On subtractive varieties, I

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Abstract. A variety $\mathcal{V}$ is subtractive if it obeys the laws $s(x,x) = 0$, $s(x,0) = x$ for some binary term s and costant 0. This means that $\mathcal{V}$ has 0-permutable congruences (namely $[0]R \circ S = [0]S \circ R$ for any congruences R, S of any algebra in $\mathcal{V}$). We present the basic features of such varieties, mainly from the viewpoint of ideal theory. Subtractivity does not imply congruence modularity, yet the commutator theory for ideals works fine. We characterize i-Abelian algebras, (i.e. those in which the commutator is identically 0). In the appendix we consider the case of a "classical" ideal theory (comprising: groups, loops, rings, Heyting and Boolean algebras, even with multioperators and virtually all algebras coming from logic) and we characterize the corresponding class of subtractive varieties.

The plan of this (short) series of papers is the following: in this one we present the basic features of subtractive varieties of algebras. In the second one we will tackle some side notions and some useful generalizations. In the third one we will present some applications, mainly to the case of definability of (principal) ideals and to Universal Algebraic logic in the Block-Pigozzi style. Other applications already appear in the joint papers [1] and [2]. Our notations and basic notions are meant to agree with [8]; specifically, if R is any binary relation on a set A, $a \in A$ and $X \subseteq A$, we put $a/R = \{x \in A : a\,R\,x\}$, $X/R = \bigcup_{a \in X} a/R$ and we will say that a subset Y of A is an R-block if $Y = a/R$ for some $a \in A$.

1. Subtractive algebras and their ideals

Here we study some features of algebras and varieties which satisfy the identities

$$s(x,x) \approx 0 \qquad\qquad s(x,0) \approx x$$

for some binary term s and zeroary term (or equationally definable constant) 0 and which will be called **subtractive**(*). Such an s will be called a **subtraction term.** Of course subtractive algebras abound in classical algebras, in algebraic logic etc.. We hope that giving them a name may be justified by the fact that those very simple identities imply a number of noteworthy

(*)If one doesn't like constants, one may alternatively assume the axioms

$$s(x,x) \approx s(y,y) \qquad\qquad s(y, s(y,y)) \approx y$$

and then define $s(x,x) = 0$.

consequences. Sometimes we will denote by $\langle \mathbf{A}, s\rangle$ a subtractive algebra, to single out the term s satisfying the basic rules above. In such an algebra, let us define the term $u(x,y,z)$ to be $s(x, s(s(x,y),z))$ and note that the following identities hold:

(1) $u(x,y,s(x,y)) \approx x$
(2) $u(x,0,0) \approx 0$
(3) $u(x,x,0) \approx x$
(4) $u(x,0,y) \approx u(x,y,0)$.

In the first of these, explicit dependence of u on x cannot be avoided, for if we had a binary term $d(x,y)$ such that $d(y,0) \approx y$ and $d(y, s(x,y)) \approx x$ then the term $t(x,y,z) = d(z, s(x,y))$ would witness permutability of congruences. Yet implication algebras and BCK algebras are subtractive but do not necessarily have permutable congruences. Truly, congruences in subtractive algebras are 0-permutable. This means that, if $\langle \mathbf{A}, s\rangle$ is subtractive and $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ and $0\,\alpha\, u\, \beta\, v$, then there is a $w \in A$ (namely $s(v,u)$) with $0\,\beta\, w\,\alpha\, v$. From now on we will always consider algebras and varieties having a constant 0 in the signature.

Let us now recall the main notions from the theory of ideals [10], [12]. If $\mathcal{K}$ is a class of similar algebras, a term $p(x_1,\ldots,x_m,y_1,\ldots,y_n)$ is a **$\mathcal{K}$-ideal term in** $\vec{y}$ (and we write $p(\vec{x},\vec{y}) \in \mathrm{IT}_{\mathcal{K}}(\vec{y})$) if the identity $p(\vec{x},0,\ldots,0) \approx 0$ holds in $\mathcal{K}$. A nonempty subset I of $\mathbf{A} \in \mathcal{K}$ is a **$\mathcal{K}$-ideal** of $\mathbf{A}$ if for any $p(\vec{x},\vec{y}) \in \mathrm{IT}_{\mathcal{K}}(\vec{y})$, for $\vec{a} \in A$ and $\vec{b} \in I$, $p(\vec{a},\vec{b}) \in I$. Under inclusion, the set $\mathrm{I}_{\mathcal{K}}(\mathbf{A})$ of all $\mathcal{K}$-ideals of $\mathbf{A}$ is an algebraic lattice; if $H \subseteq A$, the ideal $\langle H\rangle_{\mathcal{K}}$ generated by H is easily seen to be the set $\{p(\vec{a},\vec{b}) : p(\vec{x},\vec{y}) \in \mathrm{IT}_{\mathcal{K}}(\vec{y}), \vec{a} \in A, \vec{b} \in H\}$. When $\mathcal{K}$ is $\{\mathbf{A}\}$ (or, equivalently , the variety $\boldsymbol{V}(A)$ generated by $\mathbf{A}$), then a $\mathcal{K}$-ideal will be called an **ideal** and we drop all the affixes and suffixes in sight. By $\mathrm{N}(\mathbf{A})$ we denote the set $\{0/\theta : \theta \in \mathrm{Con}(\mathbf{A})\}$ and trivially $\mathrm{N}(\mathbf{A}) \subseteq \mathrm{I}(\mathbf{A}) \subseteq \mathrm{I}_{\mathcal{K}}(\mathbf{A})$ whenever $\mathbf{A} \in \mathcal{K}$. $\mathrm{N}(\mathbf{A})$ inherits in a natural way the lattice structure of $\mathrm{Con}(\mathbf{A})$. One can easily check that for any $\mathbf{A} \in \mathcal{K}$ the following are equivalent:

(1) The mapping from $\mathrm{Con}(\mathbf{A})$ into $\mathrm{I}_{\mathcal{K}}(\mathbf{A})$ defined by $\theta \longmapsto 0/\theta$ is a lattice homomorphism.
(2) $\mathrm{N}(\mathbf{A})$ is a sublattice of $\mathrm{I}_{\mathcal{K}}(\mathbf{A})$.

An algebra (resp. a class $\mathcal{K}$ of algebras) is said to be **0-permutable**, or to have **0-permutable congruences** if for $R, S \in \mathrm{Con}(\mathbf{A})$, $0/R \circ S = 0/S \circ R$ (resp. every $\mathbf{A} \in \mathcal{K}$) is 0-permutable). An algebra $\mathbf{A}$ (resp. a class $\mathcal{K}$) is said to be **0-regular**, or to have **0-regular congruences**, if for $R, S \in \mathrm{Con}(\mathbf{A})$, $0/R = 0/S$ implies $R = S$ (resp. any algebra in $\mathcal{K}$ is 0-regular). Let us recall [7] that a variety $\mathcal{V}$ is **ideal-determined** if for any $\mathbf{A} \in \mathcal{V}$, any ideal is the congruence class for exactly one congruence, namely if $\mathcal{V}$ is 0-regular and 0-permutable. In [7] it is shown that $\mathcal{V}$ is ideal determined iff for some

$n \geq 1$ there are binary terms $d_0, \ldots, d_n$ such that

$$\mathcal{V} \models d_0(x,x) \approx 0 \wedge \cdots \wedge d_n(x,x) \approx 0;$$
$$\mathcal{V} \models d_0(x,0) \approx x;$$
$$\mathcal{V} \models d_1(x,y) \approx 0 \wedge \cdots \wedge d_n(x,y) \approx 0 \quad \Longrightarrow \quad x \approx y.$$

Hence ideal determined varieties are subtractive and $\mathrm{I}_{\mathcal{V}}(A)$ and $\mathrm{Con}(\mathbf{A})$ are isomorphic; moreover it is easily seen that such varieties are congruence modular. In considering subtractive varieties and algebras we are no more requiring 0-regularity, but onle that there is a subtraction term. In case we assume the existence of an "addition" too, which of course is the case of virtually all ideal-determined varieties coming from abstract algebra and algebraic logic, then we have the situation presented in the appendix.

