Università degli Studi di Siena

DIPARTIMENTO DI INGEGNERIA DELL'INFORMAZIONE E SCIENZE MATEMATICHE
Corso di Laurea Magistrale in Matematica

Conuclear images of substructural logics

Candidato:
Giulia Frosoni

Relatore:
Prof. Franco Montagna

Anno Accademico 2013-2014

Contents

Introduction

Substructural logics are logics lacking some of the structural rules (exchange, contraction and weakening) when they are formalized in sequent systems. The lack of some of the structural rules allows substructural logics to express many concepts of the natural language that classical logic is not able to express. For example, while in classical logic we have only one conjunction, which is idempotent and whose algebraic counterpart is the lattice operation of meet, in substructural logics which do not have the contraction rule, we can define two different kinds of conjunction, one additive and one multiplicative, that reflect the meanings of conjunctions in the natural language. In terms of resources, these two conjunctions, $\wedge$ and $\cdot$, can be described in the following way: $A \wedge B$ means that I can choose A or B, but not both of them at the same time, whereas $A \cdot B$ means that I can have both A and B simultaneously. Moreover, the material implication of classical logic is far from expressing the implication of the natural language, which is characterized by a relationship cause-effect between antecedent and succedent. Also in this case, using substructural logics, we gain expressivity. Indeed there exists a substructural logic, relevance logic, where it is possible to define an implication which can be valid if and only if there is a particular logical relationship between antecedent and succedent, as happens in the natural language.

Substructural logics involve many kinds of non-classical logics such as linear logic, Łukasiewicz's many-valued logic, BCK-logic, etc... Actually, these logics were created and studied independently, mainly because of the different motivations that had led to their creation, and only later they were considered as special cases of the same concept. In this way, substructural logics aim at providing a uniform framework for very different kinds of non-classical logics.

In the thesis we present a particular construction on substructural logics. The idea comes from a famous result of McKinsey and Tarski ([16] and [17]) that provides an interpretation of intuitionistic logic into the modal logic S4. The algebraic counterpart of intuitionistic logic are Heyting algebras, while the algebraic counterpart of S4 are interior algebras, i.e., Boolean algebras endowed with an interior operator. Hence, from an algebraic point of view, McKinsey and Tarski prove a categorical equivalence between Heyting algebras and Boolean algebras with an interior operator whose image generates the Boolean algebra. In particular, the result presented above states that the image of the variety of Boolean algebras under an interior operator is the variety of Heyting algebras.

One might wonder whether it is possible to perform a similar construction starting from a logic which is different from classical logic, for instance, starting from a logic extending Full Lambek Calculus **FL**, namely a substructural logic.

In this way, it is fundamental the fact that **FL** is algebraizable and its algebraic counterpart is the variety of (pointed) residuated lattices. Furthermore, the natural generalization of an interior operator is the concept of conucleus. Therefore, in the thesis, we deal with conuclear images of substructural logics or, equivalently, conuclear images of varieties of (pointed) residuated lattices.

A conucleus σ on a residuated lattice $\mathbf{R}$ is an interior operator, namely it is contracting, idempotent and monotone, and, in addition, it satisfies the following properties: $\sigma(x) \cdot \sigma(y) \leq \sigma(x \cdot y)$ and $\sigma(1) = 1$.

The algebraic translation of an interior operator on a Boolean algebra is a special example of conucleus (here, $\cdot$ and $\wedge$ coincide). This suggests the following generalization of McKinsey and Tarski's interpretation: given a substructural logic L, we denote by L_σ the logic L added with a unary operator σ along with the axioms: [1]

1. $(\sigma(A) \cdot \sigma(B)) \to \sigma(A \cdot B)$;

2. $\sigma(A) \to A$;

[1] If L is commutative, the left and the right implications coincide and we denote both by $\to$.

3. $\sigma(A) \to \sigma(\sigma(A))$;

and the necessitation rule $\frac{A}{\sigma(A)}$.

Then we can define an interpretation $^\sigma$ as follows: $p_i^\sigma = \sigma(p_i)$ $(i = 1, \ldots, n, \ldots)$, $0^\sigma = \sigma(0)$, $1^\sigma = 1$, $(A \cdot B)^\sigma = A^\sigma \cdot B^\sigma$, $(A \vee B)^\sigma = A^\sigma \vee B^\sigma$, $(A \wedge B)^\sigma = \sigma(A^\sigma \wedge B^\sigma)$, $(A \backslash B)^\sigma = \sigma(A^\sigma \backslash B^\sigma)$ and $(A/B)^\sigma = \sigma(A^\sigma / B^\sigma)$.

The *conuclear image* of L is the logic $\sigma(\mathrm{L})$ whose theorems are precisely those formulas A such that $\mathrm{L}_\sigma \vdash A^\sigma$. Interestingly, $\sigma(\mathrm{L})$ is always a substructural logic (i.e., an extension of **FL**) and L is an extension of $\sigma(\mathrm{L})$. Moreover, the map $\mathrm{L} \mapsto \sigma(\mathrm{L})$ is an interior operator on the class of all substructural logics (thought of as sets of formulas closed under deduction and under substitutions).

For instance, if L is classical logic, $\sigma(\mathrm{L})$ is intuitionistic logic, a logic which is weaker than classical logic. As another example, by a result proved in [18], the logic of commutative and cancellative residuated lattices is the conuclear image of the logic of abelian lattice-ordered groups (both logics with 0 identified with 1). The main problem investigated in the thesis is: *What is the relationship between L and $\sigma(L)$?* In particular:

1. Which properties are excluded to hold in $\sigma(\mathrm{L})$, whatever L is?
2. Which properties may be valid in $\sigma(\mathrm{L})$ for some particular logic L but are not necessarily preserved under conuclear images?
3. Which theorems of L are preserved by the map $\mathrm{L} \mapsto \sigma(\mathrm{L})$?

As regards to (1), one of the most important results provided in the thesis is that the conuclear image of *any* substructural logic has the disjunction property. Therefore, this construction allows us to pass from a (possibly non-constructive) substructural logic to a constructive one. In view of this result, properties like excluded middle, prelinearity, weak excluded middle, etc., can never hold in a conuclear image. Moreover, by [12], the conuclear image $\sigma(\mathrm{L})$ of any substructural logic L, as well as its conuclear extension L_σ, is PSPACE-hard.

There are also other properties, like the double negation axiom DN (involutiveness of negation), which we have proved to be excluded to hold in the conuclear image of any substructural logic. Since **FL** plus DN has the disjunction property ([21]), it follows that not all substructural logics with the disjunction property are conuclear images of a substructural logic. Being a conuclear image seems to be a stronger and more constructive property than the disjunction property.

As regards to (2), interesting examples of axioms which are compatible with conuclear images but are not necessarily preserved are constituted by the axiom of divisibility and by the axiom of distributivity.

Finally, as expressed in (3), we investigate the problem of preservation under conuclear images. Our aim is to characterize substructural logics which are closed under conuclear images. In the thesis, we prove that all formulas corresponding to inequations of the form $f \leq g$ with $f \in P_2$ and $g \in N_2$ are preserved from L to σ(L). Hence, every substructural logic which is axiomatized over **FL** by an equation of this form coincides with its conuclear image.

The classes P_2 and N_2, and more in general the classes P_n and N_n, form the so-called *substructural hierarchy*, a classification of logical formulas introduced in [4] in order to find an algebraic characterization of cut elimination. It is surprising that the same classes that provide an algebraic characterization of cut elimination, are useful for inspecting properties preserved under conuclear images.

Moreover, we generalize the above result in such a way to include also preservation of cancellativity equation; we introduce new classes P_2^* and N_2^*, wider than the respective classes in the substructural hierarchy.

The provided condition is only sufficient for presevation under conuclear images but many counterexamples prove that this condition cannot be significantly generalized; indeed even slight generalizations of our classes fail to be preserved under conuclear images.

The thesis is organized as follows: in the first chapter we briefly present residuated lattices and conuclei. We cite the most important results already known and we give some examples of conuclei and conuclear images. In this way, we provide the fundamental algebraic notions that will be useful for the comprehension of the following chapters.

In the second chapter we introduce substructural logics and their sequent calculus **FL**. We analyse the role of structural rules and finally we present the construction that leads to the creation of the conuclear image of a substructural logic.

The last three chapters are devoted to the original part of the thesis. In the third chapter we discuss the disjunction property, we prove that it always holds in any conuclear image and we observe that it is helpful to outline properties that never hold in a conuclear image.

In the fourth chapter we analyse examples of properties compatible with conuclear images but that may be not preserved under conuclear images. In order to prove this, we use a result of [18].

In the fifth chapter we face up to the problem of preservation under conuclear images. After considering specific properties, we deal with the general case and we provide a sufficient condition in order that an inequation is preserved under conuclear images, as explained above. In this way, a wide class of substructural logics is proved to be closed under conuclear images.

Finally we suggest interesting topics whose investigation allows to continue and deepen the analysis started in the thesis.

Acknowledgements

I would like to express my deep gratitude to Professor Franco Montagna, for his patient guidance and valuable help in this work. I would also like to extend my thanks to Professor Aldo Ursini for his expert and helpful suggestions.

