf{P})$, respectively.

The main result of this thesis states that the functor Φ has an adjoint Φ^{-1} and that the pair (Φ, Φ^{-1}) constitutes an equivalence of categories.

Since cancellative hoops are categorically equivalent to lattice ordered abelian groups, we might obtain a similar result with cancellative hoops replaced by the better known lattice ordered abelian groups.

Moreover, if $\mathbf{P}$ has the form $\mathbf{2} \oplus \mathbf{C}$, then $\mathbf{B}(\mathbf{P}) = \mathbf{2}$, $\mathbf{C}(\mathbf{P}) = \mathbf{C}$, and $b \vee_e c = \begin{cases} c & \text{if } b = 0 \\ 1 & \text{if } b = 1 \end{cases}$. Thus in this case $\Phi(\mathbf{P})$ only depends on $\mathbf{C}$. Hence, one can easily obtain an equivalence with the category of cancellative hoops, and finally with the category of lattice ordered abelian groups. In other words, Cignoli and Torrens categorical equivalence is a special case of our result.

This thesis is organized as follows: in the first chapter we present fuzzy logic, the idea from which it arises, and we show the axiomatic system of the main propositional fuzzy logics.

In the second chapter we give some algebraic preliminaries useful for stating the following results, describing the algebraic semantics of the logics presented in the first chapter.

In the third chapter, we present a proof of the equivalence between the category of lattice ordered abelian groups and the category of cancellative hoops, and then we also prove the equivalence between the category of MV-algebras and the category of cancellative hoops with strong unit. We will hence obtain indirectly Mundici's equivalence.

In the final chapter, we present the original research of this thesis. We first prove that every product algebra $\mathbf{P}$ has a greatest boolean subalgebra, $\mathbf{B}(\mathbf{P})$ and a greatest cancellative subhoop, $\mathbf{C}(\mathbf{P})$, but these two algebras do not determine $\mathbf{P}$ up to isomorphism. Hence we investigate the properties of the restriction $\vee_e$ of join to $B(P) \times C(P)$, we introduce the category $\mathcal{T}$ of product triplets, and we define a functor Φ from the category $\mathcal{P}$ of product algebras into $\mathcal{T}$. We then introduce an adjoint, Φ^{-1}, of Φ, and finally we prove that the pair (Φ, Φ^{-1}) constitutes an equivalence of categories. We also briefly discuss some additional problems, like the restriction of the equivalence to some full subcategories of the category $\mathcal{P}$ of product algebras, and the notion corresponding, via the equivalence Φ, to the notion of filter of a product algebra.

Chapter 1

Fuzzy Logic

Logic studies reasoning. It deals with propositions, and the relation of consequence among them. Different logics diverge in their definition of sentences and notions of consequence. But the idea around which every logic evolves is the concept of truth, and the way it spreads through the propositions.

Classical logic, which is our starting point, considers truth as an absolute. It is a categorical and precise judgement, any proposition either is true or it is false. But is this kind of conception reasonable outside a Platonic hyperuranion? It will be our aim to show how, when we deal with reality, it is difficult to talk about such things as absolute truth or absolute falsity.

1.1 Vagueness: why fuzzy logic?

As Bertrand Russell states in an article published in 1923 [Ru], the problem is that even though reality is precise, things simply are what they are, and they either have some property or they do not, language is vague. Let us think again about our man with his red cup, and let us concentrate about a single word: "red". Since colours form a continuum, there are some shades of colour concerning which we shall be in doubt whether to call them red or not, and we are not able to tell in which specific shade of the spectrum red gives way to other colours. In order to have another example, suppose that our poor man went bald. It is presumed that at first he was not bald, then he starts losing his hairs one by one, and in the end he is bald. Hence there must have been one hair the loss of which turned him into a bald man. But this is absurd, since baldness is a vague concept. Some men are certainly bald, some other men surely are not, but there are other about whom we cannot tell whether they are bald or not. As Russell himself observes, the law of excluded middle is true when precise symbols are employed, but it is not true when symbols are vague, as, in fact, all symbols are.

We may think that this kind of argument does not really affect logic, since connectives seem to have a precise meaning. Yet the trouble comes with the notions of *true* and *false*. Again, we can define something as completely true or completely false only if we are talking about something which is itself a precise notion, or a precise symbol with a precisely defined property, which we already observed does not always happen. Therefore even in logical propositions we may have a certain degree of vagueness.

We may be able to imagine a clear meaning for such words as "or" and "not". But we can imagine what they would mean if only our symbolism were clear. All traditional logic assumes that precise symbols are being employed, taking for granted the accuracy of our knowledge, leaving aside all those propositions which cannot be treated with absolute concepts of true and false, just like the sentences "this cup is red" or "that man is bald". All those sentences will indeed be true "to some degree".

From this kind of need, from the necessity of dealing with statements and concepts that are not simply either completely true or completely false, arises what we are going to call fuzzy logic.

1.2 What does fuzzy logic mean?

As suggested, *fuzziness* is an attempt to express, or relate to *vagueness*, and fuzzy propositions may be true to some degree. In particular, instead of taking as truth-values only 0 and 1, corresponding respectively to "absolute falseness" and "absolute truth", fuzzy logic choses to use the whole real interval $[0, 1]$. Thus we may have propositions with a truth-value of $0, 7$. For atomically propositions, like the one already cited "that man is bald", it simply means that in a scale from 0 to 1, you think that the man in question is bald $0, 7$.

We are going to deal with *truth-functional* propositional many-valued logics, which means that the truth degree of a compound proposition is obtained by the truth degrees of the compounds, as it happens in classical logic. Precisely, for instance, every binary connective c has a truth function $f_c : [0, 1]^2 \to [0, 1]$ and for any pair of formulas φ, ψ the truth degree of the compound formula $c(\varphi, \psi)$ is determined by the truth degrees of φ and ψ following specific rules that we are going to show, that have to respect the intuition beneath each connective. We are going to follow [Ha].

We start with the requirements on a truth function which interpretates conjunction. The idea is a big truth-degree of $\varphi \& \psi$ should indicate that both the truth value of φ and the truth value of ψ are big. That is, the function should be non-decreasing in both arguments and limited by 0 and 1. We have the following definition:

Definition 1.2.1. A *t-norm* is a binary operation $*$ on $[0, 1]$ satisfying the following conditions:

i) $*$ is commutative and associative, i.e. for all $x, y, z \in [0, 1]$

$$x * y = y * x,$$

$$(x * y) * z = x * (y * z).$$

ii) $*$ is non-decreasing in both arguments:

$$x_1 \leq x_2 \text{ implies } x_1 * y \leq x_2 * y,$$

$$y_1 \leq y_2 \text{ implies } x * y_1 \leq x * y_2.$$

iii) $1 * x = x$ and $0 * x = 0$ for all $x \in [0, 1]$.

$*$ is a *continuous t-norm* if it is a t-norm and is a continuous mapping of $[0,1]^2$ into $[0,1]$.

Definition 1.2.2. The following are the most important continous t-norms:

1. *Łukasiewicz t-norm*: $x * y = max(0, x + y - 1)$,
2. *Gödel t-norm*: $x * y = min(x, y)$,
3. *Product t-norm*: $x * y = x \cdot y$ (product of reals).

We now deal with implication. In classical logic, $\varphi \to \psi$ is true if the truth value of φ is less or equal to the truth value of ψ. In order to extend this idea, we are going to consider that a big truth value of $\varphi \to \psi$ means that the truth value of φ is "not too bigger" of the truth value of ψ. Hence a truth function $x \Rightarrow y$ should be non-increasing in x and non-decreasing in y. It also have to respect the idea of *modus ponens*. Thus from (a lower bound of) the truth degree x of φ and (a lower bound of) the truth degree $x \Rightarrow y$ of $\varphi \to \psi$ we should be able to get a lower bound of the truth degree y of ψ. Moreover, the operation computing this lower bound should be non-decreasing in both arguments.

These considerations lead to the following request:

$$\text{if } a \leq x \text{ and } b \leq x \Rightarrow y \text{ then } a * b \leq y.$$

Hence,

$$\text{if } z \leq x \Rightarrow y \text{ then } x * z \leq y.$$

In order to make this rule more powerful, and define $x \Rightarrow y$ as big as possible, we also require the converse, hence we have:

$$x * z \leq y \text{ iff } z \leq x \Rightarrow y \tag{1.2.1}$$

i.e. $x \Rightarrow y$ is the maximum z satisfying $x * z \leq y$.

Lemma 1.2.3. *Let $*$ be a continous t-norm. Then there is a unique operation $x \Rightarrow y$ satisfying for all $x, y, z \in [0,1]$ the condition 1.2.1*

Proof. For each $x, y \in [0,1]$, let $x \Rightarrow y = sup\{z : x * z \leq y\}$. Let, for a fixed z $f(x) = x * z$. f is continous and non-decreasing, hence it commutes with sup, thus:

$$x * x \Rightarrow y = x * sup\{z : x * z \leq y\} = sup\{x * z : x * z \leq y\} \leq y.$$

Hence $x \Rightarrow y = max\{z : x * z \leq y\}$, and uniqueness is trivial. □

Definition 1.2.4. The operation $x \Rightarrow y$ from 1.2.3 is called *residuum* of the t-norm.

Theorem 1.2.5. *The residua of the continous t-norms defined in 1.2.2 are the following:*

$$x \Rightarrow y = 1 \text{ if } x \leq y,$$

otherwise if $x > y$

1. *Łukasiewicz implication:* $x \Rightarrow y = 1 - x + y$,
2. *Gödel implication:* $x \Rightarrow y = y$,
3. *Goguen implication:* $x \Rightarrow y = y/x$.

Proof. Assume $x > y$.

1. $x * z \leq y$ iff $x + z - 1 \leq y$ iff $z \leq 1 - x + y$. Thus $1 - x + y = max\{z : x * z \leq y\}$.
2. $x * z \leq y$ iff $min(x, z) \leq y$. Thus $y = max\{z : x * z \leq y\}$.
3. Similarly, $x * z \leq y$ iff $x \cdot z \leq y$, and hence $y/x = max\{z : x * z \leq y\}$, since $x > 0$.

□

Remark 1.2.6. Łukasiewicz implication is continous, while Gödel and Goguen are not, but the residuum of every continous t-norm is left continous in the first variable and right continous in the second variable.

Starting from the residuum, we can also define the truth function for negation.

Definition 1.2.7. The operation of *precomplement* is defined by:

$$-x = x \Rightarrow 0.$$

It easily follows that the precomplemets of the three continous t-norms are:

1. *Lukasiewicz negation*: $-x = 1 - x$,

2. *Gödel negation*: $-0 = 1$, for $x > 0, -x = 0$,

3. *Goguen negation*: $-0 = 1$, for $x > 0, -x = 0$.

Remark 1.2.8. The three t-norms presented are fundamental continous t-norms, in the sense that each continous t-norm is a combination of them (for details, see for instance [Ha]).

1.3 Many-valued propositional calculi

When we fix a continous t-norm, we fix a propositional calculus, whose set of truth-values is $[0, 1]$, where $*$ is taken as the truth function of conjunction & and his residuum $\Rightarrow$ is the truth function of implication.

Definition 1.3.1. The propositional calculus *PC** given by $*$ has propositional variables $p_1, p_2, \ldots$, connectives $\&, \rightarrow$, and the truth constants $\overline{0}$ for 0.

Formulas are defined inductively: each propositional variable and $\overline{0}$ are formulas, and if φ and ψ are formulas then $\varphi \& \psi$ and $\varphi \rightarrow \psi$ are formulas.

We can also define further connectives:

1. $\varphi \wedge \psi = \varphi \& (\varphi \rightarrow \psi)$

2. $\varphi \vee \psi = ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$

3. $\neg\varphi = \varphi \to \overline{0}$

4. $\varphi \leftrightarrow \psi = (\varphi \to \psi)\&(\psi \to \varphi)$

Definition 1.3.2. An *evaluation* of propositional variables is a mapping e assigning to each propositional variable p its truth value $e(p) \in [0,1]$. This extends to the evaluation of formulas as follows:

$$e(\overline{0}) = 0,$$

$$e(\varphi \to \psi) = e(\varphi) \Rightarrow e(\psi),$$

$$e(\varphi\&\psi) = e(\varphi) * e(\psi).$$

A formula of PC* is a 1-tautology if $e(\varphi) = 1$ for each evaluation e.

1.3.1 Basic Logic

Now we are going to give the axioms of the logic that is a common base of all the logic PC*, that hence can be shown to be 1-tautologies of every PC* (for a proof, see for instance [Ha], Lemma 2.2.6).

Definition 1.3.3. The following are the axioms of Basic Logic BL:

BL1. $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

BL2. $(\varphi\&\psi) \to \varphi$

BL3. $(\varphi\&\psi) \to (\psi\&\varphi)$

BL4. $(\varphi\&(\varphi \to \psi)) \to (\psi\&(\psi \to \varphi))$

BL5. $(\varphi \to (\psi \to \chi)) \to ((\varphi\&\psi) \to \chi)$

BL6. $((\varphi\&\psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

BL7. $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$

BL8. $\overline{0} \to \varphi$

The deduction rule is modus ponens.

Remark 1.3.4. BL1. is transitivity of implication, BL2. says that & conjunction implies its first element. $BL3$. is commutativity of &, while BL4. is commutativity of $\wedge$. BL5. expresses residuation, and BL6. expresses the idea of proof by cases. Finally, BL7. translates the "ex falso quodlibet" rule.

BL is proved to be complete with respect to BL algebras, which will be defined in the following chapter.

1.3.2 Łukasiewicz logic and Gödel logic

As we have observed, BL represents a basis for the other many-valued logics we presented. Hence, their axiomatization can be obtained as an extension of BL.

Definition 1.3.5. We define *Łukasiewicz propositional logic*, Ł, the theory given by the axioms of BL and the axiom $(\neg\neg)$ of double negation:

$$(\neg\neg) \quad \neg\neg\varphi \to \varphi.$$

A completeness theorem is given with respect to MV-algebras, which again will be presented later.

Definition 1.3.6. We shall call *Gödel propositional logic*, G, the extension of BL by the axiom:

$$(G) \quad \varphi \to (\varphi \& \varphi)$$

stating the idempotence of &.

In Gödel logic, the two conjunctions coincide, and it also results to prove all axioms of intuitionistic logic [Ha]. G is complete with respect to Gödel algebras, that as we will see, are Heyting algebras satisfying prelinearity.

We now present the many-valued logic whose algebraic semantic we are going to investigate in this thesis.

1.3.3 Product logic

Product logic, that we are going to denote by Π, is the propositional logic given by product t-norm: $x * y = x \cdot y$, where $\cdot$ is the product of reals. Recall that the corresponding implication is Goguen: if $x > y$, $x \Rightarrow y = y/x$, otherwise $x \Rightarrow y = 1$ if $x \leq y$, and the corresponding negation is Gödel negation: $-0 = 1$, for $x > 0, -x = 0$.

From now on, we are going to denote Goguen implication by $\rightarrow$ and product conjunction by $\odot$.

Definition 1.3.7. The axioms of Π are those of BL plus:

$$(\Pi 1) \quad \neg\neg\chi \rightarrow ((\varphi \odot \chi \rightarrow \psi \odot \chi) \rightarrow (\varphi \rightarrow \psi)),$$

$$(\Pi 2) \quad \varphi \wedge \neg\varphi \rightarrow \overline{0}.$$

As shown in [Ha], Łukasiewicz logic has a faithful interpretation in Π, the idea is that Łukasiewicz conjunction on $[0, 1]$ is isomorphic to restricted product $max(a, x \cdot y)$ on $[0, 1]$ for each $0 < a < 1$. In next chapter, we are going to present product algebras, which constitutes an equivalent algebraic semantic of Π, in the sense of [BP].

Chapter 2

Algebraic semantics of fuzzy logics

For all concepts of Universal Algebra we refer to [BS]. For product algebras we refer to [Ha], and to [CHN]. For MV-algebras, we also refer to [CDM]. From now on, we use boldface capital letters to denote algebras and plain text capital letters to denote their universe.

The algebras we are interested in, can be presented both in terms of residuated lattices and in terms of hoops, and we are going to show the two different constructions.

2.1 CIPRLs

Definition 2.1.1. A *commutative, integral and pointed residuated lattice* (abbreviated as *CIPRL*) is an algebra $\mathbf{L} = (L, \cdot, \rightarrow, \vee, \wedge, 0, 1)$ such that:

(1) $(L, \cdot, 1)$ is a commutative monoid.

(2) $(L, \vee, \wedge)$ is a lattice with bottom 0 and top 1 and

(3) $\rightarrow$ is a binary operation such that for all $x, y, z \in L$ the residuation property holds: $x \cdot y \leq z$ iff $x \leq y \rightarrow z$.

Definition 2.1.2. A *BL-algebra* [Ha] is a CIPRL satisfying the divisibility condition

(div) $$x \cdot (x \to y) = x \wedge y$$

and the prelinearity condition

(prel) $$(x \to y) \vee (y \to x) = 1.$$

A *product algebra* is a BL-algebra satisfying

(Π) $$\neg x \vee ((x \to x \cdot y) \to y) = 1.$$

A *Gödel-algebra* is a BL-algebra satisfying

(contr) $$x \cdot x = x.$$

A *Wajsberg algebra* is a BL-algebra satisfying

(dn) $$\neg\neg x = x,$$

where $\neg x$ is an abbreviation for $x \to 0$.