Finally an algebra $\mathbf{A}$ is called **ideal-coherent** if, for any $I \in \mathrm{I}(\mathbf{A}), \theta \in \mathrm{Con}(\mathbf{A})$, $0/\theta \subseteq I$ yields that I is a union of θ-blocks.

Proposition 1.1. *Any subtractive algebra* $\mathbf{A}$ *is ideal-coeherent, has 0-permutable congruences and* $\mathrm{N}(\mathbf{A})$ *is a sublattice of* $\mathrm{I}(\mathbf{A})$.

Proof: We have already seen that $\mathbf{A}$ has 0-permutable congruences. Assume $0/\theta \subseteq I$, $I \in \mathrm{I}(\mathbf{A})$ and $\theta \in \mathrm{Con}(\mathbf{A})$. If $b \in I$ and $(a,b) \in \theta$, then $s(a,b)\,\theta\, s(a,a) = 0$, hence $s(a,b) \in I$. Therefore $a = u(a,b,s(a,b)) \in I$, which proves ideal-coherency.

Next we show that the mapping $\theta \longmapsto 0/\theta$ is a lattice homomorphism. Assume that $a \in 0/(\theta \vee \varphi)$ for $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$; then for some $a_1, \ldots, a_n \in A$ we have

$$a\,\theta\, a_1\,\varphi\, a_2\,\ldots\, a_n\,\varphi\, 0.$$

If $n = 0$ trivially $a \in 0/\theta \vee 0/\varphi$.

In the inductive step assume that $a\,\theta\, a_1\,\varphi\, a_2\,\ldots\, a_n\,\varphi\, 0$; then

$$s(a,a_n)\,\varphi\, s(a,a_{n-1})\,\theta\,\ldots\,\theta\, s(a,a) = 0,$$

therefore, by induction hypothesis, $s(a,a_n) \in 0/\theta \vee 0/\varphi$. Since $a_n \in 0/\theta$, we get $a = u(a,a_n,s(a,a_n)) \in 0/\theta \vee 0/\varphi$. The argument is similar if $a_n\,\varphi\, 0$.

If we consider a variety of algebras, we get a number of equivalent properties collected in the proposition below. For the proof and for even more equivalent conditions see [2].

Proposition 1.2. *For a variety* $\mathcal{V}$ *the following are equivalent.*

(1) $\mathcal{V}$ *is subtractive.*
(2) *Every algebra in* $\mathcal{V}$ *is ideal-coherent.*
(3) *For any* $A \in \mathcal{V}$, *the mapping* $\theta \longmapsto 0/\theta$ *is a lattice homomorphism from* $\mathrm{Con}(\mathbf{A})$ *into* $\mathrm{I}_{\mathcal{V}}(\mathbf{A})$.
(4) $\mathcal{V}$ *is 0-permutable.*

If $\langle \mathbf{A}, s\rangle$ is a subtractive algebra and $X, Y \subseteq A$ we define

$$X + Y = \{u(a, x, y) : a \in A, x \in X, y \in Y\}.$$

PROPOSITION 1.3. *Let* $\mathbf{A}$ *belong to a subtractive variety* $\mathcal{V}$ *and let* $I, J \in \mathrm{I}_{\mathcal{V}}(\mathbf{A})$.

(1) *If* $y \in I$ *and* $s(x, y) \in I$ *then* $x \in I$.
(2) *If* $y \in \langle I + J\rangle_{\mathcal{V}}$ *then for some* $x \in I$ *we have* $s(y, x) \in J$.
(3) $I + J \in \mathrm{I}_{\mathcal{V}}(\mathbf{A})$ *and* $I + J = I \vee J$.
(4) $I \vee J = \{y :$ *for some* $x \in I, s(y, x) \in I\}$.

PROOF: (1) follows from the identities (1) and (2) regarding u. For (2) let $p(\vec{x}, \vec{y}) \in \mathrm{IT}_{\mathcal{V}}(\vec{y})$, $\vec{a} \in A, \vec{m} \in I + J, b = p(\vec{a}, \vec{m})$ and let $m_i = u(d_i, u_i, v_i)$ for some $d_i \in A, u_i \in I, v_i \in J$. Let $c = p(\vec{a}, u(d_1, u_1, 0), \ldots, u(d_k, u_k, 0))$. Since

$$p(\vec{x}, u(z_1, y_1, 0), \ldots, u(z_k, y_k, 0)) \in \mathrm{IT}_{\mathcal{V}}(\vec{y})$$

we have that $c \in I$. Moreover the term

$$s(p(\vec{x}, u(z_1, y_1, w_1), \ldots, u(z_k, y_k, w_k)), p(\vec{x}, u(z_1, y_1, 0), \ldots, u(z_k, y_k, 0)))$$

belongs to $\mathrm{IT}_{\mathcal{V}}(\vec{w})$, hence $s(b, c) \in J$.

We now prove (3). From (2), if $y \in \langle I + J\rangle_{\mathcal{V}}$, then $s(y, x) \in J$ for some $x \in I$. Then $y = u(y, x, s(y, x)) \in I + J$ and thus $I + J = \langle I + J\rangle_{\mathcal{V}}$. Moreover $I \cup J \subseteq I + J \subseteq I \vee J$, which yields $I + J = I \vee J$. Finally, (4) follows immediately.

By the well-known Mal'cev criterion for being a congruence class we get:

PROPOSITION 1.4. *If* $\mathbf{A}$ *belongs to a subtractive variety* $\mathcal{V}$ *then* $\mathrm{I}_{\mathcal{V}}(\mathbf{A}) = \mathrm{N}(\mathbf{A})$.

PROOF: Let $I \in \mathrm{I}_{\mathcal{V}}(\mathbf{A})$ and let $g(x)$ be a unary polynomial of $\mathbf{A}$. Then $g(x) = t(a_1, \ldots, a_n, x)$ for some term t and $\vec{a} \in A$. If $i, j \in I$, since $s(t(\vec{x}, y_1), t(\vec{x}, y_2)) \in \mathrm{IT}_{\mathcal{V}}(y_1, y_2)$, then $s(g(i), g(j)) \in I$. By (1) of Proposition 1.3 we conclude that if $g(i) \in I$ then $g(j) \in I$. Therefore I is a congruence class, i.e. $I \in \mathrm{N}(\mathbf{A})$.

