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BOOLEAN ALGEBRAS WITH FURTHER OPERATIONS

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Summary. Both Model theory and Algebraic Logic lead to Boolean Algebras with further operations. In this paper we will establish some basic facts for such algebras.

Riassunto. Sia la teoria dei modelli che la logica algebrica conducono ad algebre di Boole con ulteriori operazioni. Si danno alcuni fatti basilari riguardanti tali algebre.

1. Introduction.

In a previous paper [7] we have studied Boolean Algebras with Hemimorphisms and we have observed that some aspects of this theory are generalizable. This is a first generalization.

We think that further generalizations are possible: one can, for example, consider, instead of Boolean algebras, filtrable varieties, or distributive lattices, or primal algebras, and so on. It is also possible to consider infinitary operations; therefore we think that this intermediate stage of generalization is useful for algebraic logic and for model theory.

In section 2 we introduce the algebras (bafos) of the title and develop the easy theory of ideals, section 3 is a technical one, in section 4 we study *limited* and *near limited* bafos, in section 5 we study ideal classes of bafos and in section 6 we present some problems.

(added Jan. 24, 1990: Some concepts and results are already present in: *Prime Ideals in Universal Algebra*, Acta Univ. Carol. Mat. et Phis. **25** (1984) 75–87 by Aldo Ursini. Some new results are in my: *A note on bafos*, Dipt. Mat. Un. Siena, n. 204, 1989).

2. Generality, ideal theory.

2.1 Definition. A *Boolean algebra with further operations* (*bafo*) is a system $\mathcal{A} = \langle A, +, \cdot, \nu, 0, 1, f_i \rangle_{i \in I}$ where:

2.1.1 $\langle A, +, \cdot, \nu, 0, 1 \rangle$ is a boolean algebra ($+$, $\cdot$ are the lattice operations and ν the complementation);

2.1.2 The f_i are finitary operations on A.

Of course, fixing I and, for every i, the arity ρ_i of f_i we have a variety of bafos, the bafos of the "type" $\langle I, \rho \rangle$.

It follows from well known theories (see A.G. Kurosh [3] chap. III, §2, n. 4) that:

2.2 Lemma. *The equivalence classes of 0 in the congruences of a bafo are the (Boolean) ideals J such that for every $i \in I$:*

2.2.1 *If $x_1 \dot{+} y_1, x_2 \dot{+} y_2, \ldots, x_n \dot{+} y_n \in J$ then:*

$$f(x_1, x_2, \ldots, x_n) \dot{+} f(y_1, y_2, \ldots, y_n) \in J \qquad (x_i, y_i \in A)$$

where n is the arity of f and $x \dot{+} y$ stands for $x \cdot \nu y + \nu x \cdot y$.

Of course, the bafos have a *good theory of ideals* (see Aldo Ursini [8]) with the 0-ideals (or, simply, ideals) as in Lemma 2.2.

Let $\mathcal{A}$ be as in 2.1 and for every f_i (of arity, say, n) let us consider the $(2n+1)$-ary operation g so defined:

2.3 $g(p, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) =$
$$f_i(x_1, x_2, \ldots, x_n) \dot{+} f_i(x_1 + py_1, x_2 + py_2, \ldots, x_n + py_n).$$

Let us call these functions "basic idealizing functions" (bifs). It is easy to see that:

2.4 Lemma. *The ideals of a bafo are the Boolean ideals closed (with respect to the first argument) for all the bifs, i.e.: If g is a bif, $p \in J$; $x_i, y_i \in A$ then:*

$$g(p, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \in J.$$

Now applying Zorn's lemma:

2.5 Theorem. *In a bafo every proper ideal is extendable to a maximal proper ideal.*

3. Normal operations and the radical.

Let $\langle I, \rho \rangle$ be a type and V the variety of all the bafos of this type.