Wajsberg algebras are term equivalent to MV-algebras, that is, algebras $(A, \oplus, \neg, 0)$ whose operations satisfy:

MV1. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

MV2. $x \oplus y = y \oplus x$

MV3. $x \oplus 0 = x$

MV4. $\neg\neg x = x$

MV5. $x \oplus \neg 0 = \neg 0$

MV6. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$

The interpretations are respectively:

$$\neg x = x \to 0,$$

$$x \oplus y = \neg x \to y,$$

and:

$$x \cdot y = \neg(\neg x \oplus \neg y),$$
$$x \to y = \neg x \oplus y.$$

Hence, in the sequel we do not distinguish between MV-algebras and Wajsberg algebras.

As we have already said, BL-algebras, product algebras, MV-algebras and Gödel algebras constitute the algebraic semantics for Hájek's logic BL, for product logic, for Łukasiewicz logic and for the Gödel logic. CIPRLs instead constitute the algebraic semantics for FL_{ew}, the substructural logic consisting of Full Lambek calculus with weakening and exchange, see [GJKO].

In a BL-algebra, and hence in a product (resp., MV, Gödel) algebra, we can also define meet and join in terms of $\cdot$ and $\to$ by:

$$x \wedge y = x \cdot (x \to y) \tag{2.1.1}$$

$$x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x) \tag{2.1.2}$$

2.1.1 Varieties

The classes of BL-algebras, of MV-algebras (or Wajsberg algebras), of product algebras and of Gödel algebras clearly form a variety, which will be denoted by $\mathcal{BL}$, by $\mathcal{MV}$, by $\mathcal{P}$, and by $\mathcal{G}$, respectively. Since any variety determines a category whose objects are the algebras in the variety and whose morphisms are the homomorphisms, we will use the same notation for a variety $\mathcal{V}$ and for the corresponding category.

It is well known (see [Ha] and [CHN, Thm 2.4.2]) that if $\mathbf{A}$ is a member of any subvariety, $\mathcal{V}$, of $\mathcal{BL}$, then $\mathbf{A}$ is a subdirect product of totally ordered algebras in $\mathcal{V}$ (also called $\mathcal{V}$-chains in the sequel).

Moreover the varieties $\mathcal{MV}$, $\mathcal{P}$ and $\mathcal{G}$ are respectively generated by:

$$[0,1]_{MV} = ([0,1], \cdot_{MV}, \to_{MV}, \max, \min, 0, 1)$$

where $x \cdot_{MV} y = \max\{x+y-1,0\}$ and $x \rightarrow_{MV} y = \min\{1-x+y,1\}$, while the MV-operations in $[0,1]_{MV}$ are $x \oplus y = \min\{x+y,1\}$ and $\neg x = 1-x$;

$$[0,1]_\Pi = ([0,1], \cdot_\Pi, \rightarrow_\Pi, \max, \min, 0, 1)$$

where $\cdot_\Pi$ is ordinary product on $[0,1]$, $x \rightarrow_\Pi y = 1$ if $x \leq y$, and $x \rightarrow_\Pi y = y/x$ otherwise ;

$$[0,1]_G = ([0,1], \cdot_G, \rightarrow_G, \max, \min, 0, 1)$$

where $x \cdot_G y = \min\{x,y\}$, and $x \rightarrow_G y = 1$ if $x \leq y$ and otherwise
$x \rightarrow_G y = y$.

Hence, every quasiequation which is true in $[0,1]_{MV}$ (in $[0,1]_\Pi$, in $[0,1]_G$ respectively) is true in every Wajsberg algebra (resp., product algebra, Gödel algebra).

2.2 Hoops

Since lattice operations in a BL-algebra can be recovered from the operations $\cdot$ and $\rightarrow$, and since in some contexts the bottom element is not essential, the subreducts of BL-algebras and of subvarieties of $\mathcal{BL}$ in a language with $\cdot$, $\rightarrow$ and 1 only, play an important role. Hence, we briefly discuss the variety of hoops and some of its subvarieties.

Definition 2.2.1. (cf [BF]). A *hoop* is a structure $(A, \cdot, \rightarrow, 1)$ such that $(A, \cdot, 1)$ is a commutative monoid, and $\rightarrow$ is a binary operation such that

$$x \rightarrow x = 1, \quad x \rightarrow (y \rightarrow z) = (x \cdot y) \rightarrow z \quad \text{and} \quad x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x).$$

Definition 2.2.2. A hoop is said to be *basic* iff it satisfies the identity

(lin) $$(x \rightarrow y) \rightarrow z \leq ((y \rightarrow x) \rightarrow z) \rightarrow z.$$

A *Wajsberg hoop* is a hoop satisfying:

(W) $$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x.$$

A *cancellative hoop* is a hoop satisfying:

(canc) $$x \to (x \cdot y) = y.$$

The next proposition collects some well-known results proved in [AFM], in [BF] and in [Bo].

Proposition 2.2.3. *(1) Basic hoops are precisely the subreducts of BL-algebras in the language* $\{\cdot, \to, 1\}$ *of hoops.*

(2) Wajsberg hoops are precisely the subreducts of Wajsberg algebras in the language of hoops.

(3) A hoop is a partially ordered, commutative, residuated and integral monoid which is naturally ordered, i.e. $x \leq y$ *if* $\exists x : x = z \cdot y$*.*

Sometimes it is convenient to work with what we call *dual hoops.*

Definition 2.2.4. A *dual hoop* ([BF]) is an algebra $\mathbf{A} = (A, +, -, 0)$ such that $(A, +, 0, \leq)$ is a partially ordered commutative monoid with identity 0, which is the least element of A. and for all $x, y \in A$, $x - y$ is the smallest element of the set $\{z : x \leq z + y\}$.

While in hoops the partial order satisfies $x \leq y$ iff $x = z \cdot y$ for some $z \in A$ ($z = x \to y$), the partial order in dual hoops satisfies: $x \leq y$ iff $y = z + x$, for some $z \in A$.

If $\mathbf{A} = (A, \cdot, \to, 1)$ is a hoop then $\mathbf{A^d} = (A, +, -0)$ is a dual hoop, where $x + y = x \cdot y$, $x - y = y \to x$ and $0 = 1$. Conversely, if $\mathbf{A} = (A, +, -0)$ is a dual hoop then $\mathbf{A^d} = (A, \cdot, \to, 1)$ is a hoop, where $x \cdot y = x + y$, $x \to y = y - x$ and $1 = 0$. The classes of hoops and dual hoops are therefore term equivalent.

It is also possible to give a description of cancellative hoops in terms of lattice ordered abelian groups.

Definition 2.2.5. A *lattice-ordered abelian group* is an algebra $(G, +, -, \vee, \wedge, 0)$ where $(G, +, -, 0)$ is an abelian group, $(G, \vee, \wedge)$ is a lattice, and the equation $x + (y \vee z) = (x + y) \vee (x + z)$ holds.

The *negative cone* $\mathbf{G}^-$ of a lattice ordered abelian group $\mathbf{G}$ is the algebra whose domain is $\{g \in G : g \vee 0 = 0\}$ (the set of negative elements of $\mathbf{G}$), equipped with the constant 0 and with the operations $x \cdot y = x + y$ and $x \to y = (y - x) \wedge 0$.

Cancellative hoops are precisely the negative cones of lattice ordered abelian groups. Moreover, there is a categorical equivalence between the category of lattice ordered abelian groups and the category of cancellative hoops, as we are going to see in the next chapter.

We are going to present a description of a special class of product algebras, including the class of product chains.

Definition 2.2.6. Given a basic hoop $\mathbf{H}$, by $\mathbf{2} \oplus \mathbf{H}$ we denote the structure whose universe is $H \cup \{0\}$, where $0 \notin H$, and whose operations $\cdot'$ and $\to'$ are defined as follows:

$$x \cdot' y = \begin{cases} x \cdot y & \text{if } x, y \in H \\ 0 & \text{otherwise} \end{cases} \qquad x \to' y = \begin{cases} x \to y & \text{if } x, y \in H \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \in H \text{ and } y = 0, \end{cases}$$

where $\cdot$ and $\to$ denote the operations of $\mathbf{H}$ and $\wedge$ and $\vee$ are defined in terms of $\cdot$ and $\to$ as usual.

The construction $\mathbf{2} \oplus \mathbf{C}$ is the ordinal sum of the two element boolean algebra $\mathbf{2}$ and the cancellative hoop $\mathbf{C}$. We refer the reader to [BF] for the general definition and for an investigation of ordinal sums.

Proposition 2.2.7. *(1) If $\mathbf{C}$ is a cancellative hoop, then the algebra $\mathbf{2} \oplus \mathbf{C}$, is a product algebra.*

(2) Every product chain has the form $\mathbf{2} \oplus \mathbf{C}$, for some (possibly trivial) cancellative hoop $\mathbf{C}$.

Proof. For (1), it suffices to prove that the equation (Π) defining product algebras holds in $\mathbf{2} \oplus \mathbf{C}$. If $x = 0$, then $\neg x = 1$ and (Π) holds. If $x, y \neq 0$, then

$x \to x \cdot y = y$ and (Π) holds. Finally, if $x \neq 0$ and $y = 0$, then $x \to x \cdot y = 0$ and again (Π) holds.

As regards to (2), if $\mathbf{A}$ is a product chain and $x \neq 0$, then $\neg x < 1$, and by equation (Π) it must be $x \to x \cdot y \leq y$ for all y. Hence, taking $y = 0$, we obtain $\neg x = x \to 0 = x \to x \cdot 0 = 0$. This implies that $x \to 0 = 0$ for all $x > 0$. Clearly, for all z, the equation $z \cdot 0 = 0 \cdot z = 0$ holds in any BL-algebra, and hence it holds in $\mathbf{A}$. Moreover, if $x, y \neq 0$, then by (Π) we obtain that $x \to x \cdot y = y$, that is, all non-zero elements satisfy the equation defining cancellative hoops. This settles the claim. □

Notation 2.2.8. From now on, even without explicit mention, $\mathbf{P}$ will denote an arbitrary product algebra, $(\mathbf{P}_i : i \in I)$ denotes an indexed family of totally ordered product algebras such that $\mathbf{P}$ is a subdirect product of $(\mathbf{P}_i : i \in I)$. Moreover, according to Proposition 2.2.7, $(\mathbf{C}_i : i \in I)$ will denote a family of totally ordered cancellative hoops such that for all $i \in I$, $\mathbf{P}_i = \mathbf{2} \oplus \mathbf{C}_i$. Finally, if x denotes any element of $\mathbf{P}$, then for all $i \in I$, x_i will denote its i^{st} coordinate. Hence, every $x \in P$ can be written as $x = (x_i : i \in I)$, with $x_i \in P_i$. We shall write, indicating with $\hookrightarrow$ the subdirect immersion,

$$\mathbf{P} \hookrightarrow \prod_{i \in I} \mathbf{P}_i \quad \text{and for all } i \in I \quad \mathbf{P}_i = \mathbf{2} \oplus \mathbf{C}_i. \tag{2.2.1}$$

Note that since there is a monomorphism from $\mathbf{P}$ to $\prod_{i \in I} \mathbf{P}_i$, it suffices to show that an identity is valid for each $\mathbf{P}_i$ to conclude that it is valid in $\mathbf{P}$.

2.2.1 Filters

We conclude with the concept of filter. A *filter* of a CIPRL or of a hoop $\mathbf{A}$ is a subset F of A such that $1 \in F$ and if a, $a \to b \in F$, then $b \in F$. A filter is said to be *trivial* if its only element is 1, and *proper* if it does not coincide with the whole domain A of $\mathbf{A}$. A filter is *maximal* if it is proper and it is maximal wrt inclusion among all proper filters. The *co-radical*, $Rad^*(\mathbf{A})$ of a CIPRL or of a hoop $\mathbf{A}$, is the intersection of all its maximal filters. Filters stand in bijection with congruences: given a congruence θ on a CIPRL or of a hoop $\mathbf{A}$, the set $F_\theta = \{x \in A : (x, 1) \in \theta\}$ is a filter of $\mathbf{A}$, and given a filter F on $\mathbf{A}$, the set $\theta_F = \{(x, y) : x \leftrightarrow y \in F\}$ is a congruence on $\mathbf{A}$. Moreover, the maps $\theta \mapsto F_\theta$ and $F \mapsto \theta_F$ are mutually inverse isomorphisms between the filter lattice and the congruence lattice on $\mathbf{A}$.

Chapter 3

Some categorical equivalences in algebraic logic

In this chapter we outline some well-known categorical equivalences in algebraic logic. The first result in this field is Stone Representation theorem for boolean algebras, published in 1936 [St].

3.1 Stone's theorem

Stone associates to every boolean algebra **B** a topological space we shall indicate with **S(B)**, called its Stone space. The points in **S(B)** are the ultrafilters on **B**, or equivalently the homomorphisms from **B** to the two-elements boolean algebra. The closed sets which generate the topology on **S(B)** are all sets of the form $\{x \in S(B) : b \in x\}$ where b is an element of **B**. For every boolean algebra **B**, **S(B)** is a compact totally disconnected Hausdorff space: such spaces are called Stone spaces. Conversely, given any topological space X, the collection of the subsets of X that are clopen (both closed and open) is a boolean algebra.

A simple version of Stone's representation theorem states that every boolean algebra **B** is isomorphic to the algebra of clopen subsets of its Stone space **S(B)**. The isomorphism maps an element $b \in B$ to the set of all ultrafilters that contain b. This is a clopen set because of the choice of topology on $\mathbf{S(B)}$ and since $\mathbf{B}$ is a boolean algebra.

The result may be generalized to a categorical duality (equivalence with the dual) between the category of boolean algebras with morphisms the homomorphisms and the category of Stone space with morphisms the continuous functions. This duality means that every Boolean algebra is isomorphic to its Stone space, and also each homomorphism from a boolean algebra $\mathbf{A}$ to a boolean algebra $\mathbf{B}$ corresponds to a continuous function from $\mathbf{S(B)}$ to $\mathbf{S(A)}$. In other words, there is a contravariant functor that gives an equivalence between the two categories.

The theorem is a special case of *Stone duality*, a more general framework for dualities between topological spaces and partially ordered sets. In particular, a well-known extension of Stone's theorem is the one obtained for distributive lattices, that turn out to be equivalent to coherent spaces and Priestley spaces, which are ordered topological spaces, compact and totally disconnected. This representation of distributive lattices via ordered topologies is known as *Priestley's representation theorem for distributive lattices* [Pr].

As already pointed out, we are especially interested in many-valued logics, and for instance it is possible to obtain a similar result also for the category of MV algebras, which are distributive lattices. Indeed, a so called Stone-Priestley duality between the category $\mathcal{MV}$ and a particular category of Priestley spaces has been proved in [MT].

However for MV-algebras, the most important categorical equivalence is due to Daniele Mundici [Mu], who showed the equivalence with regard to the category $\mathcal{A}$ of lattice ordered abelian groups with a strong order unit, with morphisms the unit preserving homomorphisms. The category $\mathcal{L}$ of l-groups with morphisms the homomorphisms is also known to be equivalent to the category $\mathcal{CH}$ of cancellative hoops with morphisms the homomorphisms, as shown in [Fe]. This particular equivalence will be extremely interesting also for the main result of this thesis, since it allows us to express in terms of l-groups the categorical equivalence that we are going to show for the category of product algebras.

Hence, in the next section we are going to prove the equivalence between $\mathcal{L}$ and $\mathcal{CH}$, and later we also give a proof of the equivalence between the category $\mathcal{C}$ of cancellative hoops with strong unit and morphisms the unit preserving homomorphisms and the category $\mathcal{MV}$. Note that we will indirectly obtain Mundici's equivalence. In order to do this, we will refer to the proof of the equivalence between $\mathcal{MV}$ and $\mathcal{A}$ given in [CMD], and to [CT].

3.2 Cancellative hoops and l-groups

In this section we are going to see the equivalence between the category $\mathcal{CH}$ of cancellative hoops with morphisms the homomorphisms and the category $\mathcal{L}$ of l-groups with morphisms lattice homomorphisms.

Definition 3.2.1. Let $\mathbf{G}$ be an l-group. On the negative cone G^- we can define the operations $\cdot$ and $\rightarrow$ as follows:

$$x \cdot y = x + y,$$

$$x \rightarrow y = 0 \wedge (y - x)$$

$$1 = 0.$$

Lemma 3.2.2. $\boldsymbol{G^-} = (G^-, \cdot, \rightarrow, 1)$ *is a cancellative hoop.*

Proof. $(G^-, \cdot, 1)$ is clearly an abelian monoid, partially ordered, commutative and integral. Residuation property holds, since $x \rightarrow y = 0 \wedge (y - x) = sup\{z \in G^- : z \leq y - x\} = sup\{z \in G^- : z + x \leq y\} = sup\{z \in G^- : z \cdot x \leq y\}$. Cancellation law holds, indeed: $x \rightarrow (x \cdot y) = 0 \wedge (x + y - x) = y$. $\square$

Definition 3.2.3. Let Ψ be the map from $\mathcal{L}$ to $\mathcal{CH}$ defined by:

$$\Psi(\mathbf{G}) = \mathbf{G}^-$$

for every l-group G in $\mathcal{L}$, and for every h l-group homomorphism:

$$\Psi(h) = h_{|_{G^-}}$$

Lemma 3.2.4. Ψ *is a functor from* $\mathcal{L}$ *to* $\mathcal{CH}$

Proof. In lemma 3.2.2 we have showed that Ψ maps objects of $\mathcal{L}$ in objects of $\mathcal{CH}$, and clearly $\Psi(h)$ is an homomorphisms of hoops. Moreover, it preserves the identity map, since $\Psi(id_{\mathcal{L}}) = id_{|_{G^-}} = id_{\mathcal{CH}}$ and composition of morphisms, indeed $\Psi(h \circ k) = (h \circ k)_{|_{G^-}} = h_{|_{G^-}} \circ k_{|_{G^-}} = \Psi(h) \circ \Psi(k)$. □

Now we are going to define the inverted functor of Ψ.