Therefore, if $\mathbf{A}$ belongs to a subtractive variety $\mathcal{V}$, $\mathrm{I}_{\mathcal{V}}(A) = \mathrm{N}(\mathbf{A}) = \mathrm{I}(\mathbf{A})$. Thus we are allowed to denote simply by $\mathrm{I}(\mathbf{A})$ the lattice of ideals of $\mathbf{A}$, without any reference to the variety $\mathcal{V}$, and $I + J$ is the join of any two ideals I, J of $\mathbf{A}$. As observed in [2] the condition $\mathrm{I}(\mathbf{A}) = \mathrm{N}(\mathbf{A})$ for a whole variety

is not a Mal'cev condition, since it holds in the variety of pointed sets. Here is a more interesting fact:

PROPOSITION 1.5. *If* $\mathbf{A}$ *is a subtractive algebra, then* $\mathrm{I}(\mathbf{A})$ *is a modular lattice.*

PROOF: Let $I, J, H \in \mathrm{I}(\mathbf{A})$ and assume that $I \subseteq J$, $I + H = J + H$ and $I \cap H = J \cap H$. If $a \in J$, then $a \in I + H$, hence $s(a, c) \in H$ for some $c \in I$. Then $c \in J$, hence $s(a, c) \in J \cap H$ which yields $s(a, c) \in I$. Therefore $a \in I$ and $I = J$.

REMARKS: To show that the lattice of,e.g., normal subgroups of a group is modular, a short way is to prove that the congruences of a group permute. Proposition 1.5 seems to shorten further the way: having a subtraction term is enough.

It is easy to find examples which show that subtractivity does not imply congruence modularity (for such an example see [2]). Modularity of the lattice $\mathrm{I}(\mathbf{A})$ is not even a Mal'cev condition, as the variety of pointed sets shows (the ideals of a pointed set $\langle A, o\rangle$ are simply the subsets of A to which o belongs).

For any subtractive algebra $\mathbf{A}$ and $I \in \mathrm{I}(\mathbf{A})$, let I' denote the binary relation on $\mathbf{A}$ defined by

$$a\, I'\, b \qquad \text{iff} \qquad s(b, a) \in I$$

and let I^s denote the subalgebra of $\mathbf{A} \times \mathbf{A}$ generated by I'. Let us say that a reflexive subalgebra of $\mathbf{A} \times \mathbf{A}$ is a **semicongruence** of $\mathbf{A}$. The I^s is a semicongruence of $\mathbf{A}$ for any $I \in \mathrm{I}(\mathbf{A})$.

PROPOSITION 1.6. *For any subtractive algebra* $\mathbf{A}$ *and for any* $I \in \mathrm{I}(\mathbf{A})$ *the following are equivalent:*

(1) $0/I^s = I$.
(2) $I^s = I'$.
(3) I' *is a subalgebra of* $\mathbf{A} \times \mathbf{A}$.
(4) I' *is a congruence of* $\mathbf{A}$.

PROOF: First we show that (1) and (2) are equivalent. In fact from $(a, b) \in I^s$, since $(a, a) \in I^s$, we get $(0, s(b, a)) \in I^s$ and from (1), $s(b, a) \in I$ and so $(a, b) \in I'$. The converse follows from the fact that $(0, a) \in I'$ iff $a \in I$.

Assume now (1) and observe that $I' = I^s$ is a congruence. In fact, if $(a, b) \in I'$, since $(a, a) \in I'$, we get $(0, s(a, b)) \in I$ and from (1), $s(a, b) \in I$. Hence $(b, a) \in I'$. For transitivity, from $(a, b), (b, c) \in I'$, because of symmetry, we conclude that $(0, s(c, a)) \in I'$, hence $(a, c) \in I'$. Therefore (1) implies (4). Finally it is obvious that (4) implies (3) and (3) implies (2).

When the equivalent properties of Proposition 1.6 hold for a whole variety, then they are equivalent to a remarkable property which we now introduce. A subtractive variety $\mathcal{V}$ is called **d-subtractive** if for any n-ary operation f in the type of $\mathcal{V}$ there is a $3n$-ary term $r_f(\vec{x}, \vec{y}, \vec{z}) \in \mathrm{IT}_{\mathcal{V}}(\vec{z})$ such that the identity

$$s(f(\vec{x}), f(\vec{y})) \approx r_f(\vec{x}, \vec{y}, s(x_1, y_1), \ldots, s(x_n, y_n))$$

holds in $\mathcal{V}$. An algebra $\mathbf{A}$ will be called d-subtractive if it belongs to a d-subtractive variety.

PROPOSITION 1.7. *A variety $\mathcal{V}$ is d-subtractive if and only if it is subtractive and, for any $\mathbf{A} \in \mathcal{V}$ and $I \in \mathrm{I}(\mathbf{A})$, I' is a subalgebra of $\mathbf{A} \times \mathbf{A}$ (i.e. $I' = I^s$).*

PROOF: Let I be an ideal of $\mathbf{A} \in \mathcal{V}$ and let $(0, c) \in I^s$. Then for some term p and for some $(a_i, b_i) \in I'$ we have $p(\vec{a}) = 0$ and $p(\vec{b}) = c$. If $\mathcal{V}$ is d-subtractive, then there is an ideal term in $\vec{z}$, $r_p(\vec{x}, \vec{y}, \vec{z})$ satisfying the definition. Therefore

$$c = s(c, 0) = s(p(\vec{b}), p(\vec{a})) = r_p(\vec{b}, \vec{a}, s(b_1, a_1), \ldots, s(b_n, a_n)).$$

But since $s(b_i, a_i) \in I$ for all i, we conclude that $c \in I$. Therefore $0/I^s = I$ and hence, by Proposition 1.6, I' is a subalgebra of $\mathbf{A} \times \mathbf{A}$.

Conversely, given an n-ary operation f, consider the free algebra in $\mathcal{V}$ generated by x_i, y_i, $i = 1, \ldots, n$ and let I be its ideal generated by $s(x_i, y_i)$, $i = 1, \ldots, n$. If I' is a subalgebra of the square, then in particular one must have $(f(\vec{x}), f(\vec{y})) \in I'$. Hence $(s(f(\vec{x}), f(\vec{y})), 0) \in I'$ and, by Proposition 1.6, $s(f(\vec{x}), f(\vec{y})) \in I$. From here a standard argument yields a term r_f with the desired properties. Thus we can conclude that $\mathcal{V}$ is d-subtractive.

Examples of d-subtractive algebras: groups, rings, Heyting algebras. In all these cases I' is simply the congruence corresponding to I (which is the a normal subgroups, two sided ideal, ideal or filter). Ideal lattices of d-subtractive varieties have a lattice-theoretic property stronger than modularity. We will show in fact that they are Arguesian i.e. satisfy the Arguesian law. To do so we will make use of some facts already pointed out in [12]. Since that paper never appeared in print we feel it necessary to reproduce the proofs.

LEMMA 1.8. *Let X be a set and L be a subset of $\mathcal{P}(X)$ that does not contain $\emptyset$ and is a lattice under inclusion. Suppose that to each $I \in L$ there corresponds $I^\# \subseteq X \times X$ in such a way that, for $I, J \in L$ we have*

(1) *$I^\#$ is symmetric and transitive.*
(2) $I \times I \subseteq I^\#$.
(3) $(I \cap J)^\# = I^\# \cap J^\#$.
(4) *For $a \in X$, $a \in I \vee J$ iff for some $b \in J$, $(a, b) \in I^\#$.*

Then L is Arguesian.

