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Two papers on Ulam's logic with lies

AILA PREPRINT
n. 3 luglio-ottobre 1990

Direttore Responsabile: Ruggero Ferro
Iscrizione al Registro Stampa del Tribunale di Padova n. 1235 del 26.9.1990

Pubblicato con il contributo di:
Cassa di Risparmio di Padova e Rovigo
Stampa: Rotografica Padova

to appear in CAMBRIDGE

THE LOGIC OF ULAM'S GAME WITH LIES

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Someone thinks of a number between one and one million
(which is just less than 2^{20}).
Another person is allowed to ask up to twenty questions,
to each of which the first person is supposed to answer only yes or no.
Obviously the number can be guessed by asking first:
Is the number in the first half million ?
then again reduce the reservoir of numbers in the next question by one-half, and so on.
Finally the number is obtained in less than $\log_2(1000000)$.
Now suppose one were allowed to lie once or twice,
then how many questions would one need to get the right answer ?

S.M.ULAM,
Adventures of a Mathematician
Scribner's, New York, 1976
page 281

1. Playing Ulam's game The questions and answers exchanged between Questioner and Responder in Ulam's game with k lies are *propositions*. In this paper we show that the Lukasiewicz (k+2)-valued sentential calculus [14], [15] provides a natural logic for these propositions.

Throughout this paper, the Responder is identified with Pinocchio. Author and Reader will often impersonate the Questioner. *Unless otherwise stated, in this section we consider Ulam's game with at most one lie.*

Initially, Pinocchio and the Questioner agree to fix a *search space* $S = \{0,1,..., 2^n-1\}$. Writing numbers in binary notation, S is more conveniently represented by the n-cube $\{0,1\}^n$. Pinocchio arbitrarily chooses a number $x \in S$, writes it on a sheet of paper, and puts the paper in his pocket. We shall henceforth call x the *written number*. To guess the written number, the Questioner chooses his first *question* Q. In Ulam's game, Q is (uniquely determined by) a subset of S. To fix ideas, let us suppose Q is the set of even numbers in S. Thus, the Questioner asks Pinocchio

"is the number x even ?".

The *opposite question* $\overline{Q}$ is given by $S-Q$, the complement of Q in S. Pinocchio's answer to Q can either be "yes" or "no". Since a negative answer to Q is the same as a positive answer to $\overline{Q}$, let us consider without loss of generality the effect (on our state of knowledge of x) of a positive answer to Q: The original search space S is partitioned into two components A and B, where $A = Q$ and $B = \overline{Q}$ = the odd numbers in S. We cannot exclude the possibility that the written number is odd. Indeed, this can be the case if, and only if, (*iff*) Pinocchio's positive answer to Q is a lie. Now we ask our second question G, say

"is $x \leq 6$?"

Again, G is identified with the set $G = \{y \in S \mid y \leq 6\}$, and $\overline{G} = \{y \in S \mid y > 6\}$. If Pinocchio's answer to G is *positive*, in our new state of knowledge S is partitioned into three components C, D, E, where

$C = A \cap G$ are the numbers *satisfying* both answers, namely the even numbers ≤ 6;

$E = B \cap \overline{G}$ are the numbers *falsifying* both answers. These can be safely excluded from our search. For, if the written number were an odd number > 6, Pinocchio would have lied in both answers—which is impossible;

$D = S - (E \cup C) = (A \cap \overline{G}) \cup (B \cap G)$ are the numbers falsifying exactly one of Pinocchio's answers, namely the even numbers > 6, together with the odd numbers ≤ 6. Although their record is not so clean as for the numbers in C, they cannot be excluded from consideration as we did for the numbers in E. In fact, the written number is equal to a number in D iff exactly one of the two answers is a lie.

If Pinocchio's answer to G is *negative*, then C, D, and E are similarly obtained from A and B, by interchanging the roles of G and $\overline{G}$.

Proceeding by induction on the number of questions, suppose Pinocchio gives answers $a_1,...a_t$ to questions $Q_1 = Q$, $Q_2 = G$, $Q_3,..., Q_t$, respectively. Let our current state of knowledge about x be represented by a triple of pairwise disjoint sets (C_t, D_t, E_t), where

C_t is the set of numbers in S satisfying all answers $a_1,...,a_t$;

E_t is the set of numbers of S falsifying at least two answers;

D_t is the set of numbers falsifying exactly one answer.

If Pinocchio's answer a_{t+1} to the next question Q_{t+1} happens to be *positive*, then our state of knowledge is transformed into a new triple $(C_{t+1}, D_{t+1}, E_{t+1})$, where

$C_{t+1} = C_t \cap Q_{t+1}$, the numbers satisfying all answers $a_1,...,a_t, a_{t+1}$;

$E_{t+1} = E_t \cup (D_t \cap \overline{Q}_{t+1})$, the numbers falsifying at least two answers;

$D_{t+1} = (C_t \cap \overline{Q}_{t+1}) \cup (D_t \cap Q_{t+1})$, the numbers falsifying exactly one answer.

If the answer to Q_{t+1} is *negative*, one similarly obtains $(C_{t+1}, D_{t+1}, E_{t+1})$ from (C_t, D_t, E_t), by interchanging the roles of Q_{t+1} and $\overline{Q}_{t+1}$.

It is easy to see that C_{t+1}, D_{t+1}, and E_{t+1} are pairwise disjoint.

The *initial* state of knowledge (C_0, D_0, E_0), i.e., the state before the first answer, is of course given by the triplet $(S, \emptyset, \emptyset)$ with no falsified answers.

A state of knowledge (C_u, D_u, E_u) is called *final* iff all numbers of S except one, say w, are in E_u. Then we can confidently say that the written number equals w. Indeed, we have the additional information that if w belongs to C_u, then Pinocchio has never lied; if w belongs to D_u, Pinocchio has lied exactly once.

Already from this simple example, we see that Pinocchio's answers do not behave as propositions in classical logic in the following respects:

1. Two opposite answers a_i and a_j to the same (repeated) question, such as "yes, the written number is even" and "no, the written number is odd", do not lead to the inconsistent state of knowledge $(\emptyset,\emptyset,S)$, to the effect that no number can be equal to the written number. On the contrary, from a_i and a_j we obtain the information that Pinocchio's reservoir of lies is reduced by one. We conclude that the connectives in the logic of Ulam's game are so arranged that the *conjunction of two opposite answers need not express an unsatisfiable property* of numbers in the search space.

2. The conjunction of two answers, each saying "the written number is odd", is generally more informative than a single answer "the written number is odd"; for instance, under the present stipulation that Pinocchio can lie at most once, two equal answers "the written number is odd" suffice to establish that the written number is odd, while a single answer need not suffice. Specialists would say that the logic of Ulam's game with lies does not obey the contraction rule; we prefer to say that this logic obeys the *repetita juvant* (repetitions are helpful) principle.