Definition 3.2.5. Let C be a cancellative hoop negatively ordered, i.e. for $x, y \in C$, $x \leq y$ iff there is $z \in C$ such that $x = y \cdot z$. Similarly to the construction of integers from natural numbers, we define an equivalence relation $\equiv$ on the cartesian product $C \times C$. Given $a, b, c, d \in C$, we will say that $(a, b) \equiv (c, d)$ if $a \cdot c = b \cdot d$. The class of equivalence of this relation shall be denoted with $[a, b]$.

We can equip the quotient $G = C \times C/\equiv$ with a group structure defining the following operations:

$$[a, b] + [c, d] = [a \cdot c, b \cdot d],$$

$$-[a, b] = [b, a],$$

$$0 = [1, 1].$$

Definition 3.2.6. Let G^+ indicate the set $\{[a, 1] : a \in G\}$, as shown in [CT], we can define a partial order relation on G:

$$[a, b] \preccurlyeq [c, d] \text{ if } [c, d] - [a, b] \in G^+.$$

Moreover, relative meet and join operations can be defined as follows:

$$[a, b] \curlyvee [c, d] = [(a \cdot d) \vee (c \cdot b), b \cdot d],$$

$$[a, b] \curlywedge [c, d] = [(a \cdot d) \wedge (c \cdot b), b \cdot d].$$

Lemma 3.2.7. *The map* $a \mapsto [1, a]$ *is both a monoid isomorphism and an order isomorphism from* C *to* G^-.

Proof. It suffices to observe that given $a, b \in C$, we have that $[1, a] \preccurlyeq [1, b]$ iff $[a, b] \in G^+$ iff there is $c \in C$ such that $a = a \cdot 1 = b \cdot c$ iff $a \leq b$. □

Definition 3.2.8. Let Ψ^{-1} be the map from $\mathcal{CH}$ to $\mathcal{L}$ defined by:

$$\Psi^{-1}(\mathbf{C}) = \mathbf{G}$$

for $\mathbf{C}$ cancellative hoop, and given any morphism h of $\mathcal{CH}$,

$$\Psi^{-1}(h)[a, b] = [h(a), h(b)].$$

Lemma 3.2.9. Ψ^{-1} *is a functor from* $\mathcal{CH}$ *to* $\mathcal{L}$

Proof. We have already observed that Ψ^{-1} maps cancellative hoops in l-groups, and it is easy to see that $\Psi(h)$ is an l-group homomorphisms. Indeed:

$$\begin{aligned} \Psi^{-1}(h)([a, b] + [c, d]) &= \Psi^{-1}(h)[a \cdot c, b \cdot d] = [h(a \cdot c), h(b \cdot d)] \\ &= [h(a) \cdot h(c), h(b) \cdot h(d)] = [h(a), h(b)] + [h(c), h(d)] \\ &= \Psi^{-1}(h)[a, b] + \Psi^{-1}(h)[c, d] \end{aligned}$$

$$\Psi^{-1}(h)(-[a, b]) = \Psi^{-1}(h)[b, a] = [h(b), h(a)] = -[h(a), h(b)] = -(\Psi^{-1}(h)[a, b])$$

$$\begin{aligned} \Psi^{-1}(h)([a, b] \curlyvee [c, d]) &= \Psi^{-1}[(a \cdot d) \vee (c \cdot b), b \cdot d] = [h((a \cdot d) \vee (c \cdot b)), h(b \cdot d)] \\ &= [(h(a) \cdot h(d)) \vee (h(c) \cdot h(b)), h(b) \cdot h(d)] \\ &= \Psi^{-1}(h)[a, b] \curlyvee \Psi^{-1}(h)[c, d] \end{aligned}$$

Similarly it preserves $\curlywedge$. Moreover, it clearly preserves the identity map, since $\Psi^{-1}(id)[a, b] = [id(a), id(b)] = [a, b]$, and composition of morphisms:

$$\begin{aligned} \Psi^{-1}(h \circ k)[a, b] &= [(h \circ k)a, (h \circ k)b] = \Psi^{-1}(h)[k(a), k(b)] \\ &= (\Psi^{-1}(h) \circ \Psi^{-1}(k))[a, b]. \end{aligned}$$

□

We now have to show that functor Ψ defines an equivalence of categories.

Theorem 3.2.10. *The composite functor $\Psi\Psi^{-1}$ is naturally equivalent to the identity functor of $\mathcal{CH}$. In other words, for all cancellative hoops C, D and homomorphism $h : C \to D$, we have a commutative diagram:*

$$\begin{array}{ccc} C & \xrightarrow{h} & D \\ \psi_C \downarrow & & \downarrow \psi_D \\ \Psi(\Psi^{-1}(C)) & \xrightarrow{\Psi(\Psi^{-1}(h))} & \Psi(\Psi^{-1}(D)) \end{array}$$

Proof. Given $a \in C$, $\psi_C(a) = [1, a]$, and it is:

$$(\Psi(\Psi^{-1}(h)))[1, a] = [h(1), h(a)] = [1, h(a)] = \psi_D(h(a)).$$ □

Similarly, we have the following result, that settles the equivalence between $\mathcal{CH}$ and $\mathcal{L}$.

Theorem 3.2.11. *The composite functor $\Psi^{-1}\Psi$ is naturally equivalent to the identity functor of $\mathcal{L}$. In other words, for any l-group G, H and l-homomorphism $h : G \to H$, we have a commutative diagram:*

$$\begin{array}{ccc} G & \xrightarrow{h} & H \\ \psi_G \downarrow & & \downarrow \psi_H \\ \Psi^{-1}(\Psi(G)) & \xrightarrow{\Psi^{-1}(\Psi(h))} & \Psi^{-1}(\Psi(H)) \end{array}$$

Proof. Recall that Ψ takes the negative cone of G, and for all $a \in G^-$, $\psi_C(a) = [1, a]$. Hence we consider $\psi_C(a) = [1, a \wedge 0]$ and it is again:

$$(\Psi^{-1}(\Psi(h)))[1, a \wedge 0] = [h(1), h(a \wedge 0)] = [1, h(a) \wedge h(0)].$$

Indeed, observe that since h is an l-group homomorphism, it maps elements of the negative cone of G in elements of the negative cone of H. Hence,

$$\psi_H(h(a)) = (\Psi^{-1}(\Psi(h)))(\psi_G(a)) = [1, h(a) \wedge h(0)].$$ □

From Theorem 3.2.10 and 3.2.11 follows the final result:

Theorem 3.2.12. *The functor Ψ establishes a categorical equivalence between $\mathcal{L}$ and $\mathcal{CH}$.*

3.3 Cancellative hoops with strong unit and MV algebras

3.3.1 From $\mathcal{C}$ to $\mathcal{MV}$

Let C be a cancellative hoop, and consider the l-group G such that its negative cone is isomorphic to C. We invert the order of G, to deal with the positive cone instead. We will consider an element $u \in C$ a strong unit if it is such that $\forall x \in C, \exists n \in \mathbb{N} : u^n \leq x$. Hence u will be a strong unit of G, i.e. $\forall x \in G, \exists n \in \mathbb{N} : nu \geq x$ (remember that the order of G is the opposite with respect to the order of C).

Now we consider the interval $[0, u] = \{x \in G : 0 \leq x \leq u\}$, in the positive cone of G, which will be "dually" isomorphic, i.e. with inverted order, to the interval $[u, 1]$ of C.

Thus we apply Mundici's functor to the interval $[0, u]$ of G.

Definition 3.3.1. For each $x, y \in [0, u]$, we define the following operations:

$$x \oplus y = u \wedge (x + y) = u \wedge (x \cdot y),$$

$$\neg x = u - x = x \to u.$$

The structure $([0, u], \oplus, \neg, 0)$ is usually denoted with $\Gamma(G, u)$. Since the interval $[0, u]$ of G is isomorphic to the interval $[u, 1]$ of C, and C is uniquely determined from G and viceversa, we will refer to $\Gamma(G, u)$ as $\Gamma(C, u)$.

Theorem 3.3.2. $\Gamma(C, u)$ *is a MV algebra.*

Proof. We are going to show that $\Gamma(C, u)$ satisfies $MV1 \ldots MV6$. We first prove MV1. We have

$$x \oplus (y \oplus z) = u \wedge (x + (u \wedge (y + z)))$$

and

$$(x \oplus y) \oplus z = u \wedge (z + (u \wedge (x + y))).$$

If $x+y \leq u$ and $y+z \leq u$ then both members are equal to $u \wedge (x+y+z)$. Otherwise, if $u \leq y+z$ or $u \leq x+y$ then both members are equal to u. The proofs of MV2 to MV5 are straightforward. Let us prove MV6.

$$\begin{aligned}
\neg(\neg x \oplus y) \oplus y &= y \oplus \neg(y \oplus \neg x) \\
&= u \wedge (y + (u - (u \wedge (y + u - x)))) \\
&= u \wedge (y + u + (-u \vee (-y - u + x))) \\
&= u \wedge ((y + u - u) \vee (y + u - y - u + x)) \\
&= u \wedge (y \vee x) \\
&= y \vee x \\
&= \neg(\neg y \oplus x) \oplus x.
\end{aligned}$$

□

Lemma 3.3.3. *Let $A = \Gamma(C, u)$.*

1. *For all $a, b \in A, a + b = (a \oplus b) + (a \odot b)$,*
2. *for all $x_1 \dots x_n \in A, x_1 \oplus \dots \oplus x_n = u \wedge (x_1 + \dots x_n)$,*
3. *the natural order of the MV algebra A coincides with the order of $[0, u]$ inherited from G by restriction.*

Proof. 1. $a + b - (a \odot b) = a + b - \neg(\neg a \oplus \neg b) = a + b - (u - (u \wedge (u - a + u - b))) = a + b - (0 \wedge (a + b - u) = a + b \wedge u = a \oplus b$.

2. By induction on n. For $n = 1$ it is trivial, since the condition becomes $x_1 = u \wedge x_1$, valid for each $x \in A$.

   Now supposing the claim is valid for n, let us prove it for $n + 1$.

$$\begin{aligned}
x_1 \oplus \dots \oplus x_n \oplus x_{n+1} &= (u \wedge (x_1 + \dots x_n)) \oplus x_{n+1} \\
&= u \wedge (u \wedge (x_1 + \dots x_n) + x_{n+1} \\
&= u \wedge ((u \wedge (x_1 + \dots x_n)) + x_{n+1}) \\
&= u \wedge ((u + x_{n+1}) \wedge (x_1 + \dots x_{n+1})) \\
&= u \wedge x_1 + \dots + x_{n+1}
\end{aligned}$$

3. In a MV algebra, $\wedge$ and $\vee$ can be defined by:

$$x \vee y = (x \odot \neg y) \oplus y,$$

$$x \wedge y = \neg(\neg x \vee \neg y) = x \odot (\neg x \oplus y).$$

We now show that $(x \odot \neg y) \oplus y$ corresponds to the $\vee$ operation of the l-group, indeed:

$$(x \odot \neg y) \oplus y = (\neg(\neg x \oplus \neg\neg y)) \oplus y = x \vee y$$

as already seen in the proof of 3.3.2. Similarly for $\wedge$ operation, thus the two orders coincides. Recall that the order of the l-group is inverted compared with the one of the cancellative hoop C.

□

Remark 3.3.4. Let G be an l-group and $0 < u \in G$. Let $S = \{x \in G : \text{for some } 0 \leq n \in Z, |x| \leq nu\}$. Then S is a subgroup and a sublattice of G containing u, and $\Gamma(G, u) = \Gamma(S, u)$. Hence, when we consider the MV algebra $\Gamma(C, u)$ we can assume that u is a strong unit.

Definition 3.3.5. Let C and D be cancellative hoops, and let h be a cancellative hoop homomorphism from C to D.

Suppose that $u \in C$ and $v \in D$ and let $h : C \to D$ be an homomorphism such that $h(u) = v$. Then h is said to be a unit preserving homomorphism.

Let $\Gamma(h)$ denote the restriction of the l-group homomorphism associated to h to the unit interval $[0, u]$.

The following result follows straightforward from the definition given.

Lemma 3.3.6. *$\Gamma(h)$ is a homomorphism from $\Gamma(C, u)$ into $\Gamma(D, v)$.*

Theorem 3.3.7. *Let $\mathcal{A}$ denote the category whose objects are pairs $\langle C, u \rangle$ with C cancellative hoop and u a strong unit of C, and whose morphisms are unit preserving homomorphisms. Then Γ is a functor from $\mathcal{C}$ into $\mathcal{MV}$.*

Proof. We have already observed that Γ maps objects and morphisms of $\mathcal{C}$ in objects and morphism of $\mathcal{MV}$, and it is clear that it preserves the identity map and compositions of morphisms, being $\Gamma(h)$ simply the restriction of an l-group homomorphism to an interval. □

3.3.2 From $\mathcal{MV}$ to $\mathcal{C}$: settling the equivalence

In order to show the equivalence, we have to present some preliminary constructions.

Definition 3.3.8. A sequence $a = (a_1, a_2, \ldots)$ of elements of a MV algebra A is said to be *good* if for each $i = 1, 2 \ldots$, it is $a_i \oplus a_{i+1} = a_i$. Instead of $a = (a_1, \ldots, a_n, 0, 0, \ldots)$ we shall write $a = (a_1, \ldots a_n)$.

Definition 3.3.9. For any two good sequences $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ their *sum* $c = a + b$ is defined by $c = (c_1, \ldots, c_n)$, where for all $i = 1, 2, \ldots$

$$c_i = a_i \oplus (a_{i-1} \odot b_1) \oplus \ldots \oplus (a_1 \odot b_{i-1}) \oplus b_i. \tag{3.3.1}$$

Theorem 3.3.10. *The sum of two good sequences is a good sequence.*

Proof. First we observe that $(a \oplus b, a \odot b)$ is a good sequence, since in every MV algebra A $x \oplus y \oplus (x \odot y) = x \oplus y$. Indeed, recalling that every MV algebra A is a subdirect product of MV chains, we shall write $A \subseteq \prod_i A_i$, we prove the claim for a generic MV chain. If $x \oplus y = 1$ the claim follows from axiom: $x \oplus \neg 0 = \neg 0$. Otherwise, if $x \oplus y < 1$ then $\neg x \not\leq y$ (since in every MV algebra $x \leq y$ iff $\neg x \oplus y = 1$), so $y < \neg x$ and hence $y \odot x = 0$.

Note that in each MV chain $x \oplus y = x$ iff $x = 1$ or $y = 0$, thus in A_i each good sequence has the form $(1^p, a)$ for $p \geq 0, a \in A_i$, for all $i \in I$. Moreover, $a = (a_1, \ldots, a_n, \ldots)$ is a good sequence of A iff for each $i \in I$ $a_i = (\pi_i(a_1), \ldots, \pi_i(a_n), \ldots)$ is a good sequence in A_i, being π_i the projection on the i-eth component of the product. Indeed, $a_n \oplus a_{n+1} = a_n$ iff $\pi_i(a_n \oplus a_{n+1}) = \pi_i(a_n)$ for each $i \in I$.

From 3.3.1 follows that $(1^p, a) \oplus (1^q, b) = (1^{p+q}, a \oplus b, a \odot b)$, thus in each A_i the sum of two good sequences is a good sequence, and this settles the claim. □

Definition 3.3.11. Let M_A be the set of good sequences of A equipped with addition.

Theorem 3.3.12. *Let A be a MV algebra. Then M_A is an abelian monoid with the following properties:*

(i) (cancellation) *For any good sequences a, b, c, if $a+b = a+c$ then $b = c$.*

(ii) (zero-law) *If $a + b = (0, 0, \ldots) = (0)$ then $a = b = (0)$.*

Proof. From 3.3.1 we easily obtain commutativity of addition, zero-law and $a + (0) = (a)$.

Now we prove associativity and cancellation in a generic MV chain, by the usual argument on the subdirect representation follows the associativity for addition in A. Letting $a = (1^p, x)$, $b = (1^q, y)$, $c = (1^r, z)$, we have:

$$\begin{aligned}(b + a) + c &= (1^{p+q+r}, x \oplus y \oplus z, (x \odot y) \oplus ((x \oplus y) \odot z), x \odot y \odot z) \\ &= (1^{p+q+r}, x \oplus y \oplus z, (x \odot z) \oplus ((x \oplus z) \odot y), x \odot y \odot z) \\ &= b + (a + c).\end{aligned}$$

Since in every MV algebra holds $(x \odot y) \oplus ((x \oplus y) \odot z) = (x \odot z) \oplus ((x \oplus z) \odot y)$, for a proof of this fact see Proposition 1.6.2 in [CMD].