PROOF: Let $I_0, J_0, I_1, I_2, J_1, J_2 \in L$. We define $C_0 = (I_1 \vee I_2) \cap (J_1 \vee J_2)$ and cyclically C_1, C_2; let $C_0' = C_0 \cap (C_1 \vee C_2)$. Assuming that $a \in (I_0 \vee J_0) \cap (I_1 \vee J_1) \cap (I_2 \vee J_2)$ we have to show that $a \in I_1 \cap (C_0' \vee I_2)$. By (4) there are $a_0 \in J_0, a_1 \in J_1, a_2 \in J_2$ such that $(a, a_i) \in I_i^\#$ for $i = 0, 1, 2$. By (3) if $I \subseteq J$ then $I^\# \subseteq J^\#$, hence, by (2), $(a_0, a_1) \in J_0 \vee J_1$. Also, $(a_0, a_1) \in I_0^\#$ and $(a, a_1) \in I_1^\#$, so $a\,(I_0 \vee I_1)^\#\, a_0\,(I_0 \vee I_1)^\#\, a_1$; therefore $(a_0, a_1) \in C_0^\#$. Similarly we get $(a_1, a_2) \in C_0^\#$ and $(a_0, a_2) \in C_1^\#$. Then

$$a_1\,(C_1 \vee C_2)^\#\, a_0\,(C_1 \vee C_2)^\#\, a_2,$$

hence $a_1\,(C_1 \vee C_2)^\# \cap C_0^\#\, a_2$, i.e. $(a_0, a_2) \in (C_0')^\#$. Also $(a, a_1) \in I_1^\#$ and

$$a_1\,(I_2 \vee C_0')^\#\, a_2\,(I_2 \vee C_0')^\#\, a_1,$$

therefore $a\,(I_1^\# \cap (I_2 \vee C_0')^\#))\, a_1 \in J$. Now, using (3) and (4), we can conclude that $a \in I_1 \cap (I_2 \vee C_0') \vee J_1$.

Then follows the corollary:

COROLLARY 1.9. *Let $\mathbf{A}$ be an algebra. Assume that for some binary polynomials $d_1, \ldots, d_m$ we have:*

(1) *For $I \in \mathrm{I}(\mathbf{A})$, the relation I^δ defined by*

$$(a, b) \in I^\delta \text{ iff } d_i(a, b) \in I \text{ for all } i = 1, \ldots, m$$

is an equivalence relation.

(2) *For $I, J \in \mathrm{I}(\mathbf{A})$ and $a \in A$ we have that $a \in I + J$ iff, for some $a_0 \in J$, $(a, a_0) \in I^\delta$.*

Then $\mathrm{I}(\mathbf{A})$ is Arguesian.

PROOF: Since $(0, 0) \in \{0\}^\delta$, we get $d_i(0, 0) = 0$ in $\mathbf{A}$. Therefore, if $I \in \mathrm{I}(\mathbf{A})$, $I \times I \subseteq I^\delta$. Of course $(I \wedge J)^\delta = (I \cap J)^\delta = I^\delta \cap J^\delta$ and Proposition 1.8 applies.

Finally, we easily conclude:

PROPOSITION 1.10. *If $\mathbf{A}$ is d-subtractive, then $\mathrm{I}(\mathbf{A})$ is an Arguesian lattice.*

REMARKS: (a) If $\mathrm{N}(\mathbf{A})$ is a sublattice of $\mathrm{I}_{\mathcal{V}}(\mathbf{A})$ for every $\mathbf{A} \in \mathcal{V}$, then in fact $\mathrm{N}(\mathbf{A}) = \mathrm{I}_{\mathcal{V}}(\mathbf{A})$. Moreover from above it follows that, if $\mathcal{V}$ is a subtractive variety, $\mathbf{A} \in \mathcal{V}$ and I is a nonempty subset of $\mathbf{A}$, the following are equivalent.

(1) I is an ideal of $\mathbf{A}$.

(2) If $p(\vec{x}, y_1, y_2) \in \mathrm{IT}_{\mathcal{V}}(y_1, y_2)$ then for every $\vec{a} \in A, i, j \in I$ we have $p(\vec{a}, i, j) \in I$.

(3) For any term $t(\vec{x}, y_1, y_2)$ and any $\vec{a} \in A, i, j \in I$ we have

$$s(t(\vec{a}, i, j), t(\vec{a}, 0, 0)) \in I.$$

(b) Here are some examples of ideals in the context of some well-known varieties: rings (two-sised ideals), groups (normal subgroups), Heyting or Boolean algebras (ideals = 0-ideals, filters = 1-ideals), Banach algebras (ideals), C^*-algebras (ideals) etc.(*).

For semigroups, which have a well established theory of ideals, we have to cheat a bit. If **S** is a semigroup then let $\mathbf{S}^1$ be **S** with an adjoined unity (if **S** does not have one already). Then of course I is an ideal of **S** iff there is a 1-ideal of $\mathbf{S}^1$, I^1 such that $I = I^1 \cap S$. Note that the correspondence is one-to-one, modulo the admission of $\emptyset$ as an ideal, in case **S** has no unity. Of course subtractive semigroups are of a very special kind...

For lattices or semilattices, as soon as there is a decent "subtraction" (or "implication") around, one can easily embed the lattice-ideal theory into our framework (e.g. Skolem (semi)lattices of Büchi-Owens [4], relatively pseudo-complemented lattices, etc.).

2. The commutator in subtractive varieties

We assume that the reader is familiar with commutator theory for modular varieties [6]. We develop the basic facts of commutator theory for ideals in subtractive varieties. As we noted above, subtractivity does not imply congruence modularity, while trivially congruence modularity does not imply subtractivity. The commutator theory below, which has to do with ideals, works fine even if the variety is not congruence modular. Therefore this theory goes beyond commutator theory for congruence modular varieties, in particular it goes beyond ideal-determined varieties. It may be a bit surprising that the very simple identities defining subtractivity have this remarkable consequences. After all, commutator theory in [6] is a very succesful way of extending the commutator from groups (we might say: from ideal-determined varieties) to modular varieties, from normal subgroups (i.e. ideals) to congruences. In subtractive varieties we go the other way round: back to ideals, and their "product".

Let $\mathcal{V}$ be any variety. A term $t(\vec{x}, \vec{y}, \vec{z})$ is a **commutator term for $\mathcal{V}$ in** $\vec{y}, \vec{z}$, (where $\vec{y}, \vec{z}$ are disjoint sets of variables), if $t \in \mathrm{IT}_{\mathcal{V}}(\vec{y}) \cap \mathrm{IT}_{\mathcal{V}}(\vec{z})$.

(*)Of course, one-sided ideals of rings may be generalized as well, by relativizing our theory to a subset of $\mathrm{IT}_{\mathcal{V}}$ (see [13])

By $\mathrm{CT}_{\mathcal{V}}(\vec{y}, \vec{z})$ we denote the set of all commutator terms for $\mathcal{V}$ in $\vec{y}, \vec{z}$. For instance, if $\mathcal{V}$ is subtractive and $f(\vec{x}, \vec{y}, \vec{z})$ is any term, then the term

$$s(s(f(\vec{x}, \vec{0}, \vec{0}), f(\vec{x}, \vec{y}, \vec{0})), s(f(\vec{x}, \vec{0}, \vec{z}), f(\vec{x}, \vec{y}, \vec{z})))$$

is a commutator term for $\mathcal{V}$ in $\vec{y}, \vec{z}$.

Given an algebra **A** and $H, K \subseteq A$ we define

$$[H, K]_{\mathcal{V},\mathbf{A}} = \{t(\vec{a}, \vec{h}, \vec{k}) : t(\vec{x}, \vec{y}\, \vec{z}) \in \mathrm{CT}_{\mathcal{V}}(\vec{y}, \vec{z}), \vec{a} \in A, \vec{h} \in H, \vec{k} \in K\}$$

and $[H, K]_{\mathcal{V},\mathbf{A}}$ is called the **commutator** of H, K in **A** for $\mathcal{V}$. We drop the suffix $\mathcal{V}$ when $\mathcal{V}$ is clear from the context and for sure when $\mathcal{V} = \boldsymbol{V}(A)$; we drop the suffix **A** when no confusion is possible. First we prove some general facts.

