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On some Ideal Basis Theorems

PAOLO AGLIANO - ALDO URSINI

Abstract. After recalling the concept of subtractive variety we introduce generalized notions of ideal, prime ideal and radical ideal, which, for rings, coincide with the classical notions. Two concepts of gradation of an algebra over some of its subalgebras are given in a way that allows to generalize the Hilbert Basis Theorem and the following statement: "if $\mathbf{R}$ is a commutative unitary ring and all the radical ideals of $\mathbf{R}$ are of finite type, then the same holds for $\mathbf{R}[x]$".

The concept of ideal determined variety of universal algebras, though foreshadowed by Magari [6] and some others (see, for instance [3]), has been investigated by the second author of this paper who, in the early seventies gave the definition and proved the first results ([8], [9]). We deal with varieties with an equationally definable constant (which will be denoted by 0); to make a long story short, an ideal of an algebra $\mathbf{A}$ is a subset of A containig 0 and closed under certain polynomials of the algebra. The case in which any ideal of $\mathbf{A}$ is the 0-block of exactly one congruence of $\mathbf{A}$ is particularly interesting (and we say that $\mathbf{A}$ is **ideal determined**). In that case many classical results in group and ring theory have been generalized. We mention [11] and [4] (the concept of commutator of ideals), [12] (prime ideals and Cohen's Theorem) and [1] (the prime spectrum). The concept works fine as well for Boolean Algebras, algebraic structures coming from logic and in general in almost any context in which the word "ideal" appears. This paper of course is addressed mainly to a generalization of the ring-theoretic frame.

Here we deal with algebras that, though not necessarily ideal determined, are still good enough to permit a generalization of the following classics:

(A) (Hilbert Basis Theorem) If $\mathbf{R}$ is a Noetherian ring, then also $\mathbf{R}[x]$ is Noetherian.
(B) (Kaplansky's exercise [5]) If $\mathbf{R}$ is a commutative unitary ring and all the radical ideals of $\mathbf{R}$ are of finite type, then the same holds for $\mathbf{R}[x]$.

Both classical results rely heavily on the possibility of defining a notion of gradation of $\mathbf{R}[x]$ over $\mathbf{R}$, where the gradation, for a polynomial $p(x) \in \mathbf{R}[x]$, is the pair "degree of $p(x)$, leading coefficient of $p(x)$". Nevertheless a closer look at the proof reveals a substantial difference between the two arguments, difference that is probably somewhat obscured by the natural richness of a ring structure. In fact, while for proving the Hilbert Basis Theorem it is enough to use the "additive" properties of the ring as a module on itself and

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the fact that the degree of the sum of two polynomials is less or equal to the maximum degree, for Kaplansky's theorem there is a critical usage of the fact that the degree function is also well-behaved, in that, in some cases, we can divide a polynomial by another one in order to lower the degree; moreover we need to use the multiplicative structure of $\mathbf{R}[x]$ since we need to "multiply by x^n".

In view of these remarks we have decided to distinguish the two cases, since we feel that in our more general investigation we should not make use of the natural algebraic richness of the ring structure that is of course in the background, not even mentioned, in the classical cases.

Our notation is more or less standard and we refer to [4] for details. We use the vector notation $\vec{a}$ for n-tuples $(a_1, \ldots, a_n)$, and we will write $\vec{a} \in A$ instead of $\vec{a} \in A^n$.

1. Preliminaries

All algebras and varieties considered in this paper will belong to a type with a nullary operation (or will have a definable constant) denoted by 0. Most notions will in fact depend on the existence of this constant, but we will not make this dependence explicit in definitions or notations.

Definition 1.1: A variety $\mathcal{V}$ is **subtractive** [13] if there exists a binary term $s(x, y)$ in the language of $\mathcal{V}$ such that both

$$s(x,x) \approx 0 \qquad s(x,0) \approx x$$

hold in $\mathcal{V}$. An algebra $\mathbf{A}$ is **subtractive** if it belongs to a subtractive variety.

In a subtractive algebra, let us define

$$u(x,y,z) = s(x, s(s(x,y),z)).$$

Then clearly $u(x, y, z)$ satisfies

$$u(x,y,s(x,y))) = x \qquad u(x,0,0) = 0$$

$$u(x,x,0) = u(x,0,x) = x.$$

It would be nicer to have also a binary addition, i.e. a binary term $q(x, y)$ satisfying

$$q(x, s(y,x)) = y \qquad q(x,0) = x.$$

If so however, we would get permutability of congruences, since the binary term

$$t(x,y,z) = q(z, s(x,y))$$

would satisfy the Mal'cev identities

$$t(x,z,z) = t(y,y,x) = x.$$

Implication algebras, for instance, are subtractive, but do not always have permutable congruences (see [7]). The congruence of subtractive algebras however, always permute at 0: if $\mathbf{A}$ is a subtractive algebra and $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$, then $(0,x) \in \theta \vee \varphi$ implies $(0,y) \in \theta$ and $(y,x) \in \varphi$ for some $y \in A$. (Here "$\vee$" denotes of course the join of two congruences). More generally we have:

Proposition 1.2. [2] *For a variety $\mathcal{V}$ the following are equivalent:*

1. *$\mathcal{V}$ is 0-permutable.*
2. *$\mathcal{V}$ is subtractive.*
3. *$\mathcal{V}$ is ideal-coherent.*
4. *For any $\mathbf{A} \in \mathcal{V}$, the map from $\mathrm{Con}(\mathbf{A})$ to the lattice $\mathrm{I}(\mathbf{A})$ of all the ideals of $\mathbf{A}$, defined by $\theta \longmapsto 0/\theta$, is a (complete) lattice homomorphism.*

Let us recall that a nonempty subset I of an algebra $\mathbf{A} \in \mathcal{V}$ is an **ideal** of $\mathbf{A}$ if for any term $p(\vec{x}, \vec{y})$ of $\mathcal{V}$, if $p(\vec{x}, 0, \ldots, 0) \approx 0$ in $\mathcal{V}$ then, for all $a_1, \ldots, a_n \in A$, $i_1, \ldots, i_m \in I$ we have

$$p(a_1, \ldots, a_n, i_1, \ldots, i_m) \in I.$$

(Any term $p(\vec{x}, \vec{y})$ such that $p(\vec{x}, 0, \ldots, 0) \approx 0$ in $\mathcal{V}$ is called an **ideal term in** $\vec{y}$ of $\mathcal{V}$). The set $\mathrm{I}(\mathbf{A})$ of all the ideals, partially ordered by inclusion, is an algebraic lattice. The reader can consult [8], [4] and [11] for more informations. A variety $\mathcal{V}$ is **ideal-coherent** if, for any $\mathbf{A} \in \mathcal{V}$ and $I \in \mathrm{I}(\mathbf{A})$, if $0/\theta \subseteq I$ for some $\theta \in \mathrm{Con}(\mathbf{A})$, then I is a union of θ-blocks. For a proof of Proposition 1.2 see [2].

If $H \subseteq A$, by $\langle H \rangle_{\mathbf{A}}$ we mean the ideal generated by H in $\mathbf{A}$; when no confusion can arise we will drop the subscript $\mathbf{A}$. One easily shows that

$$\langle H \rangle_{\mathbf{A}} = \{p(\vec{a}, \vec{h}) : p(\vec{x}, \vec{y}) \text{ an ideal term in } y, \vec{a} \in A, \vec{h} \in H\}.$$

If $\mathbf{A}$ is a subtractive algebra and X, Y are nonempty subsets of A we define ([13])

$$X + Y = \{u(a,x,y) : a \in A, x \in X, y \in Y\}.$$

That this definition is quite relevant, it is shown by the proposition below:

Proposition 1.3. *Let $\mathbf{A}$ be an algebra in a subtractive variety $\mathcal{V}$ and let $I, J \in \mathrm{I}(\mathbf{A})$.*

1. *If $u \in I$ and $s(u,v) \in I$, then $v \in I$.*

(2) $I + J$ *is an ideal and*

$$I + J = I \vee J = \langle I \cup J \rangle_{\mathbf{A}}.$$

(3) *If* $x \in I + J$, *then there exists a* $y \in I$ *with* $s(x, y) \in J$.

PROOF: While (1) trivially follows from $u(x, y, s(x, y)) = x$, we refer to [13] for the proofs of (2) and (3).

Note that, as a consequence of Proposition 1.3, being an ideal of $\mathbf{A}$ does not depend on which subtractive variety $\mathbf{A}$ happens to belong. Moreover in [13] it is shown that $\mathrm{I}(\mathbf{A}) = \{0/\theta : \theta \in \mathrm{Con}(\mathbf{A})\}$.

2. NOETHERIAN ALGEBRAS AND THE HILBERT BASIS THEOREM

Let κ be any infinite cardinal; from now on we assume the Axiom of choice, so that all the useful properties of cardinals be available. We say that a set is κ-**finite** if its cardinality is strictly less that κ. A family $\mathcal{S}$ of sets is κ-**closed** if the union of any κ-finite chain in $\mathcal{S}$ belongs to $\mathcal{S}$. Let now A be a set and U be a closure operator on A; we say that $\langle A, U \rangle$ is κ-**Noetherian** if every closed subset of A is the closure of a κ-finite subset.

PROPOSITION 2.1. *Let* U *be an algebraic closure operator on* A. *Then the following are equivalent:*

1. $\langle A, U \rangle$ *is* κ*-Noetherian.*
2. *Every properly ascending chain of closed sets is* κ*-finite.*
3. *Every nonempty* κ*-closed family of close sets contains a maximal element.*

PROOF: (Sketch.) The proof is almost equal to the standard one; let us give a few hints. First, if $\mathcal{S}$ is a nonempty κ-closed family of closed sets, pick $I_0 \in \mathcal{S}$ and, for any $\alpha < \kappa$ define $I_{\alpha+1}$ inductively by picking a set in $\mathcal{S}$ properly containing I_α. At limit stages, take unions. Hence (2) implies (3). For the converse, if $\mathcal{C}$ is a chain of length $\beta \geq \kappa$, then take $\mathcal{S}$ to be the set of all κ-finite unions of elements of the chain, in order to deny (3).