To prove cancellation, let us assume that a, b, c defined as before are different from 1, the other cases being trivial. If $q = r$, the claim follows from the fact that in every MV chain if $x \oplus y = x \oplus z$ and $x \odot y = x \odot z$ then $y = z$. Indeed, $max(\neg x, y) = \neg x \oplus (y \odot x) = \neg x \oplus (z \odot x) = max(\neg x, z)$. Similarly, $min(\neg x, y) = min(\neg x, z)$, hence $y = z$. Instead, if $q < r - 1$ then from the identity $(1^{p+q}, a \oplus b, a \odot b) = (1^{p+q}, a \oplus c, a \odot c)$ follows $a \odot b = 1$, thus $a = b = 1$, a contradiction. If $q = r - 1$ then $a \odot b = a$ and $a \oplus b = 1$, which implies $b = 1$, again a contradiction. Similarly the cases in which $r < q$ lead to contradiction, and the proof is complete. □

Definition 3.3.13. Given $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, we define

$$a - b = (a_1, \ldots, a_n) + (\neg b_n, \ldots, \neg b_1).$$

Theorem 3.3.14. *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_m)$ be good sequences. Without loss of generality, assume $m = n$. Then the following are equivalent:*

1. *There is a good sequence c such that $b + c = a$,*

2. $b_i \leq a_i$ *for all* $i = 1, \dots, n$.

Proof. $1 \Rightarrow 2$ follows directly from 3.3.1.

$2 \Rightarrow 1$: since in a MV algebra $x \oplus y = x$ iff $\neg x \oplus \neg y = \neg y$ (see for instance Lemma 1.6.1 in [CMD]), $(\neg b_n, \dots, \neg b_1)$ is a good sequence. Let us consider $c = a - b = (a_1, \dots, a_n) + (\neg b_n, \dots, \neg b_1)$, as already defined. We shall prove that $a = c + b$. By the usual argument, we can assume that A is totally ordered, thus $a = (1^p, x), b = (1^{q,y})$. We suppose both a and b to be different from 0 and 1, such cases being trivial. Then $q \leq p$. It is $b = (1^q, y, 0^{p-q})$, from $n = p + 1$ we get $(\neg b_n, \dots, \neg b_1) = (1^{p-q}, \neg y, 0^q)$, and hence c is obtained by dropping the first $p+1$ terms from $(1^{2p-q}, x \oplus \neg y, x \ominus y)$. Now we distinguish two cases. If $y \leq x$ then $x \ominus \neg y = 1, c = (1^{p-q}, x \ominus y)$ and $c + b = (1^p, (x \ominus y) \oplus y, (x \ominus y) \odot y) = (1^p, y \vee x, 0) = (1^p, x) = a$. Otherwise, if $y > x$, then $p > q$, $x \ominus \neg y = 0, c = (1^{p-q-1}, x \oplus \neg y)$ and $c + b = (1^{p-1}, x \oplus \neg y \oplus y, (x \oplus \neg y) \odot y) = (1^p, y \wedge x) = (1^p, x) = a$. ☐

Definition 3.3.15. Given a and b good sequences of A we write $b \leq a$ if they satisfies the equivalent conditions of the previous theorem.

Theorem 3.3.16. *Let a and b be good sequences.*

1. *If $b \leq a$ there is a unique good sequence c such that $b + c = a$. This c, denoted $a - b$, is given by:*

$$c = (a_1 \dots a_n) + (\neg b_n, \dots, b_1)$$

omitting the first n terms. In particular, for each $a \in A$ we have

$$(\neg a) = (1) - (a).$$

2. *The order is translation invariant: $b \leq a$ implies $b + d \leq a + d$ for every good sequence d.*

3. *The sequence*

$$a \vee b = (a_1 \vee b_1, \ldots, a_n \vee b_n, \ldots)$$

is good and is in fact the supremum of a and b with respect to the order defined.

4. *The sequence*

$$a \wedge b = (a_1 \wedge b_1, \ldots, a_n \wedge b_n, \ldots)$$

is the infimum of a and b.

5. $((a) + (b)) \wedge (1) = (a \oplus b)$.

Proof.

1. The existence of the good sequence is assured by Theorem 3.3.14. It is unique, indeed if there exists two good sequences c_1, c_2 such that $b + c_1 = a$ and also $b + c_2 = a$ it would be $b + c_1 = b + c_2$, hence by the cancellation law $c_1 = c_2$.

2. If $b \leq a$ there exists c such that $b + c = a$. Thus $b + d + c = a + d$, hence $b + d \leq a + d$.

3. We first show that $c = a \vee b$ is a good sequence. A is a sub direct product of a family of MV chain $\{A_i\}_{i \in I}$. For each $i \in I$ $a_i = (\pi_i(a_1), \ldots, \pi_i(a_n), \ldots)$ and $b_i = (\pi_i(a_1), \ldots, \pi_i(a_n), \ldots)$ are good sequences and it is $a_i = (1^p, \alpha_i)$ and $b_i = (1^q, \beta_i)$, with $\alpha_i, \beta_i \in A_i$. Hence it is: $\pi_i(c_n) = 1$ if $n \leq max\{p, q\}$ and $\pi_i(c_n) = 0$ if $n > max\{p, q\} + 1$. For $n = max\{p, q\} + 1$ it is $\pi_i(c_n) = \alpha_i$ if $p > q$, $\pi_i(c_n) = \beta_i$ if $p < q$ and $\pi_i(c_n) = max\{\alpha_i, \beta_i\}$ when $p = q$. Thus $(\pi_i(c_1), \ldots, \pi_i(c_n), \ldots)$ is a good sequence for each $i \in I$, whence c is a good sequence of A.

   That c is the supremum follows from the second condition of Theorem 3.3.14.

4. The proof is similar to the proof of claim 3.

5. Follows from 3.3.1 and the definition of $\wedge$.

□

Lemma 3.3.17. *$(M_A, +, -, (0))$ is a cancellative hoop, and $u_A = (1)$ is a strong order unit.*

Proof. In particular, $(M_A, +, -, (0))$ is what we have defined as a dual hoop. As we have already seen, $(M_A, +, (0), \leq)$ is an abelian monoid with a partial order, it is residuated, ant it satisfies the cancellative law. It is also integral, indeed the good sequence $1_A = (1, 1, \dots)$ is such that $a \leq 1$ for all $a \in M_A$. Clearly, for each $a \in A$, there exists $m \in \mathbb{N}$ such that $mu_A = (1^m, 0, \dots)$ dominates a. □

Lemma 3.3.18. *The map $a \mapsto \varphi(a) = (a)$ defines a monomorphism from the MV algebra A onto the MV algebra $\Gamma(M_A, u_A)$.*

Proof. It follows directly form the definition of $\varphi(a)$ that it maps elements of A to the interval $((0), u_A)$. The map is clearly injective, since if $a \neq b$ then $(a) \neq (b)$.

□

We now deal with the homomorphisms. Given A, B MV algebras and $h : A \to B$ homomorphism, if $a = (a_1, a_2, \dots)$ is a good sequence of A then $(h(a_1), h(a_2), \dots)$ is a good sequence of B.

Definition 3.3.19. Let $h^* : M_A \to M_B$ be defined by:

$$h^*(a) = (h(a_1), h(a_2), \dots)$$

for all $a, b \in M_A$.

Lemma 3.3.20. *h^* is both a monoid morphism and a lattice homomorphism.*

Proof. It suffices to show that:

1. $h^*(a + b) = h^*(a) + h^*(b)$,
2. $h^*(a \vee b) = h^*(a) \vee h^*(b)$,
3. $h^*(a \wedge b) = h^*(a) \wedge h^*(b)$.

1. For each i:

$$\begin{aligned}(h^*(a+b))_i &= h(a_i \oplus (a_{i-1} \odot b_1) \oplus \ldots \oplus (a_1 \odot b_{i-1}) \oplus b_i) \\ &= h(a_i) \oplus (h(a_{i-1}) \odot h(b_1)) \oplus \ldots \oplus (h(a_1) \odot h(b_{i-1})) \oplus h(b_i) \\ &= (h^*(a) + h^*(b))_i,\end{aligned}$$

   since h is a homomorphism.

2. We have:

$$\begin{aligned}h^*(a \vee b) &= (h(a_1 \vee b_1), \ldots, h(a_n \vee b_n), \ldots) \\ &= (h(a_1) \vee h(b_1), \ldots, h(a_n) \vee h(b_n)), \ldots) \\ &= h^*(a) \vee h^*(b).\end{aligned}$$

3. Similar to the previous one. □

Definition 3.3.21. Let $\Delta : \mathcal{MV} \to \mathcal{C}$ be defined by:

$$\Delta(A) = \langle M_A, u_A \rangle,$$

for every A MV algebra in $\mathcal{MV}$, and for every h homomorphism in $\mathcal{MV}$:

$$\Delta(h) = h^*.$$

Theorem 3.3.22. *Δ is a functor from $\mathcal{MV}$ into $\mathcal{C}$.*

Proof. We have already showed that Δ maps objects and morphisms of $\mathcal{MV}$ into objects and morphisms of $\mathcal{C}$. It preserves the identity map, since $\Delta(id)(a_1, \ldots, a_n) = (id(a_1), \ldots, id_{a_n}) = (a_1, \ldots, a_n)$ and also composition of morphisms, indeed:

$$\begin{aligned}\Delta(h \circ k)(a_1, \ldots, a_n) &= ((h \circ k)(a_1), \ldots, (h \circ k)(a_n)) = \Delta(h)(k(a_1), \ldots, k(a_n)) \\ &= \Delta(h) \circ \Delta(k)(a_1, \ldots, a_n).\end{aligned}$$

□

Furthermore, we have the following result:

Theorem 3.3.23. *The composite functor* $\Gamma\Delta$ *is naturally equivalent to the identity functor of* $\mathcal{MV}$*. In other words, for all MV algebras* A, B *and homomorphism* $h : A \to B$*, we have a commutative diagram:*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \alpha_A \downarrow & & \downarrow \alpha_B \\ \Gamma(\Delta(A)) & \xrightarrow{\Gamma(\Delta(h))} & \Gamma(\Delta(B)) \end{array}$$

Proof. For each $a \in A$, $\alpha_B(h(a)) = (h(a))$ and $\alpha_A(a) = (a)$. Moreover, $\Delta(h)(a) = (h(a))$ the latter being an element of $\Gamma(\Delta(B))$. Since $\Gamma(\Delta(h))$ is the restriction of $\Delta(h)$ to $\Gamma(\Delta(B))$, we can write $\Gamma(\Delta(h))(\alpha_A(a)) = (h(a)) = \alpha_B(h(a))$, as required. □

Now we are going to show that the composite functor $\Delta\Gamma$ is also equivalent to the identity functor of the category $\mathcal{C}$.

Theorem 3.3.24. *The functor* $\Delta\Gamma$ *is naturally equivalent to the identity functor of the category* $\mathcal{C}$*. In other words, for any two cancellative hoops with strong unit* $\langle C, u\rangle$ *and* $\langle D, v\rangle$ *and unit preserving homomorphism* $f : \langle C, u\rangle \to \langle D, v\rangle$*, we have a commutative diagram*

$$\begin{array}{ccc} \langle C, u\rangle & \xrightarrow{f} & \langle D, v\rangle \\ \beta_{\langle C,u\rangle} \downarrow & & \downarrow \beta_{\langle D,v\rangle} \\ \Delta(\Gamma(\langle C, u\rangle)) & \xrightarrow{\Delta(\Gamma(f))} & \Delta(\Gamma(\langle D, v\rangle)) \end{array}$$

Proof. Let $\beta = \varphi \circ \gamma$, where φ is the function defined in Lemma 3.3.18, and for all $a \in C$, $\gamma : C \to \Gamma(\langle C, u\rangle)$ is defined by $\gamma(a) = a \vee u$; note that we can define $\gamma(a) = a \wedge u$ if we consider the $\wedge$ operation of the l-group associated to C. Hence, given $a \in C$, $\beta_{\langle C,u\rangle}(a) = (a \wedge u)$, and $\Delta(\Gamma(f))(\beta_{\langle C,u\rangle}(a)) = (f(a \wedge u)) = (f(a) \wedge f(u)) = (f(a) \wedge v)$. Similarly, $\beta_{\langle D,v\rangle}(f(a)) = (f(a) \wedge v)$. □

Theorem 3.3.25. *The functor* Γ *defines a natural equivalence between the category* $\mathcal{C}$ *of cancellative hoops with strong unit and the category* $\mathcal{MV}$ *of MV algebras.*

Proof. Follows from theorem 3.3.23 and 3.3.24. □

Chapter 4

A categorical equivalence for product algebras

We are now ready to present the original part of this thesis, and hence prove that there is a categorical equivalence between the category of product algebras and the category of product triplets, which we are going to define later on.

4.1 The greatest boolean subalgebra and the greatest cancellative subhoop of a product algebra

In this section we prove that every product algebra $\mathbf{P}$ has a greatest boolean subalgebra, $\mathbf{B}(\mathbf{P})$, and a greatest cancellative subhoop, $\mathbf{C}(\mathbf{P})$, and we investigate the relationship between $\mathbf{P}$, $\mathbf{B}(\mathbf{P})$ and $\mathbf{C}(\mathbf{P})$. We start from the following result.

Theorem 4.1.1. *Let* $\mathbf{P}$ *be a product algebra, then:*

(1) The set $B(P) = \{x \in P : \neg\neg x = x\}$ *is domain of the greatest boolean subalgebra of* $\mathbf{P}$*.*

(2) The set $C(P) = \{x \in P : \neg\neg x = 1\}$ is domain of the greatest cancellative subhoop of **P**.

(3) For all $x \in P$, $x = \neg\neg x \cdot (\neg\neg x \to x)$ where $\neg\neg x \in B(P)$ and $\neg\neg x \to x \in C(P)$. So every $x \in P$ can be represented as the product $x = b \cdot c$, $b \in B, c \in C$.

(4) $C(P)$ is the co-radical of **P**. [1]

Proof. Referring to Notation 2.2.8, for all $x \in P$ and for all $i \in I$ we have:

(1) If $x_i \in C_i \setminus \{1_i\}$, then $\neg x_i = x_i \to 0_i = 0_i$, and $\neg\neg x_i = 1_i \neq x_i$. Hence, if $x = \neg\neg x$, then for all $i \in I$, $x_i \in \{0_i, 1_i\}$. Conversely, if $x_i \in \{0_i, 1_i\}$ for all $i \in I$, then $x = \neg\neg x$. Hence, $B(P)$ is the domain of a Boolean subalegbra of $\mathbf{2}^I$. We claim that it is also the greatest boolean subalgebra of **P**. Indeed, if x belongs to any boolean subalgebra of **P**, then $x = \neg\neg x$, and $x \in B(P)$.

(2) If $\neg\neg x = 1$, then $\neg\neg x_i = 1_i$ for all $i \in I$. Since $\neg\neg 0_i = 0_i$, it follows that $x_i \neq 0_i$ for all $i \in I$. Conversely, if $x_i \neq 0_i$ for all $i \in I$, then $\neg x_i = 0_i$, $\neg\neg x_i = 1_i$, and $x \in C(P)$. Hence,

$$C(P) = \{x : \forall i \in I\,(x_i \neq 0_i)\} = \{x : \forall i \in I\,(x_i \in C_i)\}.$$

Trivially, $C(P)$ is closed under the hoop operations, and hence it is a cancellative subhoop of $\prod_{i \in I} \mathbf{C}_i$.

We claim that $C(P)$ is the domain of the greatest cancellative subhoop of **P**. Indeed, if x belongs to some cancellative subhoop of **P**, then for all $i \in I$, $x_i \in C_i$, and hence $\neg\neg x_i = 1_i$. It follows that $\neg\neg x = 1$ and $x \in C(P)$.

(3) In any BL-algebra, we have $a \cdot (a \to b) = a \wedge b$. Since we also have $x \leq \neg\neg x$, it follows $x = x \wedge \neg\neg x = \neg\neg x \cdot (\neg\neg x \to x)$. Moreover, $\neg\neg\neg\neg x = \neg\neg x$, and hence $\neg\neg x \in B(P)$. Finally, for all $i \in I$, if $x_i = 0_i$, then $\neg\neg x_i \to x_i = 1_i \in C_i$, and if $x_i \neq 0_i$, then $\neg\neg x_i \to x_i = x_i \in C_i$. Hence, $x = \neg\neg x \cdot (\neg\neg x \to x)$, with $\neg\neg x \in B(P)$ and $\neg\neg x \to x \in C(P)$, as desired.

[1] Note that the greatest cancellative subhoop of **P** may be trivial even when **P** is non-trivial. Indeed, $C(P) = \{1\}$ when **P** is a boolean algebra.

(4) Clearly, $C(P) \neq \emptyset$, because $1 \in C(P)$. Moreover, for all $x \in P$, we have $x \in C(P)$ iff for all $i \in I$, $x_i > 0_i$. Since 0_i is the bottom element of $\mathbf{P}_i$, it follows that $C(P)$ is upward closed. Moreover, if x, $x \to y \in C(P)$, then $x \cdot (x \to y) \in C(P)$, because $C(P)$ is the domain of a subhoop of $\mathbf{P}$. Since $C(P)$ is upward closed and $x \cdot (x \to y) \leq y$, it follows that $y \in C(P)$. Hence, $C(P)$ is a filter. Moreover $C(P)$ is proper, since $0 \notin C(P)$. Now for all $x \in P$ and $y \in C(P)$, if $x \cdot y = 0$, then for all $i \in I$, $x_i \cdot y_i = 0_i$, and since $y_i > 0_i$, we must have $x_i = 0$. It follows that if M is a maximal filter of $\mathbf{P}$, then the product of finitely many elements in $M \cup C(P)$ is non-zero. Hence, $M \cup C(P)$ generates a proper filter, which by the maximality of M must be equal to M.

It follows that $C(P)$ is contained in all maximal filters of $\mathbf{P}$, and hence, it is contained in the co-radical of $\mathbf{P}$.