Proposition 2.1. *If $\mathcal{V}$ is any variety, $\mathbf{A} \in \mathcal{V}$ and $H, K \subseteq A$ then:*

(1) $[H, K]_{\mathcal{V}} = [K, H]_{\mathcal{V}} \subseteq \langle H \rangle_{\mathcal{V}} \cap \langle K \rangle_{\mathcal{V}}$.
(2) $[H, K]_{\mathcal{V}} \in \mathrm{I}_{\mathcal{V}}(\mathbf{A})$.
(3) $[H, K]_{\mathcal{V}} = [\langle H \rangle_{\mathcal{V}}, \langle K \rangle_{\mathcal{V}}]_{\mathcal{V}}$.

Proof: (1) is trivial. If $p(\vec{x}, \vec{y}) \in \mathrm{IT}_{\mathcal{V}}(\vec{y})$ and $t_i(\vec{x}^i, \vec{y}^i, \vec{z}^i) \in \mathrm{CT}_{\mathcal{V}}(\vec{y}^i, \vec{z}^i)$ for $i = 1, \ldots, n$, then of course $p(\vec{x}, t_1, \ldots, t_n) \in \mathrm{CT}_{\mathcal{V}}(\vec{y}^1 * \cdots * \vec{y}^n, \vec{z}^1 * \cdots * \vec{z}^n)$. This proves (2). In (3), the inclusion from left to right is obvious; for the converse, observe that if $t(\vec{x}, \vec{y}, \vec{z}) \in \mathrm{CT}_{\mathcal{V}}(\vec{y}, \vec{z})$ and $q_i(\vec{x}^i, \vec{y}^i) \in \mathrm{IT}_{\mathcal{V}}(\vec{y}^i)$ and $p_j(\vec{u}^j, \vec{v}^j) \in \mathrm{IT}_{\mathcal{V}}(\vec{v}^j)$, then $t(\vec{x}, q_1, \ldots, q_n, p_1, \ldots, p_m) \in \mathrm{CT}_{\mathcal{V}}(\vec{y}^1 * \cdots * \vec{y}^n, \vec{v}^1 * \cdots * \vec{v}^m)$.

It follows that it is worthless to consider commutators of subsets other than ideals. Now we show that the commutator of two ideals of a subtractive algebra really depends on the algebra and not on any subtractive variety to which the algebra happens to belong. For any term $g(\vec{x}, \vec{y}, \vec{z})$ let us consider the term

$$s(s(g(\vec{x}, \vec{y}, \vec{z}), g(\vec{x}, \vec{0}, \vec{z})), s(g(\vec{x}, \vec{y}, \vec{0}), g(\vec{x}, \vec{0}, \vec{0})))$$

and call it $g'(\vec{x}, \vec{y}, \vec{z})$. Of course $g'(\vec{x}, \vec{y}, \vec{z})$ is a commutator term in $\vec{y}, \vec{z}$ **for any** class of subtractive algebras.

Proposition 2.2. *If I, J are ideals of a subtractive algebra* **A** *the commutator of I, J is equal to the ideal generated by*

$$\{g'(\vec{a}, \vec{i}, \vec{j}) : g \text{ any term}, \vec{a} \in A, \vec{i} \in I, \vec{j} \in J\}$$

Hence we are allowed to simply denote by $[I, J]$ the commutator of I, J in **A**.

Proof: Let $\mathcal{V} = \boldsymbol{V}(A)$. Then for any $\vec{a} \in A, \vec{i} \in I, \vec{j} \in J$ and any term $g(\vec{x}, \vec{y}, \vec{z})$ we have $g'(\vec{a}, \vec{i}, \vec{j}) \in [I, J]_{\mathcal{V},\mathbf{A}}$. If $t(\vec{x}, \vec{y}, \vec{z}) \in \mathrm{CT}_{\mathcal{V}}(\vec{y}, \vec{z})$, then of course the identity

$$t(\vec{x}, \vec{y}, \vec{z}) \approx t'(\vec{x}, \vec{y}, \vec{z})$$

holds in $\mathcal{V}$. This concludes the proof.

PROPOSITION 2.3. *Let* $\mathbf{A}, \mathbf{B}$ *belong to a subtractive variety* $\mathcal{V}$*; let* I, J, K_λ *for* $\lambda \in \Lambda$ *belong to* $\mathrm{I}(\mathbf{A})$, g *be an homomorphism from* $\mathbf{A}$ *onto* $\mathbf{B}$ *and* $L, M \in \mathrm{I}(\mathbf{B})$. *Then*

(1) $[I, K_\lambda + K_\mu] = [I, K_\lambda] + [I, K_\mu]$.
(2) $[I, \bigvee_{\lambda\in\Lambda} K_\lambda] = \bigvee_{\lambda\in\Lambda}[I, K_\lambda]$.
(3) $g([I, J]_\mathbf{A}) = [g(I), g(J)]_\mathbf{B}$.
(4) $[I, J]_\mathbf{A} + g^{-1}(0) = g^{-1}([g(I + g^{-1}(0)), g(J + g^{-1}(0))]_\mathbf{B})$.
(5) $g^{-1}([L, M]_\mathbf{B}) = [g^{-1}(L), g^{-1}(M)]_\mathbf{A}/ker(g)$.

PROOF: For (1) it is enough to prove that if $H, M \subseteq A$ and $a \in [I, H \vee M]$, then $a \in [I, H] + [I, M]$. Let $a = t(\vec{a}, \vec{i}, \vec{l})$ for some $t(\vec{x}, \vec{y}, \vec{z}) \in \mathrm{CT}_\mathcal{V}(\vec{y}, \vec{z})$, $\vec{a} \in A, \vec{i} \in I, \vec{l} \in H \vee M$. Assume that $\vec{l} = h_1, \ldots, h_r, m_1, \ldots, m_t$ where $h_i \in H$ and $m_j \in M$ and let $a' = t(\vec{a}, \vec{i}, 0, \ldots, 0, m_1, \ldots, m_t)$. Then $a' \in [I, M]$ and moreover

$$s(t(\vec{x}, \vec{y}, z_1, \ldots, z_r, u_1, \ldots, u_k), t(\vec{x}, \vec{y}, \vec{0}, \vec{u}))$$

is a commutator term in $\vec{y}, \vec{z}$. Therefore $s(a, a') \in [I, H]$, that yields $a \in [I, H] + [I, M]$.

The inclusion from left to right in (2) is trivial. For the converse, if $a \in [I, \bigvee_{\lambda\in\Lambda} K_\lambda]$, then for some finite $F \subseteq \Lambda$, $a \in [I, \bigvee_{\lambda\in F} K_\lambda]$. By (1) we conclude that $a \in \bigvee_{\lambda\in F}[I, K_\lambda]$ that of course implies the desired inclusion. The proof of (3) is immediate. For (4) let us observe that, if $I \in \mathrm{I}(\mathbf{A})$, $g^{-1}(g(I)) = I + g^{-1}(0)$. In fact, if $a \in g^{-1}(g(I))$, then $g(a) = g(i)$ for some $i \in I$. Since $g(s(a, i)) = s(g(a), g(i)) = 0$ and $a = u(a, i, s(a, i))$, we conclude $a \in I + g^{-1}(0)$. On the other hand, if $a \in I + g^{-1}(0)$, then $a = u(a', i, e)$ for some $a' \in A, i \in I, e \in g^{-1}(0)$. Then $g(a) = g(u(a, i, e)) = u(g(a), g(i), 0) = g(u(a, i, 0))$. But $u(a, i, 0) \in I$, hence $a \in g^{-1}(g(I))$. Now we note that it suffices to prove (4) in case I, J both contain $g^{-1}(0)$. In fact, letting $G = g^{-1}(0)$,

$$[I + G, J + G] = [I, J] + [I, G] + [J, G] + [G, G].$$

Therefore, if $G \subseteq I \cap J$, we get

$$[I + G, J + G] + G = [I, J] + G.$$

Now by the observation above and (3) we have

$$[I, J] + G = g^{-1}(g[I, J]) = g^{-1}([g(I), g(J)])$$

that concludes the proof of (4).