Next let $\mathcal{C}$ be a chain, that we may of course assume being strictly increasing. Define a_α inductively for $\alpha < \beta$. Pick any $a_0 \in I_0$; if $\alpha = \delta + 1$ pick $a_\delta \in I_\alpha - I_\delta$ and if α is a limit ordinal pick $a \in I_{\alpha+1} - I_\alpha$. Since there are at least κ successor ordinals below κ, the sequence will have at least κ distinct elements. Let I be the closure of $\{a_\alpha : \alpha < \beta\}$; if we have that I is also the closure of some $\{b_\sigma : \sigma < \tau\}$ for some $\tau < \kappa$ we get a contradiction. Hence $\langle A, U \rangle$ is not κ-Noetherian and (1) implies (2). That (2) implies (1) is trivial.

Let us remark that, in the proposition above, the assumption might be weakened to U being κ-**algebraic**, i.e. that if $x \in U(X)$, then $x \in U(Y)$ for some κ-finite subset Y of X. Actually we do not believe presently that it is worthwhile to work at that stage. In one point at least we would have to strengthen correspondingly an assumption below, at the cost of losing perspicuity.

Assume now that $\mathbf{A}$ is a subtractive subalgebra of a subtractive algebra $\mathbf{B}$. By this we mean that the operation $s(x, y)$ that witnesses subtractivity for $\mathbf{A}$ is the restriction to $\mathbf{A}$ of the operation witnessing subtractivity for $\mathbf{B}$. We will say that $\mathbf{A}$ is κ-**Noetherian** if $\langle A, \langle\ \rangle_{\mathbf{A}} \rangle$ is κ-Noetherian.

Let α be any ordinal, $\alpha > 1$. Let α' denote $\alpha \cup \{\perp\}$, where $\perp$ is a fresh object decreed to be smaller that 0 and different from α.

DEFINITION 2.2: A **gradation of B over A in** α is a pair of mappings (δ, μ) where

$$\delta : B \longrightarrow \alpha' \qquad \mu : B \longrightarrow A$$

and the following hold:

1. $\delta(b) = \perp$ if and only if $b = 0$, $\delta(b) = 0$ if and only if $b \in A - \{0\}$.
2. If $a \in \langle \mu(b) \rangle_{\mathbf{A}}, \delta(a) > 0$, then $a = \mu(c)$ for some $c \in \langle b \rangle_{\mathbf{B}}$ with $\delta(c) \leq \delta(b)$.
3. For all $a \in A$, $b, c \in B$, we have

   $$\delta(u(a, b, c)) \leq \max\{\delta(b), \delta(c)\}$$

   and $u(a, \mu(b), \mu(c))$ is either 0 or $\mu(u(a, b, c))$.
4. If $\delta(b) > 0$, $\mu(b) \in \langle \mu(b_1), \ldots, \mu(b_n) \rangle_{\mathbf{A}}$ and $\delta(b_i) \leq \delta(b)$ for $i = 1, \ldots, n$, then there is a $t \in \langle b_1, \ldots, b_n \rangle_{\mathbf{B}}$ such that $\delta(s(b, t)) < \delta(b)$.

THEOREM 2.3. *Suppose that there is a gradation* (δ, μ) *of* $\mathbf{B}$ *over* $\mathbf{A}$ *in* α. *Then, for any infinite cardinal* κ, *if* $\mathbf{A}$ *is* κ*-Noetherian, then* $\mathbf{B}$ *is* κ*-Noetherian.*

PROOF: Let $I \in \mathrm{I}(\mathbf{B})$ and define, for $\beta < \alpha$

$$I/\beta = \{b \in I : \delta(b) \leq \beta\}$$
$$I_\beta = \{\mu(b) : b \in I/\beta\}.$$

First we claim that any $I_\beta \in \mathrm{I}(\mathbf{A})$. To see that, it suffices to prove that for any natural number n, if $b_i \in I/\beta$ for $i = 1, \ldots, n$, then $\langle \mu(b_1) \rangle_{\mathbf{A}} \vee \cdots \vee \langle \mu(b_n) \rangle_{\mathbf{A}} \subseteq I_\beta$. In the language of proposition 1.2 this is equivalent to saying that, if $x \in \sum_{i=1}^{n} \langle \mu(b_i) \rangle_{\mathbf{A}}$, then there is an $x' \in I/\beta$ with $x = \mu(x')$.

We induct on n: for $n = 1$ we apply property (2) of the definition of gradation. Suppose now that $x \in \langle \mu(b_0) \rangle_{\mathbf{A}} + \sum_{i=1}^{n} \langle \mu(b_i) \rangle_{\mathbf{A}}$ with $b_i \in I/\beta$ for $i = 0, \ldots, n$. Then for some $a \in A$, $v \in \langle \mu(b_0) \rangle_{\mathbf{A}}$ and $w \in \sum_{i=1}^{n} \langle \mu(b_i) \rangle_{\mathbf{A}}$, we have $x = u(a, v, w)$. By induction hypothesis there are $v', w' \in I/\beta$ with $v = \mu(v')$ and $w = \mu(w')$, therefore $x = u(a, \mu(v'), \mu(w'))$. If $x = 0$ there is nothing to prove. If $x \neq 0$ apply property (3) of the definition of gradation to get $x = \mu(u(a, v', w'))$. Moreover

$$\delta(u(a, v', w')) \leq \max\{\delta(v'), \delta(w')\} \leq \beta,$$

since $v', w' \in I/\beta$. Hence $u(a, v', w') \in I/\beta$ as well. So $x \in I_\beta$ as desired.

