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A Theory of Sets with the Negation of the Axiom of Infinity

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Abstract. In this paper we introduce a theory of finite sets (**FST**) with a strong negation of the axiom of infinity asserting that every set is provably bijective with a natural number. We study in detail the role of the axioms of Power Set, Choice, Regularity in **FST**, pointing out the relative dependences or independences among them. **FST** is shown to be provably equivalent to a fragment of Alternative Set Theory. Furthermore the introduction of **FST** is motivated in view of a nonstandard development.

0. Introduction.

It is our opinion that Cantor's main contribution in introducing the theory of sets is the explicit formulation of the axiom of infinity, essentially stating that the collection of the natural numbers is a set. Such a notion was needed to construct the real numbers and to found the infinitesimal calculus in the manners that were emerging at the end of the past century. A large part of the mathematical community of those times was ready to accept the possibility of considering infinite procedures as completed, as finished, and to use them as a whole in the role of elements in new constructions. Since then this position has always been taken for granted in the development of set theory, and, even when the axiom of infinity was not explicitly assumed in order to study the relative strength of the different axiom systems, there was no intention to negate the above mentioned assumption.

On the contrary, we want to study a set theory in which the cantorian axiom of infinity is explicitly negated, precisely because we do not want to admit the possibility of considering a procedure going on forever as completed, as one element.

One could immediately object that thus one gives up a large part of the mathematics known today, but we reply that already in the first half of the past century calculus had a flourishing development based upon leibnitzian notions that, surely, did not include the above mentioned cantorian concept of infinity (see [**Mo**]). It is true that the foundations of the leibnitzian calculus were a little bit shaky, but nowadays most of the criticisms from that era have been overcome by some clarifications coming from mathematical logic and non standard analysis (although the latter, in its current formulations, fully accepts the cantorian point of view). Still, we think that it will be possible to develop a non standard analysis omitting the cantorian notion of infinity, but this will be the subject of further research which will find a starting point in this one: indeed, before facing infinitesimals and non cantorian infinites, it will be wise to state precisely from which set theory negating the cantorian axiom of infinity it is convenient to start.

Thus the sets that will occur in the theories that we are going to develop will be finite from the cantorian point of view, but, since the notion of finiteness cannot be made precise through formulas, this also leaves plenty of room for non standard notions of finiteness.

As far as we know, there are no works that investigate a set theory with the negation of the cantorian axiom of infinity outside the framework of Alternative Set Theory (**AST**). Indeed the theory **FST** that we will introduce in section 4 and a fragment of **AST** including only axioms for sets turn out to be provably equivalent. This equivalence will be discussed in section 6. As a matter of fact, both **AST** and **FST** negate the cantorian axiom of infinity, and both will have to face the problem of recapturing a large part of the existing mathematics. It worths recalling that **AST** finds the shortcomings of Zermelo Fraenkel set theory in the very limited notion of proper class and extends this notion to all the collections that are not well defined in any possible way, thus "too large" being an instance of "not well defined" (see [**V2**], [**H-V**], [**S1**]).

On the contrary, our non standard development will restrict the non well defined collections to those that an external observer will hardly be able to grasp because of their size. In the nonstandard development the collection of natural numbers will thus play a central role. The same will be true of the notion of being a

standard element, which we will read as "not too numerous to be reached", and of the axiom schema of separation whose application without any restriction on the use of the predicates *standard* and *internal* in the formulas gives rise to the *external* sets (for an account on these points see [**Ka**], [**N**], [**R**]).

As we said, the non standard development is out of the scope of the present paper, so the previous remark is just intended to motivate the introduction of a set theory with the negation of the cantorian axiom of infinity different from **AST**.

As a consequence of the equivalence between **FST** and a fragment of **AST**, many of the results presented in this paper are not new. However they are proved in a setting (**FST**) that seems to us most suitable for further extensions of the theory to a non standard framework.

1. Preliminaries.

We will adopt the usual notation, as found, e.g., in [**BM**].
In particular we recall some definitions.

" x is an ordinal" $\quad On(x) \equiv Trans(x) \wedge Ew(x);$

Notice that this definition of an ordinal already embodies its well-foundedness.

"x is a natural number" is defined by $Nat(x) \equiv On(x) \wedge \forall y\ (y \leq x \rightarrow \neg Lim(y))$.

ω denotes the collection of natural numbers, i.e. $x \in \omega \leftrightarrow Nat(x)$.

We will adopt the usual shortenings for "f is a Function" (Fun(f)), "the Domain of f" (Dom(f)), "the Range of f" (Ran(f)), "f is Injective" (Inj(f)).

We remark that the usual induction on the natural numbers requires the axioms of extensionality, empty set, union and replacement only. A careful review of the steps needed to prove in **ZF** that, for every formula $\varphi(x)$,

$$\varphi(\mathbf{0}) \wedge \forall \mathbf{n}\ (\varphi(\mathbf{n}) \rightarrow \varphi(\mathbf{n} \cup \{\mathbf{n}\})) \rightarrow \forall \mathbf{n}\ \varphi(\mathbf{n})$$

is a theorem, makes sure that the only axioms used are those just mentioned (see, for instance, [**BM**]).
Incidentally, notice that, under these axioms, Nat(x) is equivalent to the following

$$On(x) \wedge \forall y\, (x \supseteq y \wedge y \neq \mathbf{0} \rightarrow \exists u\, (u \in y \wedge \forall v\, (v \in y \rightarrow (v \neq u \rightarrow v \in u)))).$$

From now on, we will appeal to the principle of induction on natural numbers, without further mention, in those theories of sets including the axioms of extensionality, empty set, union and the schema of replacement.

2. Axioms for the Elementary Set Theory.

We will call *Elementary Set Theory*, and denote it by **EST**, the theory whose axioms are the following ones.