Chapter 1

Residuated lattices and conuclei

In this chapter we introduce the preliminary concepts necessary for the study of residuated lattices and their conuclear images. For a more detailed treatment about residuated lattices, we recommend the surveys [8] and [15].

1.1 Residuated lattices

A structure $\mathbf{G} = \langle G, \cdot, \leq \rangle$ is a *partially ordered groupoid* (or *pogroupoid* for short) if $\cdot$ is a binary operation on a poset $\langle G, \leq \rangle$ that is monotone, namely, if $x \leq x'$ and $y \leq y'$ then $x \cdot y \leq x' \cdot y'$.

A *residuated pogroupoid* is a structure $\mathbf{G} = \langle G, \cdot, \backslash, /, \leq \rangle$ such that $\langle G, \leq \rangle$ is a poset and the law of residuation holds, i.e., for all $x, y, z \in G$

$$x \cdot y \leq z \text{ iff } y \leq x \backslash z \text{ iff } x \leq z/y.$$

Note that in this case $\langle G, \cdot, \leq \rangle$ is a pogroupoid.

A *residuated pomonoid* is an algebra $\mathbf{G} = \langle G, \cdot, \backslash, /, 1, \leq \rangle$ such that $\langle G, \cdot, 1 \rangle$ is a monoid and $\langle G, \cdot, \backslash, /, \leq \rangle$ is a residuated pogroupoid.

A *residuated lattice ordered groupoid* (or *residuated ℓ-groupoid*) is an algebra $\mathbf{G} = \langle G, \wedge, \vee, \cdot, \backslash, / \rangle$ such that $\langle G, \wedge, \vee \rangle$ is a lattice (with associate order $\leq$) and $\langle G, \cdot, \backslash, /, \leq \rangle$ is a residuated pogroupoid.

Now we can give the definition of the algebraic structure which we will use for all the paper:

Definition 1.1. *A* residuated lattice *is an algebra* $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ *such that* $\langle A, \wedge, \vee\rangle$ *is a lattice,* $\langle A, \cdot, 1\rangle$ *is a monoid and the law of residuation holds, i.e., for all* $x, y, z \in A$

$$x \cdot y \leq z \text{ iff } y \leq x\backslash z \text{ iff } x \leq z/y.$$

An *FL-algebra* $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ is a residuated lattice, endowed with an additional constant 0, which is interpreted into an arbitrary element of A. Consequently, residuated lattices are the 0-free reducts of FL-algebras. The maps $\backslash$ and $/$ are called *left* and *right residual*. We read $x\backslash y$ as "x under y" and y/x as "y over x"; in both these expressions y is said to be the numerator and x is said to be the denominator. We adopt the usual convention of writing xy instead of $x \cdot y$ and, if there are not parentheses, we assume that multiplication is performed first, followed by the division operations and the lattice operations. Moreover, any statement about residuated lattices has a "mirror image" obtained by reading terms backwards, namely replacing $x \cdot y$ by $y \cdot x$ and $x\backslash y$ by y/x. In a residuated lattice a statement holds iff its mirror image holds, so we will use only one form.

Theorem 1.2. *The following identities hold in all residuated lattices and FL-algebras:*

1. $x(y \vee z) = xy \vee xz$ *and* $(y \vee z)x = yx \vee zx$;
2. $x\backslash(y \wedge z) = (x\backslash y) \wedge (x\backslash z)$;
3. $x/(y \vee z) = (x/y) \wedge (x/z)$;
4. $(x/y)y \leq x$;
5. $x(y/z) \leq (xy)/z$;
6. $(x/y)/z = x/(zy)$;
7. $x/y \leq xz/yz$;
8. $(x/y)(y/z) \leq x/z$;

9. $x\backslash(y/z) = (x\backslash y)/z$;

10. $(1/x)(1/y) \leq 1/yx$;

11. $(x/x)x = x$;

12. $(x/x)^2 = x/x$.

The following theorem proves that the class of all residuated lattices and the class of all FL-algebras are varieties, which we will denote by RL and FL.

Theorem 1.3 ([8]). *An algebra (of the appropriate type) is a residuated lattice or an FL-algebra if and only if it satisfies the equations defining lattices, the equations defining monoids, and the following four equations:*

1. $x(x\backslash z \wedge y) \leq z$;

2. $(y \wedge z/x)x \leq z$;

3. $y \leq x\backslash(xy \vee z)$;

4. $y \leq (z \vee yx)/x$.

Thus, RL and FL are equational classes.

Theorem 1.4. *Let $\mathbf{A}$ be a residuated lattice.*

1. *Multiplication preserves all existing joins in both arguments, i.e., if $\bigvee X$ and $\bigvee Y$ exist for $X, Y \subseteq A$, then so does $\bigvee_{x\in X, y\in Y} xy$, and*

$$\bigvee X \cdot \bigvee Y = \bigvee_{x\in X, y\in Y} xy.$$

2. *Divisions preserve all existing meets in the numerator and convert all existing joins in the denominator to meets, i.e., if $\bigvee X$ and $\bigwedge Y$ exist for $X, Y \subseteq A$, then $\bigwedge_{x\in X} z/x$ and $\bigwedge_{y\in Y} y/z$ exist for any $z \in A$, and*

$$z/\bigvee X = \bigwedge_{x\in X} z/x \text{ and } \bigwedge Y/z = \bigwedge_{y\in Y} y/z.$$

3. $x\backslash z = max\{y \in A : xy \leq z\}$ *and* $z/y = max\{x \in A : xy \leq z\}$.

A residuated lattice $\mathbf{A}$ is *commutative* if the monoid operation $\cdot$ is commutative, that is $x \cdot y = y \cdot x$ for all $x, y \in A$. In this case, we have $x \leq z/y$ iff $xy \leq z$ iff $yx \leq z$ iff $x \leq y\backslash z$. Thus, $z\backslash y = y/z$ for all $y, z \in A$; this means that the two residuals collapse to one operation which we denote by $z \to y$. Therefore, a commutative residuated lattice can be expressed as an algebra of the type $\langle A, \wedge, \vee, \cdot, \to, 1\rangle$. Commutative residuated lattices (FL-algebras respectively) form a variety which we denote by CRL (CFL respectively). For a deep investigation of commutative residuated lattices, we advise [11].

A residuated lattice $\mathbf{A}$ is *integral* if it has a greatest element and this is the multiplicative unit 1, i.e., $x \leq 1$ for all $x \in A$. We denote this subvariety of RL by IRL.

We say that a residuated lattice $\mathbf{A}$ is *contractive* if $x \leq x \cdot x$ for all $x \in A$. The corresponding variety is denoted by KRL.

A residuated lattice (or an FL-algebra) is said to be *bounded* if it has a greatest element $\top$ and a smallest element $\bot$. A *zero-bounded* FL-algebra is an FL-algebra that satisfies the identity $0 \leq x$.

The constant 0 in FL-algebras allows us to define two negation operations: $\sim x = x\backslash 0$ and $-x = 0/x$. Obviously, in commutative FL-algebras we have only one negation operation that we indicate with $\neg x = x\backslash 0 = 0/x$. An FL-algebra is called *left involutive* (*right involutive*), if it satisfies the identity $- \sim x = x$ ($\sim -x = x$, respectively). It is called *involutive* if it is both left and right involutive; it is called *cyclic* if it satisfies $\sim x = -x$.

Residuated lattices or FL-algebras are *distributive* if their lattice reducts are distributive, i.e., they satisfy the identity $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$. The corresponding varieties are denoted by DRL and DFL.

A residuated lattice is called *cancellative* if multiplication is cancellative, that is $xz = yz$ implies $x = y$ and $zx = zy$ implies $x = y$. The following lemma is taken from [2]:

Lemma 1.5. *A residuated lattice is cancellative iff it satisfies the identities*

$$xy/y = x = y\backslash yx.$$

Proof. Suppose that $\mathbf{A}$ is a cancellative residuated lattice. Since $xy/y \leq xy/y$, by the residuation law, we have $(xy/y)y \leq xy$. Moreover, in a similar way, $xy \leq xy$ implies $x \leq xy/y$, thus $xy \leq (xy/y)y$. Therefore $xy = (xy/y)y$ and the claim follows from right cancellativity. Vice versa, suppose that $xy/y = x$ holds and $ac = bc$, where $a, b, c \in A$. Then $a = ac/c = bc/c = b$ and right cancellativity is proved. The proof for left cancellativity is similar to the previous one. □

Therefore, being axiomatized by the equation $xy/y = x = y\backslash yx$, cancellative residuated lattices form a variety which we denote by $CanRL$.

We say that a residuated lattice $\mathbf{A}$ is *divisible* if, for all $x, y \in A$,

$$\text{if } x \leq y \text{ then there exist } z, u \in A \text{ such that } zy = yu = x. \tag{1.1}$$

We can easily prove that this condition is equivalent to the identity

$$x(x\backslash(x \wedge y)) = x \wedge y = ((x \wedge y)/x)x. \tag{1.2}$$

Indeed suppose that a residuated lattice $\mathbf{A}$ satisfies (1.1). Then, since $x \wedge y \leq x$, there exists u such that $x \cdot u = x \wedge y$. Therefore, from $x \cdot u \leq x \wedge y$, follows, by the residuation law, that $u \leq x\backslash(x \wedge y)$. Thus $x \wedge y = x \cdot u \leq x \cdot (x\backslash(x \wedge y))$. Moreover, since $x\backslash(x \wedge y) \leq x\backslash(x \wedge y)$, we have, by the residuation law, that $x(x\backslash(x \wedge y)) \leq x \wedge y$. In conclusion, $\mathbf{A}$ satisfies (1.2). Vice versa suppose (1.2) and take $x \leq y$. Then $x \wedge y = x$. Hence $y \cdot (y\backslash(x \wedge y)) = x \wedge y = x$ and $u = y\backslash(x \wedge y)$.