Therefore $H = \bigcup_{\beta<\alpha} I_\beta$ is the union of a chain of ideals of $\mathbf{A}$, hence it is an ideal of $\mathbf{A}$. So $H = \langle a_i : i < \lambda \rangle_{\mathbf{A}}$ for some $\lambda < \kappa$. Let now $f_i \in I$ be such that $a_i = \mu(f_i)$ and let

$$\gamma = \sup\{\delta(f_i) : i < \lambda\}$$
$$J = \langle f_i : i < \lambda \rangle_{\mathbf{B}}$$
$$I_\perp = I \cap A = \langle s_j : j < \lambda' \rangle, \quad \lambda' < \kappa.$$

Note that, since $I_\perp \in \mathrm{I}(\mathbf{A})$, it is generated by some $\lambda' < \kappa$ elements s_j. Suppose that $\gamma > 0$. Then in the chain $(I_j)_{j<\gamma}$ there are only less than κ distinct ideals: let them be I_{j_ν} for $\nu \leq \pi$ for some $\pi < \kappa$. Hence $I_{j_\pi} = \bigcup_{j<\gamma} I_j$. Moreover, for any $i < \gamma$, there is a $\nu(i) \leq \pi$ such that $j_{\nu(i)} \leq i$ and $I_i = I_{j_{\nu(i)}}$. Again any I_{j_ν} is generated by less than κ elements, so there exists a $\lambda(\nu) < \kappa$ and, for each $\rho < \lambda(\nu)$ there exists a $b^\nu_\rho \in I/j_\nu$ such that

$$I_{j_\nu} = \langle \mu(b^\nu_\rho) : \rho < \lambda(\nu) \rangle_{\mathbf{A}}.$$

Let us now define

$$L = \begin{cases} \{0\} & \text{if } \gamma = 0 \\ \langle b^\nu_\rho : \nu \leq \pi, \rho < \lambda(\nu) \rangle_{\mathbf{B}} & \text{otherwise} \end{cases}$$

and note that L is generated by less that κ elements. Finally define

$$J' = \langle J \cup L \cup \langle I \cap A \rangle_{\mathbf{B}} \rangle_{\mathbf{B}}$$

and note that J' is again generated by less than κ elements. We will show that $J' = I$.

First observe that $J' \subseteq I$ trivially. Let now $b \in I$; we induct on $\delta = \delta(b)$ to show that $b \in J'$. If $\delta \leq 0$, then $b \in I_\perp \subseteq J'$. If $\delta < \gamma$, we let $I_\delta = I_{j_{\nu(\delta)}}$ and $\nu(\delta) \leq \pi$, $j_{\nu(\delta)} \leq \delta$. Therefore $\mu(b) \in \langle \mu(b^{\nu(\delta)}_\rho) \; : \rho < \lambda(\nu(\delta)) \rangle_{\mathbf{A}}$. Therefore (being ideal generation an algebraic operator) there are finitely many $\rho_1, \ldots, \rho_n$ such that

$$\mu(b) \in \langle \mu(b^{\nu(\delta)}_{\rho_1}), \ldots, \mu(b^{\nu(\delta)}_{\rho_n}) \rangle_{\mathbf{A}}$$

and $\delta(b^{\nu(\delta)}_{\rho_\sigma}) \leq j_{\nu(\delta)} \leq \delta$ for $\sigma = 1, \ldots, n$. By property (4) of Definition 2.1 there exists a $t \in \langle b^{\nu(\delta)}_{\rho_1}, \ldots, b^{\nu(\delta)}_{\rho_n} \rangle_{\mathbf{B}}$ such taht $s(b, t) < \delta$. Since $s(b, t) \in I$, then $s(b, t) \in J'$ by induction hypothesis. But $t \in L \subseteq J'$, hence, by Proposition 1.3(1), $b \in J'$.

Finally, if $\delta \geq \gamma$, we have that $\mu(b) \in H$, hence for some finite m we have

$$\mu(b) \in \langle \mu(f_{i_1}), \ldots, \mu(f_{i_m}) \rangle_{\mathbf{A}}$$

with $i_\sigma < \lambda$ and $\delta(f_{i_\sigma}) \leq \gamma \leq \delta$ for $\sigma = 1, \ldots, m$. Apply again (4) of Definition 2.1 to get a $t \in \langle f_{i_1}, \ldots, f_{i_m} \rangle_{\mathbf{B}}$, such that $\delta(s(b, t)) < \delta$. But $t \in J \subseteq J' \subseteq I$, so $s(b, t) \in I$ and, by induction hypothesis, $s(b, t) \in J'$. But $t \in J'$, hence $b \in J'$, again by Proposition 1.3(1). Therefore $I \subseteq J'$ and so $I = J'$.