Ext: $\forall xy\, (\forall z\, (z \in x \leftrightarrow z \in y) \rightarrow x = y).$

Pair: $\forall xy\, \exists z\, \forall u (u \in z \leftrightarrow u = x \vee u = y).$

Union: $\forall x\, \exists y\, \forall z (z \in y \leftrightarrow \exists u\, (u \in x \wedge z \in u)).$

Repl: $\forall y_1,\ldots,y_k\, \forall u (\forall x \in u\, \exists! y\, \varphi(x,y) \rightarrow \exists z \forall y\, (y \in z \leftrightarrow \exists x \in u\, \varphi(x,y))).$

Emptyset: $\exists x\, \forall y\, (y \in x \leftrightarrow y \neq y).$

We will denote by **Uof2** the sentence $\forall x \forall y \exists z \forall u (u \in z \leftrightarrow (u \in x \vee u \in y))$. Remark that **Uof2** is a consequence of the axioms **Union** and **Pair**.
We will denote the empty set by the symbol **0,** the n-th numeral by the symbol **n**. **n+1** will denote $\mathbf{n} \cup \{\mathbf{n}\}$. Just from the axioms **Ext, Pair, Uof2, Emtpyset**, by metatheoretical induction on n, one can prove that 1) $\mathbf{n+1} = \{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n}\}$ for any n, and 2) $\mathbf{n} \neq \mathbf{m}$ for any $m \neq n$, and 3) every numeral is a natural number.
The operation "union with its singleton" can be applied also to any natural number x and it yields another natural number, called the successor of x, and

denoted by x'. Remark that any natural number which is not **0** is the successor of a natural number.
We will state the axiom of infinity in the following form:

Inf: $\exists x\, (\mathbf{0} \in x \wedge \forall y\, (y \in x \rightarrow y \cup \{y\} \in x)).$

Notice that **Inf** is not an axiom of **EST**.

Among the several notions of finiteness already present in the literature (see [**Ma**] where an extensive list of references is included), which are all equivalent in **ZF**, we single out three of them in order to represent three different positions in dealing with such a notion.
There are definitions of finiteness which assume the axiom of infinity, e.g.

"x is *ω-finite* ": $Fin^{\omega}(x) \equiv \neg \exists f\, (Fun(f) \wedge Inj(f) \wedge Dom(f) = \omega \wedge x \supseteq Ran(f));$

Of course we are not interested in such definitions because ω is a set only by virtue of the axiom of infinity.
There are other definitions of finiteness whose equivalence can be shown only using the axiom of choice, e.g.

"x is *f-finite* ": $Fin^{f}(x) \equiv \exists f\, \exists n (Fun(f) \wedge Inj(f) \wedge Nat(n) \wedge Dom(f) = n \wedge Ran(f) = x)$

"x is *Dedekind-finite* ": $Fin^{D}(x) \equiv \neg \exists f\, \exists y (x \supseteq y \wedge y \neq x \wedge Fun(f) \wedge Inj(f) \wedge Dom(f) = x \wedge Ran(f) = y).$

The notion of Dedekind is more geared towards a characterization of the infinite sets which turns out to be equivalent to the negation of finiteness via the axiom of choice. We will eventually accept the axiom of choice, but we do not want to stick to this assumption from the beginning. Hence we choose the notion of *f-finiteness* as our formal definition of finiteness.

3. Some initial results about finiteness.

The following results show some relationships among the three notions of finiteness that we have introduced.

The first result is well-known in the literature. It is worth noticing that it can be proved without using the full power of **ZFC**.

Lemma 1. **EST+Inf** $\vdash \forall x\, (Fin^{\omega}(x) \leftrightarrow Fin^{D}(x))$.

Proof. ($\rightarrow$) By contradiction. Let $x \supset y$ and $x \neq y$ and assume that there is a bijection f of x onto y. Let $z_0 \in x-y$. By recursion on the natural numbers, define $z_{n+1}=f(z_n)$. Let us show that if $m \neq n$ then $z_m \neq z_n$.

Assume, on the contrary, that n_0 is the least natural number such that there is $m \neq n_0$ for which $z_{n_0}=z_m$. n_0 cannot be 0, since $z_0 \in x-y$, while $z_i \in y$ for all $i>0$. For the same reason, also m can never be 0. So let $n_0=n'+1$ and $m=m'+1$, for some n' and m' respectively. Thus $z_{n_0}=f(z_{n'})$, and $z_m=f(z_{m'})$. Hence $z_{n'}=z_{m'}$, due to the injectivity of f, contradicting the minimality of n_0.

($\leftarrow$) By contradiction again. Choose x such that $Fin^{\omega}(x)$ does not hold and let f be an injective function that witnesses the property of x. Let h be the restriction of f to ω-{0}, and let s be the successor function on the natural numbers. The function $h(s(f^{-1}))$ is a bijection of Ran(f) onto Ran(h), a proper subset of Ran (f). Thus $Fin^{D}(x)$ does not hold. ⊗

The next two lemmas are easy consequences of Lemma 1.

Lemma 2. **EST** $\vdash \forall x\, Fin^{D}(x) \rightarrow \neg$**Inf**.

Proof. Let us assume **Inf**. It is a known result of **EST** that **Inf** implies the existence of the set ω of the natural numbers. The identity function on ω witnesses that $\neg Fin^{\omega}(\omega)$ holds. ⊗

Lemma 3. **EST** $\vdash \forall x\, Fin^{f}(x) \rightarrow \neg$**Inf**.

Proof. Again let us assume **Inf**. Arguing as in the previous lemma, we have that the collection ω of the natural numbers is a set. Let us prove that $\neg Fin^{f}(\omega)$.

By contradiction, assume $Fin^{f}(\omega)$. Let

$$n_0 = \min\{n:\ Nat(n) \wedge \exists f\, (Fun(f) \wedge Inj(f) \wedge Dom(f)=n \wedge Ran(f)=\omega\},$$

and let g be a bijection from n_0 onto ω. Since n_0 cannot be 0, $n_0=m_0+1$ for some natural number m_0. Let $x=\{n:\ n\in\omega \text{ and } n<g(m_0)\}$ and $y=g^{-1}[x]$. Let us define the function h from m_0 to ω as follows:

$$h(m) = \{\ g(m), \text{if } m\in y\ \ ,\ ,\ ,pred(g(m)), \text{if } m \in m_0-y.$$

where pred is the predecessor function on the natural numbers.

It is clear that h is a bijection from m_0 onto ω, against the choice of n_0. ⊗

We remark that the principle of induction on natural numbers is available in **EST** since its usual proof relies on the principle of minimum on natural numbers. In order to prove that each non empty subcollection of the natural numbers has a least element one does not need to know that such a subcollection is a set for the least element can be found in the (finite) subset whose elements are those of the given subcollection that are less or equal to an arbitrary element of the subcollection.

4. Finite Set Theory.

We will call *Finite Set Theory*, and denote it by **FST**, the theory whose axioms are those of **EST** together with the finiteness axiom

Fin: $\forall x\, Fin^{f}(x)$,

i.e. **FST** = **EST** + $\forall x\, Fin^{f}(x)$ = **EST** + **Fin**.

Let **WO** be the sentence asserting that every set can be well ordered, and **AC** the sentence asserting the existence of a choice function on any family of non empty sets. Furthermore, let **Pow** be the sentence asserting the existence of the power set of any set. Formally:

WO: $\forall x\, \exists \alpha\, \exists f\, (On\, (\alpha) \wedge Fun(f) \wedge Dom(f) = \alpha \wedge Ran(f) = x \wedge Inj(f))$.

