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AILA PREPRINT
n. 4 novembre 1990

Direttore Responsabile: Ruggero Ferro
Iscrizione al Registro Stampa del Tribunale di Padova n. 1235 del 26.9.1990

Pubblicato con il contributo di:

Cassa di Risparmio di Padova e Rovigo

Stampa: Rotografica Padova

UNIVERSAL FUNCTIONS IN PARTIAL STRUCTURES

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December, 1989

Abstract

In this work we show that every structure $\mathcal{A}$ can be expanded to a partial structure $\mathcal{A}^*$ with universal functions for the class of polynomials on $\mathcal{A}^*$. We can embed $\mathcal{A}^*$ monomorphically in a total structure $\mathcal{A}^\circ$ that preserves universal functions of $\mathcal{A}^*$ and that is universal among such structures, i.e. $\mathcal{A}^\circ$ can be homomorphically embedded in every total structure that preserves universal functions of $\mathcal{A}^*$. Universal functions are the starting point for developing recursion theoretic tools in an $\mathcal{A}^*$ that satisfies some simple additional conditions.

1 Introduction

The most interesting aspect of partial structures is expressed in Grätzer (1968) as follows: "We can say that the language of partial algebras is the natural one if we want to talk about subsets of an algebra and the properties of operations on these subsets even if the subsets are not closed under all the operations." There is, however, another reason for studying partial structures, namely, the introduction of universal functions. In par.3 we show that every structure $\mathcal{A}$ for $\mathcal{L}$ can be expanded to a structure $\mathcal{A}^*$ for $\mathcal{L}^*$ with universal functions for the class of polynomial functions, and $\mathcal{A}^*$ cannot in general be total, following Cantor's theorem. This is developed in par.4 where it is shown that in $\mathcal{A}^*$, under some fairly general conditions, a relevant part of recursion theory can be developed and a result that resembles recursion theorem can be proved. In par.5 we introduce analogues of natural numbers and show that polynomial functions are closed with respect to primitive recursion (on these analogues). In pars.6, 7, 8 we study a kind of completion of $\mathcal{A}^*$ that satisfies some particular properties. The passage from a partial to a total structure, retaining some relevant characteristics of the given partial structure, is important because it is only in the field of total structures that all the usual methods of classical logic and model theory can be freely used. In our case we are interested in completions preserving the universal functions with respect to a given monomorphism, in a sense that will be made clear in par.6. In par.8 we construct a structure $\mathcal{A}^\circ$ and a monomorphism $\chi : \mathcal{A}^* \to \mathcal{A}^\circ$ that is universal among such completions in the following sense: for every completion $\mathcal{B}$ preserving universal functions of $\mathcal{A}^*$ with respect to a monomorphism $\theta : \mathcal{A}^* \to \mathcal{B}$, there is a morphism $\psi : \mathcal{A}^\circ \to \mathcal{B}$ extending θ, i.e. such that $\theta = \psi \circ \chi$. In what follows we reserve the word 'function' for total functions, but when we speak of partial functions we include total functions as a particular case.

2 Partial structures

Let $\mathcal{L}$ be a first order language without relational symbols except for identity. We can set $\mathcal{L} = \{x_n : n \in \omega\} \cup \{e_j : j \in J\} \cup \{f_i : i \in I\} \cup \{=\}$. A function $ar : I \to \omega - 1$ assigns to every function symbol its arity. We denote with $T(\mathcal{L})$ the set of terms of $\mathcal{L}$, but we simply write T if the language $\mathcal{L}$ is clear from the context. If $\mathcal{L}$ is a first order language as above, we can introduce the notion of a partial structure $\mathcal{A}$ for $\mathcal{L}$ as follows: we set $\mathcal{A} = \langle A, \{f_i^{\mathcal{A}} : i \in I\}, \{e_j^{\mathcal{A}} : j \in J\}\rangle$ where $f_i^{\mathcal{A}}$ is an $ar(i)$-ary partial function and $e_j^{\mathcal{A}} \in A$. As total functions are particular partial functions, total structures are to be seen as particular partial structures. When we speak in general of structures, partial structures are intended. We denote with $\doteq$ the identity symbol and the identity relation. The identity relation is supposed to be a total relation in every structure. In the sequel we abbreviate with $f(x)\uparrow$ (resp.$f(x)\downarrow$) the assertion "f is undefined (resp. defined) for argument x", and with $f(x) \simeq f(y)$ the assertion 'either $f(x)\downarrow$, $g(x)\downarrow$ and $f(x) = g(x)$, or $f(x)\uparrow$ and $g(x)\uparrow$'.

We say that a term $t \in T$ is an n-ary term and write $t(x_1 \dots x_n)$, if the variables occurring in it are among $x_1, \dots, x_n$. We denote with T_n the set of n-ary terms. To every $t \in T_n$ an n-ary partial function $t^{\mathcal{A}}$ on A is associated as follows. For all $a_1, \dots, a_n \in A$,

1. If t is x_i with $(i \le n)$, then $t^{\mathcal{A}}(a_1 \dots a_n) = a_i$

2. If t is e_j, then $e_j^{\mathcal{A}}(a_1 \dots a_n) = e_j^{\mathcal{A}}$.
3. Let t be $f(t_1(x_1 \dots x_n) \dots t_k(x_1 \dots x_n))$. If for all $i \le k$ there is b_i such that $t_i^{\mathcal{A}}(a_1 \dots a_n) = b_i$, then $t^{\mathcal{A}}(a_1 \dots a_n) \simeq f^{\mathcal{A}}(b_1 \dots b_k)$; if $t_i^{\mathcal{A}}(a_1 \dots a_n)\uparrow$ for some $i \le k$, then $t^{\mathcal{A}}(a_1 \dots a_n)\uparrow$.

We say that an n-ary partial function φ on A is a partial n-ary term function on $\mathcal{A}$ if $\varphi = t^{\mathcal{A}}$ for some $t \in T_n$, and we denote with $F^{(n)}(\mathcal{A})$ the set of all n-ary partial term functions on $\mathcal{A}$. We simply write $F^{(n)}$ when the structure is clear from the context.

If $\mathcal{L} \subseteq \mathcal{L}'$ and $\mathcal{A}$ is a structure for $\mathcal{L}$, we say that a structure $\mathcal{A}'$ for $\mathcal{L}'$ is an expansion of $\mathcal{A}$ to $\mathcal{L}'$ if $\mathcal{A}'$ and $\mathcal{A}$ have the same domain and the symbols in $\mathcal{L}$ receive the same interpretation in $\mathcal{A}$ as in $\mathcal{A}'$. For every $\mathcal{A}$ we can define a language $\mathcal{L}_A = \mathcal{L} \cup \{c_a : a \in A\}$, where every c_a is new with respect to $\mathcal{L}$ and c_a is different from c_b whenever $a \neq b$. We expand $\mathcal{A}$ to a structure $\mathcal{B}$ for $\mathcal{L}_A$ setting $c_a^{\mathcal{B}} = a$. To every $t \in T_n(\mathcal{L}_A)$ an n-ary function term $t^{\mathcal{B}}$ on $\mathcal{B}$ can be associated as before and such a function is called a polynomial on $\mathcal{A}$. So every polynomial arises from a term function by parametrization. We denote with $P^{(n)}(\mathcal{A})$ the set of n-ary polynomials on $\mathcal{A}$.

There are three kinds of homomorphism between partial structures. Following Grätzer (1968) we call them homomorphism, full homomorphism and strong homomorphism. We say that a total function $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism if

1. For all $j \in J$, $\varphi(e_j^{\mathcal{A}}) = e_j^{\mathcal{B}}$
2. For all $i \in I$, if ar(i) = n, then $\varphi(f_i^{\mathcal{A}}(a_1 \dots a_n)) = f_i^{\mathcal{B}}(\varphi(a_1) \dots \varphi(a_n))$ for all $a_1, \dots, a_n \in A$ such that $f_i^{\mathcal{A}}(a_1 \dots a_n)\downarrow$.

We say that $\varphi : \mathcal{A} \to \mathcal{B}$ is a *full* homomorphism if it is a homomorphism and, for all $a_1, \dots, a_n \in A$, if $f_i^{\mathcal{B}}(\varphi(a_1) \dots \varphi(a_n))\downarrow$ and $f_i^{\mathcal{B}}(\varphi(a_1) \dots \varphi(a_n)) = \varphi(b)$ for some $b \in A$, then there are $\bar{a}_i$, $i \le n$, and $\bar{b}$ such that $\varphi(\bar{a}_i) = \varphi(a_i)$, $\varphi(\bar{b}) = \varphi(b)$ and $f_i^{\mathcal{A}}(\bar{a}_1 \dots \bar{a}_n) = \bar{b}$. We say that $\varphi : \mathcal{A} \to \mathcal{B}$ is a *strong* homomorphism if it is a homomorphism and, for all $a_1, \dots, a_n \in A$, $f_i^{\mathcal{B}}(\varphi(a_1) \dots \varphi(a_n))\downarrow$ implies $f_i^{\mathcal{A}}(a_1 \dots a_n)\downarrow$.

It is easily seen that strong homomorphisms are also full, and every full homomorphism is a homomorphism. None of these inclusions can be reversed. An injective homomorphism (full resp. strong) $\varphi : \mathcal{A} \to \mathcal{B}$ is called a monomorphism (full resp.strong). If φ is bijective, full and strong homomorphism are the same thing and φ is said to be an isomorphism. In this case we say that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic and write $\mathcal{A} \simeq \mathcal{B}$. Isomorphism is an equivalence relation between partial structures. If φ is bijective and $\varphi : \mathcal{A} \to \mathcal{B}$ is only a homomorphism, then we do not introduce an analogous notion of isomorphism, but simply say that φ is a monomorphism onto $\mathcal{B}$. In fact we reserve the name 'isomorphism' for a morphism $\varphi : \mathcal{A} \to \mathcal{B}$ that can be reversed, so that an equivalence relation $\simeq$ between partial structures results. If $\varphi : \mathcal{A} \to \mathcal{B}$ is only a monomorphism onto $\mathcal{B}$, then $\varphi^{-1} : \mathcal{B} \to \mathcal{A}$ is not in general a morphism.