REMARKS:

(1) If $\mathbf{R}$ is a commutative ring with unity, our theorem implies the Hilbert Basis Theorem for $\mathbf{R}[\vec{x}]$, the ring of polynomials in any number of variables with coefficients in $\mathbf{R}$. In this case, for $p(\vec{x}) \in \mathbf{R}[\vec{x}]$, $\mu(p) =$ leading coefficient and $\delta(p) =$ degree of p. The same is true for $\mathbf{R}[[\vec{x}]]$, the ring of formal power series with coefficients in $\mathbf{R}$, if we take, for $\sigma(\vec{x}) \in \mathbf{R}[[\vec{x}]]$, $\mu(\sigma) =$ nonzero coefficient of lowest degree and $\delta(\sigma) =$ lowest power of the variables. In these cases, of course, $s(x, y) = x - y$ and $u(x, y, z) = y + z$.

(2) If $\kappa = |A|$, then $\mathbf{A}$ is κ^+-Noetherian. Thus the existence of a gradation of $\mathbf{B}$ over $\mathbf{A}$ yields that $\mathbf{B}$ is κ^+-Noetherian as well. It might be interesting to investigate whether the converse holds.

3. The commutator and the A.C.C. for radical ideals

Drawing inspiration from the commutator theory for modular varieties, the commutator was defined in [11] for the so-called ideal determined varieties (those in in which, for every algebra $\mathbf{A}$ in the variety, $\mathrm{Con}(\mathbf{A})$ and $\mathrm{I}(\mathbf{A})$ are isomorphic under the mapping $\theta \longmapsto 0/\theta$, see [4]). In [13] it is shown that a good version of the commutator is available also in the case of subtractive varieties (that are not necessarily congruence modular). For details and the proof of Proposition 3.1 below, we refer to that paper.

Let $\mathcal{V}$ be a subtractive variety. A term $t(\vec{x}, \vec{y}, \vec{z})$ of $\mathcal{V}$ is a **commutator term in** $\vec{y}, \vec{z}$ **for** $\mathcal{V}$ if it is an ideal term in $\vec{y}$ and an ideal term in $\vec{z}$. Let $\mathbf{A} \in \mathcal{V}$ and let H, K be nonempty subsets of $\mathbf{A}$. The commutator of H and K in $\mathbf{A}$ is

$$[H, K]_{\mathbf{A}} = \{t(\vec{a}, \vec{h}, \vec{k}) : t(\vec{x}.\vec{y}, \vec{z}) \text{ a commutator term in } \vec{y}, \vec{z},\ \vec{a} \in A,\ \vec{h} \in H,\ \vec{k} \in K\}.$$

If no confusion can arise we drop the subscript $\mathbf{A}$ and, if $H = \{a\}$, $K = \{b\}$ we will write $[a, b]$ instead of $[\{a\}, \{b\}]$.

Proposition 3.1. [13] *For any $\mathbf{A} \in \mathcal{V}$, any nonempty $H, K \subseteq A$ we have:*

(1) *$[H, K]$ is always an ideal and $[H, K] = [\langle H\rangle, \langle K\rangle]$.*
(2) *$[H, K] = [K, H] \subseteq \langle H\rangle \cap \langle K\rangle$.*
(3) *$[\bigcup_{i\in I} H_i, \bigcup_{j\in J} K_j] = \bigvee_{i,j}[H_i, K_j]$.*
(4) *If φ is an epimorphism from $\mathbf{A}$ to $\mathbf{B}$, then we have $\varphi([H, K]_{\mathbf{A}}) = [\varphi(H), \varphi(K)]_{\mathbf{B}}$. If $x \in \varphi^{-1}([M, N]_{\mathbf{B}})$ for $M, N \subseteq B$, then there is a $z \in [\varphi^{-1}(M), \varphi^{-1}(N)]_{\mathbf{A}}$ with $\varphi(x) = \varphi(z)$.*
(5) *$[H, K]$ is the smallest ideal I of $\mathbf{A}$ such that the following statement holds: for any term $f(\vec{x}, \vec{y}, \vec{z})$, for any $\vec{a} \in A$, $\vec{h} \in H$, $\vec{k} \in K$, if $f(\vec{a}, \vec{h}, 0, \ldots, 0) = f(\vec{a}, 0, \ldots, 0, \vec{k}) = 0$, then $f(\vec{a}, \vec{h}, \vec{k}) \in I$ if and only if $f(\vec{a}, 0, \ldots, 0) = 0$.*

Again in [13] it is shown that $[H, K]_{\mathbf{A}}$ really does not depend on which variety $\mathbf{A}$ belongs to. As in [12], one obtains a notion of primeness as well as of radicality. An ideal I of $\mathbf{A}$ is **prime** if, whenever $[H, K] \subseteq I$, then either $H \subseteq I$ or $K \subseteq I$. Equivalently if, whenever $[a, b] \subseteq I$, then $a \in I$ or $b \in I$. An ideal is **radical** if, whenever $[H, H] \subseteq I$, then $H \subseteq I$ (equivalently $[a, a] \subseteq I$ implies $a \in I$). The radical of I, written $\mathrm{Rad}(I)$, is the intersection of all the prime ideals containing I. One easily sees that the following are equivalent:

(1) I is a radical ideal.
(2) $I = \mathrm{Rad}(I)$.
(3) I is an intersection of prime ideals.