AC: $\forall x\, \exists f\, (Fun(f) \wedge Dom(f) = x \wedge \forall y \in x (y \neq \mathbf{0} \rightarrow f(y) \in y))$.

Pow: $\forall x\, \exists y\, \forall z (z \in y \leftrightarrow x \supseteq z)$.

Remark that the usual proof of the equivalence between **WO** and **AC** needs **Pow** to show that **AC**$\rightarrow$**WO**: **Pow** guarantees that the collection of the subsets of the set for which we want to find a well order is indeed a set. Since **Pow** is not a theorem of **EST**, this proof cannot be obtained in **EST**.

WO and **AC** are provable equivalent in **FST** since we will show that **Pow** is a theorem of **FST**. Furthermore we have that:

Lemma 4. **FST** |- **WO**.
Proof. Straightforward. ⊗

So also **AC** is a theorem of **FST** in view of the following

Theorem 5 ([V1]). **FST** |- **Pow**.
Proof. It suffices to show that, for any natural number n, the collection P(n) of its subsets is a set. For, if we prove that

$$\mathbf{FST} \vdash \forall n\, (Nat(n) \rightarrow \exists x (\forall y\, (y \in x \leftrightarrow n \supseteq y))) \qquad (1)$$

then the existing bijection between any set s and some natural number n induces a bijection between P(n) and the collection of all subsets of s. Thus, by Replacement, the collection of all subsets of s is a set.
The proof of (1) is by induction which is allowed in view of the remark at the end of the previous section. The base case is trivial. For the inductive step, since the collection $y = \{x \cup \{n\}: x \in P(n)\}$ is a set by Replacement, then $P(n \cup \{n\}) = P(n) \cup y$ is a set. ⊗

Remark 6. In **EST** the cartesian product of two sets is a set. Indeed, let x and y be the two sets. For each z belonging to y, let f_z be the function of domain x defined as follows: $f_z(v)=<v,z>$. By **Repl** $Ran(f_z)$ is a set. Now let g be the function of domain y defined as follows: $g(z) = Ran(f_z)$. Again by **Repl** we have that $Ran(g) =\{Ran(f_z): z \in y\}$ is a set. Finally by **Union** we have that $\cup Ran(g)$ is a set, and it is easily seen that $\cup Ran(g) = x \times y$.

Now let us study some interesting relations among the axioms introduced so far. We need a preliminary lemma which is well known in the usual elementary theory of sets.

Lemma 7. EST |- $\forall m\, \forall n\, (Nat(m) \wedge Nat(n) \wedge \exists f\, (Fun(f) \wedge Dom(f) = m \wedge Ran(f) = n \wedge Inj(f)) \rightarrow m = n)$.
Proof. (Sketch) One easily gets a contradiction considering the least natural number m for which there are another number n, n≠m, and a bijective function from n onto m. ⊗

Notice that, as an immediate consequence of Lemma 7, we have the following

Corollary 8. FST |- $\forall x\, \exists! n\, \exists f\, (Nat(n) \wedge Fun(f) \wedge Dom(f) = n \wedge Ran(f) = x \wedge Inj(f))$.
Proof. Straightforward. ⊗

We are now in the position to prove the following

Theorem 9. EST |- (**Pow** ∧ ¬**Inf**) → $\forall x\, Fin^f(x)$.
Proof. Assume **Pow** and, by contradiction, let x be such that $\neg Fin^f(x)$ holds.
We claim that for any natural number n there exist a subset y of x and a bijection f on n onto y. If not, let n_0 be the first natural number for which the claim is false. Clearly $n_0 \neq 0$. Let $n_0=n'+1$. Then there exist a subset y of x and a bijection f from n' onto y. Furthermore $x - y \neq \mathbf{0}$, since otherwise we would have $Fin^f(x)$, against what we assumed. Let u belong to x - y. We may now consider the extension g of f such that $g(n') = u$. g is a bijection of n_0 onto the subset $y \cup \{u\}$ of x, contradicting the choice of n_0. Thus we have proved the claim.
Now let $u = \{y \in P(x): \exists n\, \exists f\, (Nat(n) \wedge Fun(f) \wedge Dom(f) = n \wedge Ran(f) = y \wedge Inj(f))\}$. u is obtained by separation (which is a consequence of **Repl**) from the collection of the subsets of x, which is a set since due to **Pow**. Thus u is a set. We may now define the function h which maps each y belonging to u into the unique natural number n for which there is a bijection from n onto y (the uniqueness of the natural number n corresponding to y is an easy consequence of Lemma 7). Applying **Repl** once more, we have that Ran(h) is a set.
Since we have just proved that for any natural number n there are a subset y of x and a bijection from n onto y, we have that $0 \in Ran(h) \wedge \forall y\, (y \in Ran(h) \rightarrow y \cup \{y\} \in Ran(h))$. Thus **Inf** holds and we have reached the desired contradiction. ⊗

We will show later that the previous result cannot be strengthened to drop **Pow** from the hypothesis.

Corollary 10. EST |- $\forall x\, Fin^f(x) \leftrightarrow$ (**Pow** ∧ ¬**Inf**).
Proof. The direction not dealt with by Theorem 9 is an immediate consequence of Lemma 3 and Theorem 5. ⊗

In the sequel it will be convenient to have the following characterization of the notion of natural number. Let Lim(x) be the formula $On(x) \wedge x \neq 0 \wedge \forall y\,(y \in x \rightarrow x \neq y \cup \{y\})$.

Lemma 11. **EST** $\vdash \forall x\,(Nat(x) \leftrightarrow On(x) \wedge \forall y \in x\, \neg Lim(y))$.

Proof. ($\rightarrow$) Let us argue by contradiction. Let x be an ordinal number for which Lim(y) holds for some ordinal $y \in x$. Since x is transitive, y is a non empty subset of x. We claim that y has no maximum. Otherwise, let u be the maximum of y, i.e. $u \in y$ and $v \leq u$ for all $v \in y$. Since y is an ordinal and u belongs to it, then $u \cup \{u\}$ either belongs to y or equals y, but it cannot equal y since Lim(y), so it belongs to y contradicting the maximality of u and proving the claim. On the other hand Nat(x) implies that any non empty subset of x, and y in particular, has a maximum. This contradiction completes the argument.

($\leftarrow$) Let x be an ordinal such that $\forall y \leq x\, \neg Lim(y)$. Let y be a non empty subset of x. Let us prove that y has a maximum element. Let $z = \cup y$. Then z is an ordinal greater than or equal to all the ordinals in y, and also smaller than or equal to x. If it is greater than all the ordinals in y, then for each v belonging to z also $v \cup \{v\}$ would belong to z. But this would imply Lim(z). Since we are assuming $\neg$Lim(y), it follows that z must be equal to an ordinal in y, and therefore z is the maximum of y. ⊗

Now we want to prove the equivalence in EST of $\forall x\, Fin^f(x)$ with **WO** $\wedge \neg$**Inf**.