The three notions of monomorphism give rise to three different notions of substructure. We say that $\mathcal{A}$ is a *weak substructure* of $\mathcal{B}$, $\mathcal{A} \subseteq \mathcal{B}$, if $id_A : \mathcal{A} \to \mathcal{B}$ is a monomorphism. In this case we have $A \subseteq B$ and $f_i^{\mathcal{A}} \subseteq f_i^{\mathcal{B}} \cap A^{n+1}$. We say that $\mathcal{A}$ is a *relative substructure* of $\mathcal{B}$, $\mathcal{A} \subseteq_r \mathcal{B}$, if $id_A : \mathcal{A} \to \mathcal{B}$ is a full monomorphism. In this case we have $f_i^{\mathcal{A}} = f_i^{\mathcal{B}} \cap A^{n+1}$. We say that $\mathcal{A}$ is a *substructure* of $\mathcal{B}$, $\mathcal{A} \subseteq_s \mathcal{B}$, if $id_A : \mathcal{A} \to \mathcal{B}$ is a strong homomorphism. In this case $f_i^{\mathcal{A}}$ is $\{(x,y) \in f_i^{\mathcal{B}} : x \in A^n\}$, i.e. the restriction of $f_i^{\mathcal{B}}$ to A^n. The main characteristic of $\mathcal{A} \subseteq \mathcal{B}$ is that we can have $A = B$ and still $f_i^{\mathcal{A}} \subset f_i^{\mathcal{B}}$, because $f_i^{\mathcal{A}}$ is less defined than $f_i^{\mathcal{B}}$: so $f_i^{\mathcal{A}}$ cannot be conceived as the restriction of $f_i^{\mathcal{B}}$ to A. If $\mathcal{A} \subseteq_r \mathcal{B}$, a function $f_i^{\mathcal{A}}$ can be less defined than $f_i^{\mathcal{B}}$ only if $f_i^{\mathcal{B}}(a) = b \in B - A$ for some $a \in A$, so total functions can become partial. If $\mathcal{A} \subseteq_s \mathcal{B}$, then A must be closed with respect to functions $f_i^{\mathcal{B}}$ of $\mathcal{B}$.

Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a monomorphism. We denote with $\varphi[\mathcal{A}]$ the structure induced by φ on $\varphi[A]$ as follows. If we set $\mathcal{C} = \varphi[\mathcal{A}]$, then we have: $f^{\mathcal{C}}(\varphi(a_1)\ldots\varphi(a_n)) \downarrow$ and $f^{\mathcal{C}}(\varphi(a_1)\ldots\varphi(a_n)) = \varphi(f^{\mathcal{A}}(a_1\ldots a_n))$ if $f^{\mathcal{A}}(a_1\ldots a_n \downarrow$, $f^{\mathcal{C}}(\varphi(a_1)\ldots\varphi(a_n)) \uparrow$ if $f^{\mathcal{A}}(a_1\ldots a_n) \uparrow$. If φ is a monomorphism, then $\mathcal{A} \simeq \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{B}$, if φ is also full then $\mathcal{C} \subseteq_r \mathcal{B}$, if φ is strong then $\mathcal{C} \subseteq_s \mathcal{B}$.

Lemma 2.1 *If $\varphi : \mathcal{A} \to \mathcal{B}$ is a morphism and $t(x_1\ldots x_n) \in T_n$, then*

$$\varphi(t^{\mathcal{A}}(a_1\ldots a_n)) = t^{\mathcal{B}}(\varphi(a_1)\ldots\varphi(a_n))$$

for all $a_1\ldots a_n \in A$ such that $t^{\mathcal{A}}(a_1\ldots a_n) \downarrow$.

Proof. By induction on t. Let $t = x_i$. Then

$$\varphi(x_i^{\mathcal{A}}(a_1\ldots a_n)) = \varphi(a_i) = x_i^{\mathcal{B}}(\varphi(a_1)\ldots\varphi(a_n)).$$

Let $t = e_j$. Then

$$\varphi(e_j^{\mathcal{A}}(a_1\ldots a_n)) = \varphi(e_j^{\mathcal{A}}) = e_j^{\mathcal{B}}(\varphi(a_1)\ldots\varphi(a_n)).$$

Let $t = f(t_1\ldots t_k)$. If $t^{\mathcal{A}}(a_1\ldots a_n) \downarrow$, then there are $b_1,\ldots,b_k$ such that $t_i^{\mathcal{A}}(a_1\ldots a_n) = b_i$, $i \leq k$, and $f^{\mathcal{A}}(b_1\ldots b_k) \downarrow$. By induction hypothesis $\varphi(b_i) = t_i^{\mathcal{B}}(\varphi(a_1)\ldots\varphi(a_n))$. Then

$$\varphi(t^{\mathcal{A}}(a_1\ldots a_n)) = \varphi(f^{\mathcal{A}}(b_1\ldots b_k)) = f^{\mathcal{B}}(\varphi(b_1)\ldots\varphi(b_k))) =$$

$$t^{\mathcal{B}}(\varphi(a_1)\ldots\varphi(a_n)). \ \square$$

3 Universal functions

In this paragraph we introduce universal functions, a concept that is closely connected with partial functions. If $\mathcal{F} = \bigcup\{\mathcal{F}_n : n \in \omega\}$, where $\mathcal{F}_n$ is a set of functions $A^n \to A$, we say that $\varphi_n : A^{n+1} \to A$ is a universal function for $\mathcal{F}_n$ if there is a surjective function $\nu_n : A \to \mathcal{F}_n$ such that, for all $a, b_1\ldots b_n \in A$, $\varphi_n(a, b_1\ldots b_n) = \nu_n(a)(b_1\ldots b_n)$. We say that the element $a \in A$ is an index of $\nu_n(a) \in \mathcal{F}_n$. If for every n there is a $\varphi_n \in \mathcal{F}$ as above, then we say that $\mathcal{F}$ is closed with respect to universal functions. It is clear that the behaviour of every n-ary function of $\mathcal{F}$ can be simulated by a parametrization of φ_n. It is easy to show, by Cantor's diagonal method, that if $\mathcal{F}$ is a set of total functions closed with respect to composition and there is a $h \in \mathcal{F}$ such that $h(a) \neq a$, for all $a \in A$, then $\mathcal{F}$ cannot be closed with respect to universal functions: if we want $\varphi_n \in \mathcal{F}$ then we must admit partial functions in $\mathcal{F}$. In fact, let $\psi(x) = h(\varphi_1(x,x))$ and let $\nu_1(a) = \psi$. Then $\varphi_1(a,a) = \psi(a) = h(\varphi_1(a,a)) \neq \varphi_1(a,a)$. As we cannot have a contradiction, we must have $\varphi_1(a,a) \uparrow$.

From the preceding argument it is clear that if $\mathcal{A}$ is a total structure for $\mathcal{L}$ and there is a polynomial function $t^{\mathcal{A}}$ such that $t^{\mathcal{A}}(a) \neq a$, for all $a \in A$, then the set of polynomial functions on $\mathcal{A}$ is not closed with respect to universal functions, i.e. the universal functions cannot be polynomials. However, if we admit partial structures, then we can show that every partial or total structure $\mathcal{A}$ for $\mathcal{L}$ can be expanded to a partial structure $\mathcal{A}^*$ for a language $\mathcal{L}^* \supset \mathcal{L}$ in which the universal functions for polynomials on $\mathcal{A}^*$ are polynomials.

In the first place we set $\mathcal{L}^* = \mathcal{L} \cup \{g_n : n \in \omega\} \cup \{c_a : a \in A\}$, where $\mathcal{L}$, $\{g_n : n \in \omega\}$ and $\{c_a : a \in A\}$ are pairwise disjoint, $c_a \neq c_b$ whenever $a \neq b$, and g_n is an n+1-ary function symbol, for all n. In the following we always suppose that $\mid A \mid$ is infinite and $\mid \mathcal{L} \mid \leq \mid A \mid$, so that $\mid \mathcal{L}^* \mid = \sup(\mid \mathcal{L} \mid, \aleph_0, \mid A \mid) = \mid A \mid$. We consider an expansion $\mathcal{A}^*$ of $\mathcal{A}$ to $\mathcal{L}^*$. Let a surjective

$\nu_n : A \to T_n(\mathcal{L}^*)$ be given, for all n. We say that $\mathcal{A}^*$ is *closed with respect to universal functions* (for the given $\{\nu_n : n \in \omega\}$) if

$$g_n^{\mathcal{A}^*}(a, b_1 \dots b_n) \simeq \nu_n(a)^{\mathcal{A}^*}(b_1 \dots b_n)$$

for all $a, b_1 \dots b_n \in A$. In this case we also say that ν_n is an indexing of $P^{(n)}(\mathcal{A}^*)$, the n-ary partial polynomials on $\mathcal{A}$. From now on we reserve symbols g_n for universal functions and symbols f_i for other functions.

We must now show that structures closed with respect to universal functions do exist. Doubts may arise as to their existence because universal functions $g_n^{\mathcal{A}^*}$ are functions of a particular kind. They presuppose the existence of all partial polynomials on $\mathcal{A}^*$, and if $\mathcal{A}^*$ is closed with respect to universal functions, then $g_n^{\mathcal{A}^*}$ is itself a partial polynomial on $\mathcal{A}^*$. This autoreferential feature can be disturbing and we could be led to think that such structures (if any) are more the result of a lucky coincidence than the result of a proper construction.

Theorem 3.1 *If $\mathcal{A}$ is a partial or total structure for $\mathcal{L}$, $\mid A \mid$ is infinite and $\mid \mathcal{L} \mid \leq \mid A \mid$, then $\mathcal{A}$ can be expanded to a partial structure $\mathcal{A}^*$ for $\mathcal{L}^*$ that is closed with respect to universal functions.*

Proof. We define by induction a sequence $\{\mathcal{A}_k : k \in \omega\}$ of structures for $\mathcal{L}^*$ as follows. For all k, $\mathcal{A}_k$ is an expansion of $\mathcal{A}$ to $\mathcal{L}^*$; so we can denote $\mathcal{A}_k$ with $(\mathcal{A}, g_n^{\mathcal{A}_k})_{n \in \omega}$, because $f_i^{\mathcal{A}_k} = f_i^{\mathcal{A}}$ and $e_j^{\mathcal{A}_k} = e^{\mathcal{A}}$. In $\mathcal{A}_0$ we set $g_n^{\mathcal{A}_0} = \emptyset$, the completely undefined function, for all $n \in \omega$. We proceed by induction; so we suppose we have defined $\mathcal{A}_k$ and we define $\mathcal{A}_{k+1}$ setting

$$g_n^{\mathcal{A}_{k+1}}(a, b_1 \dots b_n) = d \text{ iff } \nu_n(a)^{\mathcal{A}_k}(b_1 \dots b_n) = d$$

for all $a, b_1 \dots b_n, d \in A$.

We easily show that $g_n^{\mathcal{A}_k} \subset g_n^{\mathcal{A}_{k+1}}$, for all k, and so $\mathcal{A}_k \subset \mathcal{A}_{k+1}$. If $k = 0$, it is obvious. We suppose that $g_n^{\mathcal{A}_{k-1}} \subset g_n^{\mathcal{A}_k}$ and show that $g_n^{\mathcal{A}_k} \subset g_n^{\mathcal{A}_{k+1}}$. If $g_n^{\mathcal{A}_k}(a, b_1 \dots b_n) = d$, then, by definition of $g_n^{\mathcal{A}_k}$, $\nu_n(a)^{\mathcal{A}_{k-1}}(b_1 \dots b_n) = d$. As $\mathcal{A}_{k-1} \subset \mathcal{A}_k$, by lemma 2.1 we have $\nu_n(a)^{\mathcal{A}_k}(b_1 \dots b_n) = d$; so we can conclude with $g_n^{\mathcal{A}_{k+1}}(a, b_1 \dots b_n) = d$.