We have a better characterization of $\mathrm{Rad}(I)$ if the commutator is **finitary** on $\mathbf{A}$, which means that the commutator of two finitely generated ideals is again finitely generated (equivalently, $[a, b]$ is finitely generated for any $a, b \in A$). Let us define inductively $I^{(n)}$ and I_n as follows

$$I^{(1)} = I^1 = I$$
$$I^{(n+1)} = [I^{(n)}, I^{(n)}] \qquad I^{n+1} = [I^n, I].$$

If the commutator is finitary on $\mathbf{A}$, then

$$\mathrm{Rad}(I) = \{a \in A : \langle a\rangle^{(n)} \in I \text{ for some positive integer } n\}.$$

For a proof, see again [12] Proposition 3.7. For any subset X of A, the **radical ideal generated by** X, denoted by $\{X\}_{\mathbf{A}}$ is the intersection of all the radical ideals containing X. Of course, if I is an ideal, $\{I\}_{\mathbf{A}} = \mathrm{Rad}_{\mathbf{A}}(I)$.

We say that $\mathbf{A}$ is a **mild subalgebra** of $\mathbf{B}$ if $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and the contraction $I \cap A$ of any radical ideal of $\mathbf{B}$ is a radical ideal of $\mathbf{A}$.

Proposition 3.2. *Let $\mathbf{A}$ be a subalgebra of $\mathbf{B}$. Then the following are equivalent:*

(1) *$\mathbf{A}$ is a mild subalgebra of $\mathbf{B}$.*
(2) *For all $a \in A$, $\langle [a, a]_{\mathbf{A}}\rangle_{\mathbf{B}} = [a, a]_{\mathbf{B}}$.*
(3) *For all $J \in \mathrm{I}(\mathbf{B})$, $\mathrm{Rad}_{\mathbf{A}}(J \cap A) \subseteq \mathrm{Rad}_{\mathbf{B}}(J) \cap A$.*
(4) *For all $I \in \mathrm{I}(\mathbf{A})$, $\mathrm{Rad}_{\mathbf{A}}(I) \subseteq \mathrm{Rad}_{\mathbf{B}}(\langle I\rangle_{\mathbf{B}})$.*

The proof is quite easy and we leave it to the reader. Notice that if $[I^m, J] \subseteq H$ for some $m \geq 1$, then $I \cap J \subseteq \mathrm{Rad}(H)$.

Proposition 3.3. *For any $H \subseteq A$, $I, J \in \mathrm{I}(\mathbf{A})$ we have:*

(1) *$\mathrm{Rad}(\langle H\rangle) = \mathrm{Rad}(\{H\})$.*
(2) *$[\mathrm{Rad}(I), \mathrm{Rad}(J)] \subseteq \mathrm{Rad}([I, J]) = \mathrm{Rd}([\mathrm{Rad}(I), \mathrm{Rad}(J)])$.*
(3) *For all m, $\mathrm{Rad}(I^m) = I$ and $I \cap J \subseteq \mathrm{Rad}([I^m, J])$.*
(4) *If J is a radical ideal and $[I, I] \subseteq J \subseteq I$, then $J = I$.*

Proof: (1) is obvious. The first inclusion and the inclusion left-to-right in the equality of (2) are obvious as well. For the other inclusion in (2) observe that, if P is a prime ideal and $[I, J] \subseteq P$, then

$$[\{I\}, \{J\}] \subseteq \{I\} \cap \{J\} \subseteq P.$$

For (3), if P is a prime ideal and $[I^m, J] \subseteq P$, then $I^m \subseteq P$ or $J \subseteq P$. In the second case there is nothing to prove; otherwise (for $m > 1$), $[I^{m-1}, J] \subseteq P$, so we get $I \subseteq P$ sooner or later. Finally (4) follows from $\mathrm{Rad}(I^2) = I$.

We say that a radical ideal I of $\mathbf{A}$ is of **finite type** [5] if there is a finite set $F \subseteq A$ with $I = \{F\}$. As usual we can see that the following are equivalent for any algebra $\mathbf{A}$:

(1) Every radical ideal is of finite type.
(2) Every properly ascending chain of radical ideals is finite.
(3) Every nonempty set of radical ideals has a maximal element.

Moreover we have:

Proposition 3.4. *Suppose that the commutator is finitary on $\mathbf{A}$, $a \in A$ and $I \in \mathrm{I}(\mathbf{A})$. If $\{I + \langle a\rangle\}$ is of finite type, then for some $b_1, \ldots, b_m \in I$ we have*

$$\{I + \langle a\rangle\} = \{a, b_1, \ldots, b_m\}.$$

Proof: Let $\{I + \langle a\rangle\} = \{c_1, \ldots, c_k\}$. For any c_i $i = 1, \ldots, k$, either $c_i \in I$ or there exists an $m_i > 1$ with $J_i = [\langle c_i\rangle^{(m_i)}, \langle c_i\rangle^{(m_i)}] \subseteq I + \langle a\rangle$. Since the commutator is finitary any J_i is finitely generated and any generator is of the form $u(b,c,a)$ for some $c \in I$, or else it is in I, or it is equal to a. Take as b_j's all the elements of I that show up this way.