Furthermore, if the residuated lattice is integral, the condition of divisibility is equivalent to the identity

$$x(x\backslash y) = x \wedge y = (y/x)x. \tag{1.3}$$

In order to verify the equivalence, suppose that an integral residuated lattice satisfies (1.1). Since $a \wedge b \leq b$, by (1.1), there exist $\bar{z}$ and $\bar{u}$ such that $b\bar{u} =$

$\bar{z}b = a \wedge b$. Then, since $b\bar{u} = a \wedge b \leq a$, by the residuation law, $\bar{u} \leq b\backslash a$. Therefore $a \wedge b = b\bar{u} \leq b(b\backslash a)$. On the other hand, $b\backslash a \leq b\backslash a$ implies $b(b\backslash a) \leq a$. Furthermore, due to the integrality, $b(b\backslash a) \leq b \cdot 1 = b$ and so $b(b\backslash a) \leq a \wedge b$. Therefore we have proved that (1.3) holds. Vice versa, suppose that a residuated lattice satisfies (1.3) and let $a \leq b$. Then, using (1.3), $(a/b)b = a \wedge b = a = b(b\backslash a)$. Therefore, taking $u = b\backslash a$ and $z = a/b$, (1.1) holds.

A residuated lattice is said to be *prelinear* if it satisfies the prelinearity identity:

$$x\backslash y \vee y\backslash x \geq 1.$$

We now give some famous examples of residuated lattices.

A *Heyting algebra* $\mathbf{A}$ is an algebra $\langle A, \wedge, \vee, \to, 0\rangle$ such that $\langle A, \wedge, \vee, 0\rangle$ is a lattice with a minimum 0 and the law of $\wedge$-residuation holds, i.e., for all $x, y, z \in A$

$$x \wedge y \leq z \text{ iff } y \leq x \to z.$$

Therefore the variety HA of Heyting algebras is term-equivalent to the subvariety of CFL corresponding to the additional equations $xy = x \wedge y$ and $x \wedge 0 = 0$.

The element $x \to z$ is called $\wedge$-*residual* of z by x or the *pseudocomplement* of x relative to z. It is easy to see that each Heyting algebra is bounded; the minimum is 0, while $0 \to 0$, denoted by 1, is the greatest element, due to the residuation law.

Furthermore, every Heyting algebra has a distributive lattice reduct (for a proof of this result, see [8]).

Set $\neg x = x \to 0$; thus, we can observe that in a Heyting algebra the following equations hold:

$$x \wedge \neg x = 0 \text{ and } \neg(x \vee y) = \neg x \wedge \neg y.$$

A *Boolean algebra* is a Heyting algebra that also satisfies the identity

$$x \vee \neg x = 1.$$

The variety of Boolean algebras, denoted by BA, is term-equivalent to a subvariety of CFL defined by the additional equations $xy = x \wedge y$, $x \wedge 0 = 0$ and $(x \to y) \to y = x \vee y$.

A *lattice-ordered group* (or *ℓ-group* for short) is an algebra $\langle G, \wedge, \vee, \cdot, ^{-1}, 1\rangle$ such that $\langle G, \wedge, \vee\rangle$ is a lattice, $\langle G, \cdot, ^{-1}, 1\rangle$ is a group and $\cdot$ is order-preserving in both arguments. ℓ-groups form a subvariety LG of the variety of residuated lattices: indeed it is sufficient to define $x\backslash y = x^{-1}y$ and $y/x = yx^{-1}$.

Theorem 1.6. *Each of the following sets of equations forms an equational basis for LG.*

1. $(1/x)x = 1$;
2. $x = 1/(x\backslash 1)$ *and* $x/x = 1$;
3. $x = y/(x\backslash y)$;
4. $x/(y\backslash 1) = xy$ *and* $1/x = x\backslash 1$;
5. $(y/x)x = y$;
6. $x/(y\backslash z) = (z/x)\backslash y$.

An ℓ-group is said to be *abelian* if the multiplication is commutative.

An *MV-algebra* is an algebra $\mathbf{A} = \langle A, \oplus, \neg, 0\rangle$ that satisfies the following identities:

- $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- $x \oplus y = y \oplus x$;
- $x \oplus 0 = x$;
- $\neg\neg x = x$;
- $x \oplus \neg 0 = \neg 0$;
- $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

The variety of MV-algebras, denoted by MV, is term-equivalent to the subvariety of CFL with the additional equations $x \vee y = (x \to y) \to y$ and $x \wedge 0 = 0$.

1.2 Modal operators on residuated lattices: nuclei and conuclei

In this section we introduce two kinds of modal operators on residuated lattices: nuclei and conuclei. For the rest of the paper, we will focus on the concept of conucleus and conuclear image of a residuated lattice.

Definition 1.7. *A* closure operator *on a poset* $\mathbf{P} = \langle P, \leq \rangle$ *is a map* $\gamma : P \to P$ *which is expanding, monotone and idempotent, i.e., for all* $x, y \in P$, $x \leq \gamma(x)$, *if* $x \leq y$ *then* $\gamma(x) \leq \gamma(y)$, *and* $\gamma(\gamma(x)) = \gamma(x)$.

Definition 1.8. *A closure operator* γ *on a residuated lattice* $\mathbf{A}$ *is a* nucleus *if it satisfies the identity*

$$\gamma(x)\gamma(y) \leq \gamma(xy)$$

Note that, if γ is a closure operator, the condition $\gamma(x)\gamma(y) \leq \gamma(xy)$ is equivalent to the condition

$$\gamma(\gamma(x)\gamma(y)) = \gamma(xy).$$

In fact: if for all $x, y \in A$ $\gamma(x)\gamma(y) \leq \gamma(xy)$, applying γ to both these members, we obtain $\gamma(\gamma(x)\gamma(y)) \leq \gamma(\gamma(xy)) = \gamma(xy)$. Moreover, since γ is expanding, $xy \leq \gamma(x)\gamma(y)$ and, applying γ to the both members, $\gamma(xy) \leq \gamma(\gamma(x)\gamma(y))$. Vice versa, if $\gamma(\gamma(x)\gamma(y)) = \gamma(xy)$, then, since γ is expanding, $\gamma(x)\gamma(y) \leq \gamma(\gamma(x)\gamma(y)) = \gamma(xy)$.

The following theorem is taken from [18]:

Theorem 1.9. *If* $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ *is a residuated lattice and* γ *is a nucleus on it, then the algebra* $\gamma(\mathbf{A}) = \langle \gamma(A), \wedge, \vee_\gamma, \cdot_\gamma, \backslash, /, \gamma(1)\rangle$ *is a residuated lattice, where* $\wedge, \backslash$ *and* $/$ *are the same as in* $\mathbf{A}$ *and for all* $x, y \in \gamma(A)$, $x \vee_\gamma y = \gamma(x \vee y)$ *and* $x \cdot_\gamma y = \gamma(x \cdot y)$.

Proof. First, we prove that if $x \in A$ and $y \in \gamma(A)$, then $x\backslash y, y/x \in \gamma(A)$. In fact $\gamma(y/x)x \leq \gamma(y/x)\gamma(x) \leq \gamma(\gamma(y/x)\gamma(x)) = \gamma((y/x)x) \leq \gamma(y) = y$. Thus, by the residuation law, $\gamma(y/x) \leq y/x$ and consequently, since γ is expanding, $\gamma(y/x) = y/x$ and $y/x \in \gamma(A)$. In particular, for any $x, y \in A$, $\gamma(x) \in \gamma(A)$ and $\gamma(y) \in A$, thus $\gamma(y)\backslash\gamma(x) = \gamma(\gamma(y)\backslash\gamma(x))$ (and similarly for $/$). Therefore $\gamma(\mathbf{A})$ is closed under $\backslash$ and $/$. Moreover, for $x, y \in \gamma(A)$, $x \wedge y \leq x$ and $x \wedge y \leq y$, so $\gamma(x \wedge y) \leq \gamma(x) = x$ and $\gamma(x \wedge y) \leq \gamma(y) = y$. Therefore $\gamma(x \wedge y) \leq x \wedge y \leq \gamma(x \wedge y)$ and $x \wedge y$ is the greatest lower bound of x and y in $\gamma(\mathbf{A})$. Now let $x, y \in \gamma(A)$; then $x, y \leq x \vee y \leq \gamma(x \vee y)$. If $z \in \gamma(A)$ is an upper bound of x and y, then $x \vee y \leq z$ and so $x \vee_\gamma y = \gamma(x \vee y) \leq \gamma(z) = z$. Thus $x \vee_\gamma y$ is the least upper bound of x and y in $\gamma(\mathbf{A})$. Now we show that multiplication is associative. Let $x, y, z \in \gamma(A)$; $x \cdot_\gamma (y \cdot_\gamma z) = \gamma(x \cdot \gamma(y \cdot z)) = \gamma(\gamma(x)\gamma(yz)) = \gamma(x(yz)) = \gamma((xy)z) = \gamma(\gamma(xy)\gamma(z)) = \gamma(\gamma(x \cdot y) \cdot z) = (x \cdot_\gamma y) \cdot_\gamma z$. Moreover, $x \cdot_\gamma \gamma(1) = \gamma(x \cdot \gamma(1)) = \gamma(\gamma(x) \cdot \gamma(1)) = \gamma(x \cdot 1) = \gamma(x)$, therefore $\gamma(1)$ is the multiplicative unite. In the end, the residuation law, $x \cdot_\gamma y \leq z$ iff $y \leq x\backslash z$ iff $x \leq z/y$, follows from the residuation law for $\mathbf{A}$ and the observation that for all $x, y, z \in \gamma(A)$, $x \cdot y \leq z$ iff $x \cdot_\gamma y = \gamma(x \cdot y) \leq z$. Indeed if $x \cdot_\gamma y \leq z$, then $x \cdot y \leq \gamma(x \cdot y) \leq z$. Vice versa if $x \cdot y \leq z$, then $\gamma(x \cdot y) \leq \gamma(z) = z$. □