3. Although for people having direct access to the written number, each answer of Pinocchio is either true or false, this *absolute* truth-value is of little significance to the Questioner's strategy, and to his current state of knowledge. The Questioner tries to make the best use of all answers $a_1,...a_t$, since in general he cannot discriminate between informative and misleading answers. Besides, if required to minimize the length of the game—as in Ulam's problem—the Questioner will carefully balance his questions in such a way that either answer "yes" or "no" is for him equally informative, in a sense which can be made precise [3],[4]. His current state of knowledge (C_t,D_t,E_t) assigns to each point $y \in S$, a "falsity-value" given by the quantity $q(y)$ of answers falsified by y. Here, $q(y)$ ranges in the set {zero, one, "too many"}, and $C_t = q^{-1}(0)$, $D_t = q^{-1}(1)$, $E_t = q^{-1}$("too many"). We conclude that the logic of Ulam's game is *many-valued.*

Similarly, if Pinocchio is allowed to lie up to k times, his answers determine a function $q:S \to \{0,1,2,...,k,$ "too many"$\}$ assigning to each point y the quantity $q(y)$ of answers falsified by y. Given a question $Q \subseteq S$, a positive answer to Q naturally transforms q into a new assignment q', as follows:

(1) if y satisfies the answer (i.e., $y \in Q$), or $q(y) =$ "too many", then $q'(y) = q(y)$, otherwise $q'(y) = q(y)+1$, where $k+1 =$ "too many".

The effect on q of a negative answer to Q is similarly defined with reference to the opposite question $\bar{Q}$. To record the Questioner's current state of knowledge, instead of using the function q we will find it convenient to use the *relative distance* function $d : S \to \{0, 1/(k+1), 2/(k+1),..., k/(k+1), 1\}$ given by

(2) $$d(y) = (k+1-q(y))/(k+1),$$

where $k+1-$"too many" $= 0$. Thus, $d(y) = 0$ iff y falsifies more than k answers (i.e., y is an *excluded* number); $d(y) = 1/(k+1)$ iff y falsifies exactly k answers, ..., $d(y) = k/(k+1)$ iff y falsifies exactly one answer, $d(y) = 1$ iff y satisfies all answers. Intuitively, the rational number $d(y)$ is the distance (relative to $k+1$) of y from the set of excluded numbers. We shall henceforth identify each state of knowledge with its corresponding relative distance function.

2. States of knowledge are Lukasiewicz conjunctions of Post functions

For any two numbers x and y in the real unit interval [0,1], the *Lukasiewicz conjunction* $x \bullet y$ is the amount by which the sum $x+y$ exceeds 1 (this amount being 0 in case $x+y \leq 1$). In symbols,

(3) $$x \bullet y = \max(0, x+y-1).$$

The Lukasiewicz *disjunction* $x \oplus y$ is the truncated sum of x and y,

(4) $x \oplus y = \min(1, x+y) = (x^* \bullet y^*)^*$, where $x^* = 1-x$ is the *negation* function.

If x and y are only allowed to range over the two-element set $\{0,1\}$, then (3) and (4) take the more familiar form $x \bullet y = \min(x,y)$, $x \oplus y = \max(x,y)$, respectively giving the truth table of Boolean conjunction and disjunction; of course, x^* then gives Boolean negation. For

each $m = 2, 3, 4,...$, we shall denote by I_m the m-element *Lukasiewicz chain* $\{0, 1/(m-1), 2/(m-1),..., (m-2)/(m-1), 1\}$, equipped with the operations $*, \bullet, \oplus$.

In the light of [6, Theorems 16-18], any function $f:S \to I_m$ is called a *Post function* (on S) *of order m*. By (2), every relative distance function d in Ulam's game with k lies is a Post function of order $k+2$. Since states of knowledge are (uniquely determined by) conjunctions of Pinocchio's answers, the latter, too, must be representable as Post functions. According to (1) and (2), the positive answer to a question Q penalizes each $y \notin Q$ decreasing by $1/(k+1)$ its relative distance from the set of excluded numbers, unless this distance already equals 0. This motivates the following

1. DEFINITION. Given a question Q in Ulam's game with k lies, the *positive answer* to Q is the function $f_Q:S \to \{k/(k+1), 1\}$ given by

(5) $$f_Q(y) = 1 \quad \text{if } y \in Q$$
$$f_Q(y) = k/(k+1) \quad \text{if } y \notin Q.$$

The *negative answer* to Q is the positive answer $f_{\bar{Q}}$ to $\bar{Q} = S - Q$. The *Post function f of Pinocchio's answer* to Q is given by $f = f_Q$, or $f = f_{\bar{Q}}$ according as Pinocchio's answer to Q is positive or negative.

Remark. The inconsistency-tolerance property mentioned in the previous section follows from the above definition of answer: as a matter of fact, except in the 0-lie game, the negative answer $f_{\bar{Q}}$ to a question Q is different from the negation $1-f_Q$ of the positive answer to Q.

2. PROPOSITION. *Let d and d' be the relative distance functions immediately before and after Pinocchio answers a question Q. Let f be the Post function of Pinocchio's answer to Q. Then $d' = d \bullet f$.*

Proof. It is sufficient to analyze the effect of Pinocchio's positive answer f_Q. Let q and q' be the functions corresponding to d and d', as given by (2). For each $y \in S$, we have two possible cases:
Case 1: $d(y) = d'(y)$, i.e., y is not penalized by the positive answer.
Then by (1) and (2), either $d(y) = 0$ (that is, y is already in the set of excluded numbers), or $f_Q(y) = 1$ (that is, y satisfies Pinocchio's answer). Recalling (3) and (5), in the first subcase we have $d(y) \bullet f_Q(y) = 0 \bullet f_Q(y) = 0 = d(y) = d'(y)$. In the second subcase we have $d(y) \bullet f_Q(y) = d(y) \bullet 1 = d(y) = d'(y)$.
Case 2: y is penalized by the positive answer.

Then by (1) and (2), y falsifies f_Q without being an excluded number. We then have in particular, $d(y) \geq 1/(k+1)$, $q'(y) = q(y) + 1$, and $d'(y) = d(y) - 1/(k+1)$. By (3) and (5) we get $d(y) \cdot f_Q(y) = \max(0, d(y) + f_Q(y) - 1) = \max(0, d(y) - 1/(k+1)) = d'(y)$.

In any case, we have $d'(y) = d(y) \cdot f_Q(y)$, whence $d' = d \cdot f_Q$, as required.