We now look for a notion of gradation suitable for getting a finite basis theorem for radical ideals. First if α is an ordinal greater than 0, $\mathcal{V}$ a subtractive variety, $\mathbf{B} \in \mathcal{V}$, δ is a mapping of B into α, $H \subseteq B$, $b \in B$ and $\delta(b) > 0$, then we say that b **dominates** H if for any $h \in H$, there is a $b' \in B$ such that $\delta(b') < \delta(b)$ and $s(b', h) \in \langle b \rangle$.

DEFINITION 3.5: Let $\mathcal{V}$ be a variety, $\mathbf{A}$ a subalgebra of $\mathbf{B} \in \mathcal{V}$ and α an ordinal other than 0. A pair of mappings

$$\delta : B \longrightarrow \alpha \qquad \mu : B \longrightarrow A$$

is an **r-gradation of B over A in** α if:

(1) $\delta(b) = 0$ if and only if $b \in A$.
(2) For every $b, b' \in B$, if $\delta(b) > 0$, then there exists a positive integer m such that b dominates the set $[\langle \mu(b) \rangle_{\mathbf{B}}^{(m)}, \langle b' \rangle_{\mathbf{B}}]_{\mathbf{B}}$.
(3) If $b \in B - A$, then there exists a $b' \in \langle \mu(b) \rangle_{\mathbf{B}}$ with $\delta(s(b,b')) < \delta(b)$.

Before getting to the main theorem of this section, further observations are necessary. In a generic subtractive algebra $\mathbf{A}$ we do not necessarily have that, from $s(0,x) = 0$ it follows $x = 0$. (Consider the natural numbers with the difference truncated at zero.) This suggests us the following definition.

DEFINITION 3.6: An algebra $\mathbf{A}$ is **strongly subtractive** if it is subtractive and moreover, if $s(a,b) \in \langle c \rangle$, then $b \in \langle a, c \rangle$ for all $a, b, c \in A$.

It is clear that the above property is equivalent to any of the following

(1) For every $a, b \in A$, $b \in \langle a, s(a,b) \rangle$.
(2) For every $a, b \in A$, $\langle a, s(b,a) \rangle \subseteq \langle a, s(a,b) \rangle$.

As standard examples of strongly subtractive algebras one may take: rings, goups and Boolean Algebras.

The last preliminary result we need is:

PROPOSITION 3.7. *If the commutator is finitary on an algebra* $\mathbf{A}$, *then the operator sending any* $H \subseteq A$ *into* $\{H\}$ *is algebraic.*

PROOF: We have that $x \in \{H\}$ if and only if $x \in \{\langle H \rangle\}$ if an only if $\langle x \rangle^{(m)} \subseteq H$ for some positive integer m. But, being the commutator finitary, $\langle x \rangle^{(m)}$ is finitely generated and ideal generation is an algebraic operator.

THEOREM 3.8. *Let* $\mathbf{B} \in \mathcal{V}$ *be a strongly subtractive algebra, let the commutator be finitary on* $\mathbf{B}$ *and let* $\mathbf{A}$ *be a mild subalgebra of* $\mathbf{B}$. *Assume that a r-gradation of* $\mathbf{B}$ *over* $\mathbf{A}$ *exists for some ordinal* α. *Then, if every radical ideal of* $\mathbf{A}$ *is of finite type, the same holds for* $\mathbf{B}$.

PROOF: Suppose, by way of contradiction, that $\mathbf{B}$ have radical ideals not of finite type. Since radical ideal generation is algebraic we can apply Zorn's Lemma to get a radical ideal I of $\mathbf{B}$ maximal among those not of finite type. Since $\mathbf{A}$ is a mild subalgebra of $\mathbf{B}$, $I \cap A$ is a radical ideal of $\mathbf{A}$, hence it is of finite type, say $I \cap A = \{a_1, \ldots, a_n\}_{\mathbf{A}}$. Let $J = \{a_1, \ldots, a_n\}_{\mathbf{B}}$; observe that $J \subseteq I$ but they can not be equal. So pick a $b \in I - J$ such that $\delta(b)$ is minimal and let $a = \mu(b)$. Then $a \notin I$, otherwise $a \in J$ and, by property (3) of the definition of r-gradation, for some $b' \in \langle a \rangle_{\mathbf{B}} \subseteq J$, we would have $\delta(s(b,b')) < \delta(b)$. Since $b, b' \in I$, we would get $s(b,b') \in I$ and, being $\delta(b)$ minimal, $s(b,b') \in J$. But $b' \in J$, so $b \in J$ as well, that is a contradiction. Thus $a \notin I$.