We call $\gamma(\mathbf{A})$ the *nuclear image* of $\mathbf{A}$ under the nucleus γ.

Definition 1.10. *An* interior operator σ *on a poset* $\mathbf{P} = \langle P, \leq \rangle$ *is a map on* P *that is contracting, monotone and idempotent, i.e.,* $\sigma(x) \leq x$*, if* $x \leq y$ *then* $\sigma(x) \leq \sigma(y)$*, and* $\sigma(\sigma(x)) = \sigma(x)$*.*

Definition 1.11. *Let* $\mathbf{A}$ *be a residuated lattice. A* conucleus σ *on* $\mathbf{A}$ *is an interior operator that satisfies*

$$\sigma(1) = 1$$

and

$$\sigma(x)\sigma(y) \leq \sigma(xy) \text{ for all } x, y \in A.$$

We observe that for an interior operator σ, the condition $\sigma(x)\sigma(y) \leq \sigma(xy)$ is equivalent to

$$\sigma(\sigma(x)\sigma(y)) = \sigma(x)\sigma(y).$$

Indeed suppose that $\sigma(x)\sigma(y) \leq \sigma(xy)$ for all x, y in a residuated lattice $\mathbf{A}$. Since σ is contracting, $\sigma(\sigma(x)\sigma(y)) \leq \sigma(x)\sigma(y)$. In addition, by the hypothesis, $\sigma(\sigma(x)\sigma(y)) \geq \sigma(\sigma(x))\sigma(\sigma(y)) = \sigma(x)\sigma(y)$. Vice versa if $\sigma(x)\sigma(y) = \sigma(\sigma(x)\sigma(y))$, then, since $\sigma(x) \leq x$, $\sigma(y) \leq y$ and $\cdot$ is order-preserving, $\sigma(\sigma(x)\sigma(y)) \leq \sigma(xy)$, and the claim is settled.

If $\mathcal{V}$ is a variety of residuated lattices, we denote by $\mathcal{V}_\sigma$ the class which consists of algebras $\langle \mathbf{A}, \sigma \rangle$, where $\mathbf{A} \in \mathcal{V}$ and σ is a conucleus on $\mathbf{A}$. Also $\mathcal{V}_\sigma$ is a variety.

Similarly to Theorem 1.9, we can state the following theorem ([18]):

Theorem 1.12. *If* $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$ *is a residuated lattice and* σ *is a conucleus on it, then the algebra* $\sigma(\mathbf{A}) = \langle \sigma(A), \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, 1 \rangle$ *is a residuated lattice, where* $\vee, \cdot$ *and* 1 *are the same as in* $\mathbf{A}$ *and for all* $x, y \in \sigma(A)$, $x \wedge_\sigma y = \sigma(x \wedge y)$, $x \backslash_\sigma y = \sigma(x \backslash y)$ *and* $x /_\sigma y = \sigma(x/y)$.

Proof. We can easily verify that $\langle \sigma(A), \cdot, 1 \rangle$ is a monoid. Indeed $\sigma(A)$ is closed under $\cdot$: if $\sigma(x), \sigma(y) \in \sigma(A)$, $\sigma(x)\sigma(y) = \sigma(\sigma(x)\sigma(y)) \in \sigma(A)$, due to the property of conuclei. Moreover, $\sigma(1) = 1$, so $1 \in \sigma(A)$. Let $x, y \in \sigma(A)$. Since in $\mathbf{A}$ $x \leq x \vee y$, $y \leq x \vee y$, $\sigma(x) = x$ and $\sigma(y) = y$, then $x \leq \sigma(x \vee y)$ and $y \leq \sigma(x \vee y)$. Therefore $\sigma(x \vee y)$ is an upper bound of x and y. Since $x \vee y$ is the least upper bound of x and y, $x \vee y \leq \sigma(x \vee y) \leq x \vee y$. Thus $\sigma(x \vee y) = x \vee y$ is the least upper bound of x and y in $\sigma(\mathbf{A})$. Now we want to show that $x \wedge_\sigma y$ is the greatest lower bound of x and y in $\sigma(\mathbf{A})$. Since $x \wedge y \leq x$ and $x \wedge y \leq y$, $\sigma(x \wedge y) \leq \sigma(x) = x$ and $\sigma(x \wedge y) \leq \sigma(y) = y$. Let $z \in \sigma(A)$ such that $z \leq x, y$; then $z \leq x \wedge y$ and $z = \sigma(z) \leq \sigma(x \wedge y) = x \wedge_\sigma y$. Therefore, $x \wedge_\sigma y$ is the greatest lower bound of x and y in $\sigma(\mathbf{A})$. Finally we prove the residuation law: for all $x, y, z \in \sigma(A)$, $x \cdot y \leq z$ iff $y \leq x \backslash_\sigma z$ iff $x \leq z /_\sigma y$. We can observe that the inequation $x \leq \sigma(z/y)$ is equivalent to $x \leq z/y$; indeed if $x \leq \sigma(z/y)$, then $x \leq \sigma(z/y) \leq z/y$. On the other hand, if $x \leq z/y$, then $x = \sigma(x) \leq \sigma(z/y)$. Similarly, $y \leq x \backslash_\sigma z$ iff $y \leq x \backslash z$. Thus, the residuation law for $\sigma(\mathbf{A})$ follows from the residuation law for $\mathbf{A}$. □

We call $\sigma(\mathbf{A})$ the *conuclear image* of $\mathbf{A}$ under the conucleus σ.

If $\mathcal{V}$ is a variety of residuated lattices, we have already defined $\mathcal{V}_\sigma$ as the class of algebras $\mathbf{B} = \langle \mathbf{A}, \sigma \rangle$ where $\mathbf{A} \in \mathcal{V}$ and σ is a conucleus on $\mathbf{A}$. We indicate with $\sigma(\mathbf{B})$ the conuclear image of $\mathbf{A}$ under the conucleus σ. Thus $\sigma(\mathcal{V})$ is the variety generated by all the algebras $\sigma(\mathbf{B})$, where $\mathbf{B} = \langle \mathbf{A}, \sigma \rangle \in \mathcal{V}_\sigma$.

1.2.1 Examples of conuclei

In the thesis we will mostly focus on conuclear images of residuated lattices. Now we show some examples of conuclei and conuclear images. To begin with, we state the following lemma:

Lemma 1.13. *Let* $\mathbf{A}$ *be a residuated lattice and let* $\mathbf{Z}$ *be a submonoid of* $\mathbf{A}$ *such that*

$$\textit{for all } x \in A \textit{ the set } \{z \in Z : z \leq x\} \textit{ has maximum.}$$

Then the operator σ *defined by* $\sigma(x) = max\,\{z \in Z : z \leq x\}$ *is a conucleus on* $\mathbf{A}$*, and* $\sigma(A) = Z$*.*

Proof. By definition of σ, $\sigma(x) \leq x$ holds. In order to prove that σ is monotone, we suppose that $x \leq y$, where $x, y \in A$. Set $z = \sigma(x)$. By definition of σ, $z \in Z$ and $z \leq x$, so $z \leq y$. Since $\sigma(y) = max\,\{z \in Z : z \leq y\}$, $\sigma(x) = z \leq \sigma(y)$, as we wanted to prove. Now we prove that σ is idempotent: $\sigma(\sigma(x)) = \sigma(max\,\{z \in Z : z \leq x\})$. We call $k = max\,\{z \in Z : z \leq x\}$. Then $\sigma(k) = max\,\{z \in Z : z \leq k\} = k$, since $k \in Z$; therefore $\sigma(\sigma(x)) = \sigma(x)$. Moreover, $\sigma(1) = 1$ since $\mathbf{Z}$ is a submonoid and so $1 \in Z$. As regards to the condition $\sigma(x)\sigma(y) \leq \sigma(xy)$, we observe that $\sigma(xy) = max\,\{z \in Z : z \leq xy\}$. Furthermore, $\sigma(x)\sigma(y) \leq xy$ and, since $\mathbf{Z}$ is closed under $\cdot$, $\sigma(x)\sigma(y) \in Z$. Being $\sigma(xy)$ the greatest element belonging to $\mathbf{Z}$ smaller than xy, the claim is settled. For the last part, note that if $z \in \sigma(A)$, then $z = \sigma(x)$ for some $x \in A$, so $z \in Z$. Vice versa if $z \in Z$, then $\sigma(z) = z$, so $z \in \sigma(A)$. □

Lemma 1.13 is a tool for constructing several kinds of conuclei. We show some examples.