Conversely, if $x \in P \setminus C(P)$, then $x_i = 0$ for some i, and the set $M = \{x \in P : x_i > 0\}$ is a maximal filter that does not contain x.

It follows that x does not belong to the co-radical, and the claim is settled. □

Notation 4.1.2. In the sequel, the greatest boolean subalgebra and the greatest cancellative subhoop of $\mathbf{P}$ will be denoted by $\mathbf{B}(\mathbf{P})$ and by $\mathbf{C}(\mathbf{P})$, respectively. Elements of $B(P)$ will be called *boolean*, and elements of $C(P)$ will be called *cancellative*.

Theorem 4.1.3. *Let $\boldsymbol{P}$ be a product algebra, we have the following two groups of equivalent sentences, for all $x \in P$:*

1. *(a)* $x \in B(P)$

   (b) $x^2 = x$

   (c) $x \vee \neg x = 1$

   (d) $\forall i \in I \ x_i \in \{0, 1\}$.

2. *(a)* $x \in C(P)$

   (b) $x \to x^2 = x$

(c) $\neg x = 0$

(d) $\forall i \in I\ x_i > 0$.

Moreover:

3. $C \cap B = \{1\}$.

Proof. 1. See [CT] and Theorem 4.1.1.1.

2. $(a) \leftrightarrow (d)$: See Theorem 4.1.1.2

   $(a) \leftrightarrow (c)$: $\neg\neg x = 1 \Rightarrow \neg x \to 0 = 1$. We have: $\neg x_i \to 0 = 1\ \forall i$ iff $\neg x_i = 0\ \forall i$, otherwise $\forall x_i \in C\ \neg x_i \to 0 = 0$.

   Conversely, if $\neg x = 0$ then $\neg\neg x = 0 \to 0 = 1$.

   $(b) \leftrightarrow (d)$: $x \to x^2 = x$ iff $x_i \to x_i^2 = x_i$ for all i, that implies $x_i > 0$.

   Conversely, if $x_i > 0$ then $x_i \in C_i$, so $x_i \to x_i^2 = x_i$ (valid in every cancellative hoop).

3. if $x \in B \cap C$ then $x = \neg\neg x = 1$.

□

Note that there may be elements of **P** which are neither boolean nor cancellative. For instance, in the algebra $([0,1]_\Pi)^2$, the element $(0, \frac{1}{2})$ is neither boolean nor cancellative. Moreover, in any product algebra, 1 is the only element which is both boolean and cancellative.

In view of Theorem 4.1.1, a naive attempt to find a categorical equivalence for the category of product algebras (with morphism the homomorphisms) would be to try to prove an equivalence with the product of the category of boolean algebras and the category of cancellative hoops.

The first questions that come naturally to mind are: do we need any specific hypothesis on the boolean algebra and the cancellative hoop to make it possible? For instance, we may ask if they have to be linked to each other in some way. Moreover, once the product algebra is built, is it unique? That is to say, do the boolean algebra and the cancellative hoop univocally determine the product algebra on their own? In other words, we have the following conjectures:

Conjecture (1) Given a boolean algebra $\mathbf{B}$ and a cancellative hoop $\mathbf{C}$, there is a product algebra $\mathbf{P}$ such that $\mathbf{B}(\mathbf{P}) = \mathbf{B}$ and $\mathbf{C}(\mathbf{P}) = \mathbf{C}$.

Conjecture (2) Up to isomorphism, for each $\mathbf{B}$ and $\mathbf{C}$ as in conjecture (1), there is only one product algebra $\mathbf{P}$ such that $\mathbf{B}(\mathbf{P}) = \mathbf{B}$ and $\mathbf{C}(\mathbf{P}) = \mathbf{C}$).

We are going to prove that Conjecture (1) is true, while Conjecture (2) is false. Hence, the product of the categories of boolean algebras and of cancellative hoops is not equivalent to the category of product algebras.

Theorem 4.1.4. *Given any boolean algebra* $\mathbf{B}$ *and any cancellative hoop* $\mathbf{C}$, *there is a product algebra* $\mathbf{P}$ *such that* $\mathbf{B}(\mathbf{P}) = \mathbf{B}$ *and* $\mathbf{C}(\mathbf{P}) = \mathbf{C}$.

Proof. We distinguish two cases:

Case (1). $\mathbf{B}$ is finite. Then $\mathbf{B} \cong \mathbf{2}^n$ with $n > 0$. Let $\mathbf{P} = \mathbf{2}^{n-1} \times (\mathbf{2} \oplus \mathbf{C})$ if $n > 1$ and $\mathbf{P} = \mathbf{2} \oplus \mathbf{C}$ if $n = 1$. Then $\mathbf{P}$ is a product algebra (since it is product of two product algebras). Moreover $\mathbf{P} = \mathbf{2}^n \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C}$. Indeed, if $p \in P$: $p = (p_1, \ldots, p_{n-1}, p_n)$ with $p_i \in \{0, 1\}$ for $i = 1 \ldots n-1$ and $p_n \in 2 \oplus C$. Now if $p_n = 0$ then $p = p \cdot 1$, with $p \in 2^n, 1 \in C$. Else if $p_n \in C$ then $p = (p_1, \ldots, p_{n-1}, 1) \cdot (1, \ldots, 1, p_n)$ with $p = (p_1, \ldots, p_{n-1}, 1) \in 2^n$ and $(1, \ldots, 1, p_n) \in C$. It is easy to check that $\mathbf{B}(\mathbf{P}) = \mathbf{B}$ and $\mathbf{C}(\mathbf{P}) = \mathbf{C}$. Indeed, for all $i \in I$, $p_i = b_i \cdot c_i$, and since $\mathbf{B}(\mathbf{P})$ is such that for all $p \in B(P)$, for all $i \in I$, $p_i \in \{0, 1\}$ thus clearly $\mathbf{B}(\mathbf{P}) = \mathbf{2}^n$. Similarly, since $\mathbf{C}(\mathbf{P})$ is such that if $p \in C(P)$ then for all $i \in I$, $x_i > 0$ then $p_i = b_i \cdot c_i \in \{1, c_i\}$, hence $p \in C$. Moreover $C(P)$ is the greatest cancellative subhoop of P thus it must contain C, and this proves that $\mathbf{C}(\mathbf{P}) = \mathbf{C}$.

Case (2). $\mathbf{B}$ is infinite. Then $\mathbf{B}$ has a non-principal ultrafilter, U, say. Let $\mathbf{P}'$ be the product algebra $\mathbf{B} \times (\mathbf{2} \oplus \mathbf{C})$ (by abuse of language, we denote by 0 (resp., by 1) both the bottom (resp., top) element of $\mathbf{B}$ and the bottom (resp., top) element of $\mathbf{2} \oplus \mathbf{C}$. Moreover, let

$$P = \{(b, 0) : b \in B \setminus U\} \cup \{(b, c) : b \in U,\ c \in C\}.$$

It is readily seen that P contains $(0,0)$ and $(1,1)$, and is closed under $\cdot$ and under $\to$. $(0,0) \in P'$ and $(1,1) \in P''$, since $0 \in B, 0 \notin U$ and $1 \in U, 1 \in C$. P is closed under product, indeed: if $(b,0) \in P'$ and $(b',c) \in P''$ then

$$(b,0) \cdot (b',c) = (b',c) \cdot (b,0) = (b \wedge b', 0) \in P',$$

since $b \wedge b' \notin U$ with $b \notin U$.
Similarly, if $(b,0), (b',0) \in P'$ then

$$(b,0) \cdot (b',0) = (b \wedge b', 0) \in P'.$$

Finally, if $(b,c), (b',c') \in P''$ then

$$(b,c) \cdot (b',c') = (b',c') \cdot (b,c) = (b \wedge b', c \cdot c') \in P'',$$

since U is a filter thus closed under $\wedge$ and clearly $c \cdot c' \in C$ if $c, c' \in C$. P is closed under implication, indeed: if $(b,0) \in P'$ and $(b',c) \in P''$ then

$$(b,0) \to (b',c) = (b \to b', 0 \to c) = (b \to b', 1) \in P'',$$

since $b \to b' = \neg b \vee b' \in U$ with $b' \in U$.
If $(b,0), (b',0) \in P'$ then

$$(b,0) \to (b',0) = (b \to b', 1) \in P''$$

indeed again $b \to b' = \neg b \vee b' \in U$ since in a ultrafilter $b \notin U$ implies $\neg b \in U$.
If $(b,c) \in P''$ and $(b',0) \in P'$ then

$$(b,c) \to (b',0) = (b \to b', c \to 0) = (b \to b', 0) \in P'$$

indeed $b \to b \notin U$ since $\neg b, b' \notin U$.
If $(b,c) \in P''$ and $(b',c') \in P''$ then

$$(b,c) \to (b',c') = (b \to b', c \to c') \in P''.$$

Since the lattice operations are definable in terms of $\cdot$ and $\to$, P is the domain of a subalgebra, $\mathbf{P}$ say, of $\mathbf{P}'$. Moreover $\mathbf{C}$ is isomorphic to $\mathbf{C}(\mathbf{P})$

via the map $c \mapsto (1, c)$, and $\mathbf{B}(\mathbf{P})$ is the subalgebra of $\mathbf{P}$ with domain $\{(b, 0) : b \in B \setminus U\} \cup \{(b, 1) : b \in U\}$, which is isomorphic to $\mathbf{B}$ via the map

$$b \mapsto \begin{cases} (b, 0) & \text{if } b \in B \setminus U \\ (b, 1) & \text{if } b \in U. \end{cases}$$

This settles the claim. □

We now prove the failure of conjecture (2), that is, $\mathbf{B}(\mathbf{P})$ and $\mathbf{C}(\mathbf{P})$ do not determine $\mathbf{P}$ up to isomorphism.

Theorem 4.1.5. *There are non-isomorphic product algebras* $\mathbf{P}$ *and* $\mathbf{P}'$ *such that* $\mathbf{B}(\mathbf{P}) \cong \mathbf{B}(\mathbf{P}')$ *and* $\mathbf{C}(\mathbf{P}) \cong \mathbf{C}(\mathbf{P}')$*.*

Proof. Let $\mathbf{C}$ be a non-trivial totally ordered cancellative hoop, and let $\mathbf{P} = (\mathbf{2} \oplus \mathbf{C})^\omega$ and $\mathbf{P}' = \mathbf{2}^\omega \times (\mathbf{2} \oplus \mathbf{C}^\omega)$.

We have $\mathbf{B}(\mathbf{P}) = \mathbf{2}^\omega \cong \mathbf{2}^{\omega+1} = \mathbf{B}(\mathbf{P}')$, and $\mathbf{C}(\mathbf{P}) = \mathbf{C}^\omega \cong \mathbf{C}(\mathbf{P}')$.

On the other hand, $\mathbf{P}$ and $\mathbf{P}'$ are not isomorphic. To see this, let $b \in \mathbf{2}^\omega$ be defined by $b_i = 1$ for all $i \in \omega$, and consider $(b, 0) \in P'$. Note that $(b, 0)$ has the following properties:

(i) $(b, 0)$ is boolean; (ii) $(b, 0)$ is not the top; (iii) if $(b, 0) < (b', c)$, then (b', c) is cancellative; (iv) the set of upper bounds of $(b, 0)$ is not totally ordered.

If $\mathbf{P}$ and $\mathbf{P}'$ were isomorphic, there would be $x \in P$ satisfying (i), (ii), (iii) and (iv). Now let, by way of contradiction, $x \in P$ be such that (i), (ii), (iii) and (iv) are satisfied. Since x is boolean and it is not the top, there is an index $i \in \omega$ such that $x_i = 0$, and for all $j \in \omega$, $x_j \in \{0, 1\}$. If there are $i \neq j$ such that $x_i = x_j = 0$, then the element y defined by $y_i = 0$ and $y_j = 1$ for $j \neq i$ is a non cancellative element above x, and (iii) does not hold. If, say, $x_i = 0$ and $x_j = 1$ for $j \neq i$, then the set of upper bounds of x is the set of all elements z such that $z_i \in \mathbf{2} \oplus \mathbf{C}$ and $z_j = 1$ for $j \neq i$. This is a totally ordered set, being order isomorphic to $\mathbf{2} \oplus \mathbf{C}$, and (iv) cannot hold.

This settles the claim. □

4.2 External join, the category of product triplets and the functor Φ

As proved by Theorem 4.1.5, although every element of a product algebra $\mathbf{P}$ is the product of an element of $\mathbf{B}(\mathbf{P})$ and an element of $\mathbf{C}(\mathbf{P})$, the algebras $\mathbf{B}(\mathbf{P})$ and $\mathbf{C}(\mathbf{P})$ do not determine $\mathbf{P}$ up to isomorphism. This is due to the fact that in general there are many ways to define the product of an element of $\mathbf{B}(\mathbf{P})$ and an element of $\mathbf{C}(\mathbf{P})$, all leading to a product algebra.

Hence, we need additional structure in order to define product in a unique way. The idea is to introduce a third component, namely, the join of an element of $\mathbf{B}(\mathbf{P})$ and an element of $\mathbf{C}(\mathbf{P})$. Note that such join is in $\mathbf{C}(\mathbf{P})$, because, by Theorem 4.1.1, $C(P)$ is filter of $\mathbf{P}$. Hence, we have a map, called *external join* and denoted by $\vee_e$, from $\mathbf{B}(\mathbf{P}) \times \mathbf{C}(\mathbf{P})$ into $\mathbf{C}(\mathbf{P})$, satisfying suitable properties which will be presented below. The goal is to prove that such properties allow us to obtain a category which will turn out to be equivalent to the category of product algebras.

Lemma 4.2.1. *Let $\mathbf{P} = (P, \cdot, \to, \vee, \wedge, 0, 1)$ be a product algebra, and let $\vee_e$ denote the restriction of $\vee$ to $B(P) \times C(P)$. Then for all $b, b' \in B(P)$ and $c, c' \in C(P)$, the following conditions hold:*

(J1) $(b \vee_e c) \vee c' = b \vee_e (c \vee c') = (b \vee_e c) \vee (b \vee_e c')$

(J2) $b \vee_e (c \wedge c') = (b \vee_e c) \wedge (b \vee_e c')$

(J3) $(b \vee b') \vee_e c = b \vee_e (b' \vee_e c) = (b \vee_e c) \vee (b' \vee_e c)$

(J4) $(b \wedge b') \vee_e c = (b \vee_e c) \wedge (b' \vee_e c)$

(J5) $1 \vee_e c = b \vee_e 1 = 1$ *and* $0 \vee_e c = c$

(J6) $(b \vee_e c) \cdot c' = (\neg b \vee_e c') \wedge (b \vee_e c \cdot c')$

(J7) Define, for all $b \in B(P)$,

$$\theta_b = \{(c, c') : c, c' \in C(P) \text{ and } \neg b \vee_e c = \neg b \vee_e c'\}.$$

Then θ_b is a congruence of $\mathbf{C}(\mathbf{P})$. Moreover, θ_0 and θ_1 are the maximum and the minimum congruence of $\mathbf{C}(\mathbf{P})$, respectively.

Proof. With the exception of (J7), all claims have a similar proof, which refers to the subdirect representation according to Notation 2.2.8. Let $i \in I$ be arbitrary. Then clearly $b_i, b'_i \in \{0_i, 1_i\}$.

(J1) If $b_i = 0_i$, then $(b \vee_e c)_i \vee c'_i = c_i \vee c'_i = b_i \vee_e (c \vee c')_i = (b \vee_e c)_i \vee (b \vee_e c')_i$.

If $b_i = 1_i$, then $(b \vee_e c)_i \vee c'_i = 1_i = b_i \vee_e (c \vee c')_i = (b \vee_e c)_i \vee (b \vee_e c')_i$.

(J2) Similarly, if $b_i = 0_i$, then $b_i \vee_e (c \wedge c')_i = c_i \wedge c'_i = (b \vee_e c)_i \wedge (b \vee_e c')_i$.

If $b_i = 1_i$, then $b_i \vee_e (c \wedge c')_i = 1_i = 1_i \wedge 1_i = (b \vee_e c)_i \wedge (b \vee_e c')_i$.

(J3) If $b_i = b'_i = 0_i$, then $(b \vee b')_i \vee_e c_i = c_i = b_i \vee_e (b' \vee_e c)_i = (b \vee_e c)_i \vee (b' \vee_e c)_i$.

If $b_i = 1_i$ or $b'_i = 1_i$, then $(b \vee b')_i \vee_e c_i = b_i \vee_e (b' \vee_e c)_i = (b \vee_e c)_i \vee (b' \vee_e c)_i = 1_i$.

(J4) If $b_i = b'_i = 1_i$ then $(b \wedge b')_i \vee_e c_i = 1_i = 1_i \wedge 1_i = (b \vee_e c)_i \wedge (b' \vee_e c)_i$.

If $b_i = 0_i$ or $b'_i = 0_i$, then $(b \wedge b')_i \vee_e c_i = 0_i \vee_e c_i = c_i = (b \vee_e c)_i \wedge (b' \vee_e c)_i$.

(J5) Trivial, since 0 and 1 are respectively the bottom and the top of the product algebra.

(J6) If $b_i = 0_i$, then $(b \vee_e c)_i \cdot c'_i = c_i \cdot c'_i = 1_i \wedge (c_i \cdot c'_i) = (\neg b \vee_e c')_i \wedge (b \vee_e c \cdot c')_i$.

If $b_i = 1_i$, then $(b \vee_e c)_i \cdot c'_i = 1_i \cdot c'_i = c'_i = c'_i \wedge (1_i \vee c_i \cdot c'_i) = (\neg b \vee_e c')_i \wedge (b \vee_e c \cdot c')_i$.