Theorem 12. **EST** $\vdash \forall x\, Fin^f(x) \leftrightarrow$ **WO** $\wedge \neg$**Inf**.

Proof. ($\rightarrow$) This is just what was proved through the Lemma 3 and 4.

($\leftarrow$) Let x be any set and R be a well ordering of x (the existence of R is guarantied by **WO**). As it is usually done in set theory, using only the axioms of **EST**, it can be shown that every well ordered set is order isomorphic to a unique ordinal number. Let α be the ordinal number such that $(x,R) \cong (\alpha,\in)$. Since we are assuming $\neg$Inf, it follows that $\forall \beta \leq \alpha \neg Lim(\beta)$, whence $Nat(\alpha)$, due to Lemma 11. ⊗

5. The axiom of regularity.

Set theories without the axiom of regularity have been thoroughly investigated by Boffa [**B1**], [**B2**], [**B3**], and Felgner devotes to the subject several pages of his book [**F**]. The usual formulation of the axiom of regularity is the sentence

UReg: $\forall x\,(x \neq \mathbf{0} \rightarrow \exists y\,(y \in x \wedge y \cap x = \mathbf{0}))$.

Our next goal is to show that **UReg** is independent of **FST**.

Let $\mathbf{FST}^+ = \mathbf{FST} + \mathbf{UReg}$.

First let us remark that $\mathbf{FST}^+$ is consistent relative to **ZF**. To see this, consider $(R_\omega,\in)$ in the hierarchy of the well founded sets which is definable form the axioms of **ZF**. Clearly

$$(R_\omega,\in) \models \mathbf{FST}^+.$$

The problem of the independence of **UReg** does not have an equally straightforward solution. We start with the following result.

Theorem 13. **UReg** is independent of **EST** + $\neg$**Inf**.

Proof. As mentioned in [**K1**], the following result holds (see also [**F**]).

Let F be any bijection from the intended universe V of **ZFC** in itself. Define the relation E on V as follows

$$x \mathrel{E} y \Leftrightarrow x \in F(y).$$

Then, assuming the consistency of **ZFC**, we have that $(V,E) \models \mathbf{ZF}^-$, where $\mathbf{ZF}^-$ is **ZF** without the axiom of regularity.

In analogy with this result, we want to show that, if F is the bijection of R_ω onto itself such that F(0) = 1, F(1) = 0 and it is the identity elsewhere, then

$$(R_\omega,E) \models \mathbf{EST}+\neg\mathbf{Inf}+\neg\mathbf{UReg},$$

where E is defined as above.

We begin by proving that $(R_\omega,E) \models \neg$**UReg**. We have to show that

$$(R_\omega,E) \models \exists x\,(x \neq \mathbf{0} \wedge \forall y\,(y \in x \rightarrow \exists z\,(z \in y \wedge z \in x))).$$

Let us remark that $\mathbf{0}^{(R_\omega,E)} = 1$. Indeed,

$(R_\omega,E) \models \forall y\,(y \in z \leftrightarrow y \neq y)[s] \Leftrightarrow \forall r\,(r \in F(s) \leftrightarrow r \neq r) \Leftrightarrow F(s) = 0 \Leftrightarrow s = 1$,

where r and s are elements of R_ω, 0 is the empty set and 1 is its singleton in R_ω.

Thus 0 is not the interpretation of the empty set in (R_ω,E), and we prove that it is indeed an element for which **UReg** does not hold, i.e.

$(R_\omega,E) \models (x \neq \mathbf{0} \wedge \forall y\,(y\in x \rightarrow \exists z\,(z\in y \wedge z\in x)))\ [0],$

This claim is equivalent to: $(0 \neq \mathbf{0}^{(R_\omega,E)} \wedge \forall y\,(y\in F(0) \rightarrow \exists z\,(z\in F(y) \wedge z\in F(0))))$; and it holds since $0 \neq 1 \wedge \forall y\,(y\in 1 \rightarrow \exists z\,(z\in F(y) \wedge z\in 1))$, given that $F(0) = 1$ and $0\in 1$.

Next we prove that $(R_\omega,E) \models \neg$**Inf**.

To this end, first let us remark that $(x\cup\{x\})^{(R_\omega,E)} = F^{-1}(F(x^{(R_\omega,E)})\cup\{x^{(R_\omega,E)}\})$. Indeed

$(R_\omega,E) \models \forall y\,(y\in z \leftrightarrow y\in x \vee y = x)[s,t] \Leftrightarrow \forall r\,(r\in F(t) \leftrightarrow r\in F(s) \vee r = s) \Leftrightarrow$

$$F(t) = F(s)\cup\{s\} \Leftrightarrow t = F^{-1}(F(s)\cup\{s\}),$$

where r, s and t are elements of R_ω.

Thus, if, by contradiction, we had $(R_\omega,E) \models$ **Inf**, then there would be a set r belonging to R_ω such that

$$\mathbf{0}^{(R_\omega,E)}\in F(r) \wedge \forall s\in R_\omega\,(s\in F(r) \rightarrow F^{-1}(F(s)\cup\{s\})\in F(r)).$$

An easy check shows that 1, {1}, {1,{1}}, {1,{1}},{1,{1}}}, ... would belong to F(r), i.e. "the numerals built starting from 1" would belong to F(r).

In **ZF** we may define by recursion on the natural numbers a function g such that $g(0) = 1$ and $g(n+1) = g(n)\cup\{g(n)\}$.

Hence the function f of domain F(r) defined as follows

$$f(s) = \{n, \text{ if } s = g(n) \text{ for some natural number } n\ \ , , ,0, \text{ otherwise}$$

has range equal to ω.

But $(R_\omega,\in) \models$ **Repl**. Hence we will have $(R_\omega,\in) \models$ **Inf** : a contradiction.

Therefore $(R_\omega,E) \models \neg$**Inf**.

The proof that $(R_\omega,E) \models$ **EST** is simple, except, maybe, for **Repl**.

So let $a, a_1, ..., a_k \in R_\omega$ and φ be such that

$$(R_\omega,E) \models (\forall x\in u\, \exists! y\, \varphi(x,y,y_1, ..., y_k))[a, a_1, ...,a_k],$$

then $\forall r\in F(a)\, \exists! s\, \varphi^{(R_\omega,E)}(r, s, a_1, ..., a_k)$.