Example 1.1 The interior operator of a topological space (X,T), where X is a set and T is a topology on X, is obviously a conucleus on the powerset $P(X)$ of X, thought of a residuated lattice. Its conuclear image is a Heyting algebra, whose elements are the open sets of T.

Example 1.2 Another interesting example of conucleus comes from linear logic (see [9] for a survey). In fact the Lindenbaum algebra of linear logic is a pointed residuated lattice and the operator ! induces a conucleus on it. Indeed, it follows directly from the axioms of linear logic that ! is contracting, idempotent and $!1 = 1$. Moreover, using the rules of linear logic, it is easy to prove that ! is monotone, namely, the sequent $!A \Rightarrow !B$ is derivable from the sequent $A \Rightarrow B$ in the sequent calculus of linear logic. Indeed:

$$\cfrac{\cfrac{A \Rightarrow B}{!A \Rightarrow B}}{!A \Rightarrow !B}$$

Finally, $!A \otimes !B \Rightarrow !(A \otimes B)$, where $\otimes$ is the monoidal operation in the Lindenbaum algebra of linear logic, is easily derivable in the sequent calculus of linear logic. In fact:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \Rightarrow A \quad B \Rightarrow B}{A, B \Rightarrow A \otimes B}}{!A, B \Rightarrow A \otimes B}}{!A, !B \Rightarrow A \otimes B}}{!A, !B \Rightarrow !(A \otimes B)}}{!A \otimes !B \Rightarrow !(A \otimes B)}$$

Example 1.3 (**Poset sum and conuclei**). Let $\langle I, \leq \rangle$ be a poset and let $\{\mathbf{L}_i : i \in I\}$ be a family of integral residuated lattices with minimum 0: $\forall i \in I$ $\mathbf{L}_i = \langle L_i, \wedge_{L_i}, \vee_{L_i}, \cdot_{L_i}, /_{L_i}, \backslash_{L_i}, 1, 0 \rangle$. Let $\mathbf{L} = \Pi_{i \in I} L_i$ be the direct product of L_i. Then $\mathbf{L} = \langle L, \wedge_p, \vee_p, \cdot_p, /_p, \backslash_p, 1_p, 0_p \rangle$ is a residuated lattice where the operations are defined componentwise, 1_p is the function constantly equal to 1 and 0_p is the constantly null function.

Let $f \in L$. We define $\forall i \in I$

$$\sigma(f)(i) = \begin{cases} f(i) & \text{if } \forall j > i \; f(j) = 1 \\ 0 & \text{otherwise} \end{cases}$$

We prove that $\sigma : L \to L$ is a conucleus on $\mathbf{L}$. Indeed:

- σ is contracting: it follows directly from the definition of σ.
- σ is monotone: if $f \leq g$, then $f(i) \leq g(i) \; \forall i \in I$. If $\sigma(f)(i) = f(i)$, then $\forall j > i \; f(j) = 1$ but, since $\forall j > i \; 1 = f(j) \leq g(j)$ and 1 is the maximum, $\forall j > i \; g(j) = 1$, so $\sigma(g)(i) = g(i)$. Therefore $\sigma(f)(i) = f(i) \leq g(i) = \sigma(g)(i)$. If $\sigma(f)(i) = 0$, the claim is settled since 0 is the minimum.
- σ is idempotent: let $i \in I$: if $\forall j > i \; f(j) = 1$, then $\sigma(f)(i) = f(i)$ so, applying σ, $\sigma(\sigma(f)(i)) = \sigma(f(i))$. Otherwise, if there exists $j > i$ such that $f(j) < 1$, then $\sigma(f)(i) = 0$ and so $\sigma(\sigma(f)(i)) = \sigma(0)$. Since $\sigma(0) = 0$ by definition of σ, the thesis is proved.
- $\sigma(1_p) = 1_p$ by definition of σ.
- In order to prove the property $\sigma(x)\sigma(y) \leq \sigma(xy)$, we have to distinguish four cases: if $\forall j > i \; f(j) = 1$ and $g(j) = 1$, then $\forall j > i \; f(j) \cdot_{L_j} g(j) = 1$. Thus

  $$(\sigma(f) \cdot_p \sigma(g))(i) = \sigma(f)(i) \cdot_{L_i} \sigma(g)(i) = f(i) \cdot_{L_i} g(i) = (f \cdot_p g)(i) = \sigma(f \cdot_p g)(i).$$

  If there exists $k > i$ such that $f(k) < 1$ and $\forall j > i \; g(j) = 1$, then $k > i$ and $f(k) \cdot_{L_k} g(k) = f(k) \cdot_{L_k} 1 = f(k) < 1$. Therefore

  $$\sigma(f)(i) \cdot_{L_i} \sigma(g)(i) = 0 \cdot_{L_i} g(i) = 0 = \sigma(f \cdot_p g)(i).$$

  The third case is symmetrical to the second one. In the end, we suppose that there exist $k > i$ such that $f(k) < 1$ and $h > i$ such that $g(h) < 1$.

Then we have $f(k) \cdot_{L_k} g(k) < 1$, since $f(k) < 1$ and L_k is integral. Therefore the claim is settled also in this case. In fact

$$(\sigma(f) \cdot_p \sigma(g))(i) = 0 \cdot_{L_i} 0 = 0 = \sigma(f \cdot_p g)(i)$$

We observe that σ is a special conucleus; indeed, as we can see from the previous proof, it preserves the product $\cdot_p$, i.e., $\sigma(f) \cdot_p \sigma(g) = \sigma(f \cdot_p g)$, and, in addition, it preserves the meet $\wedge_p$. To prove that $\sigma(f) \wedge_p \sigma(g) = \sigma(f \wedge_p g)$, we distinguish four cases: if $\forall j > i\ f(j) = 1$ and $g(j) = 1$, then $\forall j > i\ f(j) \wedge_{L_j} g(j) = 1$. Thus

$$(\sigma(f) \wedge_p \sigma(g))(i) = \sigma(f)(i) \wedge_{L_i} \sigma(g)(i) = f(i) \wedge_{L_i} g(i) = (f \wedge_p g)(i) = \sigma(f \wedge_p g)(i).$$

If there exists $k > i$ such that $f(k) < 1$ and $\forall j > i\ g(j) = 1$, then $f(k) \wedge_{L_k} g(k) = f(k) \wedge_{L_k} 1 = f(k) < 1$. Therefore

$$\sigma(f)(i) \wedge_{L_i} \sigma(g)(i) = 0 \wedge_{L_i} g(i) = 0 = \sigma(f \wedge_p g)(i).$$

The third case is symmetrical to the previous one. In the end, if there exist $k > i$ such that $f(k) < 1$ and $h > i$ such that $g(h) < 1$, then $f(k) \wedge g(k) \leq f(k) < 1$. Therefore

$$(\sigma(f) \wedge_p \sigma(g))(i) = 0 \wedge_{L_i} 0 = 0 = \sigma(f \wedge_p g)(i).$$

Nevertheless, σ does not preserve $\vee_p$.

Now we prove that the conuclear image of $\mathbf{L}$ under this conucleus σ is the poset sum $\oplus \mathbf{L}_i$. We recall (referring to [14] and [13]) that

$$\oplus L_i = \{f \in L : \forall i \in I \text{ if } f(i) < 1, \text{ then } \forall j < i\ f(j) = 0\}.$$

The poset sum of the family $\{L_i : i \in I\}$ is a residuated lattice where the operations are defined componentwise, apart from the two residuals:

$$\oplus \mathbf{L}_i = \langle \oplus L_i, \wedge_\oplus, \vee_\oplus, \cdot_\oplus, /_\oplus, \backslash_\oplus, 1_\oplus, 0_\oplus \rangle$$

$$(f \vee_\oplus g)(i) = f(i) \vee_{L_i} g(i),$$

$$(f \wedge_\oplus g)(i) = f(i) \wedge_{L_i} g(i),$$

$$(f \cdot_\oplus g)(i) = f(i) \cdot_{L_i} g(i),$$

$$(f\backslash_{\oplus}g)(i) = \begin{cases} f(i)\backslash_{L_i}g(i) & \text{if } \forall j > i \ f(j) \leq g(j) \\ 0 & \text{otherwise} \end{cases}$$

$$(g/_{\oplus}f)(i) = \begin{cases} g(i)/_{L_i}f(i) & \text{if } \forall j > i \ f(j) \leq g(j) \\ 0 & \text{otherwise} \end{cases}$$

and $1_{\oplus}$ is the function constantly equal to 1 and $0_{\oplus}$ the constantly null function.