(J7) Let, for every $i \in I$, θ_i denote the congruence of $\mathbf{C}(\mathbf{P})$ induced by the i^{st} projection, that is, $\theta_i = \{(c, c') : c_i = c'_i\}$. If $b_i = 0$ then for all $c, c' \in C(P)$ we have $(\neg b \vee_e c)_i = (\neg b \vee_e c')_i = 1$. If $b_i = 1$, then $(\neg b \vee_e c)_i = (\neg b \vee c')_i$ iff $c_i = c'_i$, that is, iff $(c, c') \in \theta_i$. It follows that $(c, c') \in \theta_b$ iff $(c, c') \in \theta_i$ for all $i \in I$ such that $b_i = 1_i$. Hence, $\theta_b = \bigcap_{i \in I : b_i = 1_i} \theta_i$. Being an intersection of congruences, θ_b is in turn a congruence.

That θ_0 is the maximum congruence and θ_1 is the minimum congruence follows directly from the definition. □

We are ready to define a category which will turn out to be equivalent to the category of product algebras.

Definition 4.2.2. A *product triplet* is a triplet $(\mathbf{B}, \mathbf{C}, \vee_e)$, where $\mathbf{B}$ is a boolean algebra, $\mathbf{C}$ is a cancellative hoop, and $\vee_e$ is a map from $B \times C$ into C satisfying properties (J1), ...,(J7) in Lemma 4.2.1, where $\vee$ and $\wedge$ denote indifferently the join (resp., the meet) operation in $\mathbf{B}$ or in $\mathbf{C}$.

A *good pair* from a product triplet $(\mathbf{B}, \mathbf{C}, \vee_e)$ into another product triplet $(\mathbf{B}', \mathbf{C}', \vee_e')$ is a pair (h, k) where h is a homomorphism from $\mathbf{B}$ into $\mathbf{B}'$, k is a homomorphism from $\mathbf{C}$ into $\mathbf{C}'$, and for all $x \in B$ and $y \in C$, $k(x \vee_e y) = h(x) \vee_e' k(y)$.

The product triplets are the objects of a category, whose morphisms from an object $\mathbf{T} = (\mathbf{B}, \mathbf{C}, \vee_e)$ into another object $\mathbf{T}' = (\mathbf{B}', \mathbf{C}', \vee_e')$ are the good pairs from $\mathbf{T}$ into $\mathbf{T}'$ (of course, the composition $(h, k) \circ (h', k')$ of two good pairs is defined componentwise: $(h, k) \circ (h', k') = (h \circ h', k \circ k')$).

Notation 4.2.3. The category of product triplets just defined will be denoted by $\mathcal{T}$, and the category of product algebras, with morphisms the homomorphisms, will be denoted by $\mathcal{P}$.

From (J1) and (J3), we obtain that in a product triplet, the function $\vee_e$ is monotonic in both arguments. This property will be used repeatedly in the sequel.

Lemma 4.2.4. $\vee_e$ *respecting properties from (J1) to (J7) preserves* $\leq$*.*

Proof. Let $a, b \in B$ boolean algebra, and $c, d \in C$ cancellative hoop. First we prove the statement for the first component, thus suppose $a \leq b$, hence $b = a \vee b$. Using (J3) we obtain:

$$a \vee_e c \leq (a \vee_e c) \vee (b \vee_e c) = (a \vee b) \vee_e c = b \vee_e c.$$

For the second component, let us suppose $c \leq d$, hence $d = c \vee d$. From (J1) we get:

$$b \vee_e c \leq (b \vee_e c) \vee (b \vee_e d) = b \vee_e (c \vee d) = b \vee_e d.$$

□

Definition 4.2.5. We define a functor Φ from $\mathcal{P}$ into $\mathcal{T}$ as follows:

(1) For every object $\mathbf{P}$, of $\mathcal{P}$, we set $\Phi(\mathbf{P}) = (\mathbf{B}(\mathbf{P}), \mathbf{C}(\mathbf{P}), \vee_e)$, where $\vee_e$ is the restriction of join to $B(P) \times C(P)$.

(2) For every morphism f from $\mathbf{P}$ into $\mathbf{P}'$, we set $\Phi(f) = (h, k)$, where h and k are the restrictions of f to $\mathbf{B}(\mathbf{P})$ and to $\mathbf{C}(\mathbf{P})$, respectively.

Note that $\Phi(f) = (h, k)$ is a good pair, because, for all $b \in B(P)$ and for all $c \in C(P)$, $k(b \vee_e c) = f(b) \vee f(c) = h(b) \vee_e k(c)$. We have the following result:

Theorem 4.2.6. Φ *is a functor from* $\mathcal{P}$ *into* $\mathcal{T}$.

Proof. Given $\mathbf{P}$ product algebra, $(\mathbf{B}(\mathbf{P}), \mathbf{C}(\mathbf{P}), \vee_e)$ is clearly a product triple in the sense of Definition 4.2.2, since natural join satisfies properties of Lemma 4.2.1. Moreover, if $f : \mathbf{P} \to \mathbf{P}'$, $f_{\restriction_{B(P)}}$ is an homomorphism that maps $\mathbf{B}(\mathbf{P})$ to $\mathbf{B}(\mathbf{P}')$, indeed if $x \in B(P)$ $\neg\neg f(x) = f(\neg\neg x) = f(x)$. Similarly $f_{\restriction_{C(P)}}$ is an homomorphism that maps $\mathbf{C}(\mathbf{P})$ to $\mathbf{C}(\mathbf{P}')$, in fact if $x \in C(P)$ $\neg\neg f(x) = f(\neg\neg x) = f(1) = 1$. Also, trivially $f(b \vee_e c) = f(b \vee c) = f(b) \vee' f(c) = f(b) \vee_e' f(c)$. To prove that Φ is a functor, we need to show that it preserves the identity map id (which is trivial) and composition of morphisms:

$$\begin{aligned}
\Phi(f \circ g)(b, c) &= ((f \circ g)_{\restriction_{B(P)}}, (f \circ g)_{\restriction_{C(P)}})(b, c) = ((f \circ g)(b), (f \circ g)(c)) \\
&= (f(g(b)), f(g(c))) = (f_{\restriction_{B(P)}}, f_{\restriction_{C(P)}})(g(b), g(c)) \\
&= (f_{\restriction_{B(P)}}, f_{\restriction_{C(P)}})((g_{\restriction_{B(P)}}, g_{\restriction_{C(P)}})(b, c)) = (\Phi(f) \circ \Phi(g))(b, c)
\end{aligned}$$

□

4.3 Inverting the functor Φ

Our aim is to define a functor Φ^{-1} from $\mathcal{T}$ into $\mathcal{P}$ such that the pair (Φ, Φ^{-1}) is an equivalence of categories. Our first step will be the following: given a product triplet $\mathbf{T} = (\mathbf{B}, \mathbf{C}, \vee_e)$, construct a product algebra $\mathbf{P}$ such that $\Phi(\mathbf{P})$ is isomorphic to $\mathbf{T}$ (that is, there is a good pair (h, k) from $\mathbf{T}$ into $\Phi(\mathbf{P})$ such that h is an isomorphism from $\mathbf{B}$ onto $\mathbf{B}(\mathbf{P})$ and k is an isomorphism from $\mathbf{C}$ onto $\mathbf{C}(\mathbf{P})$).

We start from the following observation: every element x of a product algebra $\mathbf{P}$ can be written as $x = b \cdot c$ with $b \in B(P)$ and $c \in C(P)$. In general, the decomposition is not unique. For instance, for all $c \in C(P) \setminus \{1\}$, $0 \cdot c = 0 \cdot 1 = 0$. The next lemma provides a characterization of all $b, b' \in B(P)$ and $c, c' \in C(P)$ such that $b \cdot c = b' \cdot c'$.

Lemma 4.3.1. *For all $b, b' \in B(P)$ and $c, c' \in C(P)$, the following are equivalent:*

(i) $b \cdot c \le b' \cdot c'$

(ii) $b \le b'$ and $\neg b \vee c \le \neg b \vee c'$.

Hence, $b \cdot c = b' \cdot c'$ iff $b = b'$ and $\neg b \vee c = \neg b \vee c'$.

Proof. $(i) \Rightarrow (ii)$. Suppose $b \not\le b'$. Then, with reference to Notation 2.2.8, there is an index $i \in I$ such that $b_i = 1_i$ and $b'_i = 0_i$. It follows $0_i = (b' \cdot c)_i < c_i = b_i \cdot c_i = (b \cdot c)_i$, and hence $b \cdot c \not\le b' \cdot c'$. Now suppose $b \le b'$ and $\neg b \vee c \not\le \neg b \vee c'_i$. Then for some $i \in I$, it must be $\neg b_i \vee c_i > \neg b_i \vee c'_i$. Then clearly $b_i = 1_i$ (otherwise $\neg b_i \vee c'_i = 1_i$), and hence $b'_i = 1_i$. Moreover, $c_i = \neg b_i \vee c_i > \neg b_i \vee c'_i = c'_i$, and $c_i = (b \cdot c)_i > c'_i = (b \cdot c')_i = (b' \cdot c')_i$, a contradiction. Hence, $b \cdot c \le b' \cdot c'$ implies $b \le b'$ and $\neg b \vee c \le \neg b \vee c'$.

$(ii) \Rightarrow (i)$. Suppose (ii) holds. With reference to Notation 2.2.8, for all $i \in I$, if $b_i = 0_i$, then certainly $0_i = (b \cdot c)_i \le (b' \cdot c')_i$. If $b_i = 1$, then $(b \cdot c)_i = c_i = 0_i \vee c_i = (\neg b \vee c)_i \le (\neg b \vee c')_i = c'_i = 1_i \cdot c'_i = (b' \cdot c')_i$. □

Lemma 4.3.1 suggests the following definition:

Definition 4.3.2. Let $(\mathbf{B}, \mathbf{C}, \vee_e)$ be a product triplet. For all $(b, c), (b', c') \in B \times C$, we define $(b, c) \preceq (b', c')$ iff $b \leq b'$ and $\neg b \vee_e c \leq \neg b \vee_e c'$. Moreover, we define $(b, c) \sim (b', c')$ iff $(b, c) \preceq (b', c')$ and $(b', c') \preceq (b, c)$.

Lemma 4.3.3. *(1) The relation $\preceq$ is a preorder on $B \times C$.*

(2) The relation $\sim$ is an equivalence relation on $B \times C$.

(3) Let for all $(b, c) \in B \times C$, $[b, c]$ denote the equivalence class of (b, c) modulo $\sim$, let $B \otimes_{\vee_e} C$ be the quotient of $B \times C$ modulo $\sim$, and define, for $[b, c]$, $[b', c'] \in B \otimes_{\vee_e} C$, $[b, c] \lhd [b', c']$ iff $(b, c) \preceq (b', c')$. Then $\lhd$ is (well defined and) a partial order on $B \otimes_{\vee_e} C$.

(4) If $b \leq b'$ and $c \leq c'$, then $[b, c] \lhd [b', c']$.

Proof. That $\preceq$ is reflexive is obvious. In order to show transitivity, let $(b, c) \preceq (b', c') \preceq (b'', c'')$. Then $b \leq b' \leq b''$, and hence, $b \leq b''$. In order to conclude the proof of transitivity, it is left to prove that $\neg b \vee_e c \leq \neg b \vee_e c''$. To this purpose, note that $\neg b \vee_e c \leq \neg b \vee_e c'$ and $\neg b' \vee_e c' \leq \neg b' \vee_e c''$. But since $b \leq b'$, $\neg b' \leq \neg b$, and by using (J3) we have:

$$\neg b \vee_e (\neg b' \vee_e c') = (\neg b \vee \neg b') \vee_e c' = \neg b \vee_e c' \text{ and}$$

$$\neg b \vee_e (\neg b' \vee_e c') \leq \neg b \vee_e (\neg b' \vee_e c'') = \neg b \vee_e c''$$

Thus $\neg b \vee_e c' \leq \neg b \vee_e c''$ and finally $\neg b \vee_e c \leq \neg b \vee_e c' \leq \neg b \vee_e c''$. Hence, $\preceq$ is transitive.

Claims (2) and (3) readily follow from claim (1).

Finally, claim (4) follows from the monotonicity of $\vee_e$ (see Remark 4.2) and from the definition of $\preceq$. □

Definition 4.3.4. On $B \otimes_{\vee_e} C$, we define:

$$[b, c] \otimes [b', c'] = [b \wedge b',\ c \cdot c']$$
$$[b, c] \sqcap [b', c'] = [b \wedge b',\ c \wedge c']$$
$$[b, c] \sqcup [b', c'] = [b \vee b',\ ((\neg b \vee \neg b') \vee_e (c \vee c')) \wedge ((b \vee \neg b') \vee_e c') \wedge ((b' \vee \neg b) \vee_e c)]$$

$[b, c] \Rightarrow [b', c'] = [b \to b', \neg b \vee_e (c \to c')]$.

Moreover, we denote by $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$ the algebra whose universe is $B \otimes C$ and whose operations are $\otimes$, $\sqcap$, $\sqcup$ and $\Rightarrow$ defined above.

Our next goal is to prove that $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$ is a product algebra. To begin with, we prove that the above operations are well defined.

Lemma 4.3.5. *The operations $\otimes, \sqcap, \sqcup, \Rightarrow$ just defined are compatible with $\sim$, and hence they are well defined on $B \otimes C$.*

Proof. Due to the commutativity of $\otimes$, $\sqcup$ and $\sqcap$, it suffices to prove the following: suppose

$(*)$ $\quad (b_1, c_1) \sim (b_2, c_2)$, that is, $b_1 = b_2$ and $\neg b_1 \vee_e c_1 = \neg b_2 \vee_e c_2$.

Then we need to prove:

(1) $(b \wedge b_1,\ c \cdot c_1) \sim (b \wedge b_2,\ c \cdot c_2)$

(2) $(b \wedge b_1,\ c \wedge c_1) \sim (b \wedge b_2,\ c \wedge c_2)$.

(3) $(b \vee b_1,\ ((\neg b \vee \neg b_1) \vee_e (c \vee c_1)) \wedge ((b \vee \neg b_1) \vee_e c_1) \wedge ((b_1 \vee \neg b) \vee_e c))) \sim$
$(b \vee b_2,\ ((\neg b \vee \neg b_2) \vee_e (c \vee c_2)) \wedge ((b \vee \neg b_2) \vee_e c_2) \wedge ((b_2 \vee \neg b) \vee_e c)))$.

(4) $(b \to b_1, \neg b \vee_e (c \to c_1)) \sim (b \to b_2,\ \neg b \vee_e (c \to c_2))$

(5) $(b_1 \to b,\ \neg b_1 \vee_e (c_1 \to c)) \sim (b_2 \to b, \neg b_2 \vee_e (c_2 \to c)$.

We prove (1). Since $b_1 = b_2$, condition (1) reduces to $(\neg b \vee \neg b_1) \vee_e (c \cdot c_1) = (\neg b \vee \neg b_1) \vee_e (c \cdot c_2)$.

Then using $(*)$ and (J3) we derive $(\neg b \vee \neg b_1) \vee_e c_1 = (\neg b \vee \neg b_1) \vee_e c_2$, that is, $(c_1, c_2) \in \theta_{b \wedge b_1}$.

Since $\theta_{b \wedge b_1}$ is a congruence, we get $(c \cdot c_1, c \cdot c_2) \in \theta_{b \wedge b_1}$, which is equivalent to (1).

The proof of (2) is similar to the proof of claim (1). Indeed, we have $b_1 = b_2$, thus condition (1) reduces to $(\neg b \vee \neg b_1) \vee_e (c \wedge c_1) = (\neg b \vee \neg b_1) \vee_e (c \wedge c_2)$.

Then again from $(*)$ and (J3) we get $(\neg b \vee \neg b_1) \vee_e c_1 = (\neg b \vee \neg b_1) \vee_e c_2$, that is, $(c_1, c_2) \in \theta_{b \wedge b_1}$, and since $\theta_{b \wedge b_1}$ is a congruence, we get $(c \wedge c_1, c \wedge c_2) \in \theta_{b \wedge b_1}$, which is equivalent to (2).

We prove (3). Again, since $b_1 = b_2$, it suffices to prove:

$(\diamond)$ $((\neg b \vee \neg b_1) \vee_e (c \vee c_1)) \wedge ((b \vee \neg b_1) \vee_e c_1) \wedge ((b_1 \vee \neg b) \vee_e c) =$
$((\neg b \vee \neg b_1) \vee_e (c \vee c_2)) \wedge ((b \vee \neg b_1) \vee_e c_2) \wedge ((b_1 \vee \neg b) \vee_e c)$.

Both members of $(\diamond)$ are the meet of three terms. Since the third conjuncts in the left side and in the right side are identical, it suffices to prove:

(3i) $(\neg b \vee \neg b_1) \vee_e (c \vee c_1) = (\neg b \vee \neg b_1) \vee_e (c \vee c_2)$;

(3ii) $(b \vee \neg b_1) \vee_e c_1 = (b \vee \neg b_1) \vee_e c_2$.

By $(*)$, $\neg b_1 \vee_e c_1 = \neg b_1 \vee_e c_2$, and using (J3) we get $(b \vee \neg b_1) \vee_e c_1 = (b \vee \neg b_1) \vee_e c_2$, that is, (3ii).

By the same argument, we get $(\neg b \vee \neg b_1) \vee_e c_1 = (\neg b \vee \neg b_1) \vee_e c_2$.