We set $\mathcal{A}^* = \bigcup\{\mathcal{A}_k : k \in \omega\}$ and show that $\mathcal{A}^*$ is closed with respect to universal functions. If $g_n^{\mathcal{A}^*}(a, b_1 \dots b_n) = d$, for some d, then there is a $k > 0$ such that $g_n^{\mathcal{A}_k}(a, b_1 \dots b_n) = d$. By definition of $g_n^{\mathcal{A}_k}$, we have $\nu_n(a)^{\mathcal{A}_{k-1}}(b_1 \dots b_n) = d$. Since $\mathcal{A}_{k-1} \subset \mathcal{A}^*$, we have $\nu_n(a)^{\mathcal{A}^*}(b_1 \dots b_n) = d$ by lemma 2.1. We show that $g_n^{\mathcal{A}^*}(a, b_1 \dots b_n) \uparrow$ implies $\nu_n(a)^{\mathcal{A}^*}(b_1 \dots b_n) \uparrow$. We need the following:

Lemma 3.2 *For all $t(x_1 \dots x_n) \in T(\mathcal{L}^*)$, if $t^{\mathcal{A}^*}(a_1 \dots a_n) = b$ then, for some k, $t^{\mathcal{A}_k}(a_1 \dots a_n) = b$.*

Proof. By induction on t. If $t = x_i$, $t = e_j$ or $t = c_a$ it is obvious. Induction step. Case a). Let $t = f(t_1 \dots t_k)$, where $f \in \mathcal{L}$. If $t^{\mathcal{A}^*}(a_1 \dots a_n) = b$, then there are $b_1, \dots, b_k$ such that $t_i^{\mathcal{A}^*}(a_1 \dots a_n) = b_i$, for $i \leq k$, and $f^{\mathcal{A}^*}(b_1 \dots b_k) = b$. By induction hypothesis we can find, for all $i \leq k$, a structure $\mathcal{A}_{\varphi(i)}$ such that $t_i^{\mathcal{A}_{\varphi(i)}}(a_1 \dots a_n) = b_i$. Let $p = \max\{\varphi(i) : i \leq k\}$. Then $t_i^{\mathcal{A}_p}(a_1 \dots a_n) = b_i$, for all $i \leq k$. Since the interpretation of f does not vary in the various $\mathcal{A}_i$ and in $\mathcal{A}^*$, we have

$$b = f^{\mathcal{A}_p}(b_1 \dots b_k) = f^{\mathcal{A}_p}(t_1^{\mathcal{A}_p}(a_1 \dots a_n) \dots t_k^{\mathcal{A}_p}(a_1 \dots a_n)) = t^{\mathcal{A}_p}(a_1 \dots a_n).$$

Case b). Let $t = g_k(t_1 \dots t_{k+1})$. As before, there are $b_1, \dots, b_{k+1}$ such that $t_i^{\mathcal{A}^*}(a_1 \dots a_n) = b_i$ and $g_k^{\mathcal{A}^*}(b_1 \dots b_{k+1}) = b$. So there is a q such that $g_k^{\mathcal{A}_q}(b_1 \dots b_{k+1}) = b$. By inductive hypothesis, there are structures $\mathcal{A}_{\varphi(i)}$ such that $t_i^{\mathcal{A}_{\varphi(i)}}(a_1 \dots a_n) = b_i$. Let $p = \max\{\varphi(i) : i \leq k+1\}$ and $s = \max\{p, q\}$, then

$$b = g_k^{\mathcal{A}_s}(t_1^{\mathcal{A}_s}(a_1 \dots a_n) \dots t_{k+1}^{\mathcal{A}_s}(a_1 \dots a_n)) = t^{\mathcal{A}^s}(a_1 \dots a_n).$$

The proof of the lemma is now complete and we can return to the main proof. If $\nu_n(a)^{\mathcal{A}^*} \downarrow$ then there is a b such that

$$\nu_n(a)^{\mathcal{A}^*}(b_1 \dots b_n) = b$$

and, by the lemma, there is a k such that

$$\nu_n(a)^{\mathcal{A}_k}(b_1 \dots b_n) = b \text{ and } g_n^{\mathcal{A}_{k+1}}(a, b_1 \dots b_n) = b$$

so that $g_n^{\mathcal{A}^*}(a, b_1 \dots b_n) \downarrow$. By contraposition we have the desired result. □

So far we have proved that, for any $\mathcal{A}$, there is at least one expansion $\mathcal{A}^*$ of $\mathcal{A}$ that is closed with respect to universal functions. We reserve the notation $\mathcal{A}^*$ for the expansion obtained in theorem 3.1. The next theorem shows that every expansion of $\mathcal{A}$ that is closed with respect to universal functions must include $\mathcal{A}^*$ as a substructure.

Theorem 3.3 *If $\mathcal{A}$ is a partial structure for $\mathcal{L}$, $\mathcal{A}'$ is an expansion of $\mathcal{A}$ to $\mathcal{L}^*$ that is closed with respect to universal functions (for a given set of indexing functions $\{\nu_n : n \in \omega\}$), then $\mathcal{A}^* \subset \mathcal{A}'$.*

Proof. We show that, for all k, $\mathcal{A}_k \subset \mathcal{A}'$. If $k = 0$, it is obvious. Suppose $\mathcal{A}_k \subset \mathcal{A}'$. If $g_n^{\mathcal{A}_{k+1}}(a, b_1 \ldots b_n) = d$, then $\nu_n(a)^{\mathcal{A}_k}(b_1 \ldots b_n) = d$. As $\mathcal{A}_k \subset \mathcal{A}'$, we have

$$\nu_n(a)^{\mathcal{A}'}(b_1 \ldots b_n) = d \text{ and } g_n^{\mathcal{A}'}(a, b_1 \ldots b_n) = d.$$

This proves that $g_n^{\mathcal{A}_{k+1}} \subset g_n^{\mathcal{A}'}$, and $\mathcal{A}_{k+1} \subset \mathcal{A}'$. So $\mathcal{A}^* = \bigcup\{\mathcal{A}_k : k \in \omega\} \subset \mathcal{A}'$ □.

4 The recursion theorem

A peculiar feature of recursive definitions is that a function can occur as a part of its definition. This kind of self-reference exists, because every polynomial has an index (or better, infinitely many indices) and can be viewed as an object in the domain, therefore we can apply a polynomial to arguments which include the index of the polynomial itself. However, if we want to obtain an analogue of recursion theorem, we must start from a structure $\mathcal{A}$ that satisfies some minimal conditions. Firstly, we suppose that $\mathcal{A}$ is a structure for $\mathcal{L}$, where $\mathcal{L}$ contains a constant c_a for every $a \in A$, different constants being assigned to different objects, so that term-functions and polynomials on $\mathcal{A}$ amount to the same thing.

We now explain our second condition. If $t(x_1 \ldots x_n, x_{n+1} \ldots x_{n+m}) \in T_{n+m}$ and $a_1, \ldots, a_m \in A$, then the n-ary function $t^{\mathcal{A}}(x_1 \ldots x_n, a_1 \ldots a_m)$ that arises from a parametrization of $t^{\mathcal{A}}(x_1 \ldots x_n, x_{n+1} \ldots x_{n+m})$ is the polynomial $t^{\mathcal{A}}(x_1 \ldots x_n, c_{a_1}, \ldots, c_{a_m})$. As there is a strict connection between parametrizing a polynomial and carrying out a substitution in the corresponding term, we can define, for every m, n, an operator $\mathcal{S}_{n,m} : P^{(n+m)}(\mathcal{A}) \times A^m \to P^{(n)}(\mathcal{A})$, setting

$$\mathcal{S}_{n,m}(t^{\mathcal{A}}(x_1 \ldots x_n, x_{n+1} \ldots x_{n+m}), a_1 \ldots a_m) = t^{\mathcal{A}}(x_1 \ldots x_n, c_{a_1}, \ldots, c_{a_m}).$$

Every such operator induces a function $\varphi_{n,m} : A \times A^m \to A$, since every polynomial has an index in A. Our second condition amounts to the requirement that $\varphi_{n,m}$ is a polynomial on $\mathcal{A}$.

It can easily be shown that every $\mathcal{A}$ for $\mathcal{L}$ can be expanded to an $\mathcal{A}^s$ for $\mathcal{L}^s = \mathcal{L} \cup \{s_{n,m} : n, m \in \omega\}$ that satisfies our second condition. Firstly we define, for every n, an indexing function $\nu_n : A \to T_n(\mathcal{L}^s)$, as in the preceding paragraph. Then, by the axiom of choice, we assign to every ν_n a function $\nu_n^{-1} : T_n(\mathcal{L}^s) \to A$ such that $\nu_n(\nu_n^{-1}(t)) = t$, for all $t \in T_n(\mathcal{L}^s)$. Finally we set, for all $a_1, \ldots, a_m, b \in A$,

$$s_{n,m}^{\mathcal{A}^s}(b, a_1 \ldots a_m) = \nu_n^{-1}(\nu_{n+m}(b)(x_{n+1} \backslash c_{a_1} \ldots x_{n+m} \backslash c_{a_m}))$$

where

$$\nu_{n+m}(b)(x_{n+1} \backslash c_{a_1} \ldots x_{n+m} \backslash c_{a_m})$$

arises from

$$\nu_{n+m}(b)(x_1 \ldots x_n, x_{n+1} \ldots x_{n+m})$$

by substituting c_{a_i} to x_{n+i}, for all $i \leq m$. It can be easily verified that, for all $a_1, \ldots, a_m, e_1, \ldots, e_n, b \in A$,

$$\nu_n(s_{n,m}^{\mathcal{A}^s}(b, a_1 \ldots a_m))^{\mathcal{A}^s}(e_1 \ldots e_n) \simeq \nu_{n+m}(b)^{\mathcal{A}^s}(e_1 \ldots e_m, a_1 \ldots a_m).$$

Before stating our third condition we introduce analogues of natural numbers in $\mathcal{A}$. From our second condition we can suppose that indexing functions ν_n have been given and that parametrization functions are available in $\mathcal{A}$. In what follows, to save notations, we write $\{a\}_n$ instead of $\nu_n(a)^{\mathcal{A}}$, omitting the index when it is clear from the context. Let a be such that $\{a\}_1 = \mathrm{id}(x)$, i.e. a is an index of the identity function, then a can be seen as the number zero and we write 0 instead of a. For the same reason, we write $\bar{0}$ instead of c_a, the numeral zero. We define a successor function as follows. Let $\{d\}_2 = \pi_2^2(x_1, x_2)$. We define the term $s(x) = s_{1,1}(c_d, x)$: $s^{\mathcal{A}}$ can be seen as a successor function. In fact we have $\{s^{\mathcal{A}}(0)\} = \{s_{1,1}^{\mathcal{A}}(d, 0)\} = \pi_2^2(x_1, 0)$, i.e. the 1-ary constant function that always gives 0. If we write 1 instead of $s^{\mathcal{A}}(0)$, we have $\{1\} \neq \{0\}$ and $1 \neq 0$, because different indices correspond to different functions. In the same way we have $\{2\} = \{s^{\mathcal{A}}(s^{\mathcal{A}}(0))\}$, $\{2\} \neq \{0\}$, $\{2\} \neq \{1\}$, that implies $2 \neq 0$, $2 \neq 1$, and so on. In this way, if we denote $s^{\mathcal{A}}(\ldots s^{\mathcal{A}}(0)\ldots)$ (n times) with n, we obtain an infinite sequence $0, 1, \ldots, n, \ldots$ of objects, different in pairs, that can be viewed as the set ω of natural numbers. The whole set ω will not be useful until the next paragraph, but we immediately need 0 and 1 because our third condition on $\mathcal{A}$ runs as follows: we suppose that the characteristic function of identity is represented by a polynomial $r^{\mathcal{A}}$ such that, for all a, $b \in A$,

$$r^{\mathcal{A}}(a, b) = 0 \quad \text{if} \quad a = b$$
$$r^{\mathcal{A}}(a, b) = 1 \quad \text{if} \quad a \neq b$$

The following fact will be used in the next lemma: there is a polynomial $h^{\mathcal{A}}$ such that $h^{\mathcal{A}}(a) \neq a$, for all $a \in A$: simply define $h(x) = r(x, \bar{1})$.