First, we verify that $\sigma(L)(= \{\sigma(f) : f \in L\}) = \oplus L_i$. Let $f \in L$, then $\sigma(f) \in L$. Let $i \in I$; we suppose that $\sigma(f)(i) = f(i) < 1$. Let $j < i$, then

$$\sigma(f)(j) = \begin{cases} f(j) & \text{if } \forall q > j \ f(q) = 1 \\ 0 & \text{otherwise} \end{cases}$$

But $i > j$ and $f(i) = \sigma(f)(i) < 1$; therefore $\sigma(f)(j) = 0$, as we wanted to prove. On the other hand, if $\sigma(f)(i) = 0$, there exists $k > i$ such that $f(k) < 1$. Let $j < i$, then $k > i > j$ and $f(k) < 1$; therefore we have, also in this case, $\sigma(f)(j) = 0$. Therefore if $f \in L$, then $\sigma(f) \in \oplus L_i$ and we have proved that $\sigma(L) \subseteq \oplus L_i$.
Vice versa let $f \in \oplus L_i$, then $f \in L$ and $\forall i \in I$ if $f(i) < 1$, then $\forall j < i \ f(j) = 0$. We will prove that if $f \in \oplus L_i$, then $\sigma(f) = f$. Let $i \in I$; if $\forall j > i \ f(j) = 1$, the thesis holds by definition of σ. If there exists $j > i$ such that $f(j) < 1$, then $\sigma(f)(i) = 0$. But, if $f \in \oplus L_i$ and there exists $j > i$ such that $f(j) < 1$, $f(i)$ must be 0 and $\sigma(f)(i) = 0 = f(i)$. Therefore $f = \sigma(f)$, where $f \in L$, and consequently $f \in \sigma(L)$. Hence also the other inclusion, $\oplus L_i \subseteq \sigma(L)$, holds.

If we build the conuclear image of L, we obtain

$$\langle \sigma(L) = \oplus L_i, \wedge_{\oplus}, \vee_{\oplus}, \cdot_{\oplus}, /_{\oplus}, \backslash_{\oplus}, \sigma(1_p) = 1_p, \sigma(0_p) = 0_p \rangle$$

where, using Theorem 1.12,

$$f \wedge_{\oplus} g = \sigma(f \wedge_p g),$$

$$f \vee_{\oplus} g = f \vee_p g,$$

$$f \cdot_{\oplus} g = f \cdot_p g,$$

$$f \backslash_{\oplus} g = \sigma(f \backslash_p g),$$

$$f/_{\oplus}g = \sigma(f/_{p}g).$$

These operations are the same operations as in the poset sum. Indeed for all $i \in I$ and for all $f, g \in \oplus L_i$:

- $(f \wedge_{\oplus} g)(i) = \sigma(f \wedge_p g)(i) = (\sigma(f) \wedge_p \sigma(g))(i) = \sigma(f)(i) \wedge_{L_i} \sigma(g)(i) = f(i) \wedge_{L_i} g(i)$,
  where the second equality is due to the fact that σ preserves $\wedge_p$ and the last equality follows from the fact that $\sigma(f) = f$ if $f \in \oplus L_i$.
- $(f \backslash_{\oplus} g)(i) = \sigma(f \backslash_p g)(i) = \begin{cases} (f\backslash_p g)(i) = f(i)\backslash_{L_i} g(i) & \text{if } \forall j > i \ f(j)\backslash_{L_j} g(j) = 1 \\ 0 & \text{otherwise} \end{cases}$

  and the condition in the first clausola is equivalent to the condition $\forall j > i$ $f(j) \leq g(j)$.

Example 1.4 (**Max idempotent**). Now we build an interesting conucleus on a complete, integral, commutative residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \rightarrow, 1, 0\rangle$. (Note that any complete lattice $\mathbf{A}$ is bounded since $inf(\emptyset)$ and $inf(A)$ are the greatest and the smallest element of $\mathbf{A}$). Let $Z = \{x \in A : x \cdot x = x\}$, namely Z consists of the idempotent elements of $\mathbf{A}$. Since $\mathbf{A}$ is commutative, $\mathbf{Z} = \langle Z, \cdot, 1\rangle$ is a commutative submonoid of $\mathbf{A}$. We can easily verify that $\mathbf{Z}$ is closed under finite joins; let $x, y \in Z$, we want to prove that also $x \vee y \in Z$, namely $x \vee y$ is an idempotent element of A. The claim follows from commutativity and integrality; indeed

$$(x \vee y)^2 = (x \vee y)(x \vee y) = x^2 \vee xy \vee yx \vee y^2 = x \vee xy \vee y$$

but, since $xy \leq x$ by integrality, $(x \vee y)^2 = x \vee y$.

In addition, $\mathbf{Z}$ is closed under arbitrary joins. In fact let X be a set of idempotent elements; $\bigvee X$ exists because $\mathbf{A}$ is complete. We are going to prove that $\bigvee X$ is idempotent, so we can conclude that $\bigvee X \in Z$.

$$(\bigvee X)^2 = (\bigvee X) \cdot (\bigvee X) = \bigvee (X \cdot X) = \bigvee (X)^2.$$

To prove that $\bigvee(X)^2 = \bigvee X$, it is sufficient to verify that every element of X is smaller than an element of X^2 and vice versa every element of X^2 is smaller than an element of X. If $a \in X$, then, being a idempotent, $a \leq a \cdot a \in X^2$. Vice versa let $a, b \in X$, then $a \cdot b \in X^2$ and, because of integrality, $a \cdot b \leq a \cdot 1 = a \in X$. Therefore the claim is settled.

Finally for all $x \in A$, we define $\sigma(x)$ =sup$\{z : z \cdot z = z, z \leq x\}$. This sup exists because **A** is complete and this sup is idempotent because **Z** is closed under arbitrary joins. Since this sup belongs to the set, it is a maximum. Therefore we have

$$\sigma(x) = \max\{z : z \cdot z = z, z \leq x\}.$$

In conclusion **A** and **Z** satisfy the hypothesis of Lemma 1.13, so σ is a conucleus on **A** and $\sigma(A) = Z$.

We can also verify that $\mathbf{Z} = \langle Z, \cdot, \vee, \to', 0\rangle$ is a Heyting algebra, where

$$a \to' b = \sigma(a \to b).$$

The proof follows from the following observations:

- If 0 is the minimum of **L**, then 0 is also the minimum of **Z**.
- **Z** is closed under joins.
- **Z** is closed under multiplication (because of commutativity). Indeed, let $x, y \in Z$,

$$(x \cdot y)^2 = x \cdot y \cdot x \cdot y = x \cdot x \cdot y \cdot y = x^2 \cdot y^2 = x \cdot y.$$

- If $a, b \in Z$, then $a \to' b = \sigma(a \to b)$ is idempotent by definition of σ. Therefore **Z** is closed under $\to'$.
- In order to prove that $\langle Z, \cdot, \vee\rangle$ is a lattice, we have to prove that, given $a, b \in Z$, $a \cdot b$ is the greatest lower bound of a and b in **Z**. By integrality, $ab \leq a \cdot 1 = a$ and $ab \leq 1 \cdot b = b$, hence ab is a lower bound of a and b. Now let $c \in Z$ be a lower bound of a and b; then $c \cdot c \leq a \cdot b$ and, being c idempotent, $c \leq a \cdot b$, as we wanted to prove.

- In the end, the residuation law $a \cdot b \leq c$ iff $a \leq b \to' c$ holds for all $a, b, c \in Z$. Indeed, if $a \cdot b \leq c$, then, by the residuation law of $\mathbf{L}$, $a \leq b \to c$. Since $\sigma(b \to c)$ is the greatest idempotent smaller than or equal to $b \to c$, $a \leq b \to' c$. Vice versa if $a \leq b \to' c = \sigma(b \to c) \leq b \to c$, then, by the residuation law in $\mathbf{L}$, $a \cdot b \leq c$.

We now present two famous examples of conuclear images. Actually, the results we are going to explain show categorical equivalences between the involved varieties.

Example 1.5. By a famous result of McKinsey and Tarski ([16], [17]), it is possible to represent Heyting algebras as Boolean algebras with an interior operator. Indeed the authors prove a categorical equivalence between Heyting algebras and Boolean algebras endowed with an interior operator whose image generates the Boolean algebra. We here provide the most important steps in this construction.

Let $\mathbf{B} = \langle B, \wedge, \vee, \to, 1, 0\rangle$ be a Boolean algebra and let σ be an interior operator on $\mathbf{B}$, namely an operator $\sigma : B \to B$ that satisfies the following identities:

1. $\sigma(x \to y) \leq \sigma(x) \to \sigma(y)$;
2. $\sigma(1) = 1$;
3. $\sigma(x) \leq x$;
4. $\sigma(x) \leq \sigma(\sigma(x))$.

By the above axioms, it follows that σ satisfies the further axiom

$$\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y),$$

which implies that σ is a conucleus since in a Boolean algebra the operation of multiplication coincides with the operation of meet.