Hence, $(c_1, c_2) \in \theta_{b \wedge b_1}$, and since $\theta_{b \wedge b_1}$ is a congruence, $(c_1 \vee c, c_2 \vee c) \in \theta_{b \wedge b_1}$, which is equivalent to (3i).

We now prove (4). Since $b_1 = b_2$, it suffices to prove

$$((b \wedge \neg b_1) \vee \neg b) \vee_e (c \to c_1) = ((b \wedge \neg b_1) \vee \neg b) \vee_e (c \to c_2),$$

which is equivalent to

$$(\neg b \vee \neg b_1) \vee_e (c \to c_1) = (\neg b \vee \neg b_1) \vee_e (c \to c_2).$$

Once again, the claim follows from the fact that $\theta_{b \wedge b_1}$ is a congruence.

The proof of (5) is similar to the proof of claim (4). Hence, it suffices to prove

$$(\neg b_1 \vee \neg b) \vee_e (c_1 \to c) = (\neg b_1 \vee \neg b) \vee_e (c_2 \to c).$$

The claim similarly follows from the fact that $\theta_{b \wedge b_1}$ is a congruence. □

We are going to prove that for every product triplet $(\mathbf{B}, \mathbf{C}, \vee_e)$, the algebra $\mathbf{P} = \mathbf{B} \otimes_{\vee_e} \mathbf{C}$ is a product algebra.

Lemma 4.3.6. *$\sqcup$, $\sqcap$ are join and meet operations with respect to $\lhd$.*

Proof. We first prove the claim for $\sqcap$. To this purpose, note that $[b,c] \sqcap [b',c']$ is a lower bound for $[b,c]$ and $[b',c']$. Indeed, $b \wedge b' \leq b, b'$ and from $c \wedge c' \leq c, c'$ it follows:

$$\neg(b \wedge b') \vee_e (c \wedge c') \leq \neg(b \wedge b') \vee_e c, \quad \neg(b \wedge b') \vee_e (c \wedge c') \leq \neg(b \wedge b') \vee_e c'.$$

It remains to prove that $[b \wedge b', c \wedge c']$ is the greatest lower bound of $[b,c]$ and $[b',c']$. To this aim, suppose $[b_1, c_1] \lhd [b,c], [b',c']$. Then $b_1 \leq b$ and $b_1 \leq b'$, and hence, $b_1 \leq b \wedge b'$. Moreover $\neg b_1 \vee_e c_1 \leq \neg b_1 \vee_e c$, and $\neg b_1 \vee_e c_1 \leq \neg b_1 \vee_e c'$. Hence, $\neg b_1 \vee_e c_1 \leq \neg b_1 \vee_e (c \wedge c')$. Thus $[b_1, c_1] \lhd [b \wedge b', c \wedge c']$, and $[b \wedge b',\ c \wedge c']$ is the greatest lower bound of $[b,c]$ and $[b',c']$.

We now prove that $[b,c] \sqcup [b',c']$ is an upper bound of both $[b,c]$ and $[b',c']$. Clearly $b, b' \leq b \vee b'$, and it is left to prove

($\sqcup$1) $$\neg b \vee_e c \leq \neg b \vee_e (E_1 \wedge E_2 \wedge E_3),$$

($\sqcup$2) $$\neg b' \vee_e c' \leq \neg b' \vee_e (E_1 \wedge E_2 \wedge E_3), \text{ where}$$

$E_1 = (\neg b \vee \neg b') \vee_e (c \vee c')$, $E_2 = (\neg b \vee b') \vee_e c$ and $E_3 = (\neg b' \vee b) \vee_e c'$.

We prove ($\sqcup$1), the proof of ($\sqcup$2) being similar. Now by the monotonicity of $\vee_e$, $\neg b \vee_e c \leq E_1 \leq \neg b \vee_e E_1$, and $\neg b \vee_e c \leq E_2 \leq \neg b \vee_e E_2$. Moreover $\neg b \vee_e ((\neg b' \vee b) \vee_e c') = 1 \vee_e c' = 1 \geq \neg b \vee_e c$, and hence, $\neg b \vee_e c \leq E_3$, as desired.

We now prove that $[b,c] \sqcup [b',c']$ is the least upper bound of $[b,c]$ and $[b',c']$. Thus let $[b_1, c_1]$ be any upper bound of $[b,c]$ and $[b',c']$. Then $b, b' \leq b_1$, $\neg b \vee_e c \leq \neg b \vee_e c_1$ and $\neg b' \vee_e c' \leq \neg b' \vee_e c_1$. We need to verify that $[b,c] \sqcup [b',c'] \lhd [b_1, c_1]$.

Clearly, $b \vee b' \leq b_1$, and it is left to prove

($\sqcup$3) $$(\neg b \wedge \neg b') \vee_e (E_1 \wedge E_2 \wedge E_3) \leq (\neg b \wedge \neg b') \vee_e c_1.$$

Using (J1), (J2), (J3) and (J4) and the monotonicity of $\vee_e$, it suffices to prove

$(\sqcup 4)$ $$E_1 \wedge E_2 \wedge E_3 \leq (\neg b \vee c_1) \wedge (\neg b' \vee_e c_1).$$

Now $E_1 = (\neg b \vee \neg b') \vee_e (c \vee c') \leq (\neg b \vee \neg b') \vee_e c_1$, and $E_2 = (\neg b \vee b') \vee_e c = b' \vee_e (\neg b \vee_e c) \leq b' \vee_e (\neg b \vee_e c_1) = (b' \vee \neg b) \vee_e c_1$.

Hence, $E_1 \wedge E_2 \leq ((\neg b' \vee \neg b) \wedge (b' \vee \neg b)) \vee_e c_1 = \neg b \vee_e c_1$.

Similarly $E_1 \wedge E_3 \leq \neg b' \vee_e c_1$. Hence, $E_1 \wedge E_2 \wedge E_3 \leq (\neg b \vee_e c_1) \wedge (\neg b' \vee_e c_1)$, as desired. □

Lemma 4.3.7. $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$ *is a CIPRL.*

Proof. We have just seen that $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$ is a lattice, and it is easy to check that $[0, 1]$ and $[1, 1]$ are its minimum and maximum, respectively. Indeed, for all $[b, c] \in B \otimes_{\vee_e} C$:

$$[0, 1] \lhd [b, c] \lhd [1, 1]$$

since $0 \leq b \leq 1$, $1 \vee 1 = 1 \vee c$ and $\neg b \vee c \leq \neg b \vee 1 = 1$.

Commutativity and associativity of $\otimes$ trivially follows from commutativity and associativity of $\wedge$ and $\cdot$, and it is easy to check that $[1, 1]$ is the neutral element since $[b, c] \odot [1, 1] = [1, 1] \odot [b, c] = [b \wedge 1, c \cdot 1] = [b, c]$.

In order to verify the monotonicity of $\otimes$, suppose $[b, c] \lhd [b_1, c_1]$. We need to show that $[b, c] \otimes [b_2, c_2] = [b \wedge b_2,\ c \cdot c_2] \lhd [b_1 \wedge b_2,\ c_1 \cdot c_2] = [b_1, c_1] \otimes [b_2, c_2]$. From our hypothesis we have $b \leq b_1$ and $\neg b \vee_2 c \leq \neg b \vee_e c_1$. Thus $b \wedge b_2 \leq b_1 \wedge b_2$ is trivial. Moreover, it is $\neg b_2 \vee_e (\neg b \vee_e c) \leq \neg b_2 \vee_e (\neg b \vee_e c_1)$. Then $(\neg b \vee \neg b_2) \vee_e c \leq (\neg b \vee \neg b_2) \vee_e c_1$. But since $\theta_{b \wedge b_2}$ is a congruence on $\mathbf{C}$, thus $(\neg b \vee \neg b_2) \vee_e c \cdot c_2 \leq (\neg b \vee \neg b_2) \vee_e c_1 \cdot c_2$ which completes the proof. The commutativity of $\otimes$, implies that $\otimes$ is monotonic in both arguments.

To conclude the proof, it remains to show that the residuation property holds. We have to verify that for $[b, c]$, $[b_1, c_1]$, $[b_2, c_2]$ in $B \otimes_{\vee_e} C$, one has $[b, c] \lhd [b_1, c_1] \Rightarrow [b_2, c_2]$ iff $[b, c] \otimes [b_1, c_1] \lhd [b_2, c_2]$.

Now using the definitions of $\Rightarrow$ and of $\otimes$, along with property (J6) and with the identity $[b, c] = [b, \neg b \vee c]$, we obtain:

$$\begin{aligned} [b, c] \otimes ([b, c] \Rightarrow [b', c']) &= [b, c] \otimes [b \to b', \neg b \vee_e (c \to c')] \\ &= [b \wedge (\neg b \vee b'), c \cdot (\neg b \vee_e (c \to c'))] \end{aligned}$$

$$= [b \wedge b', (b \vee_e c) \wedge (\neg b \vee_e c \cdot (c \to c'))]$$
$$= [b \wedge b', (b \vee_e c) \wedge (\neg b \vee_e (c \wedge c')]$$
$$= [b \wedge b', ((\neg b \vee \neg b' \vee b)) \vee_e c) \wedge ((\neg b \vee \neg b') \vee_e (c \wedge c'))]$$
$$= [b \wedge b', c \wedge c'] = [b, c] \sqcap [b'c'].$$

This property, together with monotonicity of $\otimes$, shows that if $[b_1, c_1] \leq [b, c] \Rightarrow [b'c']$, then $[b, c] \otimes [b_1, c_1] \lhd [b', c']$.

In order to conclude the proof of residuation, suppose $[b, c] \otimes [b_1, c_1] = [b \wedge b_1, c \cdot c_1] \lhd [b', c']$, and let us prove that $[b_1, c_1] \lhd [b, c] \Rightarrow [b', c']$. We have:

$$(R) \qquad b \wedge b_1 \leq b' \text{ and } (\neg b \vee \neg b_1) \vee_e (c \cdot c_1) \leq (\neg b \vee \neg b_1) \vee_e c',$$

which immediately implies $b_1 \leq b \to b'$, and it is left to prove that

$$\neg b_1 \vee_e c_1 \leq \neg b_1 \vee_e (\neg b \vee_e (c \to c')) = (\neg b_1 \vee \neg b) \vee_e (c \to c').$$

Now $\theta_{b \wedge b_1}$ is a congruence of $\mathbf{C}$, and by (R) the inequality $c \cdot c_1 \leq c'$ holds in the quotient $\mathbf{C}/\theta_{b \wedge b_1}$. Hence, in such quotient we have $c_1 \leq c \to c'$. In other words, $(\neg b \vee \neg b_1) \vee_e c_1 \leq (\neg b \vee \neg b_1) \vee (c \to c')$, which immediately implies

$\neg b_1 \vee_e c_1 \leq (\neg b \vee \neg b_1) \vee_e (c \to c')$, that is, the claim. □

Theorem 4.3.8. $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$ *is a product algebra.*

Proof. We have seen in the proof of Lemma 4.3.7 that $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$ is a CIPRL which satisfies divisibility, that is, $[b, c] \otimes ([b, c] \Rightarrow [b', c']) = [b, c] \sqcap [b', c']$.

We prove that $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$ is a BL-algebra, that is, we prove that it satisfies prelinarity

$$(Prel) \qquad ([b, c] \Rightarrow [b', c']) \sqcup ([b', c'] \Rightarrow [b, c]) = [1, 1].$$

Now $([b, c] \Rightarrow [b', c']) \sqcup ([b', c'] \Rightarrow [b, c]) = [(b \to b') \vee (b' \to b), F_1 \wedge F_2 \wedge F_3]$,

where

$F_1 = ((b \wedge \neg b') \vee (b' \wedge \neg b)) \vee_e ((\neg b \vee_e (c \to c')) \vee (\neg b' \vee_e (c' \to c)))$

$F_2 = ((b \to b') \vee (b' \wedge \neg b)) \vee_e (\neg b' \vee_e (c \to c'))$

$F_3 = ((b' \to b) \vee (b \wedge \neg b')) \vee_e (\neg b \vee_e (c' \to c))$.

Since $(b \to b') \vee (b' \to b) = 1$, it is left to prove that $F_1 = F_2 = F_3 = 1$.

Now $F_1 \geq (c \to c') \vee (c' \to c) = 1$,
$F_2 \geq (b \to b') \vee \neg b' = 1$, and
$F_3 \geq (b' \to b) \vee \neg b = 1$.

In order to conclude the proof, it is left to prove the equation (Π) of product algebras. We give an indirect proof, starting from a lemma which will be used in the sequel.

Lemma 4.3.9. *(1) The maps $h : b \mapsto [b, 1]$ and $k : c \mapsto [1, c]$ are monomorphisms from $\mathbf{B}$ and from $\mathbf{C}$, respectively, into $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$.*

(2) Each element of $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$ can be written as $[b, 1] \otimes [1, c]$ for some $b \in B$, $c \in C$.

3 For all $b \in B$ and $c \in C$, $k(b \vee_e c) = h(b) \sqcup k(c)$. That is, if we identify every element of B with its isomorphic image via h and each element of C with its isomorphic image via k, then for all $b \in B$ and $c \in C$, $b \vee_e c$ represents the join of b and c in $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$.

Proof. (1) We first prove the claim for h. Injectivity of h is trivial, and it is readily seen that h preserves meet, top and bottom.

We verify that h preserves join:

$$\begin{aligned} h(b) \sqcup h(b') &= [b \vee b', ((\neg b \vee \neg b') \vee_e 1) \wedge ((\neg b \vee b') \vee_e 1) \wedge (b \vee \neg b) \vee_e 1] \\ &= [b \vee b', 1] = h(b \vee b'). \end{aligned}$$

We verify that h preserves $\neg$:

$$\neg h(b) = [b, 1] \Rightarrow [0, 1] = [\neg b, \neg b \vee_e (1 \to 1)] = [\neg b, 1] = h(\neg(b).$$

We verify that k is one-one. If $c \neq c'$, then $c = \neg 1 \vee_e c \neq \neg 1 \vee_e c' = c'$, and hence $k(c) = [1, c] \neq [1, c'] = k(c')$.

Trivially, $k(1) = [1, 1]$. Since in a cancellative hoop and in a BL-algebra the lattice operations are definable in terms of multiplication and of its residuum, in order to prove that k is a homomorphism, it suffices to show that it preserves $\cdot$ and $\to$.

Now $k(c \cdot c') = [1, c \cdot c'] = [1, c] \otimes [1, c'] = k(c) \otimes h(c')$.

Moreover

$k(c \to c') = [1, c \to c'] = [1, 0 \vee_e c \to c'] = [1, c] \Rightarrow [1, c'] = k(c) \Rightarrow k(c')$.

This concludes the proof of (1).

(2) We have $[b, c] = [b, 1] \otimes [1, c]$, as desired.

(3) We have

$$[b, 1] \sqcup [1, c] = [b \vee 1, ((\neg b \vee 0) \vee_e (c \vee 1)) \wedge ((\neg b \vee 1) \vee_e 1) \wedge ((b \vee 0) \vee_e c] = [1, b \vee_e c].$$

This concludes the proof of Lemma 4.3.9. □

We conclude the proof of Theorem 4.3.8. We already know that $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$ is a BL-algebra. Hence, it is a subdirect product of BL chains: $\mathbf{B} \otimes_{\vee_e} \mathbf{C} \subseteq_s \prod_{i \in I} \mathbf{P}_i$. Moreover, every element of $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$ is the product of an element of a boolean subalgebra and an element of a cancellative subhoop, of $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$. Hence, for all $i \in I$, the same decomposition property holds for $\mathbf{P}_i$. Since the only linearly ordered boolean algebra is $\mathbf{2}$, there is a cancellative subhoop, $\mathbf{C}_i$, of $\mathbf{P}_i$ such that every element x of P_i has the form $x = b \cdot c$, where $c \in C_i$ and either $b = 0$, and hence, $x = 0$, or $b = 1$, and hence, $x = c \in C_i$. In other words, $\mathbf{P}_i = \mathbf{2} \oplus \mathbf{C}_i$.

Hence, $\mathbf{P}_i$ is a product algebra, and $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$, being a subdirect product of product algebras, is in turn a product algebra. □

We are ready to define a functor Φ^{-1} from the category $\mathcal{T}$ of product triplets into the category $\mathcal{P}$ of product algebras.

Definition 4.3.10. For every object $\mathbf{T} = (\mathbf{B}, \mathbf{C}, \vee_e)$, of $\mathcal{T}$, we define $\Phi^{-1}(\mathbf{T}) = \mathbf{B} \otimes_{\vee} \mathbf{C}$. Moreover, for every good pair (h, k) from an object $\mathbf{T} = (\mathbf{B}, \mathbf{C}, \vee_e)$ into another object, $\mathbf{T}' = (\mathbf{B}', \mathbf{C}', \vee_e')$, of $\mathcal{T}$, we define, for all $[b, c] \in \Phi^{-1}(\mathbf{T})$, $\Phi^{-1}(h, k)([b, c]) = [h(b), 1] \otimes' [1, k(c)] = [h(b), k(c)]$, where $\otimes'$ denotes the monoid operation in $\Phi^{-1}(\mathbf{T}')$.