3.1 Definition. A *normal polynomial of type* $\langle I, \rho \rangle$ is a polynomial f for which:

3.1.1 $f(0, x_2, x_3, \ldots, x_n) = 0 \qquad (A \in V, x_i \in A)$

Of course every ideal of the bafo is closed for normal polynomials, that is:
3.2 if $p \in J$ and $x_2, x_3, \ldots, x_n \in A$ then $f(p, x_2, x_3, \ldots, x_n) \in J \qquad (A \in V,$ J ideal of $A)$

In a sense, only the normal polynomials contribute to the formation of an ideal. More precisely we have:

3.3 Lemma. *Let $\mathcal{A}$ be a bafo, J an ideal of $\mathcal{A}$ and $p \in A$. Then the ideal $\overline{J \cup \{p\}}$ generated by $J \cup \{p\}$ is:*

3.3.1 $K = \{q \in A :$ *there exists a $j \in J$, a normal polynomial f of the type and $x_2, x_3, \ldots, x_n \in A$ for which* $q = f(p + j, x_2, x_3, \ldots, x_n)\}$

Proof. First, let r be an element of K and $s \leq r$; for suitable $j \in J$, $a_2, a_3, \ldots, a_n \in A$ and f normal polynomial of the type:

$$r = f(p + j, a_2, a_3, \ldots, a_n).$$

But the g defined by:

$$g(x_1, x_2, \ldots, x_n, y) = y f(x_1, x_2, \ldots, x_n)$$

is also a normal polynomial of the type and

$$s = g(p + j, a_2, a_3, \ldots, a_n, s).$$

Now let r, s be elements of K and $u = r + s$. For suitable normal polynomials f, g and for suitable $j, l \in J$, $a_2, a_3, \ldots, a_n, b_2, b_3, \ldots, b_m \in A$ we have:

$$r = f(p + j, a_2, a_3, \ldots, a_n), \quad s = g(p + l, b_2, b_3, \ldots, b_m).$$

Now let us consider the polynomial h defined by

$$h(t, y, z, x_2, x_3, \ldots, x_n, y_2, y_3, \ldots, y_m) =$$
$$f(ty, x_2, x_3, \ldots, x_n) + g(tz, y_2, y_3, \ldots, y_m).$$

Of course, h is a normal polynomial and

$$u = r + s = h(p + j + l, p + j, p + l, a_2, a_3, \ldots, a_n, b_2, b_3, \ldots, b_m).$$

Now let j be an element of J and let us consider the polynomial f defined by:

$$f(w, x) = wx.$$

Of course, f is normal, but $j = f(p + j, j)$ so $j \in K$. The same f gives $p = f(p + j, p)$, so $p \in K$.

Finally, let $r = g(a_1, a_2, \ldots, a_n) \dot{+} g(b_1, b_2, \ldots, b_n)$ with $c_i = a_i \dot{+} b_i \in K$;

We have, for suitable elements $j_i \in K$, $d_s^{(r)} \in A$, f_i normal polynomials:

$$c_i = f_i(p + j_i, d_2^{(i)}, d_3^{(i)}, \ldots, d_m^{(i)})$$

(we can suppose that all f_i have the same arity m). Let us consider the polynomial h of arity $n(m+1)+1$ defined by:

$$h(w, z_1, z_2, \dots, z_n, x_1, x_2, \dots, x_n, \dots, y_2^{(i)}, y_3^{(i)}, \dots, y_m^{(i)}, \dots) =$$
$$g(x_1, x_2, \dots, x_n) \dot{+} g(\dots, x_i + f_i(wz_i, y_2^{(i)}, y_3^{(i)}, \dots, y_m^{(i)}), \dots).$$

This is a normal polynomial and:

$$r = h(p + \sum_i j_i, p + j_1, p + j_2, \dots, p + j_n, a_1, a_2, \dots, a_n, \dots, d_2^{(i)}, d_3^{(i)}, \dots, d_m^{(i)}, \dots).$$

So K is an ideal containing $J \cup \{p\}$. But by 3.2, $K \supseteq \overline{J \cup \{p\}}$, so

$$K = \overline{J \cup \{p\}}.$$

Of course, we can read in the above lemma "normal polynomial of arity no less than k" instead of "normal polynomials", for every fixed natural number k.

Let $\mathcal{A}$ be a bafo, f a normal polynomial, and let n be the arity of f.

For every choice of $a = (a_2, a_3, \dots, a_n) \in A^{n-1}$ we can consider a unary operation h on A (not necessarily an operation of $\mathcal{A}$) defined by:

$$hp = f(p, a_2, a_3, \dots, a_n) \qquad (p \in A).$$

For every bafo $\mathcal{A}$, let $H_{\mathcal{A}}$ be the set of all such operations (varying f, n and a). We call these "normal operators" of $\mathcal{A}$.