It is possible to prove that if $\langle \mathbf{B}, \sigma\rangle$ is a Boolean algebra with an interior operator, $\sigma(\mathbf{B}) = \langle \sigma(B), \wedge, \vee, \to', 1, 0\rangle$, the image of $\mathbf{B}$ under σ, is a Heyting algebra, where $\wedge, \vee, 1$ and 0 are the same as in $\mathbf{B}$, while the implication is defined, for all $x, y \in \sigma(\mathbf{B})$, as

$$x \to' y = \sigma(x \to y).$$

Conversely, we can prove that every Heyting algebra has the form $\sigma(\mathbf{B})$ for some Boolean algebra $\mathbf{B}$ and interior operator σ on $\mathbf{B}$. Let $\mathbf{H} = \langle H, \wedge, \vee, \to, 1, 0\rangle$ be a Heyting algebra and $\mathcal{P} = \{P : P \text{ is a prime filter of H}\}$. We recall that $F \subseteq H$ is a *filter* of $\mathbf{H}$ if it satisfies the following conditions:

- if $x, y \in F$, then $x \wedge y \in F$.
- if $x \in F$ and $x \leq y$, then $y \in F$.

A filter F is *prime* if for all $x, y \in H$

$$\text{if } x \vee y \in F, \text{ then } x \in F \text{ or } y \in F.$$

For all $a \in H$, we define

$$a^* = \{P \in \mathcal{P} : a \in P\},$$

namely a^* consists of all the prime filters containing a. We build a topology on the space taking $\{a^* : a \in H\}$ as basis of closed sets. Now we have to prove that the space is compact. Say that a family $C = \{C_i : i \in I\}$ has the *finite intersection property* (FIP) if for all $i_1, ..., i_n \in I$ $C_{i_1} \cap ... \cap C_{i_n} \neq \emptyset$. Given a family $C = \{C_i : i \in I\}$ of closed sets with the FIP, we can replace each closed set C_i with an intersection of closed sets of the basis whose intersection is exactly C_i (and hence they have the FIP). It remains to prove that each set of closed elements of the basis with the FIP have non-empty intersection. In other words, it is sufficient to prove that, given $X = \{a_i : i \in I\} \subseteq H$, if for all $i_1, ..., i_n \in I$ $a_{i_1} \wedge ... \wedge a_{i_n} \neq 0$, then there is a prime filter containing X. Moreover, by Zorn Lemma, there exists a maximal filter extending X and this maximal filter is also prime. In fact, let M be a proper, maximal filter such that $X \subseteq M$. Suppose that $a \notin M$ and $b \notin M$. Thus, due to maximality of M, if we add a or b to M, we obtain an improper filter. If M is an improper filter, it contains also 0, so there exist $m_1, ..., m_k \in M$ such that $m_1 \wedge ... \wedge m_k \wedge a = 0$ and similarly there exist

$m'_1, ..., m'_h \in M$ such that $m'_1 \wedge ... \wedge m'_h \wedge b = 0$. Set $m = (\bigwedge_{i=1}^{k} m_i) \wedge (\bigwedge_{i=1}^{h} m'_i)$. Then $m \wedge a = m \wedge b = 0$, where $m \in M$, and hence $m \wedge (a \vee b) = 0$. Therefore $a \vee b$ cannot belong to M. In conclusion the space is compact.

Now we take $\{a^* : a \in H\} \cup \left\{\mathcal{P} \backslash a^* = a^{*C} : a \in H\right\}$ as clopen basis for a new topology. We build the Boolean algebra generated by boolean combinations of elements of this new topology. Consequently, we can represent each element of the Boolean algebra in the form $\bigwedge(a_i^* \vee b_i^{*C})$. In the end, we define the interior operator on the Boolean algebra as

$$\sigma(\bigwedge(a_i^* \vee b_i^{*C})) = \bigwedge(b_i \to a_i)^*.$$

Therefore, starting from a Heyting algebra, we have built a Boolean algebra endowed with an interior operator. Combining these two constructions, we can prove a categorical equivalence between Heyting algebras and Boolean algebras, with an interior operator σ, generated by the image of σ.

Example 1.6. Montagna and Tsinakis prove in [18] a categorical equivalence between commutative, cancellative residuated lattices and abelian ℓ-groups endowed with a conucleus whose image generates the underlying group of the ℓ-group. The result is obtained restricting the functors that provide a categorical equivalence between the class ORL of Ore residuated lattices and a particular class LG_{cn} of ℓ-groups endowed with a conucleus.

An *Ore residuated lattice* is a cancellative residuated lattice whose underlying monoid is a right reversible monoid. A monoid $\mathbf{M}$ is *right reversible* if any two principal semigroup ideals of $\mathbf{M}$ have a non-empty intersection: $Ma \cap Mb \neq \emptyset$ for all $a, b \in M$. ORL contains important subvarieties of RL including the variety of commutative, cancellative residuated lattices.

Instead, LG_{cn} consists of algebras $\langle \mathbf{G}, \sigma \rangle$, where $\mathbf{G}$ is a lattice-ordered group augmented with a conucleus σ such that the underlying group of the ℓ-group $\mathbf{G}$ is the group of left quotients of the underlying monoid $\sigma(\mathbf{G})$. A group $\mathbf{G}$ is a group of *left quotients* of a monoid $\mathbf{M}$, if $\mathbf{M}$ is a submonoid of $\mathbf{G}$ and every element of $\mathbf{G}$ can be expressed in the form $a^{-1}b$ for some $a, b \in M$.

It is known ([6]) that a cancellative monoid has a group of left quotients if and only if it is right reversible and, in addition, a right reversible monoid uniquely determines its group of left quotients.

Let $\mathbf{L}$ be an Ore residuated lattice and let $\mathbf{G}(\mathbf{L})$ be the group of left quotients of the underlying monoid of $\mathbf{L}$. It is proved that there exists a lattice order on $G(L)$ that extends the order of $\mathbf{L}$ and with respect to which $\mathbf{G}(\mathbf{L})$ becomes an ℓ-group. Indeed, if $\leq$ is the lattice order of $\mathbf{L}$, we indicate with $\preceq$ the binary relation on $G(L)$ defined, for all $a, b, c, d \in L$, by

$$a^{-1}b \preceq c^{-1}d \text{ iff there exist } x, y \in L \text{ such that } xb \leq yd \text{ and } xa = yc.$$

Then $\preceq$ is the unique lattice order on $G(L)$ that extends $\leq$ and with respect to which $\mathbf{G}(\mathbf{L})$ is a lattice-ordered group.

In conclusion, let $\mathbf{L}$ be an Ore residuated lattice and let $\mathbf{G}(\mathbf{L})$ be its lattice-ordered group of left quotients (which is uniquely determined by $\mathbf{L}$). We define the following conucleus $\sigma_{\mathbf{L}} : G(L) \to G(L)$: for all $a, b \in L$

$$\sigma_{\mathbf{L}}(a^{-1}b) = a \backslash b.$$

Therefore, starting from an Ore residuated lattice, we have obtained an ℓ-group endowed with a conucleus.

Conversely, it is proved that if $\langle \mathbf{G}, \sigma \rangle$ is a lattice-ordered group with a conucleus, $\sigma(\mathbf{G})$, the image of $\mathbf{G}$ under the conucleus σ, is an Ore residuated lattice.

Combining these two constructions, we obtain a categorical equivalence. Moreover, restricting the previous maps (precisely functors), we obtain the result for the subcategories $CCanRL$ of commutative, cancellative residuated lattices and CLG_{cn} of abelian ℓ-groups endowed with a conucleus. In conclusion, it follows from the previous result that commutative, cancellative residuated lattices are equivalent to abelian ℓ-groups $\langle \mathbf{G}, \sigma \rangle$ with a conucleus σ such that $\sigma(G)$ generates $\mathbf{G}$.

1.3 Galois connections

This section is devoted to sum up the most important notions about Galois connections which will be useful for the explanation of a particular construction, involving residuated lattices and conuclei, in the next section. Firstly, we list some of the principal results about the general residuation theory which are closely related to Galois theory. The following results are taken from [8].

Definition 1.14. *Let* **P** *and* **Q** *be posets. A map* $f : P \to Q$ *is* residuated *if there exists a map* $f^* : Q \to P$ *such that for any* $p \in P$ *and* $q \in Q$

$$f(p) \leq q \Leftrightarrow p \leq f^*(q).$$

In this case we say that f and f^* form a *residuated pair* and f^* is a *residual* of f.

Lemma 1.15. *Let* **P** *and* **Q** *be posets. If* $f : P \to Q$ *and* $f^* : Q \to P$ *form a residuated pair, then* $f^* \circ f$ *is a closure operator and* $f \circ f^*$ *is an interior operator.*

Proof. Let $x \in P$. Since $f(x) \leq f(x)$, by the residuation property, $x \leq f^*(f(x))$. In a similar way, $f(f^*(x)) \leq x$. Hence $f^* \circ f$ is expanding and $f \circ f^*$ is contracting. Now we suppose that $x \leq y$, where $x, y \in P$. Then $x \leq y \leq f^*(f(y))$, so, by residuation, $f(x) \leq f(y)$ and f is monotone. Similarly the monotonicity of f^* is proved. Moreover, for all $x \in P$, $f \circ f^*(f(x)) \leq f(x)$ since $f \circ f^*$ is contracting. Thus, by monotonicity of f^*, $f^*(f(f^*(f(x)))) \leq f^*(f(x))$. The reverse inequality is due to the fact that $f^* \circ f$ is expanding, hence $f^* \circ f$ is idempotent. The proof of the idempotence of $f \circ f^*$ is very similar. □

Lemma 1.16. *If* $f : P \to Q$ *and* $f^* : Q \to P$ *form a residuated pair, then*

1. $f^*(q) = max\ \{p \in P : f(p) \leq q\}$;

2. $f(p) = min\ \{q \in Q : p \leq f^*(q)\}$;

3. $f \circ f^* \circ f = f$ *and* $f^* \circ f \circ f^* = f^*$.