At this point we can expand a structure $\mathcal{A}$ satisfying the three conditions above to a structure $\mathcal{A}^*$ for $\mathcal{L}^* = \mathcal{L} \cup \{g_n : n \in \omega\}$ as in the preceding paragraph: $\mathcal{A}^*$ is closed with respect to universal functions and we can show that an analogue of the recursion theorem holds in $\mathcal{A}^*$.

Lemma 4.1 *The completely undefined 1-ary function is a polynomial on $\mathcal{A}^*$.*

Proof. As $\mathcal{A}^*$ is a partial structure, there is at least a term $t(c_b)$ such that $t^{\mathcal{A}^*}(c_b)\uparrow$. For instance, let $p(x) = h(g_1(x,x))$, where h is such that $h^{\mathcal{A}^*}$ is total and $h^{\mathcal{A}^*}(a) \neq a$, for all $a \in A$. (Remember the remark after our third condition on $\mathcal{A}$.) Let $\{b\}_1 = p^{\mathcal{A}^*}$, then we must have $g_1^{\mathcal{A}^*}(b,b)\uparrow$, otherwise

$$g_1^{\mathcal{A}^*}(b,b) = h^{\mathcal{A}^*}(g_1^{\mathcal{A}^*}(b,b)) \neq g_1^{\mathcal{A}^*}(b,b).$$

Thus we can set $t(c_b) = g_1(c_b, c_b)$. If we can show that there is a term $q(x)$ such that, for all $a \in A$, $q^{\mathcal{A}^*}(a) = b$, then $t^{\mathcal{A}^*}(q^{\mathcal{A}^*}(a))\uparrow$, for all a, and so $t(q(x))$ denotes the completely undefined polynomial on $\mathcal{A}^*$. But the constant c_b can be conceived as an 1-ary term $q(x)$ such that $q^{\mathcal{A}^*}(a) = b$ for all a. We can also give a more formal definition of $q(x)$. Let $d \in A$ be such that $\{d\}_2 = \pi_2^2(x_1, x_2)$, the 2-ary projection function that gives the second coordinate. We have

$$\{s_{1,1}^{\mathcal{A}^*}(d,b)\}_1(e) = \pi_2^2(e,b) = b$$

for all $e \in A$. So we can set $q(x) = g_1(s_{1,1}(c_d, c_b), x)$, because

$$g_1^{\mathcal{A}^*}(s_{1,1}^{\mathcal{A}^*}(d,b), e) = \{s_{1,1}^{\mathcal{A}^*}(d,b)\}_1(e). \ \square$$

Let $\{a_\xi : \xi < \alpha\}$ be an enumeration of A, where $\alpha = \mid A \mid$. As every a_ξ can be viewed as an index of a 1-ary polynomial $\{a_\xi\}$, we can associate to every a_ξ the following α-termed sequence of polynomials:

$$\{\{a_\xi\}(a_0)\}, \ldots, \{\{a_\xi\}(a_\eta)\}, \ldots \quad (\eta < \alpha).$$

If $\{a_\xi\}(a_\eta)\uparrow$, we place the completely undefined function $\emptyset$ at the η-th place, and $\emptyset$ is a polynomial by the lemma above.

The following lemma shows that if $\{a_\xi\}$ is not total, then there is a total $\{a_k\}$ that gives rise to the same sequence, and the index a_k can be recovered from a_ξ by means af a total polynomial.

Lemma 4.2 *There is a total polynomial $q^{\mathcal{A}^*}(x)$ such that, for all $a \in A$,*

1. *$\{q^{\mathcal{A}^*}(a)\}$ is a total polynomial*
2. *$\{\{q^{\mathcal{A}^*}(a)\}(b)\} = \{\{a\}(b)\}$, for all $b \in A$.*

Proof. Let $\{e\}_3 = g_1^{\mathcal{A}^*}(g_1^{\mathcal{A}^*}(x_2,x_3),x_1)$. Then by parametrization,

$$\{s_{1,2}^{\mathcal{A}^*}(e,a_1,a_2)\}(a_3) \simeq \{\{a_1\}(a_2)\}(a_3)$$

for all a_1, a_2, $a_3 \in A$. Let $\{d\}_2 = s_{1,2}^{\mathcal{A}^*}(e, x_2, x_3)$. Then

$$\{s_{1,1}^{\mathcal{A}^*}(d,a_1)\}(a_2) \simeq s_{1,2}^{\mathcal{A}^*}(e,a_1,a_2)$$

for all $a_1,a_2 \in A$. We can therefore conclude that

$$\{\{s_{1,1}^{\mathcal{A}^*}(d,a_1)\}(a_2)\}(a_3) \simeq \{s_{1,2}^{\mathcal{A}^*}(e,a_1,a_2)\}(a_3) \simeq \{\{a_1\}(a_2)\}(a_3)$$

for all a_1, a_2, $a_3 \in A$. Then we have $\{\{s_{1,1}^{\mathcal{A}^*}(d,a)\}(b)\} = \{\{a\}(b)\}$, for all a, $b \in A$, and this proves (2) setting $q(x) = s_{1,1}(c_d, x)$, because $s_{1,1}^{\mathcal{A}^*}(d,x)$ is total. As for (1), we observe that $\{q(a)\} = s_{1,2}^{\mathcal{A}^*}(e,a,x_2)$, a total function. $\square$

Theorem 4.3 (Recursion Theorem) *For any $t(x) \in T_1(\mathcal{L}^*)$ such that $t^{\mathcal{A}^*}$ is a total polynomial, there is an $a \in A$ such that $\{a\} = \{t^{\mathcal{A}^*}(a)\}$.*

Proof. Take as given an enumeration $\{a_\xi : \xi < \alpha\}$ of A, where $\mid A \mid = \alpha$. As we have seen before, every $\{a_\xi\}$ gives rise to an enumeration of polynomials $\{\{a_\xi\}(a_\eta)\}$, as η varies in α. If we allow the variation of ξ in α, we obtain a square matrix of polynomials. When $\{a_\xi\}(a_\eta)\uparrow$, we set the 1-ary polynomial $\emptyset$ at the intersection of the ξ-th row with the η-th column. We observe that the diagonal of the matrix is the enumeration given by $g_1^{\mathcal{A}^*}(x,x)$. Let $\{a_k\} = g_1^{\mathcal{A}^*}(x,x)$: then the diagonal occurs as the k-th row. If $q(x)$ is as in the lemma above, then $\{q^{\mathcal{A}^*}(a_k)\}$ is total and gives rise to the same enumeration as $\{a_k\}$. So the diagonal occurs as the $q^{\mathcal{A}^*}(a_k)$- th row and

$$\{\{q^{\mathcal{A}^*}(a_k)\}(a_\eta)\} = \{\{a_k\}(a_\eta)\} = \{\{a_\eta\}(a_\eta)\} \tag{1}$$

by point (2) of the lemma above. Let $p(x) = t(g_1(q(c_{a_k}), x))$, then $\{a_\sigma\} = p^{\mathcal{A}^*}$, for some $\sigma < \alpha$, and $p^{\mathcal{A}^*}$ is total by its definition. Then

$$\{\{a_\sigma\}(a_\sigma)\} = \{t^{\mathcal{A}^*}(g_1^{\mathcal{A}^*}(q^{\mathcal{A}^*}(a_k), a_\sigma))\} = \{t^{\mathcal{A}^*}(\{q^{\mathcal{A}^*}(a_k)\}(a_\sigma))\}$$

by definition of $p(x)$, and

$$\{\{a_\sigma\}(a_\sigma)\} = \{\{q^{\mathcal{A}^*}(a_k)\}(a_\sigma)\}$$

by (1). The theorem follows if we set $a = \{q^{\mathcal{A}^*}(a_k)\}(a_\sigma)$. $\square$

The following Corollary shows that the index of a polynomial can occur in the definition of the polynomial itself.

Corollary 4.4 *For any* $p(x_1, x_2) \in T_2(\mathcal{L}^*)$*, there is an* $a \in A$ *such that* $\{a\}(b) \simeq p^{\mathcal{A}^*}(b, a)$*, for all* $b \in A$*.*

Proof. Let $\{e\}_2 = p^{\mathcal{A}^*}$. If we can prove that there is an a such that $\{a\} = \{s^{\mathcal{A}^*}_{1,1}(e, a)\}$, then we have achieved our objective. Let $t(x) = s_{1,1}(c_e, x)$, then $t^{\mathcal{A}^*}$ is total and by the theorem above there is an a such that $\{a\} = \{t^{\mathcal{A}^*}(a)\}$. (This corollary can obviously be generalized to $p(x_1 \dots x_n)$, for all n.) □

Let Γ be an operator on $\mathcal{A}^*$ such that $\Gamma : P^{(1)}(\mathcal{A}^*) \to P^{(1)}(\mathcal{A}^*)$, then Γ is said to be a polynomial operator if there is a total polynomial $t^{\mathcal{A}^*} : A \to A$ that describes the behaviour of Γ of as follows: for any polynomial $q^{\mathcal{A}^*}$ on $\mathcal{A}^*$ and any $a \in A$ such that $q^{\mathcal{A}^*} = \{a\}$, $\Gamma(q^{\mathcal{A}^*}) = t^{\mathcal{A}^*}(a)$.