QED

The *initial* state of knowledge is the Post function on S constantly equal to 1. A Post function $g:S \to I_m$ is a *final* state of knowledge iff for exactly one $x \in S$ we have $g(x) \neq 0$. The Questioner's states of knowledge are given by Pinocchio's answers as follows

3. COROLLARY. *The relative distance function* d *after Pinocchio's answers to questions* $Q_1,\ldots,Q_t$ *is given by the Lukasiewicz conjunction of the Post functions of the answers.*

3. Formalization of Ulam's game in Lukasiewicz logic

For each $m = 2,3,4,\ldots$, the set of *variable* and *connective* symbols in the m-valued sentential calculus of Lukasiewicz [14], [15], and the set of *formulas* are exactly the same as for the 2-valued (Boolean) calculus. Just as in the 2-valued calculus every formula represents a Boolean function, in the m-valued calculus each formula p with variables $X_1,\ldots,X_n$ *represents the function* $f_p:(I_m)^n \to I_m$, according to the following inductive definition: Each variable X_i represents the projection onto the i-th axis; if we know f_q and f_r, then $f_{\text{not } q} = 1-f_q$, $f_{q \text{ and } r} = f_q \cdot f_r$, and $f_{q \text{ or } r} = f_q \oplus f_r$. Two formulas p and q are *logically equivalent* iff $f_p = f_q$. A *tautology* is a formula $p = p(X_1,\ldots,X_n)$ such that f_p is the constant function 1 on $(I_m)^n$. For example, "(not X) or X" is a tautology in each m-valued calculus, while "(not (X or X)) or X" is a tautology only in the 2-valued calculus. In fact, for each $m \geq 3$, the formula "X or X" is not logically equivalent to the formula "X" in the m-valued calculus. This is just a reformulation of the *repetita juvant* principle.

We shall formalize Ulam's game with at most k lies in the $(k+2)$-valued sentential calculus of Lukasiewicz. By Corollary 3, we must only code Post functions of Pinocchio's answers by means of formulas of the Lukasiewicz calculus. Our formalization is then a natural generalization of the familiar representation of Boolean functions by formulas in the 2-valued calculus. For the importance of the formalization procedure, already in the 2-valued case, see [2, p. 5].

We cannot identify the search space $\{0,1\}^n$ with the set of extreme points of the cube $(I_{k+2})^n$. As a matter of fact, McNaughton's fundamental representation theorem [8] states that the set of functions $\{f_p \mid p$ is a formula in the $(k+2)$-valued calculus$\}$ is the set of restrictions of McNaughton functions to the n-dimensional cube $(I_{k+2})^n$. By definition, a *McNaughton*

function $f:[0,1]^n \to [0,1]$ is a piecewise linear (continuous) functions all of whose pieces have integral coefficients. Let $x = (x_1 \ldots x_n) \in (I_{k+2})^n$. Write each rational coordinate x_i as a fraction a_i/b_i, where a_i, b_i are integers, $\gcd(a_i, b_i) = 1$, $a_i \geq 0, b_i > 0$. Let v be the least common multiple of the denominators b_i. Let V_x be the set of possible values of McNaughton functions at x. Then a straightforward computation shows that

(6) $$V_x = \{t/v \mid t = 0,1,\ldots,v\}.$$

Thus in particular, a McNaughton function can only take the values 0 and 1 on the Cantor cube $\{0,1\}^n$, and hence its restriction to $\{0,1\}^n$ cannot be a Post function of order > 2. To find a substitute for the original search space $\{0,1\}^n$, let us consider the cube $C(n,k) = \{1/(k+1), k/(k+1)\}^n$. Note that $C(n,k)$ is indeed a cube for each $k \geq 0$, except $k = 1$. In order to avoid notational complications, *in the rest of this paper we shall assume* $k \neq 1$.

4. PROPOSITION. (i) *For each function* $f: C(n,k) \to I_{k+2}$ *there is a formula* p *such that* f *equals the restriction of* f_p *to* $C(n,k)$.

(ii) *For each formula* r, *the restriction of* f_r *to* $C(n,k)$ *is a Post function of order* $k+2$ *on* $C(n,k)$.

Proof. (i) For each $x \in C(n,k)$ let $g_x:[0,1]^n \to [0,1]$ be any McNaughton function such that $g_x(x) = f(x)$. The existence of g_x is ensured by (6). Let U_x be an open neighbourhood of x. We can safely assume $U_x \cap U_y = \emptyset$ whenever $x \neq y$. By [9, 4.17], for each x there is a McNaughton function h_x such that $h_x(x) = 1$, and $h_x = 0$ outside U_x. Therefore, the McNaughton function $h = \sup\{g_x \wedge h_x \mid x \in C(n,k)\}$ equals f on $C(n,k)$. By McNaughton's theorem, there is a formula p such that $f_p = h$.

(ii) Trivial, using (6).

QED

We are now ready to formalize Ulam's game with k lies in the $(k+2)$-valued calculus. Let the map $\mu:\{0,1\} \to \{1/(k+1), k/(k+1)\}$ be defined by: $\mu(0) = 1/(k+1)$, and $\mu(1) = k/(k+1)$. Then μ induces a one-one correspondence $x = (x_1 \ldots x_n) \to x^\mu = (\mu(x_1) \ldots \mu(x_n))$ from the original search space $S = \{0,1\}^n$ onto $C(n,k)$, in symbols, $\mu:\{0,1\}^n \cong C(n,k)$. By Proposition 4, we have a natural one-one correspondence

(7) $$f \to f^\mu, \quad \text{where } f(x) = f^\mu(x^\mu) \text{ for all } x \in \{0,1\}^n$$

between Post functions on S of order $k+2$, and restrictions to $C(n,k)$ of McNaughton functions. Under this correspondence, the *initial* state of knowledge corresponds to the function constantly equal to 1 on $C(n,k)$. The latter is in turn represented by any formula p such that f_p equals 1 on $C(n,k)$. For instance, any tautology will do. Similarly, a *final* state of knowledge is represented by any formula r which is *uniquely satisfiable* in $C(n,k)$, in the sense that there is precisely one $x \in C(n,k)$ such that $f_r(x) \neq 0$.

Let $Q \subseteq S$ be a *question*. Then the correspondence $\mu:\{0,1\}^n \cong C(n,k)$ canonically transforms Q into a subset Q^μ of $C(n,k)$, by the stipulation $x \in Q$ iff $x^\mu \in Q^\mu$. In the light of Definition 1, together with the correspondence (7), Pinocchio's *positive answer* f_Q is then represented by any formula a such that $f_a(x) = 1$ for all $x \in Q^\mu$, and $f_a(x) = k/(k+1)$ for all $x \in C(n,k) - Q^\mu$. Formulas representing *negative answers* are similarly defined, with $\overline{Q}$ in place of Q. Formulas representing the actual *answer* to Q are now defined according as the answer is "yes" or "no". Suppose d is the Questioner's state of knowledge after Pinocchio's answers to questions $Q_1,...,Q_t$. Suppose formula a_i represents the answer to Q_i, for each $i = 1,...,t$. Then by Corollary 3, the (Lukasiewicz) conjunction $s = (a_1$ and a_2 and...and $a_t)$ represents d, in the sense that the Post function d^μ coincides with the restriction of f_s to $C(n,k)$.
Summing up our analysis, we have the following table:

ULAM GAME	*LUKASIEWICZ LOGIC*
maximum number k of lies	k+2 truth-values $0, 1/(k+1),..., k/(k+1), 1$
search space $S = \{0,1\}^n$	$C(n,k) = S^\mu = \{1/(k+1), k/(k+1)\}^n$
number y in S	point y^μ in $C(n,k)$
initial state of knowledge	tautology
final state of knowledge	formula uniquely satisfiable in $C(n,k)$
current state of knowledge d	formula s such that $d^\mu = f_s$ in $C(n,k)$
question Q	corresponding subset Q^μ of $C(n,k)$
opposite question $\overline{Q}$	complementary subset of $\overline{Q}$ in $C(n,k)$
positive answer f_Q to Q	formula p with $(f_Q)^\mu = f_p$ in $C(n,k)$
negative answer $f_{\overline{Q}}$ to Q	formula p with $(f_{\overline{Q}})^\mu = f_p$ in $C(n,k)$
state d after answers to $Q_1,...,Q_t$	conjunction s of corresponding formulas
set of excluded numbers, $d^{-1}(0)$	points of $C(n,k)$ falsifying s

3. Concluding remarks

1. A long-standing problem in m-valued logic—one to which Lukasiewicz himself devoted considerable attention—is to give natural interpretations to truth-values when $m \geq 3$ [15, p.275]. In our interpretation, truth-values are distances (measured in units of m–1) from the set of excluded numbers, in Ulam's game with m–2 lies. The contraction rule fails because

the *repetita juvant* principle holds—just as in everyday life. Noncontradictory co-existence of opposite answers, which is inevitable in the presence of lies, can be handled using more than two truth-values.

2. In the 0-lie case, an optimal strategy [1, pp.6 and 62] to guess a number $x \in \{0,1\}^n$, is by definition a sequence of questions, *alias* Boolean functions $f_1,...,f_n$ of n variables, such that for each choice of the parameters $\varepsilon_1,...,\varepsilon_n \in$ {yes, no}, the function $f_1^{\varepsilon_1} \wedge ... \wedge f_n^{\varepsilon_n}$ is nonzero in exactly one point. Here, $f^{yes} = f$, and $f^{no} = 1$–f. Equivalently, $f_1,...,f_n$ is an *independent* set of n elements in the free Boolean algebra with n generators [12, p. 39]. Note that the binary notation system is a by-product of the particular optimal strategy $b_1,...,b_n$, where each Boolean function b_i asks "is the i-th digit of x equal to 1 ?".

Considering now the case when at most k lies are allowed, in the (nonadaptive, static, predetermined [1, p.9]) case when all questions precede all answers, optimal searching strategies immediately yield optimal k-error correcting codes. However [7], very few optimal such codes are known when $k \geq 2$. The situation is much better in the (adaptive, dynamical, sequential) case of Ulam's game, where questions may depend on Pinocchio's answers. For small values of k optimal strategies are known [11], [3], [4], [10]. These strategies can also be used in the equivalent version of Ulam's game where Pinocchio need not know when he lied. We can for instance, assume that a honest Pinocchio is sending us his answers from a distant place, using a low-power transmitter. Distortion can affect Pinocchio's transmission. On the other hand, using powerful transmitters, we can send our questions to Pinocchio without any distortion. We must correct the distortions (lies ?) of Pinocchio's answers, by asking the minimum number of questions. In this way, Ulam's game naturally fits in the theory of communication with feedback [5], [13]. The representation of searching strategies in terms of formulas is a starting point to measure the complexity of the underlying coding and decoding procedures.

3. Since the many-valued sentential calculus of Lukasiewicz is deeply related to AF C^*-algebras [9], and since AF C^*-algebras are useful in the description of quantum spin systems, our interpretation of Ulam's game may be of help in the analysis of the logical aspects of such systems.

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Proceedings SILFS Congress, Viareggio 1990, to appear

ALGEBRAS OF ULAM'S GAMES WITH LIES

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Someone thinks of a number between one and one million (which is just less than 2^{20}).
Another person is allowed to ask up to twenty questions,
to each of which the first person is supposed to answer only yes or no.
Obviously the number can be guessed by asking first:
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Now suppose one were allowed to lie once or twice,
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S.M.ULAM,
Adventures of a Mathematician
Scribner's, New York, 1976
page 281

0. Prologue In this paper we continue the logical analysis [11] of Ulam's game. We shall conveniently identify with Pinocchio the "first person" in the above quotation.

Ulam's game has an alternative interpretation in the framework of communication with feedback [6], [16]. Here, a low power transmitter P (a sincere Pinocchio who does not know which of his answers may turn out to be false) is sending binary numbers to a receiver Q (the Questioner). Distortion may transform into $1-b$ any bit b traveling from P to Q; however, after receiving each bit, Q can communicate with no distortion to P whether b or $1-b$ was actually received. We naturally expect that a long binary number x contains more distorted bits than a short one. Accordingly, in this paper we consider the generalization of Ulam's game where each number x is associated with a maximum number $m(x)$ of lies, or distortions, depending on x. We also give an algebraic analysis of the simplest generalizations of Ulam's game with an infinite search space. We shall describe a one-one correspondence between a fairly large class of Ulam games and of MV algebras. We assume familiarity with MV algebras. We refer to [3], [4], and [8, § 3] for background.

1. Notation and terminology [11] A *Ulam game* is a triple $G = (S, m, L)$, where S is a structure, m is a function from S to $\mathbf{N} = \{0,1,2,...\}$, and L is a set of subsets of S. S is called the *search space*, m is the *lie bound*, and L is the *language* of the game. Each $Q \in L$ is called a *question*, and is canonically identified with the question "does the unknown x belong to the set Q ?"; in particular, the empty set corresponds to the trivial question "is x outside the search space ?". Apart from the case of no lies, which is widely considered in combinatorial search theory [1], in the literature one finds examples of Ulam's games with $m \neq 0$ where S is a linearly ordered set and L is the set of initial segments, or S is a topological space and questions are measurable subsets of S; see, e.g., [13],[14]. The above quotation deals with the Ulam game $G = (S, m, L)$ with S the set of numbers between one and one million, m the function constantly equal to 2, and L the powerset of S. Optimal searching strategies are described in [5] and [12].

Games $G = (S, m, L)$ and $G' = (S', m', L')$ are *isomorphic* iff there exists an isomorphism $\theta : S \cong S'$ such that $m(x) = m'(\theta(x))$ for each $x \in S$, and for every subset X of S we have that $X \in L'$ iff $\{\theta(x) \mid x \in X\} \in L'$.

