Duality for metrically complete Abelian ℓ-groups

Marco Abbadini[1]*, Vincenzo Marra[2], and Luca Spada[1]

[1] Dipartimento di Matematica, Università di Salerno
{mabbadini, lspada}@unisa.it
[2] Dipartimento di Matematica, Università di Milano
vincenzo.marra@unimi.it

Various representation theorems concerning the algebra of continuous functions over a compact Hausdorff space were proved at the beginning of the forties. In 1941, Kakutani [3] gave an order-theoretic characterisation of the unital lattice-ordered real Banach lattices of the form $C(X, \mathbb{R})$ (= continuous functions from X into $\mathbb{R}$). In the same year, Yosida showed in the landmark paper [5] that a vector lattice with an order unit is isomorphic to $C(X, \mathbb{R})$ if and only if it is archimedean and metrically complete. Similarly, Stone proved in [4] that an abelian lattice ordered group with a strong order unit (henceforth unital abelian ℓ-group) is isomorphic to $C(X, \mathbb{R})$ if and only if it is divisible, archimedean, and metrically complete. In contrast with Kakutani's result, the metric in the two latter results is not a primitive operator, but it is induced by the order unit. Finally, in 1943, on the way to a representation theorem for complex C^*-algebras, Gelfand and Neumark [1] proved that a complex unital C^*-algebra can be represented as the family of all continuous $\mathbb{C}$-valued functions on a compact Hausdorff space if and only if it is commutative. As in Kakutani's representation result, the metric is a primitive element in the structure of a C^*-algebra. All the aforementioned results extend to dualities with the category of compact Hausdorff spaces and continuous functions among them.

In this work we are concerned with a generalisation of Stone's result to unital abelian ℓ-groups that are not necessarily divisible. In this direction, an important result was proved by Goodearl and Handelman [2, Theorem 5.5]:

Theorem 0.1. *Let X be a compact Hausdorff space and $C(X, \mathbb{R})$ be the set of continuous functions from X into $\mathbb{R}$. For each $x \in X$, let A_x be either $\mathbb{R}$ or $\mathbb{Z}_n$ for some positive integer n. Set*

$$D = \{f \in C(X, \mathbb{R}) \mid f(x) \in A_x \text{ for all } x \in X\},$$

and give to D the structure of a unital abelian ℓ-group inherited from $C(X, \mathbb{R})$. Then D is a metrically complete unital archimedean abelian ℓ-group. Conversely, any metrically complete unital archimedean abelian ℓ-group is isomorphic to one of this form.

The crucial restriction to functions satisfying $f(x) \in A_x$ can be understood as a *labelling* on the space X that must be respected by the continuous functions considered.

If no conditions that relate topology and the labels $(A_x)_{x\in X}$ are imposed, "non-isomorphic" labelled spaces may give rise to isomorphic unital abelian ℓ-groups. This serves as motivation for the definition of an a-normal space in what follows.

For $q \in [0, 1]$, we write $\mathtt{den}(q)$ to denote the denominator of q (in its irreducible form) if $q \in \mathbb{Q}$, and 0 otherwise. Equipping the topological space $[0, 1]$ with the denominator function $\mathtt{den}\colon [0, 1] \to \mathbb{N}$ gives a prime example of what we call an *a-normal space*.

Definition 0.2. An *a-normal space* (for arithmetically-normal space) is a pair (X, ζ) where X is a compact Hausdorff space and $\zeta\colon X \to \mathbb{N}$ is a function with the following properties.

*Presenting author.

1. For every $n \in \mathbb{N}$, the preimage $\zeta^{-1}[\mathtt{div}(n)]$ under ζ of the set $\mathtt{div}(n)$ of divisors of n is closed.
2. For any two distinct points x and y of X, there exist disjoint open neighbourhoods U and V of x and y respectively such that, for every $z \in X \setminus (U \cup V)$, $\zeta(z) = 0$.

The appropriate notion of a morphism from an a-normal space (X, ζ_X) to an a-normal space (Y, ζ_Y) is a continuous function $f\colon X \to Y$ that *decreases denominators*, i.e. such that, for any $x \in X$, $\zeta_Y(f(x))$ divides $\zeta_X(x)$.

Our main result is the following duality.

Theorem 0.3. *The category of metrically complete unital archimedean abelian ℓ-groups with lattice-group homomorphisms preserving the order unit is dually equivalent to the category of a-normal spaces with continuous denominator-decreasing maps.*

References

[1] I. Gelfand and M. Neumark. On the imbedding of normed rings into the ring of operators in Hilbert space. *Rec. Math. [Mat. Sbornik] N.S.*, 12(54):197–213, 1943.

[2] K. R. Goodearl and D. E. Handelman. Metric completions of partially ordered abelian groups. *Indiana Univ. Math. J.*, 29(6):861–895, 1980.

[3] S. Kakutani. Concrete representation of abstract (M)-spaces. (A characterization of the space of continuous functions.). *Ann. of Math. (2)*, 42:994–1024, 1941.

[4] M. H. Stone. Applications of the theory of Boolean rings to general topology. *Trans. Amer. Math. Soc.*, 41(3):375–481, 1937.

[5] K. Yosida. On vector lattice with a unit. *Proc. Imp. Acad. Tokyo*, 17:121–124, 1941.