Next we claim that

$$[\langle a \rangle_{\mathbf{A}}, I]_{\mathbf{B}} \subseteq \{J + \langle b \rangle_{\mathbf{B}}\}_{\mathbf{B}}.$$

In fact, let $i \in I$. By (2) of the definition of r-gradation there is a natural m such that b dominates the set $[\langle a \rangle_{\mathbf{A}}^{(m)}, \langle i \rangle_{\mathbf{B}}]_{\mathbf{B}}$. Hence, if $c \in [\langle a \rangle_{\mathbf{A}}^{(m)}, \langle i \rangle_{\mathbf{B}}]_{\mathbf{B}}$, then there exists a d with $\delta(d) < \delta(b)$ and $s(c,d) \in \langle b \rangle_{\mathbf{B}} \subseteq I$. Since $c \in \langle i \rangle_{\mathbf{B}} \subseteq I$ we may conclude that $d \in I$ and again that $d \in J$, by minimality of $\delta(b)$. Since $\mathbf{B}$ is strongly subtractive, we get that $c \in \langle d, b \rangle_{\mathbf{B}}$. Therefore

$$[\langle a \rangle_{\mathbf{A}}^{(m)}, \langle i \rangle_{\mathbf{B}}]_{\mathbf{B}} \subseteq \{J + \langle b \rangle_{\mathbf{B}}\}_{\mathbf{B}}$$

and, since $i \in I$ was generic we get $[\langle a \rangle_{\mathbf{A}}^{(m)}, I]_{\mathbf{B}} \subseteq \{J + \langle b \rangle_{\mathbf{B}}\}_{\mathbf{B}}$, whence the claim follows.

Now, since $\{I + \langle a \rangle_{\mathbf{B}}\}_{\mathbf{B}}$ is of finite type, by Proposition 3.4, there are $r_1, \ldots, r_m \in I$ such that

$$\{I + \langle a \rangle_{\mathbf{B}}\}_{\mathbf{B}} = \{a, r_1, \ldots, r_m\}_{\mathbf{B}}$$

and moreover the ideal

$$K = \{\{J + \langle b \rangle_{\mathbf{B}}\}_{\mathbf{B}} + \langle r_1, \ldots, r_m \rangle_{\mathbf{B}}\}_{\mathbf{B}}$$

is of finite type essentially because J is. So we compute

$$\begin{aligned}
[I, I] &\subseteq \{[I, I + \langle a \rangle_{\mathbf{B}}]\}_{\mathbf{B}} \\
&\subseteq \mathrm{Rad}([I, \{I + \langle a \rangle_{\mathbf{B}}\}_{\mathbf{B}}]) \\
&= \mathrm{Rad}([I, \{a, r_1, \ldots, r_m\}_{\mathbf{B}}]) \\
&= \mathrm{Rad}([\mathrm{Rad}(I), \mathrm{Rad}(\{a, r_1, \ldots, r_m\}_{\mathbf{B}}]) \\
&= \mathrm{Rad}([I, \langle a, r_1, \ldots, r_m \rangle_{\mathbf{B}}]) \\
&= \mathrm{Rad}([I, a] + \langle r_1, \ldots, r_m \rangle_{\mathbf{B}}) \\
&\subseteq K \subseteq \mathrm{Rad}(\mathrm{Rad}(I)) = I.
\end{aligned}$$

Therefore $K = I$, via Proposition 3.3(4). But this is a contradiction, since I was not of finite type. Hence the theorem is proved.

REMARKS:

(1) It would have been possible to present the above result at the same level of generality as the previous one on ideals, i.e. with any infinite cardinal κ. But many notions would have been too cumbersome to formulate, so we leave this further task to the reader.

(2) Let us now explain how our results apply to the classical case of commutative rings. Showing that the commutator (i.e. the product) of ideals of rings is finitary is a standard exercise in ring theory. Moreover it is obvious that any variety of commutative rings is a strongly subtractive variety (interpreting of course $s(x, y)$ as $x - y$). If $\mathbf{R}$ is any commutative ring then it is again easy to see that $\mathbf{R}$ is a mild subalgebra of $\mathbf{R}[x]$. Finally let us interpret δ and μ of the definition of r-gradation as the degree and the leading coefficient of a polynomial respectively. The condition (1) of Definition 3.5 is satisfied. Let now $p(x), q(x) \in R[x]$ with $\delta(p(x)) > 0$. The reader can easily check that the algorithm of division yields the desired domination, so that condition (2) is fulfilled. Finally let $p(x) \in R[x]$ and let $a = \mu(p(x))$, $n = \delta(p(x))$. If we let $p'(x) = ax^n$, then $p'(x) \in \langle a \rangle_{\mathbf{R}[x]}$ and $\delta(p(x) - p'(x)) < \delta(p(x))$. Hence condition (3) of Definition 3.5 is fulfilled and we can apply Theorem 3.8.

Here is a partial list of classical algebras to which our result may apply: loops, groups, (multi)operator groups, rings (associative or nonassociative, commutative or non commutative), group rings, Heyting algebras, Boolean algebras (possibly with operators), algebras, Lie algebras. It is only fair to say that some specifications are necessary in some cases in order to make the assumptions valid. Moreover it is not true that we know the meaning of our result in each of those cases; this, however, is due to our ignorance and, hopefully, not to the intrinsic weakness of the results. Presently we aimed mainly to establish the "true" reason why the Hilbert basis theorem works.

The first trigger to this work was an old manuscript of the second author [10], which essentially contained THeorem 2.3. Theorem 3.8 was essentially in charge of the first author.

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Dipartimento di Matematica, Università di Siena, 53100 Siena, Italy.