A state of knowledge of the Questioner (the second person in the above quotation) is completely described by a function assigning to each point $x \in S$ the quantity $q(x)$ of answers currently falsified by x, where $q(x) \in \{0,1,2,...,m(x),$ "too many"$\}$. Upon identifying "too many" with $m(x)+1$, and dividing $q(x)$ by $m(x)+1$, we can represent the current state of knowledge by the relative number of answers falsified by x, or, dually, by the quantity $d(x) = 1 - q(x)/(m(x)+1)$. By definition, a *state of knowledge* of the game $G = (S, m, L)$ is a function $d{:}S \to \mathbf{Q}$ such that for each $x \in S$, $d(x) \in \{0, 1/(m(x)+1),..., m(x)/(m(x)+1), 1\}$. The quantity $d(x)$ is the *relative distance* (in units of $m(x)+1$) of x from the condition of falsifying too many answers. The *initial* state of knowledge is the function constantly equal to one (no x falsifies any answer). After Pinocchio has answered questions $Q_1,...,Q_t$, the state of knowledge d is uniquely determined by his answers, as follows: For any subset Q of S, the *positive answer* $Q^{\text{yes}}: S \to \mathbf{Q}$ is defined by $Q^{\text{yes}}(x) = 1$ whenever $x \in Q$, and $Q^{\text{yes}}(x) = 1 - 1/(m(x)+1)$, otherwise. This accounts for the fact that if $x \in Q$, the number of answers falsified by x is unchanged, while if $x \notin Q$, x takes a step towards the condition of falsifying too many answers. Also define $Q^{\text{no}} = \overline{Q}^{\text{yes}}$, where $\overline{Q} = S \setminus Q$ is the *opposite* question. Then the state of knowledge after questions $Q_1,...,Q_t$ have respectively been answered $e_1,..., e_t$, (where $e_j \in \{\text{yes, no}\}$ for each $j = 1,...,t$) is the function $Q_1^{e_1} \bullet ... \bullet Q_t^{e_t}$, where $\bullet$ is Lukasiewicz conjunction, $a \bullet b = \max(0, a + b - 1)$. A state of knowledge d is *final* iff $d(x) \neq 0$ holds for exactly one element x in the search space.

Given a Ulam game $G = (S, m, L)$ *the MV algebra* A_G *associated to* G is the MV algebra of rational valued functions on S, with pointwise operations, generated by the answers to the questions of L.

2. Bounded numbers of lies A Ulam game $G = (S, m, L)$ is *finite* iff S is a finite set—or, a finite discrete topological space—and L is the powerset of S. It follows that conjunctions of answers exhaust all possible functions $f : S \to \mathbf{Q}$ such that $f(x) \in \{0, 1/(m(x) + 1), ..., m(x)/(m(x) + 1), 1\}$ for every $x \in S$. The MV algebra A_G is finite.

Conversely, let A be a finite MV algebra. Let S be the set of maximal ideals of A; for each $J \in S$ let A/J be the quotient algebra of A. Then $A/J \cong \{0, 1/(k_J+1),..., k_J/(k_J+1), 1\}$, for some $k_J \in \mathbf{N}$. By the well-known representation theorem of finite MV algebras (see for instance [10, Proof of 5.1]), A is isomorphic to the finite product $\prod\{A/J \mid J \in S\}$. The latter is the MV algebra of all rational valued functions f over S such that for each $J \in S$, $f(J) \in \{0, 1/(k_J+1),..., k_J/(k_J+1), 1\}$. Let $m{:}S \to \mathbf{N}$ be defined by $m(J) = k_J$. The game $G_A = (S, m, \text{powerset}(S))$ has the property that $A \cong A_{G_A}$. Up to isomorphism, the maps $A \to G_A$ and $G \to A_G$ are inverse of each other. In conclusion we have

2.1 THEOREM. *The map* $G \to A_G$ *induces a one-one correspondence between isomorphism classes of finite Ulam games, and isomorphism classes of finite MV algebras.*

An application [10, Proposition 5.1] of Grothendieck's functor K_0 yields the following

2.2 COROLLARY. *Isomorphism classes of finite Ulam games are in one-one correspondence with isomorphism classes of finite dimensional C*-algebras.*

For each $n = 2,3,4,...$, we let I_n be the Lukasiewicz chain with n elements, i.e., the MV algebra $\{0, 1/(n-1), 2/(n-1), ..., (n-2)/(n-1), 1\}$ with natural MV operations. Let X be a Boolean (i.e., a totally disconnected, compact, Hausdorff) space. We denote by $C(X, I_n)$ the MV algebra of all continuous functions from X into I_n, the latter being equipped with the discrete topology. In the light of Epstein's representation theorem [7, Theorem 16], we say that an MV algebra B is a *Post* MV *algebra of order* n iff $B \cong C(X, I_n)$ for some Boolean space X. Thus in particular, Post MV algebras of order 2 are the same as Boolean algebras.

Generalizing the notion of finite game, we say that a Ulam game $G = (S, m, L)$ is *Boolean* iff S is a Boolean space, m is a continuous function from S to $\mathbf{N}$, the latter being equipped with the discrete topology, and L is the set of all clopen subspaces of S. We say that G has *constant lie bound* iff, in addition, m is a constant function, say $m(x) = k$ for all $x \in S$. It follows that, for each $Q \in L$, Q^{yes} is a continuous function from S into I_{k+2}. Since the pointwise MV operations preserve continuity, A_G will be an algebra of continuous functions from S into I_{k+2}. On the other hand, it is easy to see that for each continuous function $f{:}S \to I_{k+2}$ there exist answers to questions of L whose conjunction equals f. Therefore, A_G is a Post MV algebra of order $k+2$. Conversely, given a Post MV algebra A of order $k+2$, say $A \cong C(X, I_{k+2})$ with X Boolean, let m be the function constantly equal to k over X, and let L be the set of clopen subspaces of X. Let $H = (X, m, L)$. Then it is easy to show that $A \cong A_H$. Trivially, isomorphic MV algebras correspond to isomorphic games. We have proved

2.3 THEOREM. *For each $k = 0, 1, \ldots$, the map $G \to A_G$ induces a one-one correspondence between isomorphism classes of Boolean Ulam games with constant lie bound equal to k, and isomorphism classes of Post MV algebras of order $k+2$.*

We can now specialize Theorem 2.1 to the case of constant lie bound

2.4 COROLLARY. *Under the above map, finite Ulam games with constant lie bound correspond to finite Post MV algebras of finite order.*

If $G = (S, m, L)$ is a Boolean game where m is no more assumed to be a constant function, still by compactness it follows that the range of m is finite, and S is the union of finitely many pairwise disjoint clopen subsets $X_1, \ldots, X_r$ such that m is constant over each X_j. Thus, G splits into a finite number of Boolean Ulam games, each with a constant lie bound. An application of Theorem 2.3 immediately yields

2.5 COROLLARY. *The map $G \to A_G$ induces a one-one correspondence between isomorphism classes of Boolean Ulam games, and isomorphism classes of finite products of Post MV algebras of finite order.*

3. Arbitrarily large numbers of lies Generalizing the examples of the previous section, we say that a Ulam game $G = (S, m, L)$ is *quasiboolean* iff G obeys the following three conditions:

(i) S is a set, and L is a field of subsets of S. In addition, L is *reduced*, i.e., for any two points x' and x'' in S, there is a question Q in L such that $x' \in Q$ and $x'' \notin Q$, [15, p. 18].

