Comparable binary relations and the amalgamation property

Paolo Lipparini

Dipartimento di Matematica, Università di Roma "Tor Vergata", ROME, ITAILA
lipparin@axp.mat.uniroma2.it

The theory of two binary relations has the strong amalgamation property, when the first relation is assumed to be coarser than the second relation, and each relation satisfies a chosen set of properties from the following list: transitivity, reflexivity, symmetry, antireflexivity and antisymmetry. The result fails, for general comparability conditions, when three or more binary relations are taken into account; however, the result does generalize when all the relations are supposed to be transitive. The amalgamation property is maintained when we add families of unary operations preserving all the relations.

If $\mathcal{K}$ is a class of structures closed under isomorphism, $\mathcal{K}$ is said to have the *strong amalgamation property* (SAP) if, for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ such that $\mathbf{C} \subseteq \mathbf{A}$, $\mathbf{C} \subseteq \mathbf{B}$ and $C = A \cap B$, there is a structure $\mathbf{D} \in \mathcal{K}$ such that $\mathbf{A} \subseteq \mathbf{D}$ and $\mathbf{B} \subseteq \mathbf{D}$. The amalgamation property is important in algebra, model theory and algebraic logic, among other.

Given two binary relations R and S on the same domain, we say that S is *coarser* than R if $R \subseteq S$, more explicitly, if $a \ R \ b$ implies $a \ S \ b$, for all a and b in the domain. If this is the case, we shall also say that R is *finer* than S. Notice that we use the expression "coarser" in the non-strict sense of "coarser than or equal to". A unary operation f is *R-preserving* (respectively, *R-reversing*) if $x \ R \ y$ implies $f(x) \ R \ f(y)$ (respectively, $f(y) \ R \ f(x)$).

Theorem. *Consider the following properties of a binary relation R.*

1. *R is transitive;*
2. *R is reflexive;*
3. *R is symmetric;*
4. *R is antireflexive, that is, $x \ R \ x$ never holds;*
5. *R is antisymmetric.*

Then the following statements hold.

(A) For every pair $P, Q \subseteq \{1, 2, 3, 4, 5\}$, the class $\mathcal{K}_{P,Q}$ of structures with

  (a) a binary relation R satisfying the properties from P and

  (b) a coarser relation S satisfying the properties from Q

  has SAP.

(B) SAP is maintained if we add families of

  (i) unary operations which are both R- and S-preserving;

  (ii) unary operations which are both R- and S-reversing;

(iii) unary operations which are R-preserving;

(iv) unary operations which are R-reversing.

(C) On the other hand, the class of structures with a transitive relation R, a coarser binary relation S and an S-preserving function f has not the amalgamation property (here we do not include the condition that f is R-preserving).

(D) The theory of an antisymmetric relation S with two partial orders $\leq$ and $\leq'$ both finer than S has not the amalgamation property.

As usual [1], under SAP and for finite languages, we get Fraïssé limits for the class of finite structures; moreover, in case (A) the first-order theory of the Fraïssé limit is ω-categorical, has quantifier elimination and is the model-completion of the original theory.

In spite of (D), the theory of a family of *transitive* relations with some given set of comparability conditions has SAP. See [2] for many more theorems, examples and counterexamples.

References

[1] Hodges, W., *Model theory*, Encyclopedia of Mathematics and its Applications **42**, Cambridge University Press, Cambridge, 1993.

[2] Lipparini, P., *The strong amalgamation property into union*, arXiv:2103.00563 (2021).