Of course, we can also read Lemma 3.3 as follows:

$\overline{J \cup \{p\}} = \{q$: there exists $j \in J$ and a normal operator h for which $q = h(p+j)\}$.

Now we can find the radical (the meet of maximal proper ideals, or the whole of A if there are not maximal proper ideals).

3.4 Theorem. *The radical R of $\mathcal{A}$ is:*

$\{p$: for every j, if there exists a normal operator h for which $h(p+j) = 1$, then also exists a normal operator k for which $kj = 1\}$.

Mutatis mutandis, the elements of R have Frattini's property. 3.4 is substantially a particular case of the following general theorem, which is an easy generalization of well known facts regarding the Frattini subgroup of a group.

If K is a Moore algebraic operator on a set M (that is: $K : \mathcal{P}(M) \to \mathcal{P}(M)$ and, for $X, Y \subset M$ and $x \in M$, $X \subset KX$, $KK = K$, $K(X \cup Y) \supset KX$, if $x \in KX$ then a finite $Z \subset X$ exists for which $x \in KZ$) then the "non-generators" (the x for which, for all $X \subset M$, if $KX = M$ then $K(X \setminus \{x\}) = M$) form a closed set, and precisely the meet of M with all maximal proper closed sets.

Proof (of 3.4). Let p be in R, and suppose that, for suitable h, j, $h(p+j) = 1$. Then the ideal generated from j is all A (otherwise, it generates a proper ideal and it is possible to find a maximal ideal J with $p \notin J$) and so there exists a normal operator k, with $kj = 1$.

If $p \notin R$ then there is a maximal ideal J with $p \notin J$ and so $\overline{J \cup \{p\}} = A$. Then, for suitable $h, j \in J$ we have $h(p+j) = 1$ but, J being proper, there is no operator k with $kj = 1$.

If $p \in A$, an element $j \in A$ is a *pseudocomplement* of p if a normal operator k exists such that $k(p+j) = 1$. Then the element of the radical are the p such that for every pseudocomplement j of p there is a normal operator h for which $hj = 1$.

If, for example, $\mathcal{A}$ is a topological algebra $\langle A, +, \cdot, \nu, 0, 1, k\rangle$ then the minimum pseudocomplement of p is νkp and we have, as in [6, Theor. 9] and in [7, n. 3]:

$$R = \{p : k\nu kp = 1\}.$$

4. The limited bafo.

4.1 Definition. A *limited bafo (lbafo)* is a bafo for which normal operators have a maximum, k.

4.2 Theorem. *If in a lbafo, $\mathcal{A}$, k is the maximum normal operator, then:*

4.2.1 $k0 = 0$

4.2.2 $p \leq kp$ $\qquad (p \in A)$

4.2.3 $kkp = kp$ $\qquad (p \in A)$

4.2.4 *if* $p \leq q$ *then* $kp \leq kq$ $\qquad (p, q \in A)$

Proof. 4.2.1 is obvious. The h defined by:

$$hp = p + kp \qquad (p \in A)$$

is also a normal operator and $kp \leq hp \leq kp$ (because k is the maximum) and so $kp = hp \geq p$. kk is also a normal operator: but by 4.2.2 $k \leq kk$ and being k the maximum $kk \leq k$, so $k = kk$.

Now suppose $p < q$. The operator h defined by: $hx = k(px) + kx$ is also normal and so $h \leq k$. Now $kq \geq hq = k(pq) + kq = kp + kq \geq kp$.

In [7] we have seen that the semisimple lbahs form a variety; this is no longer true for lbafos:

4.3 Example. Let S be $\{a,b,c,d\}$ (four different elements) and A the Boolean algebra of the subsets of S with the unary Moore operator k which has as closed sets $\emptyset$, $\{a,b\}$, $\{c\}$, S. This algebra has the maximal ideals $J = \{\emptyset, \{c\}\}$ and $\{\emptyset, \{a\}, \{b\}, \{a,b\}\}$. The algebra A/J is isomorphic with the algebra of the subsets of $\{a,b,d\}$ with the operator h in which the closed sets are $\emptyset$, $\{a,b\}$, $\{a,b,d\}$. A is semisimple but A/J is not.