For (5), let $g(a) \in [L, M]_\mathbf{B}$, i.e. $g(a) = t(\vec{b}, \vec{l}, \vec{m})$ for some $t(\vec{x}, \vec{y}, \vec{z}) \in \mathrm{CT}_\mathcal{V}(\vec{y}, \vec{z})$, $\vec{b} \in B, \vec{l} \in L, \vec{j} \in M$. Choose $a_i \in g^{-1}(b_i)$, $i_r \in g^{-1}(l_r)$ and $j_k \in g^{-1}(m_k)$. Then $a' = t(\vec{a}, \vec{i}, \vec{j}) \in [g^{-1}(L), g^{-1}(M)]$ and $g(a) = g(a')$. Conversely, if $g(a) = g(a')$ and $a' = t(\vec{a}, \vec{i}, \vec{j})$ for some $t(\vec{x}, \vec{y}, \vec{z}) \in \mathrm{CT}_\mathcal{V}(\vec{y}, \vec{z})$, $\vec{a} \in A, \vec{i} \in g^{-1}(L), \vec{j} \in g^{-1}(M)$, take $b_i = g(a_i)$. Then

$$g(a) = g(a') = t(\vec{b}, g(\vec{i}), g(\vec{j})) \in [L, M]_\mathbf{B}.$$

REMARKS:

(1) In d-subtractive varieties the commutator behaves in a way analogous to the case of groups and rings. What we mean is that the commutator there can be characterized in terms of some "term condition" in Freese-McKenzie fashion. As a matter of fact one may reproduce in this case the argument in Chapter 1 of [6] almost literally. We leave this task to the interested reader.
(2) (At the referee's suggestion.) A natural equation to consider is the following

$$(MC) \qquad 0/[R, S] = [0/R, 0/S] \qquad R, S \in \mathrm{Con}(\mathbf{A})$$

which involves the commutator for ideals and the commutator for congruences and may be called the mixed commutator identity. We will deal with it in the second paper of the series, in the framework of varieties which are both congruence modular (to have a good commutator for congruences) and subtractive (to have a good commutator for ideals). Of course (MC) holds in any ideal-determined variety.

3. i-ABELIAN ALGEBRAS

An algebra $\mathbf{A}$ will be called **i-abelian** if $[A, A] = \{0\}$. It is clear from the properties of the commutator that, in any variety the class of i-abelian algebras is a subvariety. For a variety $\mathcal{V}$, let $\boldsymbol{AB}(\mathcal{V})$ denote the largest i-abelian subvariety of $\mathcal{V}$.

PROPOSITION 3.1. *For any subtractive variety* $\mathcal{V}$, $\boldsymbol{AB}(\mathcal{V})$ *is d-subtractive.*

PROOF: We will show that if $\mathbf{A} \in \boldsymbol{AB}(\mathcal{V})$, $I \in \mathrm{I}(\mathbf{A})$, then I' is a subalgebra of $\mathbf{A} \times \mathbf{A}$. First observe that if $t(\vec{x}, \vec{y}) \in \mathrm{IT}_\mathcal{V}(\vec{y})$, then the identity

$$s(t(\vec{x}, \vec{y}), t(\vec{z}, \vec{y})) \approx 0$$

holds in $\boldsymbol{AB}(\mathcal{V})$, simply because the shown term is a commutator term in $\vec{x} * \vec{z}, \vec{y}$. Let f be an n-ary operation; then

$$s(f(u(x_1, y_1, 0), \ldots, u(x_n, y_n, 0)), f(\vec{y}))$$

is an ideal term in $\vec{y}$. Therefore

$$s(f(u(x_1, y_1, 0), \ldots, u(x_n, y_n, 0)), f(\vec{y})) \approx 0$$

which means that

$$s(f(u(x_1, y_1, z_1), \ldots, u(x_n, y_n, z_n)), f(\vec{y}))$$

is an ideal term for $\boldsymbol{AB}(\mathcal{V})$ in $\vec{z}$. Therefore, if $(a_i, b_i) \in I'$, also $(f(\vec{a}), f(\vec{b})) \in I'$.

Next we want to characterize $\boldsymbol{AB}(\mathcal{V})$ when $\mathcal{V}$ is already d-subtractive. For $\mathbf{A}$ in a d-subtractive variety $\mathcal{V}$ we define Δ to be $\{0\}'$, i.e. $(a, b) \in \Delta$ iff $s(a, b) = 0$. Since $\mathcal{V}$ is d-subtractive, $\Delta \in \mathrm{Con}(\mathbf{A})$; in the quotient $\mathbf{A}/\Delta$ the Δ-block containing a will be denoted by a_Δ. For a subtractive variety $\mathcal{V}$, let $\boldsymbol{AF}(\mathcal{V})$ denote the largest affine subvariety of $\mathcal{V}$

PROPOSITION 3.2. *Let $\mathcal{V}$ be d-subtractive. Then*

$$\mathbf{A} \in \boldsymbol{AB}(\mathcal{V}) \qquad \text{iff} \qquad \mathbf{A}/\Delta \in \boldsymbol{AF}(\mathcal{V}).$$

PROOF: Of course $\boldsymbol{AF}(\mathcal{V}) \subseteq \boldsymbol{AB}(\mathcal{V})$. Let $\mathcal{K} = \boldsymbol{I}(\{\mathbf{A}/\Delta : \mathbf{A} \in \boldsymbol{AB}(\mathcal{V})\})$; then $\mathcal{K} \subseteq \boldsymbol{AB}(\mathcal{V})$ and it is routine to check that $\mathcal{K}$ is a variety. Now $\mathcal{K}$ satisfies the axiom

$$\forall x \forall y (x \approx y \longleftrightarrow s(x, y) \approx 0),$$

hence it is 0-regular. Therefore $\mathcal{V}$ is ideal determined and so $\mathcal{K} \subseteq \boldsymbol{AF}(\mathcal{V})$ (see [7]).