Since **Repl** holds in $(R_\omega,\in)$, there is $b\in R_\omega$ such that

$$\forall s\,(s\in b \leftrightarrow \exists r\in F(a)\, \varphi^{(R_\omega,E)}(r, s, a_1, ..., a_k)\,).$$

Therefore $(R_\omega,E) \models \forall y\,(y\in w \leftrightarrow \exists x\in u\, \varphi(x,y,y_1,..., y_k)\,)\ [a,F^{-1}(b),a_1,...,a_k]$, whence the immediate conclusion. ⊗

Now let us prove that **UReg** is also independent of **FST**.

Theorem 14. **UReg** is independent of **FST**.

Proof. In view of the proof of Theorem 13, it is enough to show that **Pow** holds in (R_ω,E), i.e.

$$(R_\omega,E) \models \forall x\, \exists y\, \forall z\,(z\in y \leftrightarrow \forall w\,(w\in z \rightarrow w\in x).$$

Let r be any set belonging to R_ω, we have to prove that there is s belonging to R_ω such that

$$\forall t\,(t\in F(s) \leftrightarrow \forall q\,(q\in F(t) \rightarrow q\in F(r)\,),$$

i.e. $\forall t\,(t\in F(s) \leftrightarrow F(r) \supseteq F(t)\,)$.

Let $s = F^{-1}(\{t: F(r) \supseteq F(t)\,\})$. s is well defined and it is the set we were looking for.

Now, as a consequence of Corollary 10 and of Theorem 13, we have that

$$(R_\omega,E) \models \forall x\, Fin^f(x) \wedge \neg\textbf{UReg},$$

whence the desired independence result, keeping in mind that **FST** = **EST** + $\forall x Fin^f(x)$. ⊗

Introducing the sentence **UReg**, we wanted to distinguish between the usual notion of regularity and a stronger one. So now we introduce a sentence, that we will call *strong regularity*, shortly **SReg**, remarking that it can be stated only within theories in which the usual hierarchy of the well founded sets (the hierarchy of R_α's) can be defined. Of course, a sufficient condition for the possibility of stating **SReg** is the provability of **Pow** in the given theory.

SReg: $\forall x\, \exists\alpha\,(\, On(\alpha) \wedge R(\alpha) \supseteq x)$.

The following theorem holds.

Theorem 15. **FST** ⊢ **SReg** → **UReg**.

Proof. First remark that **SReg** can be stated in **FST**, since **Pow** is a theorem of **FST** (see Theorem 10). Then the proof proceeds as in ZF, once the notion of rank of a set has been introduced. ⊗

Theorems 14 and 15 give an immediate proof of the following Corollary.

Corollary 16. **SReg** is independent of **FST**.

The two notions of regularity introduced are equivalent in **ZF**. The proof in **ZF** of the implication which is not dealt with in Theorem 15, is based on the possibility of showing that for every set x there is a transitive set containing x. This proof is immediate once one has **Inf**.

Since **Inf** is not an axiom in any of the theories we are dealing with, let us introduce an axiom asserting that for every set there is a transitive set containing it.

Trcl : $\forall x\, \exists y\, (\mathrm{Trans}(y) \wedge y \supseteq x)$.

At first sight it could seem that **Trcl** is provable in **FST** (after all, in **FST** every set is f-finite). Actually, we have the following

Theorem 17. Trcl is independent of **FST**.

Proof. Under the hypothesis that **ZF** is consistent, we will produce a model of **EST + Pow + ¬Inf + ¬Trcl**.

We take advantage of the following result (see [**K1**]): If **ZF** is consistent, then also **ZF** + "there is a countable set $\{x_n: n\in \omega\}$ of distinct elements such that, for all n, $x_n = \{x_{n+1}\}$" is consistent. So the latter theory has a model, say (M,E), and let y be the set $\{x_n: n\in \omega\}$ of pairwise distinct elements x_n's in M such that $x_n = \{x_{n+1}\}$, for all natural numbers n.

Let us use the following notation: $y_0 = y$, $y_{n+1} = \{x: y_n \supseteq x \text{ and } |x| < \omega\}$.

Finally let $N = \cup\{y_n: n\in \omega\}$ and let E' be the restriction of E to N.

It is easy to check that (N,E') $\models$ **EST** ∧ **¬Inf** ∧ **Pow** ∧ **UReg** ∧ **¬Trcl**. ⊗

The above theorem has already been proved (within the framework of Alternative Set Theory) in [**S3**]. In the same paper, the author proves also that **UReg** is independent of **FST +Trcl**, while it is easy to show that

EST ⊢ (**UReg** ∧ **Trcl**) ↔ **SReg**,

by using the same proof of the equivalence given in **ZF**.

Corollary 18. UReg and **SReg** are not equivalent in **FST**.

Proof. With the notation of Theorem 17, let x_n be any chosen element of y. It is obvious that x_n cannot be obtained iterating the power set operation starting from the empty set. ⊗

In the literature the name "axiom of weak foundation", **WReg**, is reserved for the following sentence:

$$\exists z\, \forall x\, (x\neq \mathbf{0} \rightarrow \exists y\, (y\in x \wedge (y\cap x=\mathbf{0} \vee (y\in z \wedge y=\{y\})))).$$

This kind of regularity is compatible with the existence of an infinite set of reflexive elements. We will use this fact in the next section.

6. A comparison of FST with a fragment of AST.

As stated in the introduction, **FST** is equivalent to a Fragment of Alternative Set Theory including only axioms for sets. In this section we are going to describe this fragment and sketch the proof of the equivalence.

Let us call **VF** the fragment of Alternative Set Theory (see [**S1**]):

$\forall x\forall y\, (x = y \leftrightarrow \forall z\, (z\in x \leftrightarrow z\in y))$ (Extensionality);

$\exists x\forall y\, (y \notin x)$ (Emptyset);

$\forall x\forall y\, \exists z\forall w\, (w\in z \leftrightarrow w\in x \vee w = y)$ (Union with a singleton);

$(\varphi(\mathbf{0}) \wedge \forall x\forall y\, (\varphi(x) \rightarrow \varphi(x\cup\{y\})) \rightarrow \forall x\varphi(x)$ (Schema of induction on sets),

where we use the notation $x\cup\{y\}$ for the set whose existence is ensured by the axiom of union with a singleton.

Notice that **VF** captures within **ZF** the collection of hereditarily finite sets.

It is proved in [**V2**] that from the axioms of **VF** one can derive those of **FST**, once the definition of natural number is given. In particular, to prove **Fin**, one applies the schema of induction to the formula $\mathrm{Fin}^f(x)$ (see section 2).