It is useful to introduce also the following:

4.4 Definition. A *near limited bafo* (*nlbafo*) is a bafo in which there exists an $n \geq 1$ and a normal polynomial f of arity $n+1$ such that for every normal polynomial g, of arity, say, $m+1 > 1$ and for every choice af some $a_0, a_1, \dots, a_m$ there exist $b_1, b_2, \dots, b_n$ for which:

4.4.1 $g(a_0, a_1, \dots, a_m) < f(a_0, b_1, b_2, \dots, b_n)$

Of course an lbafo is a nlbafo but:

4.5 Example. Let A be an infinite Boolean algebra and $0 = a_0 < a_1 < \dots < a_n < \dots$ elements of A and $B = \{a_i : i \in \omega\}$. Let us define a binary function f on A putting:

4.5.1 $f(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x \notin B \text{ or } y \notin B \\ y & \text{if } x \neq 0,\ x \in B,\ y \in B \end{cases}$

In the bafo $\langle A, f \rangle$ the ideal generated by a_1 is, of course, $\{x \in A :$ there exists an $i \in \omega$ with $x < a_i\}$. Every normal operator maps a_1 in some a_n and there is not a maximum, but f satisfies the condition for having a nlbafo.

5. Ideality.

Let X be a class of similar algebras and $A \subset \prod_i A_i$ a subalgebra of the product of certain $A_i \in X$. Let L_i be the congruence lattice of A_i, $L = \prod_i L_i$ and for $x, y \in A$ put:

5.1 $$(\Delta(x,y))_i = \overline{\{(x_i, y_i)\}},$$

where $\overline{M}$ stands for "the congruence of A generated by M".

If J is an ideal of L then the R defined by:

5.2 $$xRy \text{ iff } \Delta(x,y) \in J \qquad (x,y \in A)$$

is a congruence of A.

Let us remember that X is an *ideal* class iff every congruence of every subdirect product of algebras of X is linked with an ideal by 5.2 (see R. Franci [1] or R. Magari [5]).

An algebra A is *ideal* iff $\{A\}$ is ideal.

The variety $\mathbf{V}X$ generated by an ideal class X is called idealizable.

If X has only simple algebras then the ideals of L are linked with the filters on I and we have a filtral class and the generated variety is filtrable.

Let us remember also Lemma 4 of [4].

5.3 Proposition. *Let $(A_i)_{i \in I}$ be a family of similar algebras and let A be a subalgebra of $\prod_i A_i$. A necessary and sufficient condition for A to have only ideal congruences (that is congruences "linked" with ideals as in 5.2) is that for every $n \in \omega$, $n \neq 0$ and for every choice of pairs $p, p^{(0)}, p^{(1)}, \dots, p^{(n-1)}$ of elements of A if:*

$$\Delta(p) < \bigvee_{i \in n} (p^{(i)})$$

then:

$$p \in \overline{\{p^{(i)} : i \in n\}}$$

(where of course M stands for "congruence generated by M in A").

Now let us suppose that the A_i belong to a variety of algebras with a good theory of ideals (as the bafos) and suppose that every finitely generated ideal is principal (this is the case for bafos).

We have:

5.4 Corollary. *Under the above hypotheses, an equivalent condition for A to have only ideal congruences is: for every choice of $x, y \in A$, if y_i belongs to the ideal generated by x_i ($i \in I$), then: y belongs to the ideal generated by x.*

Now we can generalize a theorem of [7] on bahs:

5.6 Lemma. *Every variety of nlbafos is ideal.*

Proof. Let X be a variety of nlbafos and let F be the free algebra of X on $\aleph_0$ generators $(a_i)_{i \in \omega}$. F itself is a nlbafo and let f, n be as in Definition 4.4, so for every normal polynomial g_i of arity, say, $ni+1 > 1$ there exist $b_1, b_2, \dots, b_n$ for which:

$$g_i(a_0, a_1, \dots, a_n) < f(a_0, b_1, b_2, \dots, b_n)$$

Now let I be a set, for $i \in I$ let $A_i \in X$, let A be a subalgebra of $\prod_i A_i$ and $x, y \in A$ with:

$$y_i \in \overline{\{xi\}} \qquad (i \in I)$$

($\overline{}$ = ideal generated by ...)

