A computable analog of the theory of Borel equivalence relations

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This is joint work with Uri Andrews.

The study of the complexity of equivalence relations has been a major thread of research in diverse areas of logic. A reduction of an equivalence relation E on a domain X to an equivalence relation F on a domain Y is a function $f : X \to Y$ which induces an injection on the quotient sets, $X/_E \to Y/_F$. In the literature, there are two main definitions for this reducibility:

- In descriptive set theory, *Borel reducibility* is defined by assuming that X and Y are Polish spaces and f is Borel;
- In computability theory, *computable reducibility* is defined by assuming that X and Y coincide with the set ω of natural numbers and f is computable.

Despite the clear analogy between the two notions, for a long time the study of Borel and computable reducibility were conducted independently. Yet, a theory of computable reductions which blends ideas from both computability theory and descriptive set theory is rapidly emerging. In this talk, we will discuss differences and similarities between the Borel and the computable settings as we provide computable, or computably enumerable, analogs of fundamental concepts from the Borel theory. In particular, we concentrate on natural effectivizations of two classic concepts: orbit equivalence relations and the Friedman-Stanley jump.

Orbit equivalence relations

Feldman and Moore [4] proved that every countable Borel equivalence relation arises as the orbit equivalence relation of a Borel action of a countable group. The proof relies on Luzin-Novikov Uniformization, which ensures that every countable Borel equivalence relation has a uniform Borel enumeration of each class. Coskey, Hamkins, and Miller [3] proposed to effectivize this study by focusing on computable groups G acting on the c.e. subsets of ω. The resulting orbit relations are denoted by E_G^{ce}. In [3], it is shown that there is no computable analog of the Feldman-Moore theorem. Here, we expand this investigation by proving, e.g., that the universal element for the class of the E_G^{ce}'s admits many natural realizations (up to computable reducibility).

The computable Friedman-Stanley jump

Clemens, Coskey, and Krakoff [2] recently proposed the following computable analog of the Friedman-Stanley jump studied in descriptive set theory for gauging the complexity of Borel isomorphism relations [5]. For any equivalence relation E on the natural numbers, E^+ is given by

$$x \, E^+ \, y \Leftrightarrow [W_x]_E = [W_y]_E.$$

Here, we present many new results about this jump [1]. In particular, we show that the computable Friedman-Stanley jump gives benchmark equivalence relations going up the hyperarithmetic hierarchy. That is, for every hyperarithmetic equivalence relation E, there is a notation a for a computable ordinal so that E is computably reducible to Id^{+a}, where Id denotes the identity on the natural numbers.

References

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