Exact Morita equivalence of doctrines and applications

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The notion of topos is central in category theory as well as in categorical logic and in algebraic geometry. The archetypical examples are the topos of sheaves $\mathbf{Sh}(\mathcal{A})$ on a local $\mathcal{A}$, also called localic topos, and the Effective topos **Eff** [4].

Over the years, several authors have employed free constructions of categories and Lawvere doctrines as tools to construct toposes from simpler structures, namely exact completions.

A key example is the *tripos-to-topos construction* [5, 14]. This is an instance of an exact, free construction that, given a Lawvere doctrine [7], produces an exact category which happens to be a topos, see [10, 8]. Another construction that might lead to a topos (under suitable hypotheses on the base category [12]) is the so-called ex/lex completion which produces an exact category from a lex category $\mathcal{C}$, see [2]. As shown in [10, 9], this construction happens to be also an instance of the tripos-to-topos construction applied to the tripos given by the weak subobject doctrine of $\mathcal{C}$.

Some toposes, like Hyland's Effective topos **Eff** or certain localic toposes described in [8], may be presented both as a tripos-to-topos construction and as a ex/lex completion. Instead, for most locales, the topos of sheaves $\mathbf{Sh}(\mathcal{A})$ can be only presented in as a tripos-to-topos.

The main purpose of this talk is to show when these two free constructions coincide, that means showing when a tripos-to-topos is also a ex/lex completion.

To achieve this goal, we introduce the notion of exact-Morita equivalence in the context of Lawvere doctrines.

Recall that the notion of *Morita equivalence* was originally introduced by Kiiti Morita in the context of ring theory [13] and, nowadays, this term is applied in different, but closely related senses, in a wide range of mathematical fields. For example, in the algebraic setting, two algebraic theories are called Morita equivalent if corresponding varieties of algebras are equivalent, while two geometric theories are said to be Morita equivalent if they have equivalent classifying toposes [6] (a concept widely used in [1]).

In our setting, we say that two Lawvere doctrines are exact-Morita equivalent if their tripos-to-topos are equivalent.

Combining this version of Morita equivalence for doctrines with a second free construction called full existential completion [15, 11], we show that a tripos-to-topos of a Lawvere doctrine is an instance of the exact completion of a lex category if and only if the starting doctrine is exact-Morita equivalent to a doctrine obtained as full existential completion.

Therefore, we can conclude that any tripos-to-topos admits a presentation as exact completion of a lex category precisely when the starting doctrine is exact-Morita equivalent to a full existential completion.

Employing our characterisation, we provide new examples of toposes having both these presentations, such as the localic topos $\mathbf{Sh}(\mathcal{A})$ on a supercoherent locale $\mathcal{A}$.

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Finally, thanks to our results, we can conclude that both supercoherent toposes and realizability toposes provide counterexamples to the question regarding the existence of non-equivalent triposes which produce the same tripos-to-topos construction left open in [3]. In particular, the notion of exact-Morita equivalence we introduced in the setting of doctrine provides a conceptual explanation of why different triposes may raise to the same topos.

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