Theorem 4.3.11. *Φ^{-1} is a functor from $\mathcal{T}$ into $\mathcal{P}$.*

Proof. By Theorem 4.3.8, for every object $\mathbf{T}$ of $\mathcal{T}$, $\Phi^{-1}(\mathbf{T})$ is a product algebra. Moreover, if (h,k) is a good pair from $\mathbf{T}$ into $\mathbf{T}'$, then $\Phi^{-1}(h,k)$ is clearly a map from $\Phi^{-1}(\mathbf{T})$ into $\Phi^{-1}\mathbf{T}'$. In order to prove that it is a homomorphism, since in a product algebra the lattice operations are definable from the monoid operation and its residual, it suffices to prove that $\Phi^{-1}(h,k)$ preserves $\otimes$ and $\Rightarrow$. Now using the superscript $'$ to denote operations in $\mathbf{B}'$ or in $\mathbf{C}'$ or in $\mathbf{B}' \otimes_{\vee'_e} \mathbf{C}'$,

$$\begin{aligned}\Phi^{-1}(h,k)([b_1,c_1] \otimes [b_2,c_2]) &= \Phi^{-1}(h,k)([b_1 \wedge b_2, c_1 \cdot c_2]) \\ &= [h(b_1) \wedge' h(b_2), k(c_1) \cdot' k(c_2)] \\ &= [h(b_1, k(c_1)] \otimes' [h(b_2), k(c_2)] \\ &= \Phi^{-1}(h,k)([b_1,c_1]) \otimes' \Phi^{-1}(h,k)([b_2,c_2]).\end{aligned}$$

Moreover,

$$\begin{aligned}\Phi^{-1}(h,k)([b_1,c_1] \Rightarrow [b_2,c_2]) &= \Phi^{-1}(h,k)([b_1 \to b_2, \neg b_1 \vee_e (c_1 \to c_2]) \\ &= [h(b_1) \to' h(b_2), \neg' h(b_1) \vee'_e (k(c_1) \to' k(c_2)].\end{aligned}$$

On the other hand,

$$\Phi^{-1}(h,k)([b_1,c_1]) \Rightarrow' \Phi^{-1}(h,k)([b_2,c_2]) = [h(b_1), k(c_1)] \Rightarrow' [h(b_2), k(c_2)]$$
$$= [h(b_1) \to h(b_2), \neg' h(b_1) \vee'_e (k(c_1) \to' k(c_2)],$$

and the claim is proved.

Clearly Φ^{-1} preserves identity morphisms, and it is left to prove that it preserves compositions. Now a straightforward computation shows that if (h,k) is a good pair from $\mathbf{T} = (\mathbf{B}, \mathbf{C}, \vee_e)$ into $\mathbf{T}' = (\mathbf{B}', \mathbf{C}', \vee'_e)$ and (h',k') is a good pair from $\mathbf{T}'$ into $\mathbf{T}'' = (\mathbf{B}'', \mathbf{C}'', \vee''_e)$, then for all $[b,c] \in B \otimes C$, we have

$$\Phi^{-1}((h',k') \circ (h,k))([b,c]) = (\Phi^{-1}(h',k')) \circ (\Phi^{-1}(h,k))([b,c]) = [h'(h(b)), k'(k(c))].$$

□

4.4 Concluding the proof of the categorical equivalence

In this section we prove the main result of this thesis, that is, the categories $\mathcal{P}$ and $\mathcal{T}$ are equivalent. We start from the following lemma.

Lemma 4.4.1. *(a) Let* $\mathbf{T} = (\mathbf{B}, \mathbf{C}, \vee_e)$ *be an object of* $\mathcal{T}$*. Then the sets* $B_0 = \{[b, 1] : b \in B\}$ *and* $C_0 = \{[1, c] : c \in C\}$ *are the domains of the maximum boolean subalgebra and of the maximum cancellative subhoop of* $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$*, respectively.*

Proof. Given $[b, c] \in B \otimes_{\vee_e} C$, we have $\neg[b, c] = [b, c] \Rightarrow [0, 1] = [\neg b, 1]$, and hence, $\neg\neg[b, c] = [b, 1]$. It follows that $\neg\neg[b, c] = [b, c]$ iff $[b, c] = [b, 1]$. This means that B_0 is the domain of the greatest boolean subalgebra of $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$. Moreover, $\neg\neg[b, c] = [1, 1]$ iff $b = 1$. Hence, C_0 is the domain of the greatest cancellative subhoop of $\mathbf{B} \otimes_{\vee_e} \mathbf{C}$. □

We are ready to prove our main result.

Theorem 4.4.2. *The pair* (Φ, Φ^{-1}) *provides an equivalence between the categories* $\mathcal{P}$ *and* $\mathcal{T}$*.*

Proof. By Theorem 1, page 93 of [ML], it will suffice to prove the following:

(a) The functor Φ^{-1} is faithful and full.

(b) For every object $\mathbf{P}$ in $\mathcal{P}$, $\Phi^{-1}(\Phi(\mathbf{P}))$ is isomorphic to $\mathbf{P}$.

As regards to (a), we prove first that Φ^{-1} is faithful. Let (h, k), (h', k') be two good pairs from $\mathbf{T} = (\mathbf{B}, \mathbf{C}, \vee_e)$ into $\mathbf{T}' = (\mathbf{B}', \mathbf{C}', \vee_e')$, and suppose $\Phi^{-1}(h, k) = \Phi^{-1}(h', k')$. Then for every $[b, c] \in B \otimes C$, $[h(b), k(c)] = [h'(b), k'(c)]$. This implies that $h(b) = h'(b)$ for all $b \in B$. Moreover fixing $b = 1$, we obtain that $(1, k(c)) \sim (1, k'(c))$ for all $c \in C$, and hence, $k(c) = k'(c)$ for all $c \in C$. Hence, $(h, k) = (h', k')$, and Φ^{-1} is faithful.

In order to prove that Φ^{-1} is full, let f be a morphism $\Phi^{-1}(\mathbf{T})$ into $\Phi^{-1}(\mathbf{T}')$. Then for $b \in B$ and for $c \in C$, $f([b, 1])$ is an element of a boolean subalgebra of $\Phi^{-1}(\mathbf{T}')$, and hence, by Lemma 4.4.1, has the form $[b', 1]$ for a (necessarily unique) $b' \in B'$. We set $h(b) = b'$. Likewise, $f([1, c]) = [1, c']$ for some (necessarily unique, because θ_1 is the identity congruence) $c' \in C'$. We set $k([1, c]) = c'$.

We prove that (h, k) is a good pair. That h and k are homomorphisms on $\mathbf{B}$ and on $\mathbf{C}$ follows from the definitions of h and k. For instance, k preserves $\rightarrow$, because

$$\begin{aligned}[1, k(c \rightarrow c')] &= f([1, c] \Rightarrow [1, c']) = f([1, c]) \Rightarrow' f([1, c']) \\ &= [1, k(c)] \Rightarrow [1, k(c')] = [1, k(c) \rightarrow k(c')].\end{aligned}$$

We verify that for $b \in B$ and for $c \in C$, one has $k(b \vee_e c) = h(b) \vee_e k(c)$.

Now by Lemma 4.3.9 (3),

$[1, k(b \vee_e c)] = f([b, 1] \sqcup' [1, c]) = f([b, 1]) \sqcup' f([1, c]) = [h(b), 1] \sqcup' [1, k(c)] = [1, h(b) \vee_e k(c)]$,

which implies $k(b \vee_e c) = h(b) \vee_e' k(c)$.

Hence, (h, k) is a good pair. Moreover, $f([b, c)]) = f([b, 1] \otimes [1, c]) = [h(b), 1] \otimes [1, k(c)] = [h(b), k(c)]$, and $f = \Phi^{-1}(h, k)$. Hence, Φ^{-1} is full.

We now prove claim (b). Let $\mathbf{P}$ be any product algebra. Then every element $p \in P$ can be written as $p = b \cdot c$, with $b \in B(P)$ and $c \in C(P)$. We set $g(p) = [b, c]$. Since $b \cdot c = b' \cdot c'$ iff $b = b'$ and $\neg b \vee_e c = \neg b \vee_e c'$, iff $[b, c] = [b', c']$, g is a well defined and one-one map from P into $\Phi^{-1}(\Phi(P))$. That g is onto is clear. Hence, g is a bijection, and since residuals may be expressed in terms of order and of the monoid operation, it is left to prove:

(b1) g is an order isomorphism, and

(b2) g is a monoid isomorphism.

For (b1), we have, for all $b, b' \in B(P)$ and for all $c, c' \in C(P)$, $b \cdot c \leq b' \cdot c'$ iff $b \leq b'$ and $\neg b \vee_e c \leq \neg b \vee c'$ iff $g(b \cdot c) = [b, c] \lhd [b', c'] = g(b' \cdot c')$.

For (b2), we have $g(b \cdot c \cdot b' \cdot c') = [b \wedge b', c \cdot c'] = [b, c] \otimes [b', c'] = g(b \cdot c) \otimes g(b' \cdot c')$.

This concludes the proof of Theorem 4.4.2 □

4.5 Additional results

4.5.1 Special cases of the equivalence theorem

Theorem 4.4.2 specializes to an equivalence between the full subcategory $\mathcal{P}_2$ of all product algebras of the form $\mathbf{2} \oplus \mathbf{C}$, $\mathbf{C}$ a cancellative hoop. Indeed, if $\mathbf{P} = \mathbf{2} \oplus \mathbf{C}$ is an object of $\mathcal{P}_2$, then $\mathbf{B}(\mathbf{P}) = \mathbf{2}$, $\mathbf{C}(\mathbf{P}) = \mathbf{C}$, and with reference to Notation 2.2.8,

$$(b \vee_e c)_i = \begin{cases} 1_i & \text{if } b_i = 1_i \\ c_i & \text{otherwise} \end{cases}$$

Hence, $\Phi(\mathbf{P})$ only depends on $\mathbf{C}(\mathbf{P})$.

Likewise, if f is a homomorphism from $\mathbf{P} = \mathbf{2} \oplus \mathbf{C}$ to $\mathbf{P}' = \mathbf{2} \oplus \mathbf{C}'$, the restriction h of f to $\mathbf{2}$ is the map sending 1 into 1 and 0 into 0, and hence, $\Phi(f)$ really depends on its restriction k to $\mathbf{C}$. Hence, we may replace product triplets by cancellative hoops and the good pairs by the homomorphisms on cancellative hoops, thus obtaining that the category $\mathcal{P}_2$ is equivalent to the category of cancellative hoops. Since this category is in turn equivalent to the category of lattice ordered abelian groups, we obtain Cignoli and Torrens equivalence as a special case of Theorem 4.4.2.

Note also that boolean algebras may be characterized as those product algebras $\mathbf{P}$ such that $\mathbf{C}(\mathbf{P})$ is a trivial hoop, and $\vee_e$ is defined by $b \vee_e 1 = 1$ for all $b \in B(P)$. In this case, $\Phi(\mathbf{P})$ only depends on $\mathbf{B}(\mathbf{P})$ and a morphism on $\mathbf{P}$ only depends on its restriction to $\mathbf{B}(\mathbf{P})$. But since $\mathbf{B}(\mathbf{P}) = \mathbf{P}$, the result is just a triviality.

4.5.2 Filters

We have noticed in Section 2 that filters are in bijection with congruences, and hence, with epimorphisms. Since a homomorphism f on a product algebra $\mathbf{P}$ is uniquely determined by its restrictions h and k to $B(P)$ and to $C(P)$, we may expect that a filter F of $\mathbf{P}$ is uniquely determined by its intersection F_B with $B(P)$ and by its intersection F_C with $C(P)$. Moreover, since h and k

are not arbitrary homomorphisms on $\mathbf{B}(\mathbf{P})$ and on $\mathbf{C}(\mathbf{P})$, respectively, but they must satisfy the equation $k(b \vee_e c) = h(b) \vee_e k(c)$, we may expect that F_B and F_C are related by a suitable property. This property is described in the next definition.

Definition 4.5.1. Let $\mathbf{T} = (\mathbf{B}, \mathbf{C}, \vee_e)$ be a good triplet. A *good filter pair* of $\mathbf{T}$ is a pair (F_1, F_2) such that F_1 is a filter of $\mathbf{B}$, F_2 is a filter of $\mathbf{C}$, and for all $b \in F_1$ and for all $c \in C$, if $\neg b \vee_e c \in F_2$, then $c \in F_2$.

Theorem 4.5.2. *(1) If* $\mathbf{P}$ *is a product algebra and* F *is a filter of* $\mathbf{P}$*, then* (F_B, F_C) *is a good filter pair, and* $F = \{b \cdot c : b \in F_B, c \in F_C\}$.

(2) For every good filter pair (F_1, F_2) *of* $(\mathbf{B}(\mathbf{P}), \mathbf{C}(\mathbf{P}), \vee_e)$*, the set* $F_1 \cdot F_2 = \{b \cdot c : b \in F_1,\ c \in F_2\}$ *is a filter of* $\mathbf{P}$.

Proof. (1) That F_B is a filter of $\mathbf{B}(\mathbf{P})$ and F_C is a filter of $\mathbf{C}(\mathbf{P})$ is clear. Moroever, if $b \in F_1$, then $b \in F$, and $(\neg b, 0) \in \theta_F$. Moreover, if $\neg b \vee_e c \in F_C$, then $\neg b \vee_e c \in F$, and $(\neg b \vee_e c, 1) \in \theta_F$. Since $(\neg b, 0) \in \theta_F$, we conclude that $(c, 1) \in \theta_F$, and hence, $c \in F \cap C(P) = F_C$.

Now if $b \in F_B$ and $c \in F_C$, then $b, c \in F$, and hence $b \cdot c \in F$, because filters are closed under $\cdot$. Conversely, if $x \in F$, then x is the product of an element $b \in B(P)$ and an element $c \in C(P)$. Moreover, $b \cdot c \leq b \cdot 1 = b$, and $b \cdot c \leq 1 \cdot c = c$, and hence, $b, c \in F$. Moreover $b \in F \cap B(P) = F_P$, and $c \in F \cap C(P) = F_C$. Hence, $x = b \cdot c$ with $b \in B(P)$ and $c \in C(P)$.

(2) We start from the following lemma:

Lemma 4.5.3. *If* $b \cdot c = b' \cdot c'$*, if* $b \in F_1$ *and* $C \in F_2$*, then* $b' \in F_1$ *and* $c' \in F_2$.

Proof. If $b \cdot c = b' \cdot c'$, then $b = b'$ and $\neg b \vee_e c = \neg b \vee_e c'$. Hence, $b' = b \in F_1$. Moreover, since $\neg b \vee_e c \geq c \in F_2$, we get $\neg b \vee c' = \neg b \vee c \in F_2$. From this, since $b \in F_1$, we obtain $c' \in F_2$. □

We conclude the proof of Theorem 4.5.2. Since $1 = 1 \cdot 1$ and $1 \in F_1 \cap F_2$, we have $1 \in F_1 \cdot F_2$.

Now suppose x, $x \to y \in F_1 \cdot F_2$. Then there are $b_1, b_2 \in F_1$, $c_1, c_2 \in F_2$, $b' \in B(P)$ and $c' \in C(P)$ such that $x = b_1 \cdot c_1$, $x \to y = b_2 \cdot c_2$, and $y = b' \cdot c'$. Now $b_2 \cdot c_2 = x \to y = (b_1 \to b') \cdot (\neg b_1 \vee_e (c_1 \to c'))$, with $b_1 \to b' \in B(P)$ and $\neg b_1 \vee_e (c_1 \to c') \in C(P)$. By Lemma 4.5.3, $b_1 \to b' \in F_1$, and $\neg b_1 \vee_e (c_1 \to c') \in F_2$. Finally, since $b_1 \in F_1$, we get $b' \in F_1$ and $c_1 \to c' \in F_2$, which, together with $c_1 \in F_2$, yields $c' \in F_2$ and $y = b' \cdot c' \in F_1 \cdot F_2$.

This settles the claim. □

Conclusions and open problems

After our way through the foundaments of fuzzy logic and the study of some important categorical equivalences in algebraic logic, at last we came to the final achievement of this thesis, i.e. the categorical equivalence for product algebras we have developed in the last chapter. This is an original result, and to the best of our knowledge it is the first equivalence involving the whole variety of product algebras.

We have showed how any product algebra $\mathbf{P}$ can be expressed by means of a precise boolean algebra $\mathbf{B}(\mathbf{P})$ and a precise cancellative hoop $\mathbf{C}(\mathbf{P})$, and that in order to have a categorical equivalence we needed to consider the external join between bolean and cancellative elements.

It is possible to think about other ways to obtain different categorical equivalences in this area of research.

For instance, let us consider the following construction. Given a product algebra $\mathbf{P}$, and given an element $a \in P$, and let $\mathbf{P[a]}$ be the interval $[a, 1]$ with the following operations:

$$x \odot y = (x \cdot y) \vee a,$$

$$x \Rightarrow y = x \to y,$$

$$0 = a,$$

$$1 = 1.$$

If $a \in C(P)$, $\mathbf{P[a]}$ is a MV algebra, and with this construction we have a categorical equivalence if a is a strong order unit.

Hence we may ask what do we obtain if we consider a boolean element instead of a cancellative element, or if we consider an element which is neither boolean nor cancellative.

We may wonder what happens if we start from a cancellative hoop and consider the construction $\mathbf{C}[\mathbf{a}]$ as before. Furthermore, we may wonder what is the connection between $\mathbf{C}$ and the family of MV algebras $\{C[a] : a < 1, a \in C\}$, and see whether and how they are related one to the other, or if it is possible to obtain a categorical equivalence with a category of MV algebras starting from this construction. The same problem can clearly be formulated in terms of l-groups, and thus for each l-group $\mathbf{G}$ one can study the family of MV algebras $\{\Gamma(G, a) : a \in G\}$.

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