Conversely, let $\mathbf{A}/\Delta \in \boldsymbol{AF}(\mathcal{V})$, so that $\langle \mathbf{A}/\Delta, +, 0\rangle$ is a commutative group for some binary operation $+$ and every operation of $\mathbf{A}/\Delta$ is affine. Let g be any term; to simplify the notation we may assume that g is ternary, but it is clear that the argument holds in general. Let r_g be the 9-ary term witnessing d-subtractivity for g. Then $r_g(\vec{x}, \vec{y}, \vec{z})$ is an ideal term in $\vec{z}$. Now affinity yields

$$\begin{aligned} r_g(x, y, z, x, 0, z, 0, y, 0) &= r_g(x, y, 0, x, 0, 0, 0, y, 0) + r_g(0, 0, z, 0, 0, z, 0, 0, 0) \\ &= r_g(x, y, 0, x, 0, 0, 0, y, 0). \end{aligned}$$

But (even in $\mathbf{A}$)

$$\begin{aligned} s(g(x, y, z), g(x, 0, z)) &= r_g(x, y, z, x, 0, z, 0, y, 0) \\ s(g(x, y, 0), g(x, 0, 0)) &= r_g(x, y, 0, x, 0, 0, 0, y, 0). \end{aligned}$$

Therefore in $\mathbf{A}/\Delta$

$$s(g(x, y, z), g(x, 0, z))_\Delta = s(g(x, y, z), g(x, 0, 0))_\Delta$$

which really means (cfr. Proposition 2.2) that $g'(x, y, z) = 0$. But then $[A, A] = \{0\}$ and $\mathbf{A}$ is i-abelian.

4. APPENDIX: THE VERY CLASSICAL CASE

As we have already noticed the notion of subtractivity is implicit in the study of ideal-determined varieties. By itself the notion is admittedly pretty weak (for some enrichments of this notion one may refer to [5] or to some results in [2]). Here we want to consider the kind of subtractivity which is common to virtually all ideal-determined structures of modern abstract algebra and algebraic logic; that is what we mean by the "very classical case".

In all varieties of classical algebras that are ideal determined (e.g. groups, operator groups, loops, rings, Boolean algebras, relatively pseudocomplemented lattices, Lie algebras, Banach algebras etc.) we find as a common feature a "strong additive structure", in the sense that there are binary terms $d_1(x, y), \ldots, d_m(x, y)$ and an $m + 1$-ary term $p(x_1, \ldots, x_{m+1})$ such that

$$(CI_m) \qquad \begin{aligned} & d_1(x, x) \approx \ldots d_m(x, x) \approx 0 \\ & p(y, d_1(x, y), \ldots, d_m(x, y)) \approx x \end{aligned}$$

are identities of the variety. Note that from CI_m we get at once :

(1) $p(x, 0, \ldots, 0) \approx x$.
(2) Permutability of congruences, since the term

$$t(x, y, z) = p(z, d_1(x, y), \ldots, d_m(x, y))$$

satisfies the Mal'cev identities.
(3) Ideal determinacy, since 0-permutability is obvious from (2) and 0-regularity follows from the fact that if $d_i(x, y) \approx 0$ for $i = 1, \ldots, m$ then

$$y = p(y, 0, \ldots, 0) = p(y, d_1(x, y), \ldots, d_m(x, y)) = x.$$

Therefore it seems worth seeking the general algebraic meaning of the Mal'cev condition CI_m.

A variety satisfying CI_m for some m is called **classically ideal determined** (in [11] they were called "BIT speciali"). For an algebra $\mathbf{A}$, a subalgebra $\mathbf{S}$ of $\mathbf{A} \times \mathbf{A}$ is called **classical** if $(x,y) \in S$ implies $(x,x) \in S$ for all $x, y \in A$. It is easily seen that the family $\mathrm{Cs}(\mathbf{A})$ of all classical subalgebras of $\mathbf{A} \times \mathbf{A}$ is an algebraic lattice under inclusion. Since clearly any intersection of classical subalgebras is still a classical subalgebra it makes sense to define, for $H \subseteq A \times A$, $\langle H \rangle_{\mathrm{Cs}}$, the classical subalgebra generated by H. A description of $\langle H \rangle_{\mathrm{Cs}}$ is easily obtained if we define

$$H^{\Delta} = \{(x,x) : \text{ there is a } y \in A \text{ with } (x,y) \in H\}$$
$$H_0 = H$$
$$H_{n+1} = \text{subalgebra generated by } H_n \cup H_n^{\Delta} \text{ in } \mathbf{A} \times \mathbf{A}.$$

and we observe that a subalgebra $\mathbf{S}$ of $\mathbf{A} \times \mathbf{A}$ is classical if and only if $S^{\Delta} \subseteq S$.

PROPOSITION 4.1. *For any $H \subseteq A \times A$ we have*

$$\langle H \rangle_{\mathrm{Cs}} = \bigcup_{n \in \omega} H_n.$$

PROOF: Trivially $H' = \bigcup_{n \in \omega} H_n$ is a subalgebra of $\mathbf{A} \times \mathbf{A}$. It is also classical, since $(x,y) \in H'$ implies $(x,y) \in H_n$ for some n, implying $(x,x) \in H_{n+1} \subseteq H'$. Moreover $H \subseteq H'$ and and easy induction shows that, if $\mathbf{S} \in \mathrm{Cs}(\mathbf{A})$ and $H \subseteq S$, then $H' \subseteq S$. Hence $\langle H \rangle_{\mathrm{Cs}} = H'$.

PROPOSITION 4.2. *Let $\mathbf{S} = \langle H \rangle_{\mathrm{Cs}}$ and $\mathbf{S}' \in \mathrm{Cs}(\mathbf{A})$. Assume that for any u, v with $(u,v) \in H$ it follows $(u,u) \in S'$ (i.e. $H^{\Delta} \subseteq S'$). Then $S^{\Delta} \subseteq S'$.*

PROOF: By induction on n, one sees that $H_n^{\Delta} \subseteq S'$. For $n = 0$ that holds by hypothesis. Assume now that $H_n^{\Delta} \subseteq S'$ and take $(u,v) \in H_{n+1}$. Then, for some term $t(x_1, \ldots, x_{r+s})$ we have

$$u = t(a_1, \ldots, a_r, b_1, \ldots, b_s)$$
$$v = t(a_1, \ldots, a_r, b_1', \ldots, b'_s)$$

for some $(a_1, a_1), \ldots, (a_r, a_r) \in H_n^{\Delta}$ and $(b_1, b'_1), \ldots, (b_s, b'_s) \in H_n$. But then, $(b_j, b'_j) \in H_n^{\Delta}$ for all $j = 1, \ldots, s$, therefore $(u,v) \in S'$.

DEFINITION 4.3:

(1) An algebra $\mathbf{A}$ is **classically** 0-**regular** if, whenever $\mathbf{S}, \mathbf{S}' \in \mathrm{Cs}(\mathbf{A})$, from $0/S = 0/S'$, $S^{\Delta} \subseteq S'$ and $(S')^{\Delta} \subseteq S$, it follows $S = S'$. An equivalent formulation is: for all $S, S' \in \mathrm{Cs}(\mathbf{A})$

$$0/S = 0/S' \quad \text{and} \quad S^{\Delta} = (S')^{\Delta} \quad \text{imply} \quad S = S'.$$

(2) An algebra $\mathbf{A}$ is 0-**coherent** if for any $\theta \in \mathrm{Con}(\mathbf{A})$ and any subalgebra $\mathbf{B}$ of $\mathbf{A}$, if $0/\theta \subseteq B$, then $b/\theta \subseteq B$ for all $b \in B$. Equivalently if any subalgebra containing $0/\theta$ for some $\theta \in \mathrm{Con}(\mathbf{A})$ is a union of θ-blocks.

(3) A variety $\mathcal{V}$ is **classically** 0-**regular** (0-**coherent**) if any $\mathbf{A} \in \mathcal{V}$ is classically 0-regular (0-coherent).