Theorem 4.5 *If* Γ *is a polynomial operator on* $\mathcal{A}^*$*, then* Γ *has a fixed point.*

Proof. Immediate, by the preceding corollary. □

5 Primitive recursion

As we have just seen in the preceding paragraph, there is a set of analogues of natural numbers in $\mathcal{A}^*$ and this fact can be used to introduce primitive recursion. We say that a partial function $f : A^2 \to a$ is defined by primitive recursion from the partial functions $q : A \to A$ and $h : A^3 \to A$ if the restriction of (the domain of) f to $\omega \times A$ is such that

$$\begin{aligned} f(0, z) &\simeq g(z) \\ f(n+1, z) &\simeq h(f(n.z), n, z) \end{aligned}$$

We'll prove that the set of partial polynomials on $\mathcal{A}^*$ is closed with respect to primitive recursion i.e. if g and h are partial polynomials on $\mathcal{A}^*$, so is f.

In the first place we must show that a predecessor function and a pairing function are available as polynomials. We observe that the polynomial $\alpha(x) = g_1^{\mathcal{A}^*}(x, 0)$ is a predecessor function for our natural numbers, because

$$\alpha(0) = \{0\}(0) = id(0) = 0$$

and

$$\alpha(n+1) = \{s^{\mathcal{A}^*}(n)\}(0) = \pi^2_2(0, n) = n.$$

Before defining our pairing function we introduce a sign function $\beta(x)$ such that

$$\begin{aligned} \beta(0) &= 0 \\ \beta(n+1) &= 1 \end{aligned}$$

We set $\beta(x) = r^{\mathcal{A}^*}(x, \alpha(x))$, where $r^{\mathcal{A}^*}$ is the characteristic function of identity. (Remember our third condition on $\mathcal{A}$ in the preceding paragraph.) If $x \notin \omega$, then $\beta(x)$ may be 0, 1 or it may diverge.

Finally we define our pairing function $\delta(x, y, z)$ such that

$$\begin{aligned} \delta(x, y, z) = x \quad &\text{if} \quad z = 0 \\ \delta(x, y, z) = y \quad &\text{if} \quad z = n+1 \end{aligned}$$

as follows. Let $\{d\}_2 = \pi^2_2(x_1, x_2)$ and $\theta(x) = s^{\mathcal{A}^*}_{1,1}(d, x)$. We have, for all a, $b \in A$, $\{\theta(a)\}(b) = a$. We define

$$\delta(x, y, z) = g_1^{\mathcal{A}^*}(g_1^{\mathcal{A}^*}(\beta(z), \theta(x)), g_1^{\mathcal{A}^*}(\theta(y), z)).$$

If $z = 0$, we have

$$\begin{aligned} \delta(a, b, 0) &= \{\{\beta(0)\}(\theta(a))\}(\{\theta(b)\}(0)) \\ &= \{\{0\}(\theta(a))\}(b) = \{\theta(a)\}(b) = a. \end{aligned}$$

If $z = n+1$, we have

$$\begin{aligned} \delta(a, b, n+1) &= \{\{\beta(n+1)\}(\theta(a))\}(\theta(b)\}(n+1)) \\ &= \{\{1\}(\theta(a))\}(b) = \{0\}(b) = b. \end{aligned}$$

If $z \notin \omega$ then $\delta(a, b, z)$ may be a, b or it may diverge: it depends on $\beta(z)$.

At this point we can introduce the traditional definition of primitive recursion by means of the recursion theorem (see, for instance, theorem 2.10 in Hinman (1978)). Suppose that f is defined by primitive recursion from g and h, as in the schema above, and that g and h are polynomials on $\mathcal{A}^*$. Let e_1, e_2 be such that

$$\{e_1\} = q(\pi^3_3(x_1, x_2, x_3))$$

and

$$\{e_2\} = h(g_2^{\mathcal{A}^*}(x_1, \alpha(x_2), x_3), \alpha(x_2), x_3).$$

We can define a polynomial

$$\eta(x_1, x_2, x_3) = g_3^{\mathcal{A}^*}(\delta(e_1, e_2, x_2), x_1, x_2, x_3).$$

By the corollary 4.4 there is an index i such that $\{i\}(x_2, x_3) \simeq \eta(i, x_2, x_3)$. We can prove by induction on ω that $\{i\} = f$. In fact,

$$\begin{aligned} \{i\}(0, x_3) &\simeq \eta(i, 0, x_3) \simeq \{\delta(e_1, e_2, 0)\}(i, 0, z) \\ &\simeq \{e_1\}(i, 0, z) \simeq g(z) \end{aligned}$$

and

$$\{i\}(n+1,x_3) \simeq \eta(i,n+1,x_3) \simeq \{\delta(e_1,e_2,n+1)\}(i,n+1,x_3)$$
$$\simeq \{e_2\}(i,n+1,x_3) \simeq h(\{i\}(n,x_3),n,x_3) \simeq h(f(n.x_3),n,x_3)$$

by inductive hypothesis.

6 Completions of partial structures

These remaining paragraphs are devoted to the passage from partial to total structures. First, we define the concept of completion of a partial structure. If $\mathcal{B}$ is partial structure for $\mathcal{L}$, we say that a total structure $\mathcal{C}$ for $\mathcal{L}$ is a completion of $\mathcal{B}$ if there is a full monomorphism $\varphi : \mathcal{B} \to \mathcal{C}$. The completion of a partial structure can be conceived, roughly speaking, as a total structure in which the partial structure is included as a relative substructure: the values for previously undefined functions can be found among the new elements. For instance, we can define a completion $\bar{\mathcal{B}}$ simply by adding a new element as the value for all instances of indefiniteness. We set $\bar{B} = B \cup \{u\}$, with $u \notin B$ and define $f^{\bar{\mathcal{B}}}$ as follows: for all $b \in B$, $f^{\bar{\mathcal{B}}}(b) = f^{\mathcal{B}}(b)$ if $f^{\mathcal{B}}(b)\downarrow$, $f^{\bar{\mathcal{B}}}(b) = u$ if $f^{\mathcal{B}}(b)\uparrow$, $f^{\bar{\mathcal{B}}}(u) = u$. We say that $\bar{\mathcal{B}}$ is the trivial completion of $\mathcal{B}$.

Before constructing a different kind of completion, we analyse what happens in the trivial completion. The main characteristic of $\bar{\mathcal{B}}$ is that every equation $t^{\mathcal{B}}(b) = q^{\mathcal{B}}(b)$ gives raise to an equation $t^{\bar{\mathcal{B}}}(b) = q^{\bar{\mathcal{B}}}(b)$, for all $b \in B$. In fact, if $t^{\mathcal{B}}(b)\downarrow$ and $q^{\mathcal{B}}(b)\downarrow$, we have, for some $c \in B$, $t^{\mathcal{B}}(b) = c = q^{\mathcal{B}}(b)$, so that $t^{\bar{\mathcal{B}}}(b) = c = q^{\bar{\mathcal{B}}}(b)$. If $t^{\mathcal{B}}(b)\uparrow$ and $q^{\mathcal{B}}(b)\uparrow$, we have, by definition of $\bar{\mathcal{B}}$, $t^{\bar{\mathcal{B}}}(b) = u = q^{\bar{\mathcal{B}}}(b)$. (This preservation result no longer holds in general when we take a completion $\mathcal{C}$ of $\bar{\mathcal{B}}$, because we can have $t^{\mathcal{C}}(b) \neq q^{\mathcal{C}}(b)$ when $t^{\mathcal{B}}(b)\uparrow$ and $q^{\mathcal{B}}(b)\uparrow$.) This happy state of affairs is compensated by a lacking of generality of the trivial completion. $\bar{\mathcal{B}}$ cannot generally be embedded in total structures in which $\mathcal{B}$ is embeddable, because too many identifications of objects have taken place in $\bar{\mathcal{B}}$: so $\mathcal{B}$ is not properly represented by $\bar{\mathcal{B}}$ in the realm of total structures. In Grätzer (1968) a completion with good embedding properties is constructed and we'll obtain an analogous completion with some additional properties related to the preservation of universal functions. We cannot be satisfied with Grätzer's completion, because we want universal functions to be still at work in the completion, even if we know that they can act as universal functions only on a subdomain (by Cantor's theorem).

In the preceding paragraphs we have shown that every structure $\mathcal{A}$ for $\mathcal{L}$ can be expanded to a structure $\mathcal{A}^*$ for $\mathcal{L}^*$ that is closed with respect to universal functions. We know that $\mathcal{A}^*$ cannot be total, under fairly general conditions, but we can look for a good representation $\mathcal{A}^\circ$ of $\mathcal{A}^*$ in the realm of total structures in the following sense. We firstly require that $\mathcal{A}^\circ$ be a completion of $\mathcal{A}^*$, so that (an isomorphic image of) $\mathcal{A}^*$ can be found as a relative substructure of $\mathcal{A}^\circ$. (So the initial structure $\mathcal{A}$ is a substructure of the restriction of $\mathcal{A}^\circ$ to $\mathcal{L}$.)

Before formulating the second requirement, the embedding property, we must introduce some definitions. Starting from $\mathcal{A}^*$ we define a set Σ of axioms as follows:

$$\Sigma = \{g_n(c_a, x_1 \dots x_n) = \nu_n(a)(x_1 \dots x_n) : a \in A, n \in \omega\}.$$

In Σ the behaviour of universal functions and the meaning of the indices of polynomial functions are represented, for a given set of indexing functions ν_n. (Remember that $\nu_n(a)$ is a term of $\mathcal{L}^*$ for every choice of a.) Let a structure $\mathcal{B}$ for $\mathcal{L}^*$ and a monomorphism $\varphi : \mathcal{A}^* \to \mathcal{B}$ be given. We say that a structure $\mathcal{B}$ preserves universal functions (of $\mathcal{A}^*$) with respect to φ, if

$$\mathcal{B} \models F(x_1 \dots x_n)[\varphi(a_1) \dots \varphi(a_n)]$$

for all $a_1, \dots, a_n \in A$, for all $F(x_1 \dots x_n) \in \Sigma$, for all $n \in \omega$. This means that $\varphi(a)$ can be conceived as an index for the polynomial function $\nu_n(a)^{\mathcal{B}}$ as long as we apply $\nu_n(a)^{\mathcal{B}}$ only to arguments in $\varphi[\mathcal{A}]$. In fact

$$\mathcal{B} \models g_n(c_a, x_1 \dots x_n) = \nu_n(a)(x_1 \dots x_n)[\varphi(a_1) \dots \varphi(a_n)]$$

so that

$$g_n^{\mathcal{B}}(\varphi(a), \varphi(a_1) \dots \varphi(a_n)) = \nu_n(a)^{\mathcal{B}}(\varphi(a_1) \dots \varphi(a_n)),$$

because φ is a morphism and $c_a^{\mathcal{B}} = \varphi(a)$, for all $a \in A$. This behaviour of universal functions cannot in general be extended beyond the subdomain $\varphi[A]$ by Cantor's theorem.