(ii) For each $n = 0, 1, 2, \ldots$, the set $m^{-1}(n) = \{x \in S \mid m(x) = n\}$ is a member of L.

(iii) For every ultrafilter (maximal filter) U of L which is not determined by any point of S (i.e., $\cap U = \emptyset$, [15, p. 15]) we have $\lim_{x \to U} m(x) = \infty$, in the sense that for each $r \in \mathbb{N}$ there is $Y \in U$ such that $m(y) > r$ for all $y \in Y$.

Intuitively, Condition (i) means that any two distinct points of S can be distinguished by the questions available in L. Condition (ii) enables the Questioner to ask Pinocchio "is x among the elements for which the rules of the game allow you to lie at most n times ?". Condition (iii) states that any "nonstandard" point x^* outside S determined by a maximal consistent set U of questions of L is actually outside the scope of any searching strategy, because the number of lies Pinocchio can tell about the "standard" points near x^* tends to infinity.

FROM G TO A_G.

Given a quasiboolean Ulam game $G = (S, m, L)$, since A_G is a subdirect product of finite Lukasiewicz chains, A_G is *Archimedean* (i.e., the intersection of maximal ideals of A_G is zero. Note that Chang [3], [4] uses the adjective "representable" instead of "Archimedean"). The positive answer $\emptyset^{yes}$ to the trivial empty question takes the value $1 - 1/(m(x)+1)$ for each x in S. Its dual, namely the function s given by

(1) $$s(x) = 1/(m(x)+1), \quad \text{for each } x \in S,$$

turns out to be a *singular* element of A_G (i.e., $s > 0$, and whenever $a \oplus b = s$ with $a \cdot b = 0$, and $a, b \in A_G$, then $a \wedge b = 0$. Here, as usual, $\oplus$ denotes truncated addition, $a \oplus b = \min(1, a+b)$. It is easy to verify that an element t of an MV algebra A is singular iff t is singular, in the usual sense [2, p. 232] as an element of the abelian l-group with strong unit corresponding to A via the functor Γ, as in [8, 3.10]). Further, s is the greatest singular element of A_G. For each question $Q \in L$, the element $1 - Q^{yes}$ coincides with s over $S \setminus Q$, and is constantly equal to zero over Q. Thus in particular, $1 - Q^{yes}$ is a singular element of A_G. By definition of A_G, it follows that A_G is generated by its singular elements, together with the constant function 1. Clearly, if G' and G'' are isomorphic games, then so are $A_{G'}$ and $A_{G''}$.

FROM A TO G_A.

Conversely, let A be an Archimedean MV algebra that is generated by its singular elements (together with the constant element 1) , and has a greatest singular element s. Then an obvious generalization of [9, 3.1] yields a cardinal κ, and a compact Hausdorff subspace V of the Tichonoff cube $[0,1]^\kappa$, such that A can be identified with the MV algebra of restrictions to V of the (McNaughton) functions in the free MV algebra L_κ over κ generators. The functions in A are continuous, and separate points in V. As in [8, §8], there is a canonical one-one correspondence between maximal ideals J in A and points x_J in V; J and x_J satisfy the relation $J = \{f \in A \mid f(x_J) = 0\}$. Under this correspondence, the quotient map $a \to a/J$ can be safely identified with the evaluation map $a \to a(x_J)$, $a \in A$. We partition V into the disjoint sets S and T as follows:

$$S = \{x \in V \mid s(x) > 0\}, \text{ and } T = \{x \in V \mid s(x) = 0\}.$$

Claim 1. T has at most one element.

As a matter of fact, each singular $t \in A$ is constantly zero over T, while the element 1 constantly takes the value 1. Since A is generated by its singular elements, together with the constant 1, each function of A is constant over T. If there were two distinct points x' and x'' in V, then by the above mentioned separation property of A, there would exist a function $f \in A$ such that $f(x') > 0$ and $f(x'') = 0$, whence f would not be constant over T, a contradiction.

Claim 2. For each $b \in A$, if $b(x) = 0$ for all $x \in S$, then $b = 0$.

As a matter of fact, assume $b > 0$ (absurdum hypothesis). Then $b(i) > 0$, where i is the only point of T. Since b is generated via a finite number of applications of pointwise MV operations starting from the singular elements of A and the constant 1, and since all singular elements take value zero at point i, it follows that $b(i) = 1$. Letting I be the maximal ideal of A corresponding to i, the quotient A/I is isomorphic to the two-element MV algebra $\{0,1\}$. It follows that b is singular. However, the fact that $1 = b(i) > s(i) = 0$ contradicts our assumption that s is the greatest singular element in A.

Claim 3. The map restricting to S each function of A is an isomorphism of A onto an algebra A^* of rational valued functions over S.

Indeed, Claim 2 already shows that the restriction map to S is one-one. To prove that all functions in A^* (equivalently, all functions in A) are rational valued, it is sufficient to show that for every maximal ideal J, the quotient A/J is a finite Lukasiewicz chain, say $A/J = \{0, 1/(k+1), ..., k/(k+1), 1\}$ for some $k = k_J \in \mathbf{N}$. This is true by [3, 3.19], because A/J is a subalgebra of the unit real interval [0,1], and the singular element $s/J = s(x_J)$ is the atom of A/J, in symbols, $s(x_J) = 1/(k_J+1)$.

After the proof of Claim 3, let the Ulam game $G_A = (S, m, L)$ be defined as follows: The lie bound m of G_A is given by

(2) $$m(x) = 1/s(x) - 1, \quad \text{for each } x \in S.$$

We have just proved that $m(x) \in \mathbf{N}$. Further, a subset X of S is an element of L iff either X is the empty set, or X is the support of a singular element of A, in symbols, $X = \mathrm{supp}(t) = \{x \in S \mid t(x) > 0\}$, for some singular element $t \in A$. Note that $S = \mathrm{supp}(s)$. L is closed under unions and intersections, because sups and infs of singular elements are singular. Whenever X is in L, say $X = \mathrm{supp}(t)$, for some singular t, then also $S \setminus X$ is in L, since $S \setminus X = \mathrm{supp}(s-t)$, and $s-t$ is a well defined singular element of A, namely $s-t = (1-t)\bullet s$, where $\bullet$ is Lukasiewicz conjunction [3, 3.15]. We have proved that L is a field of subsets of S. To prove that L is reduced, given any two distinct points x' and x'' in S, since the functions in A separate points, there is $g \in A$ such that $g(x') = 0$, and $g(x'') > 0$. Letting $h = g \wedge s$, we have $h(x') = 0$ and $h(x'') > 0$, because $g(x'') \geq s(x'') > 0$. Since h is singular, L is reduced.

Claim 4. For each $k = 0, 1, 2, ...$, the inverse image $m^{-1}(k)$ is a member of L.