The main result of this section is the following:

THEOREM 4.4. *For a variety $\mathcal{V}$ the following are equivalent.*

(1) *$\mathcal{V}$ is classically 0-regular.*

(2) *$\mathcal{V}$ is classically ideal determined.*

(3) *$\mathcal{V}$ is 0-coherent.*

PROOF: Let us assume (1). Let M be the set of all $d(x,y) \in \mathbf{F}_{\mathcal{V}}(x,y)$ (the algebra freely generated in $\mathcal{V}$ by x, y) such that $d(x,x) = 0$ holds and let f be the endomorphism of $\mathbf{F}_{\mathcal{V}}(x,y)$ defined by $f(x) = f(y) = x$. We define

$$H = \{(x,x)\} \cup \{(0, d(x,y)) : d \in M\}$$
$$S_M = \langle H \rangle_{\mathrm{Cs}}$$
$$K = \{(x,x)\} \cup \{(0,u) : f(u) = 0\}$$
$$S_f = \langle K \rangle_{\mathrm{Cs}} \qquad S = \langle \{(x,y)\} \rangle_{\mathrm{Cs}}.$$

First we show that

$$(i) \qquad S_f \subseteq \mathrm{Ker}(f),$$

i.e. that $(u,v) \in S_f$ implies $f(u) = f(v)$. To show that, we prove that $(u,v) \in K_n$ implies $f(u) = f(v)$ by induction on n. For $n = 0$ it is trivial. If $(u,v) \in K_{n+1}$ then we must have a term $t(x_1, \ldots, x_{r+s})$ such that

$$u = t(a_1, \ldots, a_r, b_1, \ldots, b_s)$$
$$v = t(a_1, \ldots, a_r, b_1', \ldots, b'_s)$$

for some $(a_1, a_1), \ldots, (a_r, a_r) \in K_n^{\Delta}$ and $(b_1, b'_1), \ldots, (b_s, b's) \in K_n$. But then, $f(u) = t(\overrightarrow{f(a)}, \overrightarrow{f(b)}) = t(\overrightarrow{f(a)}, \overrightarrow{f(b')}) = f(v)$. It follows that

$$(ii) \qquad 0/S_f = \{u : f(u) = 0\} = f^{-1}(0).$$

Moreover we can show that

$$(iii) \qquad S \subseteq Ker(f)$$

with an inductive argument similar to the one above. Next we want to show that

$$(iv) \qquad S_f = S.$$

To this aim it is enough to show that

$$(v) \qquad 0/S_f = 0/S$$

$$(vi) \qquad S_f^{\Delta} \subseteq S \qquad\qquad S^{\Delta} \subseteq S_f.$$

To prove (v) we assume first that $u \in 0/S_f$, i.e. $f(u) = 0$. Then $u = t(x,y)$ for some binary term t and $t(x,x) = f(t(x,y)) = f(u) = 0$. Since $(x,x) \in S$, we get $(0,u) \in S$. The reverse inclusion follows from (ii) and (iii). To prove (vi) we apply Proposition 4.2. So assume that $(u,v) \in K$; then either $u = 0$ and $f(v) = 0$ or $u = v = x$. By (v), $(0,0) \in S$ and $(x,x) \in S$, therefore $(u,u) \in S$, hence $S_f^{\Delta} \subseteq S$. For the other inclusion, if $(u,v) = (x,y)$ then $u = x$ and, since $(x,x) \in S_f$ we are done. So $(v), (vi)$ are proved and (iv) follows.

We now show that

$$(vii) \qquad S_f = S_M.$$

Again it is enough to prove that

$$(viii) \qquad 0/S_f = 0/S_M$$

$$(ix) \qquad S_f^{\Delta} \subseteq S_M \quad ; \quad S_M^{\Delta} \subseteq S_f.$$

Let $d(x,y) \in M$. We have $f(d(x,y)) = d(x,x) = 0$, so $(0, d(x,y)) \in S_f$. Also $(x,x) \in S_f$, therefore $H \subseteq K$ and hence $S_M \subseteq S_f$ and, *"a fortiori"*, $0/S_M \subseteq 0/S_f$. The reverse inclusion is proved using (ii). Assume $f(u) = 0$, say $u = t(x,y)$. Then

$$t(x,x) = f(t(x,y)) = f(u) = 0,$$

so $t(x,y) \in M$ and $(0,u) \in S_M$. Thus we have proved $(viii)$. For (ix), again we use Proposition 4.2 and the proof is similar to the one above for (vi). Hence (vii) holds. Observe that, from (iv) and (vii) one gets

$$S_f = S = S_M = \bigvee_{d \in M} (\langle\{(0, d(x,y))\}\rangle_{\mathrm{Cs}} \vee \langle (x,x)\rangle_{\mathrm{Cs}}).$$

But S is a compact element of $\mathrm{Cs}(\mathbf{F}_{\mathcal{V}}(x,y))$, hence there are $d_1, \ldots, d_m \in M$ with

$$S = \bigvee_{i=1}^{m} (\langle\{(0, d_i(x,y))\}\rangle_{\mathrm{Cs}} \vee \langle (x,x)\rangle_{\mathrm{Cs}}.$$

As soon as one notes that $(x,y) \in S$, the usual Mal'cev argument yields the existence of polynomials satisfying CI_m. Hence (1) implies (2).

Next let us assume (2), i.e. that CI_m holds for some m. Suppose that $\mathbf{A} \in \mathcal{V}$, $\mathbf{S}, \mathbf{S}' \in \mathrm{Cs}(\mathbf{A})$, $0/S = 0/S'$ and $S^{\Delta} = (S')^{\Delta}$. Take $(u,v) \in S$; since S is classical $(u,u) \in S$, therefore $(0, d_i(b,a)) \in S$ for all i. Hence $(0, d_i(b,a)) \in S'$ for all i and moreover $(u,u) \in (S')^{\Delta} \subseteq S'$, hence $(u,u) \in S'$. Therefore, since

$$u = p(u, 0, \ldots, 0)$$

$$v = p(u, d_1(v,u), \ldots, d_m(v,u))$$

we conclude that $(u,v) \in S'$. Hence $S \subseteq S'$ and specularly $S' \subseteq S$. So (2) implies (1).

The equivalence of (2) and (3) was already shown in [3]; for completeness' sake we give the easy proof. Assume (2) and let $\mathbf{A} \in \mathcal{V}$, $\theta \in \mathrm{Con}(\mathbf{A})$, $\mathbf{B}$ a subalgebra of $\mathbf{A}$ and $0/\theta \subseteq B$. Let $b \in B$ and let $a \in b/\theta$. Then $d_i(a,b)\ \theta\ d_i(b,b)\ \theta\ 0$ for all i, hence $d_i(a,b) \in B$ for all i. But then

$$a = p(b, d_1(a,b), \ldots, d_m(a,b)) \in B$$

so $b/\theta \subseteq B$ and (3) is proved. Finally, assume (3). Let once again f be the endomorphism of $\mathbf{F}_{\mathcal{V}}(x,y)$ defined by $f(x) = f(y) = x$ and $f(0) = 0$. Let $\mathbf{B}$ be the subalgebra of $\mathbf{F}_{\mathcal{V}}(x,y)$ generated by $\{y\} \vee \{d(x,y) : d(x,y)\ \mathrm{Ker}(f)\ 0\}$. Then $0/\mathrm{Ker}(f) \subseteq B$ and $y \in B$, hence by hypothesis $y/\mathrm{Ker}(f) \subseteq B$. But then $x/\mathrm{Ker}(f) \subseteq B$ as well, hence $x \in B$. Then apply the usual Mal'cev argument to get terms satisfying CI_m. Hence (3) implies (2) and the proof is finished.

A further condition equivalent to classical ideal determinacy was investigated in [3].

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