We can now state our second requirement for the completion $\mathcal{A}^\circ$ of $\mathcal{A}^*$. There is a full monomorphism $\chi : \mathcal{A}^* \to \mathcal{A}^\circ$ such that, for every total structure $\mathcal{B}$ for $\mathcal{L}^*$, for every monomorphism $\theta : \mathcal{A}^* \to \mathcal{B}$ such that $\mathcal{B}$ preserves universal functions (of $\mathcal{A}^*$) with respect to θ, there is a morphism $\psi : \mathcal{A}^\circ \to \mathcal{B}$ such that, for all $a \in A$, $\theta(a) = \psi(\chi(a))$. The construction of an $\mathcal{A}^\circ$ with such properties, carried out in the final paragraph, requires an analysis of the computation process of partial polynomials, particularly with respect to universal functions. This is undertaken in the next paragraph.

7 Analysis of the computation process

Let $\mathcal{L}$ and $\mathcal{L}^*$, $\mathcal{A}$ and $\mathcal{A}^*$, be as in par.3, let $CT(\mathcal{L}^*)$ be the set of closed terms of $\mathcal{L}^*$. We define three binary relations contr, $\Rightarrow$, $\rightarrow$ on $CT(\mathcal{L}^*)$ as follows.

Definition 1. For all t_1,t_2 we set t_1 contr t_2 iff

a) there is $i \in I$ and $a_1, \dots, a_n, b \in A$ such that $ar(i) = n$, $f_i^{\mathcal{A}^*}(a_1 \dots a_n) = b$ and

$$t_1 = f_i(c_{a_1} \dots c_{a_n}) \text{ and } t_2 = c_b$$

or

b) there is $n \in \omega - 1$ and $a, b_1 \dots b_n \in A$ such that

$$t_1 = g_n(c_a, c_{b_1} \dots c_{b_n}) \text{ and } t_2 = \nu_n(a)(c_{b_1} \dots c_{b_n})$$

or

c) there is a constant symbol $e \in \mathcal{L}$ such that $e^{\mathcal{A}} = a$ and $t_1 = e$, $t_2 = c_a$.

We say that t is contractible if t contr t' for some t': in this case we say that t' is the contractum of t. (There is at most one such t'.)

Definition 2. For all t_1, t_2, we set $t_1 \Rightarrow t_2$ iff t_2 is obtained from t_1 by replacing an occourrence of a contractible subterm of t_1 with its contractum.

Definition 3. For all t_1, t_2, we set $t_1 \rightarrow t_2$ iff there is a finite sequence $s_1, \dots, s_n$ such that $t_1 = s_1$, $t_2 = s_n$ and $s_i \Rightarrow s_{i+1}$, for all $i < n$.

Remarks

1. The relation contr is univocal; $\Rightarrow$ is not univocal because a term can contain several different occourrences of contractible subterms; $\rightarrow$ is not univocal.
2. The relations contr and $\Rightarrow$ are irriflexive; $\rightarrow$ is reflexive. Infact $t \rightarrow t$ holds because there is a sequence $s_1 \dots s_n$ such that $n = 1$, $s_1 = t_1$, $s_n = t$ and $s_i \Rightarrow s_{i+1}$, for all $i < n$, holds vacuously.
3. If f is not a universal function symbol and $f^{\mathcal{A}}(a_1 \dots a_n) \uparrow$, then there is no b such that $f(c_{a_1} \dots c_{a_n})$ contr c_b. But if $g_n^{\mathcal{A}}(a, b_1 \dots b_n) \uparrow$, then we always have $g_n(c_a, c_{b_1} \dots c_{b_n})$ contr $\nu_n(a)(c_{b_1} \dots c_{b_n})$. So every term like $g_n(c_a, c_{b_1} \dots c_{b_n})$ is contractible.
4. If $\mathcal{A}$ does not contain universal functions, then we can define an analogous relation contr omitting clause b) in definition 1 above. As for $\Rightarrow$ and $\rightarrow$ we can retain definition 2 and 3.

We say that t is in normal form (n.f.) if there is no t' such that $t \Rightarrow t'$, i.e. if no contraction can take place in it. We say that t is in strong normal form if it is in n.f. and t is c_a for some $a \in A$; otherwise, t is said to be in weak normal form. It is easy to give an inductive definition of weak normal form. We define a subset $W \subset CT(\mathcal{L}^*)$ as follows: $t \in W$ iff

1. t is $f_i(c_{a_1} \dots c_{a_n})$, for some $i \in I$ and $a_1, \dots, a_n \in A$ such that $f_i^{\mathcal{A}^*}(a_1 \dots a_n) \uparrow$

or

2. t is $f_i(s_1 \dots s_n)$, for some $i \in I$ and $s_1 \dots s_n \in CT(\mathcal{L}^*)$, such that, for all $i \leq n$, $s_i \in W$ or s_i is c_a, for some $a \in A$.

It is easy to prove that t is in weak n.f. iff $t \in W$. On the one hand we show that if $t \in W$ then t is in n.f., and no element of W is of the kinf c_a. On the other, we show by induction on t that if t is in n.f. and is not in strong n.f., then it must be in W. From remark 3 above, t must be of the kind $f(s_1 \dots s_n)$. If t is of minimal complexity, all s_i are c_a, for some $a \in A$, and then $t \in W$ from 1 in definition above. If t is not of minimal complexity, then there are some s_i not of the kind c_a. Every such s_i must be in n.f. and is not in strong n.f. by hypothesis. So they must be in W by inductive hypothesis and $t \in W$ from 2 in the above definition of W.

We say that t has (strong resp. weak) n.f. if there is a t' in (strong resp. weak) n.f. such that $t \rightarrow t'$. It is easy to see that there are terms without n.f. (strong or weak). For instance, let $\nu(a) = g_1(x, x)$. Then $g_1(c_a, c_a) \Rightarrow g_1(c_a, c_a) \Rightarrow \dots$.

We show that computations of partial polynomials on $\mathcal{A}^*$ can be represented by relation $\rightarrow$ in the following sense. Let $p \in P^{(n)}(\mathcal{A}^*)$. We cannot speak of the computation of a function without giving an intensional characterization of the function itself. So we choose $t \in T_n(\mathcal{A}^*)$ such that $t^{\mathcal{A}^*} = p$. We prove that, for all $a_1, \dots, a_n \in A$, if $t^{\mathcal{A}^*}(a_1 \dots a_n) \downarrow$ and $t^{\mathcal{A}^*}(a_1 \dots a_n) = b$, for some $b \in A$, then $t(c_{a_1} \dots c_{a_n}) \rightarrow c_b$. The sequence $s_1, \dots, s_n$, with $s_1 = t(c_{a_1} \dots c_{a_n})$ and $s_n = c_b$, given by $t(c_{a_1} \dots c_{a_n}) \rightarrow c_b$, is to be understood as a computation of the function $t^{\mathcal{A}^*}$, applied to arguments $a_1, \dots, a_n$, leading to its value b.

In the first place we define, corresponding to every $\mathcal{A}_k$ introduced in theorem 3.1, the binary relations contr$_k$, $\stackrel{k}{\Longrightarrow}$, $\stackrel{k}{\longrightarrow}$ on $CT(\mathcal{L}^*)$ as follows. We define $t_1$contr$_k t_2$ like $t_1$contr t_2, but changing clause b) of Definition 1 into the following:

b'). There is $n \in \omega - 1$, $a, b_1, \dots, b_n \in A$ such that $g_n^{\mathcal{A}_k}(a, b_1 \dots b_n) \downarrow$ and

$$t_1 = g_n(c_a, c_{b_1} \dots c_{b_n}), \quad t_2 = \nu(a)(c_{b_1} \dots c_{b_n}).$$

We define $t_1 \stackrel{k}{\Longrightarrow} t_2$ and $t_1 \stackrel{k}{\to} t_2$ on the basis of contr_k as above. We need the following lemmas.

Lemma 7.1 *a) If $j < i$, then $t_1 \stackrel{j}{\longrightarrow} t_2$ implies $t_1 \stackrel{i}{\longrightarrow} t_2$.*
b) For all i, $t_1 \stackrel{i}{\longrightarrow}$ implies $t_1 \to t_2$

Proof. a) It is immediate if we observe that,

$$\text{if} \qquad g_n^{\mathcal{A}_j}(a, b_1 \dots b_n) \downarrow, \qquad \text{then} \qquad g_n^{\mathcal{A}_i}(a, b_1 \dots b_n) \downarrow .$$

So $t_1 \text{contr}_j t_2$ implies $t_1 \text{contr}_i t_2$.
b) Trivial.

Lemma 7.2 *For all $t \in T_n(\mathcal{L}^*)$, for all $a_1, \dots, a_n, b \in A$, if $t^{\mathcal{A}^*}(a_1 \dots a_n) = b$, then there is a k such that $t^{\mathcal{A}_k}(a_1 \dots a_n) = b$.*

Proof. By induction on t. If $t = x_i$ or $t = e_j$, then the lemma holds for any k. If $t = f(t_1 \dots t_h)$, then there are $b_1, \dots, b_h$ such that $t_i^{\mathcal{A}^*}(a_1 \dots a_n) = b_i$, ($i \le h$), and $f^{\mathcal{A}^*}(b_1 \dots b_n) = b$. By induction hypothesis, we can choose, for every $i \le h$, a $\varphi(i)$ such that $t^{\mathcal{A}_{\varphi(i)}}(a_1 \dots a_n) = b_i$. Let $j = \max\{\varphi(i) : i \le h\}$. Then $t_i^{\mathcal{A}_j}(a_1 \dots a_n) = b_i$, for all $i \le h$, and $f^{\mathcal{A}_j}(b_1 \dots b_h) = b$. So $t^{\mathcal{A}_j}(a_1 \dots a_n) = b$. If $t = g_h(t_1 \dots t_{h+1})$, then there are $b_1, \dots, b_{h+1}$ such that $t_i^{\mathcal{A}^*}(a_1 \dots a_n) = b_i$, for every $i \le h+1$, and $g_h^{\mathcal{A}^*}(b_1 \dots b_{h+1}) = b$. By definition of $g_h^{\mathcal{A}^*}$, there is a p such that $g_h^{\mathcal{A}_p}(b_1 \dots b_{k+1}) = b$. We can prove, as above, that there is a j such that $t_i^{\mathcal{A}_j}(a_1 \dots a_n) = b_i$, for all $i \le h+1$). If we set $k = \max\{p, j\}$, then $t^{\mathcal{A}_k}(a_1 \dots a_n) = b$.