For the converse, the only non trivial step is to show that the schema of induction on sets follows from the axioms of **FST**. That is done by combining the induction on natural numbers with the axiom of finiteness. For, assume that for some formula φ, $\varphi(\mathbf{0})$ and $\forall x\forall y\, (\varphi(x) \rightarrow \varphi(x\cup\{y\})$ hold, while $\forall x\varphi(x)$ does not hold. One easily gets a contradiction by considering the least natural number n for which there exists an x bijective with n such that $\varphi(x)$ does not hold.

7. The independence of FST from EST + ¬Inf.

Our plan is to show that **FST** is independent of **EST** + ¬ **Inf**, relative to the consistency of **ZF**$^-$, consists in exhibiting a model of **EST** + ¬**Inf** in which $\forall x \mathrm{Fin}^f(x)$ fails.

We need some preliminaries. Given any set A, we define

$$R_0(A) = A$$

and, for any ordinal α,

$$R_{\alpha+1}(A) = A;$$

$$R_\alpha(A) = A, \text{ when a is a limit ordinal.}$$

We let $\quad N(A) = \cup\{R_n(A): \mathrm{Nat}(n)\}$

and $\quad WF(A) = \cup\{R_\alpha(A): \alpha \in \mathrm{On}\}$.

Recall that a set x is *reflexive* if $x = \{x\}$. Of course, the existence of a reflexive set is inconsistent with the usual axiom of Regularity of **ZF**.

It is well known that if **ZF**$^-$ is consistent, then **ZF**$^-$ + "there exists a countable set A of reflexive elements such that V = WF(A)" is consistent (see, for instance, [**F**]).

We will construct our model within the latter theory. So, let $(M,E) \models$ **ZF**$^-$ +"V=WF(A)", with A a denumerable set of reflexive elements in (M,E). Since from now on our intended universe of set theory will be (M,E), we will use $\in$ in place of E. Furthermore, since the structures that we will introduce are substructures of $(M,\in)$, we will also denote by $\in$ the membership relation restricted to such substructures.

One might be tempted to prove that

$$(N(A), \in) \models \mathbf{EST} + \neg\, \mathbf{Inf} + \neg\forall x\, \mathrm{Fin}^f(x).$$

Unfortunately, this is not true. It can be proved that all the axioms of **EST** but **Repl** are true in $(N(A), \in)$. Also ¬ **Inf**, **Pow** and $\neg\forall x\ \mathrm{Fin}^f(x)$ hold in $(N(A),\in)$.

If **Repl** were true in $(N(A),\in)$, then Theorem 9 would lead to a contradiction.

So it is necessary to "refine" $(N(A), \in)$.

The following model, Theorem and its Lemmas have been suggested by Kunen [**K2**].

We start with the description of a model that will lead us to the results that we want.

In $(N(A),\in)$, there is an isomorphism ξ between the group of permutations of A and the group of permutations of N(A) that preserve the membership relation, such that, for each permutation σ of A, $\xi(\sigma)$ is the unique membership preserving permutation of N(A) that extends σ. Given a subset F of A, let $H_0(F)$ be the set of all permutations of A that fix the elements of F, i.e.

$$H_0(F) = \{\sigma: \sigma \text{ is a permutation of A and } \sigma(x)=x \text{ for all } x\in F\}.$$

Let H(F) be the set of all the extensions to N(A) of the permutations of $H_0(F)$, i.e. $H(F) = \{\sigma: \sigma=\xi(\sigma') \text{ for some } \sigma'\in H_0(F)\}$. We will refer to a permutation of H(F) by saying that it is a permutation that fix F. So H(**0**) is the set of all extensions to N(A) of the permutations of A, i.e. the set of all permutations of N(A) that preserve the membership relation. From now on, we will use the words "permutation of N(A)" to mean a permutation of N(A) that preserves the membership relation.

Let $HB_k = \{x\in N(A)$: for each y belonging to $Tc(x)\cup\{x\}$ there is a subset F_y of A with k elements such that if $\sigma\in H(F_y)$ then $\sigma(y)=y$ }.

Let $M = \cup\{HB_k: k\in\omega\}$. $(M,\in)$ is our candidate for the required "refinement".

Before proceeding towards our goals, let us remark a basic feature of this structure.

Lemma 19. The HB_k's are all transitive, so M is transitive.

Proof. Let k be any natural number and let $x\in y\in HB_k$. Then $Tc(x)\cup\{x\} \supseteq Tc(y)\cup\{y\}$, so that for each element t belonging to $Tc(x)\cup\{x\}$ there is a subset F_t of A with k elements such that if $\sigma\in H(F_t)$ then $\sigma(t)=t$. Thus x belongs to HB_k. ⊗

Theorem 20. i) $(M,\in) \models \neg\forall x\ \mathrm{Fin}^f(x)$,

ii) $(M,\in) \models \neg$**Inf**, and

iii) $(M,\in) \models$ **EST**.

Proof.

i) A belongs to HB_1, and hence to M, and $\neg\mathrm{Fin}^f(A)$.

ii) It is not difficult to show by induction on the index n of the $R_n(A)$'s that every set belonging to N(A) without reflexive elements in its transitive closure is f-finite. A fortiori the result holds for M. Assuming that we have already proved iii), it follows that $\neg$**Inf**, since, otherwise, by **Inf** and Separation one could assert the existence of a set without reflexive elements belonging to its transitive closure which is not f-finite.

iii) The only point that involves some difficulties is to show that $(M,\in) \models$ **Repl**. So let X be a set belonging to M and $\varphi(x,y)$ a formula in the language of **ZF** enriched with the names $\underline{m}$ for all elements $m\in M$. Remark that only a finite number of names will occur in the formula $\varphi(x,y)$. Assume that $(M,\in,M) \models \forall x\,(x\in X \rightarrow \exists! y\ \varphi(x,y))$. For each $x\in X$, call $y(x)$ such a unique y. Let $Y = \{y(x): x\in X\}$. We have to prove that Y belongs to M, i.e. it belongs to $R_p(A)\cap HB_k$ for some convenient natural numbers p and k . Since each y(x) is an element of M, it suffices to show that

a) there is a finite subset F of A such that $\sigma(Y) = Y$ for all $\sigma\in H(F)$, and

b) there are uniform bounds p_0 and k_0 such that , for all $x\in X$, y(x) belongs to $R_{p_0}(A)\cap HB_{k_0}$.

Since $X\in M$ there are n and k such that $X\in R_{n+1}(A) \cap HB_k$ (hence $R_n(A) \cap HB_k \supseteq X$ due to the definition of $R_p(A)$ and Lemma 19).

Let F be a finite subset of M such that, for all permutations $\sigma\in H(F)$, we have that $\sigma(X) = X$ and also $\sigma(m) = m$ for all elements m whose names occur in $\varphi(x,y)$.

