Posetal partial applicative structures

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Partial applicative structures (PAS) are very elementary mathematical entities. They just consist of a non-empty set R on which a partial binary operation $\cdot : R \times R \rightharpoonup R$ is defined. Such a structure $\mathcal{R} = (R, \cdot)$ can be seen at least from three different points of view. First, it can be seen simply as an algebraic structure which is a generalization of the algebraic structure called magma (in fact it is its “partial” version). We will call this point of view *algebraic*. Second, such a structure can also be thought of as carrying an abstract notion of computability; indeed, every element r of R represents (by currying) a partial unary function $\{r\} : R \rightharpoonup R$ which can be thought as $\mathcal{R}$-computable. We will call this point of view *computational*. Third, one can consider the elements of R as “realizers” and the subsets of R can be seen as “propositions” which are “realized” by their elements. This point of view gets richer when one allows propositions to depend on arbitrary elements of a set, that is if one considers the sets $\mathcal{P}(R)^I$ where I is a set, and when an adequate notion of “entailment” between “dependent propositions” is introduced, that is $\varphi \vdash_I \psi$ if and only if

$$\bigcap_{i \in I} \{r \in R | \forall a \in \varphi(i)\, (r \cdot a \downarrow \wedge r \cdot a \in \psi(i))\} \neq \emptyset$$

The idea behind this definition is that φ entails ψ if there is a uniform effective way to transform realizers of φ into realizers of ψ. We could call this point of view *logical*. From this logical perspective, every partial applicative structure gives rise to an indexed binary relation $\pi_{\mathcal{R}}$, that is a contravariant functor from the category of sets to the category of sets endowed with binary relations and maps preserving them.

Among the examples of partial applicative structures one subfamily is very well studied, namely the family of partial combinatory algebras (PCA), see e.g. [3]. They are partial applicative structures $\mathcal{R}$ for which there exist $k, s \in R$ such that

1. $k \cdot a \cdot b = a$ for every $a, b \in R$;
2. $s \cdot a \cdot b \downarrow$ for every $a, b \in R$;
3. $s \cdot a \cdot b \cdot c \simeq a \cdot c \cdot (b \cdot c)$ for every $a, b, c \in R$.

In such a case, $\pi_{\mathcal{R}}$ is a very nice functor, since it is in fact a tripos, that is it is a contravariant functor from the category of sets to the category of Heyting prealgebras such that each reindexing map has left and right adjoints satisfying the so-called Beck-Chevalley condition and, moreover, it has a so-called generic element. These properties allow interpreting first-order typed logic in $\pi_{\mathcal{R}}$ and to construct a topos, called realizability topos, by applying the so-called tripos-to-topos construction ([1]) to $\pi_{\mathcal{R}}$.

In the general case of a PAS obviously one does not have all these good properties. This is why it is interesting to look for conditions on $\mathcal{R}$ making $\pi_{\mathcal{R}}$ satisfy specific parts of these good properties, or to investigate properties which do not hold, if not in the trivial case, in the PCA case, but which could still be obtained when the much more general case of a PAS is considered.

In this talk we consider one of these properties, namely we study those PASs giving rise to indexed posets. We will call this PASs *posetal*. Reflexivity and transitivity in each fiber of $\pi_{\mathcal{R}}$ can be easily characterized in terms of the operation $\cdot$, or better from a computational point of view. Reflexivity just amounts to the identity function id_R to be $\mathcal{R}$-computable, while transitivity amounts to ask that if f and g are $\mathcal{R}$-computable partial functions, then there exists an $\mathcal{R}$-computable function h such that $g \circ f \subseteq h$. The antisymmetry of fibers cannot be as simply characterized in terms of $\mathcal{R}$-computable functions. One can easily notice that it is equivalent to ask the fiber over a singleton to be antisymmetric. However, to provide a meaningful characterization in terms of the operation $\cdot$, one needs to introduce two properties:

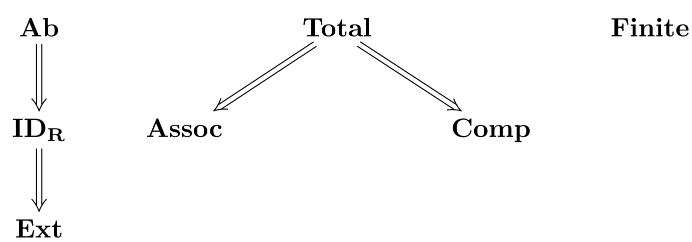
(1W) for every $r, s, a, b \in R$, if $r \cdot a = b$ and $s \cdot b = a$, then $a = b$;

(TL) for every $r, x \in R$, $\{r\}(x) = x$ or there exists $n \in \mathbb{N}^+$ such that $\{r\}^n(x) \not\downarrow$.

We will show that $\mathcal{R}$ is posetal if and only if each fiber of $\pi_{\mathcal{R}}$ is reflexive and transitive and **1W** and **TL** hold.

Although $\mathbf{1W} \wedge \mathbf{TL}$ is equivalent to the antisymmetry of fibers (**Ant**), in presence of reflexivity and transitivity, if we discard these assumptions the equivalence is no longer valid; indeed in the general case no direction of the implication holds. However one can show that in the general case **Ant** implies **1W** and a weaker for of **TL**.

Once one has characterized posetal PASs, it is natural to ask what is the status of standard computational and algebraic properties in the posetal case. Some of them, like the $\mathcal{R}$-computability of constant functions, are simply incompatible with the posetal case, or better they hold only in the trivial case. Some hold in any posetal PAS. Other standard properties are related as represented in the following diagram in the posetal case



where **Ab** and **Assoc** are abelianity and associativity of $\cdot$, $\mathbf{ID_R}$ is the existence of a right identity, **Ext** is extensionality ($\{r\} = \{s\}$ implies $r = s$), **Comp** is the closure under composition of the set of partial functions represented in $\mathcal{R}$, while **Total** and **Finite** are simply the requests that $\cdot$ is total and R is finite, respectively. Adequate examples show that all these implications are strict.

We will also consider a particular class of posetal PASs which are *generated* by a partial function satisfying the property of $\{r\}$ in **TL** and present the relation between the properties above by restricting to this case.

This presentation is based mainly on [2].

References

[1] J. M. E. Hyland, P. T. Johnstone, and A. M. Pitts. Tripos theory. Math. Proc. Cambridge Philos. Soc., 88(2):205–231, 1980.

[2] S. Maschio. On posetal partial applicative structures. Submitted.

[3] J. van Oosten. Realizability: an introduction to its categorical side, volume 152 of Studies in Logic and the Foundations of Mathematics. Elsevier B. V., Amsterdam, 2008.