Lemma 7.3 *For all $t \in T_n(\mathcal{L}^*)$, for all $a_1 \dots a_n, b \in A$, if $t^{\mathcal{A}_k}(a_1 \dots a_n) = b$ then $t(c_{a_1} \dots c_{a_n}) \stackrel{k}{\longrightarrow} c_b$.*

Proof. By induction on the complexity of t and on k. If $k = 0$, then the lemma holds trivially. If $t = x_i$ or $t = e_i$, then the lemma holds for any k. Let $t = f(t_1 \dots t_h)$. If $t^{\mathcal{A}_k}(a_1 \dots a_n) = b$, then there are $b_1, \dots, b_h$ such that $t_i^{\mathcal{A}_k}(a_1 \dots a_n) = b_i$, for all $i \le h$, and $f^{\mathcal{A}_k}(b_1 \dots b_h) = b$. By induction hypothesis on t, $t_i(c_{a_1} \dots c_{a_n}) \stackrel{k}{\longrightarrow} c_{b_i}$. As $f(c_{b_1} \dots c_{b_h}) \stackrel{k}{\longrightarrow} c_b$, we also have $t(c_{a_1} \dots c_{a_n}) \stackrel{k}{\longrightarrow} c_b$. Let $t = g_h(t_1 \dots t_{h+1})$. If $t^{\mathcal{A}_k}(a_1 \dots a_n) = b$, then there are $b_1, \dots, b_{h+1}$ such that $t_i^{\mathcal{A}_k}(a_1 \dots a_n) = b_i$, for all $i \le h+1$, and $g_h^{\mathcal{A}_k}(b_1 \dots b_{h+1}) = b$. By induction hypothesis on t, we have

(1) $\quad t_i(c_{a_1} \dots c_{a_n}) \stackrel{k}{\longrightarrow} c_{b_i}$, for all $i \le h+1$.

By definition of $\stackrel{k}{\longrightarrow}$ we have

(2) $\quad g_h(c_{b_1} \dots c_{b_{h+1}}) \stackrel{k}{\longrightarrow} \nu_h(b_1)(c_{b_2} \dots c_{b_{h+1}})$

By definition of $g_h^{\mathcal{A}_k}$,

$$g_h^{\mathcal{A}_k}(c_{b_1} \dots c_{b_{h+1}}) = b \qquad \text{implies} \qquad \nu_h(b_1)^{\mathcal{A}_{k-1}}(c_{b_2} \dots c_{b_{h+1}})$$

So by induction hypothesis on k,

$$\nu_h(b_1)(c_{b_2} \dots c_{b_{h+1}}) \stackrel{k-1}{\longrightarrow} c_b.$$

Then

$$\nu_h(b_1)(c_{b_2} \dots c_{b_{h+1}}) \stackrel{k}{\longrightarrow} c_b$$

by lemma 7.1 a), and $t(c_{a_1} \dots c_{a_n}) \stackrel{k}{\longrightarrow} c_b$ follows from (1), (2), (3).

Theorem 7.4 *For all $t \in T_n(\mathcal{L}^*)$, for all $a_1, \dots, a_n, b \in A$,*

$$\text{if} \qquad t^{\mathcal{A}^*}(a_1 \dots a_n) = b \qquad \text{then} \qquad t(c_{a_1} \dots c_{a_n}) \to c_b.$$

Proof. By lemma 7.2 , there is a k such that $t^{\mathcal{A}_k}(a_1 \dots a_n) = b$. By lemma 7.3 , we can prove that $t(c_{a_1} \dots c_{a_n}) \stackrel{k}{\longrightarrow} c_b$. The desired result $t(c_{a_1} \dots c_{a_n}) \stackrel{k}{\longrightarrow} c_b$ follows from lemma 7.4 b).

Theorem 7.4 still holds if we substitute $\mathcal{A}^*$ with any partial structure $\mathcal{A}$ for $\mathcal{L}$: the proof can be extracted from the proof of the above theorem. However theorem 7.4 does not hold for all expansions of $\mathcal{A}$ to a structure $\mathcal{A}'$ for $\mathcal{L}^*$ closed with respect to universal functions. but only if $\mathcal{A}'$ is $\mathcal{A}^*$, the canonical expansion. The converse of 7.4 will be proved in the next paragraph as corollary 8.5.

8 The completion $\mathcal{A}^\circ$

The analysis of the computation process given in the preceding paragraph is a first step in the construction of a total structure $\mathcal{A}^\circ$ in which $\mathcal{A}^*$ can be monomorphically embedded. The domain A° of $\mathcal{A}^\circ$ is obtained as follows. We define a binary relation $\equiv$ on $CT(\mathcal{L}^*)$ setting $t_1 \equiv t_2$ iff there is a t_3 such that $t_1 \to t_3$ and $t_2 \to t_3$. We prove in theorem 8.3 that $\equiv$ is an equivalence relation. We set $A^\circ = CT(\mathcal{L}^*)/\equiv$ and denote with $\mid t \mid$ the class of t modulo $\equiv$. For every constant symbol $e \in \mathcal{L}^*$, we set $e^{\mathcal{A}^\circ} = |c_a|$, where $a = e^{\mathcal{A}^*}$.

For every n-ary function symbol $h \in \mathcal{L}^*$, be it a universal function or not, we define

$$h^{\mathcal{A}^\circ}(| t_1 | \dots | t_n |) = | h(t_1 \dots t_n) |$$

for all $t_1, \dots, t_n \in CT(\mathcal{A}^*)$. We must show that this is a good definition, i.e.that $s_i \equiv t_i$, for all $i \leq n$, implies $h(s_1 \dots s_n) \equiv h(t_1 \dots t_n)$. If $s_i \equiv t_i$, then there is u_i such that $s_i \to u_i$ and $t_i \to u_i$, for all $i \leq n$. So we have $h(s_1 \dots s_n) \to h(u_1 \dots u_n)$ and $h(t_1 \dots t_n) \to h(u_1 \dots u_n)$, so that $h(s_1 \dots s_n) \equiv h(t_1 \dots t_n)$. The definition of $\mathcal{A}^\circ$ is now complete.

Theorem 8.1 *The function $\varphi : \mathcal{A}^* \to \mathcal{A}^\circ$, where $\varphi(a) = | c_a |$, is a monomorphism.*

Proof. φ is injective because if $a \neq b$, then the symbol c_a is different from c_b; c_a and c_b are terms in n.f. and so there is no t such that $c_a \to t$ and $c_b \to t$, i.e. $| c_a | \not\equiv | c_b |$. φ is a morphism. In fact, let $e_j^{\mathcal{A}^*} = a$, then

$$\varphi(e_j^{\mathcal{A}^*}) = \varphi(a) = | c_a | = e_j^{\mathcal{A}^\circ}.$$

Suppose now that $h^{\mathcal{A}^*}(a_1 \dots a_n) \downarrow$, then we have $\varphi(h^{\mathcal{A}^*}(a_1 \dots a_n)) = | c_a |$, for some a in A such that $h^{\mathcal{A}^*}(a_1 \dots a_n) = a$. So we have

$$h^{\mathcal{A}^\circ}(\varphi(a_1) \dots \varphi(a_n)) = h^{\mathcal{A}^\circ}(| c_{a_1} | \dots | c_{a_n} |) = | h(c_{a_1} \dots c_{a_n}) | .$$

By theorem 7.4

$$h^{\mathcal{A}^*}(a_1 \dots a_n) = a \quad \text{implies} \quad h(c_{a_1} \dots c_{a_n}) \to c_a.$$

Thus, from $c_a \to c_a$, we have $| h(c_{a_1} \dots c_{a_n}) | = | c_a |$. □

What has still to be proved is that $\equiv$ is an equivalence relation. We need the following:

Lemma 8.2 *For any $t_1, t_2, t_3 \in CT(\mathcal{L}^*)$, if $t_1 \to t_2$ and $t_1 \to t_3$, then there is $t_4 \in CT(\mathcal{L}^*)$ such that $t_2 \to t_4$ and $t_3 \to t_4$.*

Proof. Case a). We suppose $t_1 \Rightarrow t_2$ and $t_1 \Rightarrow t_3$, i.e $t_1 \to t_2$ and $t_1 \to t_3$ in one step. We introduce the following notation for replacement. For every subterm s of t, we suppose that the occurrences of s in t are numbered in a sequence $s^1, \dots, s^n$ (where n depends on s). So we note with $t[s^i/u]$ the replacement of the i-th occurrence of s in t with u, and with $t[s^i/u, p^j/q]$ the simultaneous replacement of the i-th occurrence of s with u and of the j-th occurrence of p with q. From our hypothesis we see that $t_2 = t_1[s^i/u]$ and $t_3 = t_1[p^j/q]$, where s contr u and p contr q. If $s = p$ and $i = j$, then $t_2 = t_3$, and we can set $t_4 = t_2 = t_3$. In any other case we have

$$t_2[p^j/q] = t_1[s^i/u, p^j/q] = t_1[p^j/q, s^i/u] = t_3[s^i/u],$$

because the order in which replacements take place does not affect the result. So we can set $t_4 = t_1[s^i/u, p^j/q]$ and conclude with $t_2 \Rightarrow t_4$ and $t_3 \Rightarrow t_4$.
Case b). We suppose $t_1 \Rightarrow t_2$ and $t_1 \to t_3$ in n steps, so there is a sequence $s_1, \dots, s_{n+1}$ such that $s_1 = t_1$, $s_{n+1} = t_3$ and $s_i \Rightarrow s_{i+1}$ for all $i \leq n$. By n applications of case a), we can find a sequence $u_1 \dots u_n$ such that $t_2 \Rightarrow u_1$ and $s_2 \Rightarrow u_1, \dots, u_{n-1} \Rightarrow u_n$ and $s_{n+1} \Rightarrow u_n$. So $t_2 \to u_n$ and $s_{n+1} = t_3 \to u_n$, and we can set $t_4 = u_n$.
Case c). We suppose $t_1 \to t_2$ in m steps and $t_1 \to t_3$ in n steps. Then we obtain case c) with m applications of case b), as we have obtained case b) from case a). □

Lemma 8.3 *The relation $\equiv$ is an equivalence.*

Proof. $\equiv$ is obviously reflexive and symmetric. We suppose $t_1 \equiv t_2$ and $t_2 \equiv t_3$, then we have, for some t_4, t_5, $t_1 \to t_4$, $t_2 \to t_4$ and $t_2 \to t_5$, $t_3 \to t_5$. By the lemma, there is t_6 such that $t_4 \to t_6$ and $t_5 \to t_6$. So, by transitivity of $\to$, $t_1 \to t_6$ and $t_3 \to t_6$, and $t_1 \equiv t_3$ follows. □

In the next theorems we show that $\mathcal{A}^\circ$ properly represents $\mathcal{A}^*$ in the realm of total structures and satisfies the requirements formulated in par.6.

Lemma 8.4 *If $\mathcal{B}$ is a structure for $\mathcal{L}^*$, $\varphi : \mathcal{A}^* \to \mathcal{B}$ is a morphism and $\mathcal{B}$ preserves universal functions (of $\mathcal{A}^*$) with respect to φ, then, for all t, $q \in TC(\mathcal{L}^*)$,*

1. *t contr q implies $t^{\mathcal{B}} = q^{\mathcal{B}}$*
2. *$t \Rightarrow q$ implies $t^{\mathcal{B}} = q^{\mathcal{B}}$*
3. *$t \to q$ implies $t^{\mathcal{B}} = q^{\mathcal{B}}$*
4. *$t \equiv q$ implies $t^{\mathcal{B}} = q^{\mathcal{B}}$*

Proof. 1. There are three cases, corresponding to a), b) and c) in definition 1 of par.7.

