An overview on the Taylor expansion of programs

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1 The differential λ-calculus

At the end of the 80's a major discovery in logic and computer science appeared: Girard's linear logic [6]. This opened a whole new field of research, in which the common line is the deep role reserved to resources in a computation/proof. In particular, we are interested here in the correspondence between linearity in analysis (polynomials of degree 1) and linearity in logic and computer science, which means the use of the hypothesis exactly once in a proof, or the use of the argument exactly once in a computation.

In analysis, a sufficiently regular function can be seen (locally around a point) as a series of polynomials weighted via some rational coefficient: its Taylor expansion at that point. Now, a polynomial in x of degree k can be thought of as a function which uses its argument x exactly k times (in order to compute x^k). This analogy can be made rigorous by considering the so called *differential λ-calculus* [5], introduced by Ehrhard and Regnier. The novelties of this syntax are the presence of non-determinism (handled as formal sums) and the presence of a constructor $(\mathsf{D}M) \bullet M$, which has to be seen as the directional derivative of M along N. The calculus is obtained by endowing this syntax with the usual β-reduction plus a reduction contracting redexes of the form $\mathsf{D}(\lambda x.M)\bullet N$. The most interesting feature of this calculus is that it becomes now possible to Taylor expand an ordinary λ-term via an inductively defined Taylor map Θ (with a codomain which we do not precise here), whose crucial case of definition is: $\Theta(MN) = \sum\limits_{n\in\mathbb{N}} \frac{1}{n!}(\mathsf{D}^n\Theta(M)\bullet\Theta(N)^n)0$. As in analysis happens for function, now a λ-term is associated with an infinite series of approximants weighted via the usual factorial coefficients.The Taylor map Θ also satisfies a direct characterization: $\Theta(M) = \sum\limits_{t\in\mathcal{T}(M)} \frac{1}{m(t)}t$, where $m(t) \in \mathbb{N}$ is a certain number dealing with the combinatorics of some group handling permutations of resources, and $\mathcal{T}$ is a very easily inductively defined function, called the *qualitative Taylor expansion*.

From a categorical point of view, the differential λ-calculus corresponds to a particular kind of Cartesian closed differential categories [3], called Cartesian closed differential λ-categories [4]. Very briefly, in a Cartesian differential category, each homset is equipped with a structure of commutative monoid (i.e. we can add morphisms and there is a 0 morphism), it is Cartesian closed (with the closed structure compatible with the additive structure), and it is equipped with a *differential operator* D satisfying a series of axioms reminiscent of the ones satisfied by the "real" differential operator. A Cartesian closed differential λ-category (a model of the differential λ-calculus) is now simply a Cartesian closed differential category satisfying and additional property, called the "D-Curry axiom". An example of such categories is the one of convenient vector spaces with smooth maps, where the differential operator is indeed the "real" differential $Df : \mathbb{V} \times \mathbb{V} \to \mathbb{W}$, $Df(x,u) := \frac{d}{dt}\Big|_{t=0} f(x+tu)$, of smooth functions $f : \mathbb{V} \to \mathbb{W}$.

2 Taylor subsumes Böhm

In [5], Ehrhard and Regnier mention that: *Understanding the relation between the term and its full Taylor expansion might be the starting point of a renewing of the theory of approximation.* When they say "theory of approximation" they basically mean the theory of Böhm trees and the $\lambda\bot$-calculus. Without going into details, let us just say that in the $\lambda\bot$-calculus we have a new symbol $\bot$ (representing the absence of computational information) and define the preorder generated by setting $\bot \leq M$ for all M. One also endows the syntax with a certain reduction and defines a set of approximants of a term M. Finally, the Böhm tree of M appears as the *sup* of such approximants. In [2, Chapter 14], this notions, together with the technique known as "labelled reductions", are used to prove some fundamental properties of λ-calculus: Monotonicity, Genericity property of the unsolvables, Scott continuity lemma, Stability property and Perpendicular Lines property. They respectively say that: Böhm trees are contextual (thus define a λ-theory); that the notion of unsolvable term is the right one in order to model the intuitive notion of "computationally meaningless"; that for each approximant P of FM, there is an approximant Q of M which is already enough for F to use in order to compute P (this is the key property which entails the Scott continuity); that terms are stable w.r.t. taking intersections, in a certain sense; that if a term, seen as a function of n arguments, is constant (mod Böhm trees) on n "perpendicular lines", than it is constant (mod Böhm trees) everywhere. We will not explain the precise meaning of those classic results. Instead, we say that in [1], we (together with our – at that time – doctoral supervisor) showed that the Taylor expansion and the resource calculus actually provides a technique allowing to prove those results in an arguably more satisfactory way. Our work was crucially based on two elements:

- first, the fact that the resource λ-calculus satisfies strong properties, namely linearity (i.e. no argument can be duplicated nor erased during reduction, but only used exactly once), strong normalisation and confluence (this is the only one which already holds in the λ-calculus);

- second, the so-called Ehrhard and Regnier's *commutation formula* $\mathcal{T}(\mathrm{BT}(M)) = \mathrm{NF}(\mathcal{T}(M))$ relating Böhm trees and Taylor normal forms (a notion that we do not define here) in order to translate the statements from the λ-calculus to the resource version.

At the end of the day we can say that, in λ-calculus, resource approximation (which is at the basis of Taylor expansion) "subsumes" Böhm trees approximation, answering positively to Ehrhard and Regnier's proposal that we mentioned. In other words, Taylor expansion allowed us to basically rewrite Chapter 14 of the famous Barendregt's book.

References

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