Notice that such a finite F does exist since the parameters in $\varphi(x,y)$ are finitely many.

In this situation, we have also that $\sigma(Y) = Y$ for all $\sigma\in H(F)$. Indeed, let $y\in Y$, so that $\sigma(y)\in\sigma(Y)$ and $y = y(x)$ for some $x\in X$, and use the following lemma.

Lemma 21. For every natural number p, for every p-tuple $m_1,...,m_p$ of elements of M, for every formula $\alpha(x_1, ...,x_n,\underline{m}_1,...,\underline{m}_p)$ in the language of **ZF** plus the names $\underline{m}_1,...,\underline{m}_p$ of the elements $m_1,...,m_p$, and for every permutation $\sigma\in H(G)$, where G is a finite subset of A such that H(G) fixes $m_1,...,m_p$,

$$(M,\in,m_1,...,m_p) \models \alpha(x_1, ...,x_n,\underline{m}_1,...,\underline{m}_p) \Leftrightarrow$$
$$(M,\in,m_1,...,m_p) \models \alpha(\sigma(x_1), ..., \sigma(x_n),\underline{m}_1,...,\underline{m}_p).$$

Proof. By induction on the construction of the formula α. No point involves any difficulty. ⊗

Resuming our proof of Theorem 20, we make use of this Lemma by letting G be the set F previously defined and letting $\alpha(x_1, ...,x_n,\underline{m}_1,...,\underline{m}_p)$ be the formula

$\varphi(x,y)$. For all $\sigma\in H(F)$, since $\sigma(x)\in X$ then also $\sigma(y)\in Y$ and we have that $Y \supseteq \sigma(Y)$. The same argument applied to σ^{-1} yields the other inclusion. So $\sigma(Y) = Y$ for all $\sigma\in H(F)$, and a) is proved.

To prove b), we argue as follows. For any finite subset G of A and for any $x\in M$, we define

$$orb_G(x) = \{y\in M: \exists\sigma\ (\sigma\in H(G) \wedge \sigma(y) = x)\}.$$

$orb_G(x)$ will be called the G-orbit of x. Notice that, for each G, the relation of belonging to the same G-orbit is an equivalence relation on M. As a consequence of Lemma 21, if an element x of X belongs to $orb_F(z)$ for some $z\in X$, then y(x) belongs to $orb_F(y(z))$: so the number of F-orbits that can be found in Y is not greater than the number of F-orbits that can be found in X. Now we will be done if we prove that

*) for any finite subset G of A, for any p and k, and any $z\in R_p(A)\cap HB_k$, $orb_G(z)$ is contained in $R_p(A)\cap HB_k$ (this follows from Lemma 22 below), and that

**) for any finite subset G of A and for any $Z\in M$, Z is the union of finitely many G-orbits (this follows from Lemma 23 below).

Indeed, **) applied to our F and X implies not only that X is partitioned into finitely many F-orbits, but also that Y is partitioned into finitely many F-orbits, say $orb_F(y(x_i))$ for i=1,...,q, in view of the already remarked consequence of Lemma 21. On the other hand *) tells us that each F-orbit partitioning Y, say $orb_F(y(x_i))$, is included in some $R_{p_i}(A)\cap HB_{k_i}$.

So let $p_0 = \max\{p_i: 1 \leq i \leq q\}$ and $k_0 = \max\{k_i: 1\leq i \leq q\}$. Thus p_0 and k_0 are the uniform bounds that we were looking for, and Y is included in $R_{p_0}(A)\cap HB_{k_0}$. ⊗

Lemma 22. For any natural numbers p and k and for any permutation $\sigma\in H(\mathbf{0})$, $\sigma(R_p(A)\cap HB_k) = R_p(A)\cap HB_k$.

Proof. Since intersections are preserved by the permutations, we prove separately that

i) if x belongs to $R_p(A)$ then $\sigma(x)$ belongs to $R_p(A)$, and that

ii) if x belongs to HB_k then also $\sigma(x)$ belongs to HB_k.

i) is proved by induction on p. The base step is obvious. Thus let us consider $R_{p+1}(A) = P(R_p(A))$. Let $\sigma\in H(\mathbf{0})$, then $\sigma(R_{p+1}(A)) = \{\sigma(x): R_p(A) \supseteq x\}$. By inductive hypothesis $R_p(A) \supseteq \sigma(x)$; hence $R_{p+1}(A) \supseteq \sigma(R_{p+1}(A))$. The same argument, with σ^{-1} instead of σ, yields the other inclusion.

ii) The hypothesis that $x \in HB_k$ means that for all $y \in Tc(x) \cup \{x\}$ there is a subset G_y of cardinality k of A such that, for all $\tau \in H(G_y)$, $\tau(y) = y$. To prove that $\sigma(x) \in HB_k$, we have to show that for all $z \in Tc(\sigma(x)) \cup \{\sigma(x)\}$ there is a subset L_z of cardinality k of A such that, for all $\tau \in H(L_z)$, $\tau(z) = z$.

First notice that $Tc(\sigma(x)) = \sigma(Tc(x))$ and that $\sigma(\{x\}) = \{\sigma(x)\}$, so that the z's that belong to $Tc(\sigma(x)) \cup \{\sigma(x)\}$ are none other than the $\sigma(y)$'s for $y \in Tc(x) \cup \{x\}$.

We claim that, for every such $\sigma(y)$ and every $\tau \in H(\sigma(G_y))$, $\tau\sigma(y) = \sigma(y)$. This is obvious, since $\sigma^{-1}\tau\sigma \in H(G_y)$.

Thus if we take $L_{\sigma(y)} = \sigma(G_y)$ then, for all $z \in Tc(\sigma(x)) \cup \{\sigma(x)\}$, $L_{\sigma(y)}$ is a subset of cardinality k of A such that, for all $\tau \in H(L_{\sigma(y)})$, $\tau(z) = z$, and ii) is proved. ⊗

Lemma 23. For all natural numbers m and k, and for each finite subset G of A, the permutations $\sigma \in H(G)$ partition $R_m(A) \cap HB_k$ into a finite number of G-orbits.

Proof. First we remark that it is correct to say that the G-orbits partition, not only M, but also $R_p(A) \cap HB_k$, for all p and k. Indeed every permutation, and in particular the permutations that fix G, maps $R_p(A)$ and HB_k onto themselves respectively, as is stated in Lemma 22.

So we are left with the proof that the number of orbits partitioning $R_p(A) \cap HB_k$ is finite. We argue by induction on p and show that for all numbers k and all finite subsets G of A, $R_p(A) \cap HB_k$ is partitioned into finitely many G-orbits.