Using (2), we shall equivalently prove that for each $n = 1, 2, ...$, there is a singular element $s_n \in A$ such that $\mathrm{supp}(s_n) = s^{-1}(1/n)$. Let $(H, 1)$ be the abelian l-group with strong unit corresponding to A via the functor Γ, as given by [8, 3.10]. H is the l-group of rational valued functions over V generated by A, with the constant function 1 as the strong unit. Let $g \in H$ be defined by $g = |ns-1|$, where $|\ |$ is absolute value. Then, for every $x \in V$ we have $g(x) = 0$ iff $s(x) = 1/n$. Therefore, $s - (s \wedge g)$ is a singular element of A having the required property.

Claim 5. For each nondetermined ultrafilter U in L, $\lim_{x \to U} m(x) = \infty$.

Suppose to the contrary that there is a natural number n and a nondetermined ultrafilter U of L such that every $X \in U$ has nonempty intersection with the set $M = m^{-1}(0) \cup m^{-1}(1) \cup ... \cup m^{-1}(n)$. For any such X there is a singular element $t \in A$ such that $t^{-1}(0) = X \cup T$. Since t is a continuous function over V, then $X \cup T$ is a closed subspace of V. Similarly, for each $k = 1, 2, ...$, the inverse image $s^{-1}(1/k)$ is a closed subspace of V, whence by (2), so is the set M. The compactness of V now yields a point $y \in M \subseteq S$ contained in each $X \in U$; thus, U is determined by y, a contradiction.

$G \to A_G$ IS THE INVERSE MAP OF $A \to G_A$.

Starting from an Archimedean MV algebra A generated by its singular elements (together with the constant element 1), and possessing a greatest singular element s, we have obtained a quasiboolean Ulam game G_A. It is clear from the construction that if $A \cong B$ then $G_A \cong G_B$. We shall prove that, up to isomorphism, the map $A \to G_A$ is the inverse of the map $G \to A_G$.

Claim 6. $A_{G_A} = A^* \cong A$.

As a matter of fact, since the search space S of G_A consists of all points $x \in V$ such that $s(x) > 0$, the domain of all functions in A_{G_A} is the same as for the functions in A^*. Further, by (1) and (2), the greatest singular element of A^* is the same as the greatest singular element of A_{G_A}. Every singular element $t \in A^*$ is also a singular element of A_{G_A}, since the set $\{x \in A^* \mid t(x) > 0\} = \{x \in A \mid t(x) > 0\}$ belongs to the language L of G_A. Therefore, A^* is a subalgebra of A_{G_A}. Conversely, A_{G_A} is generated by singular elements of the form

(3) $\quad s_X(x) = s(x)$, if $x \in X$, and $s_X(x) = 0$, otherwise,

letting X range over all questions of the language L of G_A. Now, each s_X is also a member of A^*. Recalling Claim 3, the proof of Claim 6 is complete.

In order to prove that $G_{A_G} \equiv G$, assume $G = (S, m, L)$ to be quasiboolean. For each $X \in L$, define the singular element $s_X \in A_G$ as in (3). Let s be the greatest singular element of A_G. Also, for each $f \in A_G$, let $\mathrm{supp}(f) = \{x \in S \mid f(x) > 0\}$. For every $n = 1, 2, 3, \ldots$, let S_n and $S_{n,\infty}$ be defined by

$$S_n = s^{-1}(1/n) = \{x \in S \mid s(x) = 1/n\}, \text{ and}$$
$$S_{n,\infty} = S_n \cup S_{n+1} \cup S_{n+2} \cup \ldots .$$

Claim 7. For each singular $t \in A_G$ and each $n = 1, 2, \ldots$, there is $X \in L$ such that $\mathrm{supp}(t) \cap S_n = X$.

We first observe that for each $f \in A_G$ and each $x \in S$, there is $Y \in L$ such that $x \in Y$ and $f(y) = f(x)$ for all $y \in Y$. This is trivially true of the constant 1, and it is also true of each singular $s_X \in A_G$ $(X \in L)$, by Condition (ii) in the definition of quasiboolean games. Therefore, by definition of A_G, it is true of f. By Condition (iii), each ultrafilter of L containing S_n must be determined. Therefore the family $\{Z \cap S_n \mid Z \in L\}$ is a clopen basis making S_n into a Boolean space. Hausdorffness follows from Condition (i). For each $x \in \mathrm{supp}(t) \cap S_n$ there is a clopen $Y_x \subseteq S_n$ such that $x \in Y_x$, $Y_x \in L$, and $t(y) = 1/n$ for all $y \in Y_x$. Thus, $\mathrm{supp}(t) \cap S_n$ is open in S_n, indeed it is clopen, being the complement in S_n of the open set $\mathrm{supp}(s-t) \cap S_n$. There are sets $Y_{x_1}, \ldots, Y_{x_t}$ whose union Y coincides with the compact set $\mathrm{supp}(t) \cap S_n$. Since $Y \in L$, the proof of Claim 7 is complete.

Claim 8. For each $f \in A_G$ there is $n > 0$ such that $S_{n,\infty}$ is the union of finitely many pairwise disjoint elements of L, say $X_1, \ldots, X_k$, and there are integers $m_1, \ldots, m_k \geq 0$ such that over each X_i, either $f = m_i s_{X_i}$, or $f = 1 - m_i s_{X_i}$.

The proof is by induction on the number of basic MV operations in f. It is sufficient to only consider negation $a \to 1 - a$, and truncated addition $(a, b) \to a \oplus b = \min(1, a+b)$.

Basis. For each $X \in L$, the singular element s_X satisfies the claim, and so does the constant 1, by Condition (ii).

Negation step. Trivial.

Addition step. Here, $f = g \oplus h$ and, by induction hypothesis, g has a number $n' > 0$, together with sets $Y_1, \ldots, Y_u$ and integers $p_1, \ldots, p_u$ satisfying the claim. Similarly, h has n'', $Z_1, \ldots, Z_v$, $q_1, \ldots, q_r$. Let $n = \max(n', n'')$. Let $X_1, \ldots, X_k$ be the coarsest partition of $S_{n,\infty}$ refining both partitions induced by the Y's and the Z's. For each $i = 1, \ldots, k$, X_i is a member of L, since X_i is the intersection of $S_{n,\infty}$ with some $Y \in \{Y_1, \ldots, Y_u\}$ and some $Z \in \{Z_1, \ldots, Z_v\}$. Also, there are natural numbers $a_1, \ldots, a_k$ for g, and $b_1, \ldots, b_k$ for h such that over X_i, g and h satisfy the conditions of the claim. Note that $\{a_1, \ldots, a_k\} \subseteq \{p_1, \ldots, p_u\}$, and $\{b_1, \ldots, b_k\} \subseteq \{q_1, \ldots, q_r\}$. We examine the restriction of f over each X_i. It is no loss of generality to assume that $n > a_1 + \ldots + a_k + b_1 + \ldots + b_k$.

Case 1. $g = a_i s_{X_i}$ and $h = b_i s_{X_i}$.