For every $i \in I$ there exists a normal polynomial g_i of arity, say, $n_i + 1 > 1$ and $a_i^{(k)} \in A_i$ with:

$$y_i = g_i(x_i, a_i^{(1)}, a_i^{(2)}, \dots, a_i^{(n_i)})$$

Now let $\phi_i \colon F \to A_i$ be such that $\phi_i a_0 = x_i$, $\phi_i a_k = a_i^{(k)}$; we have:

$$g_i(x_i, a_i^{(1)}, a_i^{(2)}, \dots, a_i^{(n_i)}) < f(x_i, \phi_i b_1, \phi_i b_2, \dots, \phi_i b_n)$$

and so:

$$y \leq f(x, (\phi_i b_1)_{i\in I}, (\phi_i b_2)_{i\in I}, \ldots, (\phi_i b_n)_{i\in I})$$

and $y \in \overline{\{x\}}$.

So X is ideal.

5.7 Lemma. *Every ideal bafo is near limited.*

Proof. Let A be an ideal bafo, M the set of the normal polynomials of the type; if $g \in M$ let its arity be $n_g > 1$. Let us set:

$$I = \{\langle g, a_1, a_2, \ldots, a_{n_g}\rangle : g \in M, a_i \in A\}$$

and consider $B = A^I$.

Let be $p, q \in B$ defined by:

$$\begin{aligned} p_i &= a_1 \\ q_i &= g(a_1, a_2, \ldots, a_{n_g}) \end{aligned} \qquad (i = \langle g, a_1, a_2, \ldots, a_{n_g} >\in I)$$

Of course $q_i \in \overline{\{pi\}}$ and so, being A ideal, $q \in \overline{\{p\}}$. Hence there exists an $n \geq 1$, a normal polynomial f of arity $n+1$ and some $b^{(i)} \in B$ with:

$$q = f(p, b^{(1)}, b^{(2)}, \ldots, b^{(n)})$$

and so, for every $i \in I$:

$$g(a_1, a_2, \ldots, a_n) = f(a_1, b_i^{(1)}, b_i^{(2)}, \ldots, b_i^{(n)}).$$

Now we can state the following:

5.8 Theorem. *A variety of bafos is ideal if and only if it is a variety of near limited bafos.*

Proof. By the previous lemmas.

Let us remember that a (universal) algebra A is *superprincipal* iff for every compact congruence R of A and for every x there exists an $y \in A$ such that R is generated by $\{(x, y)\}$. Since, of course, every bafo is superprincipal we have:

5.9 Corollary. *Every variety of lbafos falls into case (b) of* [5], *that is, it is superprincipally idealizable.*

Let us remember that in the classification of [5] we have examined the following properties for a variety X.

(a,1) X is filtrable i.e., there exists a filtral class Y for which $X = \mathbf{V}Y$.

(a,2) $X = \mathbf{V}(\Sigma \cap X)$ (where Σ is the class of simple algebras) and $\Sigma \cap X$ is filtral.

(b,1) X is ideal and generated by $\Pi \cap X$ (where Π is the class of superprincipal algebras).

(b,2) X is superprincipally idealizable.

(c,1) X is ideal.

(c,2) X is idealizable and regular (that is, if $A, B \in X$, A subalgebra of B and R is a congruence of A, then there exists a congruence S of B with $R = A^2 \cap S$).

(d) X is idealizable.

In [5] we have seen that properties with the same letter are equivalent and (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) (properly).

Now, being the bafos superprincipal, we have:

5.10 Theorem. *Every idealizable variety of bafos is superprincipally idealizable and so it is ideal itself and its algebras are near limited.*

Now we have seen that the varieties of bafos which have property (d) have also (b) and are varieties of lbafos.

We can ask for property (a); it is easy to see that:

5.11 Theorem. *A variety of bafos is filtrable iff it is a semisimple variety of lbafos.*

(Remember that a filtrable variety is semisimple, see again [5]).

6. Problems.

6.1 Let X be a variety of algebras with property (d) and add further operations. Can we generalize the results of this paper?

(If not, suppose that X have (c) or (b) or (a).)

6.2 A particular case of 6.1 (with property (a)): suppose X a variety of distributive lattices.

6.3 Suppose X is a variety of groups (see Kurosh *loco citato*).

6.4 The Boolean algebras with a Moore operator are a paradigmatic case for lbafo: it will be interesting to study these algebras.

(added Feb. 7, 1990: we have studied this problem in a short paper: *Boolean Algebras with a Moore Operator* Dipt. Mat. Un. Siena n. 207, 1989).

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