If p = 0 then $R_0(A) \cap HB_k = A \cap HB_k$ which equals either the empty set, if k=0, or A, if k≠0. In the first case there is nothing to prove. Otherwise, let $G = \{a_{j_1}, ..., a_{j_h}\}$, thus $R_0(A) \cap HB_k = A$ is partitioned in h+1 G-orbits, namely $\{a_{j_1}\}$, ..., $\{a_{j_h}\}$ and A-G.

So let p = q+1 and assume the inductive hypothesis true up to q. We have to count the G-orbits of $R_{q+1}(A) \cap HB_k$, with G of cardinality h, say. We claim that the following hold:

1) for any set K of cardinality h+k such that $A \supseteq K \supseteq G$, there are finitely many elements $r \in R_{q+1}(A) \cap HB_k$ that are fixed by the permutations of H(K);

2) every $s \in R_{q+1}(A) \cap HB_k$ is in the G-orbit of some r as in point 1.

To prove 1), notice first of all that, by Lemma 19, $r \in R_{q+1}(A) \cap HB_k$ implies that $R_q(A) \cap HB_k \supseteq r$. Now, by Lemma 22, $R_q(A) \cap HB_k \supseteq orb_K(t)$, for all $t \in r$. Therefore r is the union of K-orbits in $R_q(A) \cap HB_k$ and, by the inductive

hypothesis applied to the K-orbits, there are finitely many such orbits, so there are finitely many r that are fixed by H(K).

To prove 2), let s be an arbitrary element of $R_{q+1}(A) \cap HB_k$. So s is fixed by H(L), for some L of cardinality k. By considering $G \cup L$ and by adding further elements, if necessary, we get a superset J of G of cardinality h+k such that s is fixed by H(J).

Choose K any superset of G of cardinality h+k and σ any permutations that maps J onto K and fixes G. Then the element $r = \sigma(s)$ is fixed by H(K). Indeed, for any $\tau \in H(K)$, $\sigma^{-1}\tau\sigma \in H(J)$, and therefore $\sigma^{-1}\tau\sigma(s) = s$. So 2) is proved.

Now, since every $s \in R_{q+1}(A) \cap HB_k$ is in the G-orbit of some $r \in R_{q+1}(A) \cap HB_k$ that is fixed by all the permutations of H(K), with $K \supseteq G$ of cardinality h+k, and, since by 1) there are finitely many such r's, then there are finitely many possibilities for $orb_G(s)$.

That proves the Lemma. ⊗

We can now state and prove the desired independence result.

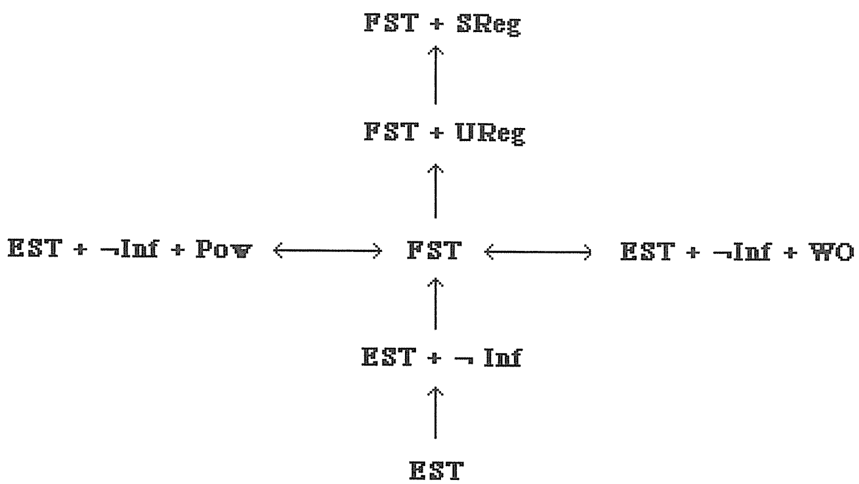
Corollary 24. If $\mathbf{ZF^-}$ is consistent, then **Pow**, **WO** and $\forall x\, Fin^f(x)$ are independent of **EST** + ¬**Inf**.

Proof. The independence of $\forall x\, Fin^f(x)$ of **EST** + ¬**Inf** follows immediately from Theorem 20. The independence of **Pow** and **WO** of **EST** + ¬**Inf** is straightforward from Corollary 10 and Theorem 12. ⊗

Notice that another consequence of Corollary 10, Theorem 12 and Theorem 20 is that $(M,\in)$ satisfies neither **Pow** nor **WO**.

8. Conclusion.

The relationships among the different theories that we have considered so far are summarized in the following picture. The presence of an arrow from one theory to another means proper inclusion of the former into the latter. The presence of a double arrow stands for equivalence



Since the present work has been carried out in view of the development of a finite non-standard set theory, it remains to choose a theory for finite sets that we consider the most adequate for that purpose. We feel that it is very natural to think of sets as well determined by the rank construction starting from the empty set. So we will assume the strong axiom of regularity and we will choose **FST** + **SReg** as our theory for finite sets, now being aware of the relative strength of its axioms.

We will name this theory *Regular Finite Set Theory* and we will denote it by **RFST**.

It is interesting to note that, if Con(**PA**) and Con(**RFST**) are, respectively, the sentences asserting the consistency of the theory of formalized arithmetic (**PA**) and the consistency of **RFST**, then it can be proved finitistically that

$$\mathrm{Con}(\mathbf{PA}) \leftrightarrow \mathrm{Con}(\mathbf{RFST}).$$

A sketch of the proof is as follows. Notice first of all that the usual interpretation of **PA** within **ZF** - **Inf** implies that Con(**ZF** - **Inf** + ¬**Inf**) → Con(**PA**).

Moreover, it can be proved quite easily the existence of a primitive recursive relation E on natural numbers such that (ω,E) and $(R_\omega,\in)$ are isomorphic structures (see exercise 5 in chapter III of [**K1**]). The isomorphism between the above structures allows us to define a translation * of the formulas in the language of **ZF** into formulas in the language of **PA** such that , for every sentence φ in the language of **ZF**,

$$\mathbf{ZF} - \mathbf{Inf} + \neg\mathbf{Inf} \vdash \varphi \quad \Rightarrow \quad \mathbf{PA} \vdash \varphi^* .$$

From the above metaimplication we get that Con(**PA**)→ Con(**ZF** - **Inf** + ¬**Inf**). Since **ZF** - **Inf** + ¬**Inf** and **RFST** have the same strength, Con(**PA**) ↔ Con(**RFST**) can therefore be proved finitistically.

As stated in the introduction, the development of **RFST** in a non-standard framework will be pursued in future works.

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