Then $g \oplus h = (a_i + b_i) s_{X_i}$.

Case 2. $g = 1 - a_i s_{X_i}$ and $h = 1 - b_i s_{X_i}$.

Then $g \oplus h = 1$ (over X_i).

Case 3. $g = a_i s_{X_i}$ and $h = 1 - b_i s_{X_i}$.

Then if $a_i \geq b_i$, $g \oplus h = 1$, while if $a_i < b_i$, $g \oplus h = 1 - (b_i - a_i) s_{X_i}$.

This completes the proof of Claim 8.

Claim 9. For each singular $t \in A_G$ there is $X \in L$ such that $t = s_X$.

It is sufficient to find $X \in L$ such that $X = \mathrm{supp}(t)$. To this purpose, firstly, using Claim 8 we choose an $n > 0$, a partition $X_1, \ldots, X_k$ of $S_{n,\infty}$, and integers $m_1, \ldots, m_k \geq 0$ such that over each X_i, t coincides with $m_i s_{X_i}$. By the assumed singularity of t, we can safely exclude the case $t = 1 - m_i s_{X_i}$; further, $m_i \in \{0,1\}$. Therefore, $\mathrm{supp}(t) \cap S_{n,\infty}$ coincides with the union R of those X_j such that $m_j = 1$. Since $X_1, \ldots, X_k \in L$, it follows that $R \in L$. Secondly, by Claim 7 there are $Y_1, \ldots, Y_{n-1}$ in L such that for each $j = 1, \ldots, n-1$, $\mathrm{supp}(t) \cap S_j = Y_j$. Then $\mathrm{supp}(t) = X$, where $X = R \cup Y_1 \cup \ldots \cup Y_{n-1}$, and X is an element of L, as required.

Claim 10. The map $\psi: x \in S \to J_x = \{f \in A_G \mid f(x) = 0\}$ is a one-one correspondence between S and the set of maximal ideals of A_G not containing s.

As a matter of fact, for each $x \in S$, J_x is a maximal ideal of A_G. By Condition (i) in the definition of quasiboolean games, if $x' \neq x''$ then $J_{x'} \neq J_{x''}$. Then ψ is a one-one map from S into the set of maximal ideals of A_G not containing s. To prove that ψ is onto this set, let J be an arbitrary maximal ideal of A_G such that $s \notin J$. The set $\{X \in L \mid s_X \notin J\}$ is an ultrafilter U of L. If U were not determined by any point of S (absurdum hypothesis), then, since A_G is Archimedean and $s \notin J$, identifying A_G/J with a subalgebra of the real MV algebra $[0,1]$, there is a natural number $n > 0$ such that the singular element s/J equals $1/n$. On the other hand, Condition (iii) yields an $X \in U$ such that, say, $s(x) < 1/(3n)$ for all $x \in X$. It follows that $s_X/J < 1/n$. By definition of U, $s_X/J > 0$, whence $s_X/J = s/J = 1/n$, a contradiction. Therefore, U is determined, i.e., $U = \{X \in L \mid y \in X\}$ for some $y \in S$. We shall prove that $J = J_y$. For each $X \in L$ we have $s_X \notin J$ iff $s_X/J > 0$ iff $X \in U$ iff $y \in X$ iff $s_X(y) > 0$ iff $s_X \notin J_y$. Thus by Claim 9, J and J_y contain the same singular elements of A_G. If $f \in J$, then the singular element $f \wedge s \in J$, and hence $f \wedge s \in J_y$. From $s(y) > 0$ we get $f(y) = 0$. Thus, $J \subseteq J_y$, and by the assumed maximality of J, $J = J_y$. The proof is complete.

Claim 11. $G_{A_G} \cong G$.

As noted above, A_G is Archimedean, has a greatest singular element s, and is generated by its singular elements, together with the constant 1. There is an isomorphism η of A_G onto an MV algebra A of continuous functions over a compact Hausdorff space V, and maximal ideals of A canonically correspond to points in V. By Claim 10, it follows that ψ induces a one-one correspondence θ between S and the set $S^* = \{y \in V \mid (\eta(s))(y) > 0\} = \mathrm{supp}(\eta(s))$, in such a way that for each $x \in S$ and $f \in A_G$ we have $f(x) = (\eta(f))(\theta(x))$. By construction, the search space of G_{A_G} is S^*, since $\eta(s)$ is the greatest singular element of A. By (1) and (2), the lie bound function m^* of G_{A_G} satisfies the identity $m^*(\theta(x)) = m(x)$ for all $x \in S$. For each $X \in L$, the set $\theta(X) = \{\theta(x) \mid x \in X\}$ is in the language of G_{A_G}, being the support of the singular element $\eta(s_X) \in A$. Conversely, for each Y in the language of G_{A_G} we can write $Y = \mathrm{supp}(t)$ for some singular $t \in A$. By Claim 9, $t = \eta(s_X)$ for some $X \in L$. Then θ is an isomorphism between G and G_{A_G}, and the proof of Claim 11 is complete.

In conclusion, we have proved

3.1 THEOREM. *The map* $G \to A_G$ *induces a one-one correspondence between isomorphism classes of quasiboolean Ulam games, and isomorphism classes of Archimedean MV algebras with greatest singular element, and which are generated by their singular elements together with the constant element* 1.

An application of the results of [8, §3] yields

3.2 COROLLARY. *The map* $(H,u) \to \Gamma(H,u) \to G_{\Gamma(H,u)}$ *induces a one-one correspondence between isomorphism classes of Archimedean l-groups with strong unit, with a greatest singular element, generated by their singular elements and by the strong unit* u, *and isomorphism classes of quasiboolean Ulam games.*

Let us finally recapitulate our algebraization:

ULAM GAME	*MV-ALGEBRA*
game $G = (S, m, L)$	algebra A
search space S	maximal ideal space of A (*)
element $x \in S$	maximal ideal J in A (*)
lie bound for x is $m(x)$	cardinality of quotient chain A/J is $m(x)+2$
G is finite with k lies	A is a finite Post MV algebra of order $k+2$
G is finite with variable lie bound	A is a finite MV algebra
G is infinite Boolean with k lies	A is an infinite Post MV algebra of order $k+2$
G is Boolean	A is a finite product of Post MV algebras of finite order
G is Boolean with no lies	A is a Boolean algebra
quasiboolean, possibly unbounded lie bound	Archimedean, generated by singulars, with greatest singular
initial state of knowledge	the constant element 1
arbitrary state of knowledge	arbitrary element of A
question	subset of maximal ideal space (*)
opposite question	complementary subset
question in L	support of singular element of A
answer	dual of singular element
answer "yes, x is outside the search space"	dual $1 - s$ of greatest singular element
state of knowledge after some answers	conjunction of corresponding duals of singular elements
final state of knowledge	nonzero multiple of an atom

(*) When the number of lies is unbounded, elements in the search space correspond to maximal ideals of the algebra, with the exception of the only maximal ideal containing the greatest singular element.

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