Case a). $t = f(c_{a_1} \dots c_{a_n})$, $q = c_a$, $f^{\mathcal{A}^*}(a_1 \dots a_n) = a$. As φ is a morphism and $t^{\mathcal{A}^*} \downarrow$, $q^{\mathcal{A}^*} \downarrow$, we have

$$t^{\mathcal{B}} = \varphi(t^{\mathcal{A}^*}) = \varphi(f^{\mathcal{A}^*}(a_1 \dots a_n)) = \varphi(a)$$

and

$$q^{\mathcal{B}} = \varphi(q^{\mathcal{A}^*}) = \varphi(c_a^{\mathcal{A}^*}) = \varphi(a)$$

Case b). $t = g_n(c_a, c_{b_1} \dots c_{b_n})$, $q = \nu_n(a)(c_{b_1} \dots c_{b_n})$. Then

$$t^{\mathcal{B}} = g_n^{\mathcal{B}}(\varphi(a), \varphi(b_1) \dots \varphi(b_n)) = \nu_n(a)^{\mathcal{B}}(\varphi(b_1) \dots \varphi(b_n)) = q^{\mathcal{B}}$$

as $\mathcal{B}$ preserves universal functions.

Case c). $t = e$ and $q = c_a$, where e is a constant of $\mathcal{L}$ and $e^{\mathcal{A}^*} = a$. As φ is a morphism, we have $t^{\mathcal{B}} = \varphi(e^{\mathcal{A}^*}) = \varphi(a)$ and $q^{\mathcal{B}} = \varphi(c_a^{\mathcal{A}^*}) = \varphi(a)$.

2. We adopt the notation of lemma 8.2. If $t \Rightarrow q$ then we have $q = t[s^i/u]$ and s contr u, where s^i is the i-th occurrence in t of the subterm s of t. We have $q^{\mathcal{B}} = (t[s^i/u])^{\mathcal{B}} = t^{\mathcal{B}}$, because $s^{i\mathcal{B}} = u^{\mathcal{B}}$ by part 1 of this lemma.

3. If $t \to q$, then there is a finite sequence $t_1, \dots, t_n$ such that $t = t_1$, $q = t_n$ and $t_i \Rightarrow t_{i+1}$, for $i < n$. By part 2, $t_1^{\mathcal{B}} = \dots = t_n^{\mathcal{B}}$ and $t^{\mathcal{B}} = q^{\mathcal{B}}$.

4. If $t \equiv q$, then there is a s such that $t \to s$ and $q \to s$, by 8.2, so, by point 3 above, $t^{\mathcal{B}} = s^{\mathcal{B}} = q^{\mathcal{B}}$. □

Now we can prove the converse of 7.4:

Corollary 8.5 *For all $t \in T_n(\mathcal{L}^*)$, for all $a_1, \dots, a_n, b \in A$,*

$$\text{if} \quad t(c_{a_1} \dots c_{a_n}) \to c_b \quad \text{then} \quad t^{\mathcal{A}^*}(a_1 \dots a_n) = b.$$

Proof. We observe that $id : \mathcal{A}^* \to \mathcal{A}^*$ is a morphism and $\mathcal{A}^*$ obviously preserves universal functions with respect to id. So, by lemma 8.4 (3), $t(c_{a_1} \dots c_{a_n}) \to c_b$ implies $t(c_{a_1} \dots c_{a_n})^{\mathcal{A}^*} = c_b^{\mathcal{A}^*}$, i.e. $t^{\mathcal{A}^*}(a_1 \dots a_n) = b$. □

Lemma 8.6 *For all t, $q \in CT(\mathcal{L}^*)$, $t^{\mathcal{A}^\circ} = q^{\mathcal{A}^\circ}$ implies $t^{\mathcal{A}^*} = q^{\mathcal{A}^*}$.*

Proof. We remember that, by definition of $\mathcal{A}^\circ$, for all $t \in CT(\mathcal{L}^*)$, $t^{\mathcal{A}^\circ} = \mid t \mid$. We prove the contraposition of the lemma and suppose $t^{\mathcal{A}^*} \neq q^{\mathcal{A}^*}$.

Case a). $t^{\mathcal{A}^*} = a_1$, $q^{\mathcal{A}^*} = a_2$, $a_1 \neq a_2$. In this case $t \to c_{a_1}$, and $q \to c_{a_2}$, by 7.4, and $t \not\equiv q$ (otherwise $c_{a_1} \equiv t \equiv q \equiv c_{a_2}$, which is absurd). Therefore $\mid t \mid \neq \mid q \mid$ and $t^{\mathcal{A}^\circ} \neq q^{\mathcal{A}^\circ}$, because $t^{\mathcal{A}^\circ} = \mid t \mid$ and $q^{\mathcal{A}^\circ} = \mid q \mid$, by definition of $\mathcal{A}^\circ$.

Case b). $t^{\mathcal{A}^*} = a_1$ and $q^{\mathcal{A}^*} \uparrow$. (If $t^{\mathcal{A}^\circ} \uparrow$ and $q^{\mathcal{A}^*} = a_2$, the proof is the same.) As φ is a morphism, $\varphi(t^{\mathcal{A}^*}) = t^{\mathcal{A}^\circ}$, so $t^{\mathcal{A}^\circ} = \varphi(a_1) = c_{a_1}^{\mathcal{A}^\circ} = \mid c_{a_1} \mid$. If $t^{\mathcal{A}^\circ} = q^{\mathcal{A}^\circ}$, then $\mid t \mid = \mid q \mid$ and $\mid c_{a_1} \mid = \mid t \mid = \mid q \mid$, because $t^{\mathcal{A}^\circ} = \mid t \mid$ and $q^{\mathcal{A}^\circ} = \mid q \mid$. So we have $q \to c_{a_1}$ and $q^{\mathcal{A}^*} = c_{a_1}^{\mathcal{A}^*}$, by 8.5. So $q^{\mathcal{A}^*} = a_1 = t^{\mathcal{A}^*}$, what is absurd. □

Theorem 8.7 *The morphism φ of theorem 8.1 is full.*

Proof. Let $h^{\mathcal{A}^\circ}(\varphi(a_1) \dots \varphi(a_n)) = \varphi(b)$. We have $\varphi(b) = c_b^{\mathcal{A}^\circ}$ and

$$h^{\mathcal{A}^\circ}(\varphi(a_1) \dots \varphi(a_n)) = h^{\mathcal{A}^\circ}(\mid c_{a_1} \mid \dots \mid c_{a_n} \mid) = \mid h(c_{a_1} \dots c_{a_n}) \mid = h(c_{a_1} \dots c_{a_n})^{\mathcal{A}^\circ}.$$

So $c_b^{\mathcal{A}^\circ} = h(c_{a_1} \dots c_{a_n})^{\mathcal{A}^\circ}$, and $b = c_b^{\mathcal{A}^*} = h(c_{a_1} \dots c_{a_n})^{\mathcal{A}^*} = h^{\mathcal{A}^*}(a_1 \dots a_n)$, by 8.6. □

In this way we have shown that $\mathcal{A}^\circ$ is a completion of $\mathcal{A}^*$ and this satisfies the first requirement stated in par.6. It can easily be shown that $\mathcal{A}^\circ$ preserves universal functions with respect to φ, i.e. for all $a_1 \dots a_n \in A$, for all $F(x_1 \dots x_n) \in \Sigma$,

$$\mathcal{A}^\circ \models F(x_1 \dots x_n)[\varphi(a_1) \dots \varphi(a_n)]$$

If $F(x_1 \dots x_n)$ is $g_n(c_a, x_1 \dots x_n) = \nu_n(a)(x_1 \dots x_n)$, we have

$$g_n^{\mathcal{A}^\circ}(c_a, \varphi(a_1) \dots \varphi(a_n)) = g_n^{\mathcal{A}^\circ}(\mid c_a \mid, \mid c_{a_1} \mid \dots \mid c_{a_n} \mid) = \mid g_n(c_a, c_{a_1} \dots c_{a_n}) \mid$$

and

$$\nu_n(a)^{\mathcal{A}^\circ}(\varphi(a_1) \dots \varphi(a_n)) = \nu_n(a)^{\mathcal{A}^\circ}(\mid c_{a_1} \mid \dots \mid c_{a_n} \mid) = \mid \nu_n(a)(c_{a_1} \dots c_{a_n}) \mid$$

but $g_n(c_a, c_{a_1} \dots c_{a_n})$ contr $\nu_n(a)(c_{a_1} \dots c_{a_n})$, so

$$\mid g_n(c_a, c_{a_1} \dots c_{a_n}) \mid = \mid \nu_n(a)(c_{a_1} \dots c_{a_n}) \mid .$$

We conclude our work by showing that $\mathcal{A}^\circ$ also satisfies the second requirement.

Theorem 8.8 *There is a full monomorphism $\chi : \mathcal{A}^* \to \mathcal{A}^\circ$ such that, for every total structure $\mathcal{B}$ for $\mathcal{L}^*$, for every monomorphism $\theta : \mathcal{A}^* \to \mathcal{B}$ such that $\mathcal{B}$ preserves universal functions (of $\mathcal{A}^*$) with respect to θ, there is a morphism $\psi : \mathcal{A}^\circ \to \mathcal{B}$ such that, for all $a \in A$, $\theta(a) = \psi(\chi(a))$.*

Proof. We define $\chi(a) = \mid c_a \mid$, for all $a \in A$, and define $\psi(\mid t \mid) = t^{\mathcal{B}}$, for all $t \in CT(\mathcal{L}^*)$. Firstly we show that ψ is a function. If $\mid t \mid = \mid q \mid$, then $t \equiv q$ and, by lemma 8.4 (4), $t^{\mathcal{B}} = q^{\mathcal{B}}$. Then we show that $\theta(a) = \psi(\chi(a))$. In fact, $\theta(a) = c_a^{\mathcal{B}}$, as θ is a morphism, and $\psi(\chi(a)) = \psi(\mid c_a \mid) = c_a^{\mathcal{B}}$. Finally we show that ψ is a morphism. For all $\mid t_1 \mid, \dots, \mid t_n \mid \in A^\circ$, for all n-ary $h \in \mathcal{L}^*$, we have

$$h^{\mathcal{A}^\circ}(\mid t_1 \mid \dots \mid t_n \mid) = \mid h(t_1 \dots t_n) \mid .$$

So we have, by definition of ψ,

$$\psi(h^{\mathcal{A}^\circ}(\mid t_1 \mid \dots \mid t_n \mid)) = \psi(\mid h(t_1 \dots t_n)) \mid = h(t_1 \dots t_n)^{\mathcal{B}} =$$

$$h^{\mathcal{B}}(t_1^{\mathcal{B}} \dots t_n^{\mathcal{B}}) = h^{\mathcal{B}}(\psi(\mid t_1 \mid) \dots \psi(\mid t_n \mid)). \quad \square$$

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