# Equality is coalgebraic

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**Abstract**

The aim of the talk is to illustrate a result about the comonadicity of elementary doctrines in categorical logic. I shall discuss the logical significance of this result, and its connections with equality, quotients and the elimination of imaginary elements.

Lawvere's hyperdoctrines [5]mark the beginning of applications of category theory in logic, and they provide a very clear algebraic tool to work with syntactic theories and their extensions in logic. A ***doctrine*** [6] consists of a family of posets indexed on a category with finite products. More precisely, it is a functor $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{Pos}$ into the category $\mathcal{Pos}$ of posets, such that the base category $\mathcal{C}$ has finite products. The controvariant action $P(f)\colon P(Y) \to P(X)$ induced by an arrow $f\colon X \longrightarrow Y$ is called ***reindexing along*** $f$. A (possibly multi-sorted) logical theory $T$ gives rise to a doctrine $\mathsf{P}_T$ as follows. The base category consists contexts and context morphisms, *i.e.* finite lists of sorted variables and finite lists of sorted terms. Composition is given by substitution of terms in terms and product is concatenation of contexts. The poset $\mathsf{P}_T(x_1\colon X_1, \ldots, x_n\colon X_n)$ is the Lindenbaum-Tarksi algebra of formulas in context $(x_1\colon X_1, \ldots, x_n\colon X_n)$. Reindexing along a context morphism $(t_1, \ldots, t_n)$ is given by substitution of terms in formulas:

$$\varphi \in \mathsf{P}_T(y_1\colon Y_1, \ldots, y_n\colon Y_n) \longmapsto \varphi[{}^{t_1}/_{y_1}, \ldots, {}^{t_n}/_{y_n}] \in \mathsf{P}_T(x_1\colon X_1, \ldots, x_m\colon X_m).$$

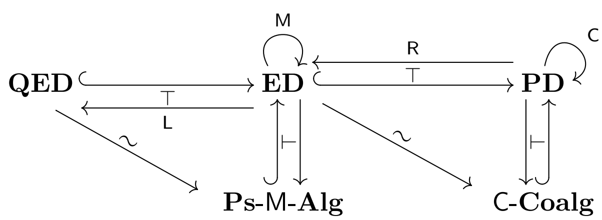
Extending the theory amounts to equip the doctrine with additional structure which, in the spirit of functorial semantics, it is done by requiring certain structural functors to be adjoints. For example, theories with conjunctions correspond to those doctrines, called ***primary***, whose fibres have binary meets which are preserved by reindexing. Adding equality predicates amounts to require that every reindexing along a diagonal $\mathrm{pr}_{1,2,2}\colon Z \times X \longrightarrow Z \times X \times X$ has a left adjoint which satisfies two natural conditions. These are known as ***elementary doctrines***.

Morphisms between doctrines can be understood as interpretations of a theory into another one, and these can be equipped with a notion of transformation too, giving rise to a 2-category **Doc**. Consider for instance the power set doctrine $\mathcal{P}$ on $\mathcal{Set}$, whose fibre over a set $S$ is the poset of subsets of $S$, and reindexing is given by counter-image. Morphisms $\mathsf{P}_T \longrightarrow \mathcal{P}$ are precisely models à la Tarski of $T$ and a model of $T$ will soundly interpret some logical constant if and only if it preserves the relevant structure on $\mathsf{P}_T$ as a morphism. In particular, when $T$ is a theory in classical first order logic, elementary embeddings between models of $T$ appear as those transformations between morphisms $\mathsf{P}_T \longrightarrow \mathcal{P}$ that preserve the first order structure.

The algebraic character of the theory of doctrines makes it a suitable context where to address the question: "What is the theory obtained by (co)freely adding logical structure?" or the closely related question: "How to express additional logical structure in terms of what is already available?". More precisely, in the first case we ask whether a certain forgetful functor is adjoint and, in the second case, whether the adjunction obtained in this way is (co)monadic.

As a case in point, the forgetful functor from from elementary doctrines into primary ones is comonadic, meaning that elementary doctrines are equivalent to coalgebras for a certain

2-comonad C on the 2-category **PD** of primary doctrines [3]. This is depicted in the right-hand triangle in the diagram below. It is also known that elementary doctrines with quotients are (pseudo) monadic over elementary ones [10] and, interestingly enough, the 2-monad M canonically induced by C on the 2-category **ED** of elementary doctrines turns out to be the one presenting elementary doctrines with quotients as (pseudo) algebras. In particular, the functors R, M and L are completely determined by C.




After reviewing background and motivations briefly sketched above, I shall describe the comonadicity result for elementary doctrines and its relevance to logic. In particular, several important constructions in categorical logic can be described using the action of C. These include exact completions [1, 2], the pretopos completion [7] and the tripos-to-topos construction [4]. Thinking in terms of theories, the comonad C produces a theory $\overline{T}$ from a theory $T$ by adding quotients for definable equivalence relations. Furthermore, every model of the original theory $T$ can be turned functorially into a model of $\overline{T}$, and the categorical setting provides a neat description of the induced model $\mathsf{P}_{\overline{T}} \longrightarrow \mathcal{P}$ as a composition of morphisms of doctrines. As one would expect, when $T$ has equality this extension is conservative. As an interesting byproduct, the theory $\overline{T}$ produced by this functor eliminates imaginaries in the sense of Poizat [8], and I shall compare this construction with the one by Shelah [9].

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