Homotopy setoids as elementary quotient completion

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Setoids, or Bishop sets, provide a notion of set in constructive mathematics. In *intensional* Martin-Löf intuitionistic type theory [8], they provide a way to reconcile the good computational properties of the system with the desirable extensional constructs [5]. In particular, setoids formally add quotients of equivalence relations to the system. A setoid is a pair (X, R) where X is a closed type and R is a dependent type of the form

$$x_1, x_2 : X \vdash R(x_1, x_2)$$

which is an equivalence relation. In categorical logic, there are various constructions for completing a category with quotients, and setoids appear as an instance of the main construction called the *exact completion*. Given a weakly left exact category (wlex) $\mathscr{C}$, the exact completion $\mathscr{C}_{ex}$ of $\mathscr{C}$ is an *exact* category [2]. Closed types and functions up to functional extensionality form a category **ML** which has finite products and weak pullbacks. Setoids and functions preserving relations form a category denoted with **Std**. It turns out that the category **Std** is equivalent to the exact completion of **ML**.

In order to study the properties of the exact completion $\mathscr{C}_{ex}$, one can verify if the category $\mathscr{C}$ shares a weaker version of these properties. For instance, in [1, Theorem 3.3] and [3, Theorem 3.6] the authors characterize the categories whose exact completion leads to a *local cartesian closed* category (lcc). In [4, Proposition 2.1], the authors characterize the categories whose exact completion is an *extensive* category. These results apply to the category of setoids which is a well-known lcc *pretopos*, see [7]. Hence, we have the following facts.

Facts. *The category* ***Std*** *is an lcc pretopos and* $\boldsymbol{ML}_{ex} \cong \boldsymbol{Std}$.

In this work, we have considered a homotopical version of setoids in view of ideas from the homotopy type theory [9]. We have defined a *homotopy setoids* as a setoid (X, R) such that the base type X is an *h-set* and the equivalence relation R is an *h-proposition.* By definition, h-props and h-sets are types such that the following types are inhabited

$$\mathsf{is\text{-}prop}(R) := \prod_{x,y:R} \mathsf{Id}_R(x, y) \qquad \mathsf{is\text{-}set}(X) := \prod_{x,y:X} \mathsf{is\text{-}prop}(\mathsf{Id}_X(x, y)).$$

Intuitively, h-propositions are types that are empty or contractible and h-sets are types that are *discrete*. This is a natural restriction to consider since the set-based mathematics can be formalized using only these two homotopy levels. We denote with $\mathbf{Std}_0$ the full subcategory of **Std** of h-setoids and with $\mathbf{ML}_0$ the full subcategory of **ML** of h-sets. Our main objective was to prove that $\mathbf{Std}_0$ shares properties similar to **Std**.

Problem. *The category* $\boldsymbol{Std}_0$ *is not exact.*

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A possible solution to this problem is to study homotopy setoids in the more general context of *elementary doctrines*, which were introduced by Maietti and Rosolini in [6]. An elementary doctrine consists of a suitable functor $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{Pos}$ form a category $\mathscr{C}$ with finite products to the category of *partially ordered sets (posets)* and monotone functions. The key feature is that for each object $X \in \mathscr{C}$, there exists an element $\delta_X \in \mathsf{P}(X \times X)$, called the *fibered equality*, which plays the role of the equality predicate.

An example is given by the functor $F^{ML} : \mathbf{ML}^{op} \to \mathsf{Pos}$ which associates to each closed type X, the poset of the types depending on X up to *logical equivalence*. The action of F^{ML} on arrows is given by substitution of terms. Similarly, we can define the functor $F^{ML_0} : \mathbf{ML}_0^{op} \to \mathsf{Pos}$ which sends an h-set X to the poset of the types depending on X which are h-propositions.

If P is an elementary doctrine, it is possible to define P-eq. relations and the corresponding notion of well-behaved quotient. In [6], the authors provide a construction which associates to each elementary doctrine P an elementary doctrine $\overline{\mathsf{P}} : \overline{\mathscr{C}}^{op} \to \mathsf{Pos}$, called the *elementary quotient completion* of P, with well-behaved quotients in a suitable universal way. The following are examples of elementary quotient completion:

1. The exact completion of a category with finite products and weak pullbacks is an instance of the elementary quotient completion for the elementary doctrine of *weak subobjects*.
2. The category $\mathbf{Std}$ is equivalent to the base category $\overline{\mathbf{ML}}$ and the category $\mathbf{Std}_0$ is equivalent to the base category $\overline{\mathbf{ML}_0}$.

Hence, in order to study the properties of $\mathbf{Std}_0$, we have provided a generalization of [1, Theorem 3.3] and [4, Proposition 2.1] to the context of elementary doctrines and elementary quotient completion. We have defined a *relative pretoposes* as an elementary doctrines $\mathsf{P} : \mathscr{C}^{op} \to \mathsf{Pos}$ with well-behaved quotients such that the category $\mathscr{C}$ is extensive, (in this case the category $\mathscr{C}$ is called a pretopos relative to P). We have applied the results to F^{ML_0} and we have obtained the following property for h-setoids.

Theorem. *The category* $\boldsymbol{Std}_0$ *is a lcc pretopos relative to* $\overline{F^{ML_0}}$.

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