Quantale Modules,
with Applications to
Logic and Image Processing

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Ph.D. Thesis

Abstract

The approach to deductive systems by means of complete residuated lattices and complete posets has been proposed by Nikolaos Galatos and Constantine Tsinakis in [4], and we revisit it in terms of quantales and quantale modules, adding some new contributions. The results shown in [4], recalled in Chapters 1 and 5, together with our contributions presented in Chapter 5, are unquestionably promising and — in our opinion — open a new and fruitful perspective on Mathematical Logic, besides proving, once more, its strong relationship with Algebra.

Briefly, in [4] the authors prove that equivalences and similarities between deductive systems, over a propositional language, can be treated with categorical and algebraic tools, by representing each deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash \rangle$ as a pair $\langle \mathbf{P}, \gamma_\vdash \rangle$, where $\mathbf{P}$ is a complete poset over which an action from a complete residuated lattice is defined, and $\gamma_\vdash$ is a *structural closure operator*, i.e. a closure operator that is invariant, in a precise sense, under the action of the residuated lattice.

The second motivation that stimulated our investigation on quantale modules comes from the area of Image Processing. Indeed, in the literature of Image Processing, several suitable representations of digital images as $[0, 1]$-valued maps are proposed. Such representations are the starting point for defining both compression and reconstruction algorithms based on fuzzy set theory, also called *fuzzy algorithms*, and mathematical morphological operators, used for shape analysis in digital images. Most of the fuzzy algorithms use a suitable pair of operators, one for compressing the image and the other one for approximating the original image starting from the compressed one; see, for instance, [1, 5, 6]. The idea is similar to that of the so-called "integral transforms" in Mathematical Analysis: every map can be discretized by means of a *direct transform* and then approximated through the application of a suitable *inverse transform*. Moreover, as well as a direct integral transform is defined as an integral-product composition, a fuzzy compression operator is defined as a join-product composition (where the product is actually a left-continuous triangular norm), and its inverse operator has the form of a meet-division composition, where the division is the residual operation of the same triangular norm. In mathematical morphology, the operators of *dilation* and *erosion* — whose action on an image can easily be guessed by their names — that are translation invariant can be expressed, again, as compositions join-product and meet-residuum respectively. All these methods can be placed under a common roof by essentially abstracting their common properties. Indeed they are all examples of $\mathcal{Q}$-*module transforms*, that we define in Chapter 4 and that turn out to be precisely the homomorphisms between free $\mathcal{Q}$-modules.

All these considerations show that a deep categorical and algebraic study of quantale modules, will probably push progresses in the fields of Logic and Image Processing and the aim of this thesis is right to study quantale modules keeping their possible applications as a constant inspiration and a further intention. Our

hope is also to give impulse to an extensive study of quantale modules, by showing a glimpse of their great potential.

Structure of the Work

The thesis is organized in three distinct parts and seven chapters.

Part I contains most of the preliminary notions and results, and is divided in three chapters.

Chapter 1. We recall some definitions and results regarding algebraizable logics and deductive systems, useful for motivating the study of quantale modules and necessary for the comprehension of the results of Logic that follow as applications of the properties of quantale modules.

Chapter 2. This chapter is dedicated to a brief overview of the categorical notions involved in the theory developed.

Chapter 3. We show some results on residuated maps, sup-lattices and quantales. A quantale module can be thought of as an object similar to a ring module, where we have a quantale instead of a ring and a sup-lattice instead of an Abelian group. Then, in order to make the thesis as self-contained as possible, it is necessary to include some preliminary notions and results, also considering that such structures may not be as familiar to the reader as rings and groups.

Part II can be considered as the main (theoretical) part, and is divided in two chapters.

Chapter 4. This chapter contains all the main results and constructions on quantale modules. In Sections 4.1 and 4.2 we present the categories of quantale modules and start establishing the first results. So we define objects, morphisms, subobjects, free objects, hom-sets, and we show that the product and the coproduct of a family of quantale modules are both the Cartesian product equipped with coordinatewise defined operations and with, respectively, canonical projections and their left adjoints as the associated families of morphisms. Sections 4.3 and 4.4 are devoted to the study of two classes of operators, that we already mentioned, on quantale modules: the *structural closure operators* (also called *nuclei*) and the *transforms*. The importance of such operators for applications has been already underlined, but it will be clear soon also their centrality for the theory of quantale modules. Indeed, among other things, we show that each transform is a $\mathcal{Q}$-module homomorphism of free modules and vice versa. On the other hand, $\mathcal{Q}$-module nuclei are strongly connected with homomorphisms as well; we can say that, somehow, $\mathcal{Q}$-module morphisms, transforms and nuclei are three different points of view of the same concept.

Projective (and injective) quantale modules are investigated in Section 4.5. Apart from free modules, that are obviously projective, we show a characterization of projective cyclic modules and we prove that the product of projective (respectively, injective) objects is projective (resp., injective); several results of this section are due to N. Galatos and C. Tsinakis. In Section 4.6, we prove that the categories of quantale modules have the strong amalgamation property while, in Section 4.7, we show the existence of tensor products of quantale modules. Their construction and properties are similar to the analogues for ring modules, and this analogy include also the use of tensor products for extending the set of scalars of a module. Also in the case of tensor products, like for projectivity and injectivity, we can apply our results to Logic. Moreover, we show that any module obtained by extending the quantale of scalars of a coproduct of cyclic projective modules, is the coproduct of cyclic projectives itself, thus projective.

Chapter 5. The exposition of how quantale modules are connected to Logic is the content of this chapter. At the beginning, we abstract the definitions of consequence relations and deductive systems in the algebraic frameworks of sup-lattices and quantale modules. This approach follows, even if from a slightly different point of view, the aforementioned work by N. Galatos and C. Tsinakis. The main novelty, here, consists of the extension of this approach to the comparison of deductive systems defined on different languages.

Part III contains the applications of quantale modules — and especially of $\mathcal{Q}$-module transforms — to Image Processing; it is composed of two chapters.

Chapter 6. After a brief overview on the literature on fuzzy image compression and mathematical morphology, we show how parts of these areas fall within the formal theory we have established in Part II.

Chapter 7. We present an example of $\mathcal{Q}$-module transform together with a concrete application of it. The results of the application have been compared with those obtained by using JPEG, the most famous algorithm for image compression. The operator shown, called Łukasiewicz transform, is defined between free modules on the quantale reduct of the MV-algebra $[0,1]$; the algorithm based on it is called, not surprisingly, ŁTB — "Łukasiewicz Transform Based" (see [1, 2]).

References

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