The algebraic theory of $\mathrm{C}(X)$ and its logic

Luca Reggio

Department of Computer Science, University College London, United Kingdom
l.reggio@ucl.ac.uk

Algebras of continuous functions on compact Hausdorff spaces play a central role in functional analysis, e.g. in the theory of commutative (complex, or real) C*-algebras. In fact, by the Gelfand–Naimark theorem, the category of commutative unital C*-algebras is equivalent to $\mathsf{KH}^{\mathrm{op}}$, the opposite of the category of compact Hausdorff spaces and continuous maps. The aim of this talk is to discuss two distinct, yet related, aspects of the category $\mathsf{KH}^{\mathrm{op}}$.

The first one concerns its *axiomatisability*. It has long been known that $\mathsf{KH}^{\mathrm{op}}$ cannot be axiomatised in various fragments of first-order logic, cf. [2, 1, 9]. Recently, these results were improved to the effect that $\mathsf{KH}^{\mathrm{op}}$ is not axiomatisable in first-order logic—even if infinitary conjunctions and disjunctions are allowed [5].

By contrast, considering all continuous functions between Tychonoff cubes

$$[0,1]^m \to [0,1]^n,$$

with m and n any cardinals, we obtain an algebraic theory in the sense of Lawvere–Linton [4, 6]. Its models are precisely the algebras of the form

$$\mathrm{C}(X) \coloneqq \{f \colon X \to [0,1] \mid f \text{ is continuous}\},$$

for X a compact Hausdorff space. In [3], Isbell proved that, though infinitary, this algebraic theory can be generated using a single operation of countably infinite arity along with finitely many finitary operations.

The problem of axiomatising this algebraic theory was solved in collaboration with Vincenzo Marra in [7], where a *finite* axiomatisation of a (necessarily infinitary) variety V equivalent to $\mathsf{KH}^{\mathrm{op}}$ is provided. I shall outline the main ideas underlying this result; these rely to a large extent on the theory of Chang's MV-algebras, which are to Łukasiewicz many-valued logic as Boolean algebras are to classical propositional logic.

The second aspect pertains to the logic counterpart to the algebraic theory of $\mathrm{C}(X)$. I shall present an infinitary propositional logic $\mathcal{L}$ (given by a Hilbert-style calculus augmented with an infinitary inference rule) that corresponds to the equational consequence relation associated with the variety V, in the sense that a strong completeness theorem holds.

In the same way that, by Stone duality for Boolean algebras, the spaces of models of classical propositional theories are the zero-dimensional compact Hausdorff spaces, the spaces of models of $\mathcal{L}$-theories are precisely the compact Hausdorff spaces.

I shall establish the Beth definability property for $\mathcal{L}$ and show it is equivalent to the Stone–Weierstrass theorem for compact Hausdorff spaces. The second part of this talk is based on [8].

References

[1] B. Banaschewski. More on compact Hausdorff spaces and finitary duality. *Canad. J. Math.*, 36(6):1113–1118, 1984.

[2] P. Bankston. Some obstacles to duality in topological algebra. *Canad. J. Math.*, 34(1):80–90, 1982.

[3] J. Isbell. Generating the algebraic theory of $C(X)$. *Algebra Universalis*, 15(2):153–155, 1982.

[4] F. William Lawvere. Functorial semantics of algebraic theories. *Proc. Nat. Acad. Sci. U.S.A.*, 50:869–872, 1963.

[5] M. Lieberman, J. Rosický, and S. Vasey. Hilbert spaces and C*-algebras are not finitely concrete. Preprint available at arXiv:1908.10200, 2019.

[6] F. E. J. Linton. Some aspects of equational categories. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 84–94. Springer, New York, 1966.

[7] V. Marra and L. Reggio. Stone duality above dimension zero: axiomatising the algebraic theory of C(X). *Adv. Math.*, 307:253–287, 2017.

[8] L. Reggio. Beth definability and the Stone–Weierstrass Theorem. *Ann. Pure Appl. Logic*, 172(8):102990, 2021.

[9] J. Rosický. Elementary categories. *Arch. Math. (Basel)*, 52(3):284–288, 1989.