Lemma 1.17. *If f and f^* form a residuated pair, then f preserves existing joins and f^* preserves existing meets.*

Definition 1.18. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be two posets. The maps* $^{\triangleright} : P \to Q$ *and* $^{\triangleleft} : Q \to P$ *form a* Galois connection *if for all* $p \in P$ *and* $q \in Q$

$$q \leq p^{\triangleright} \Leftrightarrow p \leq q^{\triangleleft}.$$

Note that a Galois connection from $\mathbf{P}$ to $\mathbf{Q}$ is a residuated pair from $\mathbf{P}$ to the dual $\mathbf{Q}^{\delta}$ of $\mathbf{Q}$.

An example of Galois connection is obtained taking the map Th, which denotes the equational theory of a class of algebras, and the map Mod, which produces all algebraic models of a set of equations.

Moreover, the relation between sets of polynomials and their zero sets forms a Galois connection. Fix a natural number n and a field K. Let A be the set of all subsets of the polynomial ring $K[x_1, ..., x_n]$, ordered by inclusion, and let B be the set of all subsets of K^n, ordered by inclusion. If S is a set of polynomials, we associate to S the set of elements of K^n which are zeros of all the polynomials in S. Conversely, if U is a subset of K^n, we associate to U the set of polynomials in $K[x_1, ..., x_n]$ vanishing in the elements of U. These two maps form a Galois connection.

In addition, one of the most important examples of Galois connection is the following: let R be a relation between two sets A and B. Let $X \subseteq A$ and $Y \subseteq B$. We define the maps $^{\triangleright} : P(A) \to P(B)$ and $^{\triangleleft} : P(B) \to P(A)$ in the following way:

$$X^{\triangleright} = \{y \in B : \forall x \in X (xRy)\},$$

$$Y^{\triangleleft} = \{x \in A : \forall y \in Y (xRy)\}.$$

In other words, $X^{\triangleright}$ consists of the elements of B that are related to all the elements of X, while $Y^{\triangleleft}$ consists of the elements of A that are related to all the elements of Y. The pair $(^{\triangleright}, ^{\triangleleft})$ is called the Galois connection *induced by R*.

The following lemma about Galois connections is a consequence of the corresponding results about residuated pairs:

Lemma 1.19. *Suppose that* $^{\triangleright} : P \to Q$ *and* $^{\triangleleft} : Q \to P$ *form a Galois connection between the posets* $\mathbf{P}$ *and* $\mathbf{Q}$. *Then the following properties hold:*

1. *The maps* $^{\triangleright}$ *and* $^{\triangleleft}$ *are order-reversing and they convert existing joins into meets, i.e., if* $\bigvee X$ *exists in* P *for some* $X \subseteq P$, *then* $\bigwedge X^{\triangleright}$ *exists in* Q *and* $(\bigvee X)^{\triangleright} = \bigwedge X^{\triangleright}$, *and likewise for* $^{\triangleleft}$.

2. *The maps* $^{\triangleright\triangleleft} : P \to P$ *and* $^{\triangleleft\triangleright} : Q \to Q$ *are both closure operators.*

3. $^{\triangleright\triangleleft\triangleright} = ^{\triangleright}$ *and* $^{\triangleleft\triangleright\triangleleft} = ^{\triangleleft}$.

4. *For all* $q \in Q$, $q^{\triangleleft} = max\ \{p \in P : q \leq p^{\triangleright}\}$ *and for all* $p \in P$, $p^{\triangleright} = max\ \{q \in Q : p \leq q^{\triangleleft}\}$.

5. $\mathbf{P}^{\triangleright\triangleleft} = \mathbf{Q}^{\triangleleft}$ *and* $\mathbf{Q}^{\triangleleft\triangleright} = \mathbf{P}^{\triangleright}$.

Therefore, by Property 2 of the previous lemma, the Galois connection induced by R, which we have just presented above, gives rise to a closure operator $\gamma_R : P(A) \to P(A)$ associated with R, where

$$\gamma_R = X^{\triangleright\triangleleft} = \{k \in A : \forall u \in B((\forall x \in X(xRu)) \to (kRu))\}.$$

Lemma 1.20. *Let* A *and* B *be sets.*

1. *If* R *is a relation between* A *and* B, *then* γ_R *is a closure operator on* $P(A)$.

2. *If* γ *is a closure operator on* $P(A)$, *then* $\gamma = \gamma_R$ *for some relation* R *with domain* A.

Now we introduce a characterization of nuclei on $P(\mathbf{A})$.

Definition 1.21. *A relation* $N \subseteq A \times B$ *is called* nuclear *on a groupoid* $\mathbf{A}$ *if for every* $a_1, a_2 \in A, b \in B$, *there exist subsets* $a_1 \backslash\backslash b$ *and* $b//a_2$ *of* B *such that*

$$a_1 \cdot a_2\ N\ b \text{ iff } a_1\ N\ b//a_2 \text{ iff } a_2\ N\ a_1\backslash\backslash b.$$

Theorem 1.22. *([8]) If* $\mathbf{A}$ *is a groupoid and* $N \subseteq A \times B$, *then* γ_N *is a nucleus on* $P(\mathbf{A})$ *iff* N *is a nuclear relation.*

1.4 Construction of conuclei

In this section we introduce a general method to build, given a residuated lattice and a subpomonoid of it, another residuated lattice and a conucleus on it.

We take a residuated lattice $\mathbf{M} = \langle M, \wedge, \vee, \cdot, \backslash, /, 1\rangle$. We denote by $P(M)$ the powerset of M. Then $P(\mathbf{M}) = \langle P(M), \cap, \cup, \cdot, \backslash, /, \{1\}\rangle$ is a complete residuated lattice, where $X \cdot Y = \{x \cdot y : x \in X, y \in Y\}$, $X/Y = \{z : \{z\} \cdot Y \subseteq X\}$ and $Y\backslash X = \{z : Y \cdot \{z\} \subseteq X\}$.
We define on $P(\mathbf{M})$ the following maps:

$$X \uparrow = \{z \in M : \forall x \in X (x \leq z)\}$$

$$X \downarrow = \{z \in M : \forall x \in X (z \leq x)\}.$$

We observe that $\uparrow$ and $\downarrow$ form a Galois connection; it is exactly the Galois connection induced by $\leq \subseteq M \times M$.
We define the map $\gamma : P(M) \to P(M)$ by

$$\gamma(X) = (X \uparrow) \downarrow \text{ for all } X \subseteq M.$$

Therefore, using Galois theory, $\gamma = \gamma_{\leq}$ is a closure operator. Moreover, $\gamma = \gamma_{\leq}$ is a nucleus since the relation $\leq \subseteq M \times M$ is nuclear; to prove this, it is sufficient to define $z//x = \{z/x\}$, $z\backslash\backslash x = \{z\backslash x\}$ and the nuclearity of $\leq$ follows from the residuation law of $\mathbf{M}$.
Now we build

$$M^* = \{\gamma(X) : X \subseteq M\};$$

in other words, we define the residuated lattice $\mathbf{M}^*$ as the γ-image of $P(\mathbf{M})$ (that is the nuclear image of $P(\mathbf{M})$ under γ). The operations on $\mathbf{M}^*$ are defined as Theorem 1.9 explains. Indeed

$$X \vee_\gamma Y = \gamma(X \cup Y),$$

$$X \wedge_\gamma Y = X \cap Y,$$

$$X \cdot_\gamma Y = \gamma(X \cdot Y),$$

$$X\backslash_\gamma Y = \{z \in M : X \cdot \{z\} \subseteq Y\},$$

$$Y/_\gamma X = \{z \in M : \{z\} \cdot X \subseteq Y\}.$$

Therefore $\mathbf{M}^* = \langle M^*, \vee_\gamma, \wedge_\gamma, \cdot_\gamma, \backslash_\gamma, /_\gamma, \gamma(\{1\})\rangle$ is a complete residuated lattice.

Now we consider a submonoid $\mathbf{N}$ of $\mathbf{M}$. We define

$$N^* = \{\gamma(Y) : Y \subseteq N\}.$$

(We observe that, also in this case, γ is calculated in $\mathbf{M}$, namely $\uparrow$ and $\downarrow$ are calculated in $P(\mathbf{M})$). Note that $\mathbf{N}^*$ is a submonoid of $\mathbf{M}^*$. Indeed:

- $\gamma(\{1\}) \in N^*$ because $1 \in N$, since $\mathbf{N}$ is a monoid.
- $\mathbf{N}^*$ is closed under $\cdot_\gamma$; let $Y, Y' \subseteq N$.

  $$\gamma(Y) \cdot_\gamma \gamma(Y') = \gamma(\gamma(Y) \cdot \gamma(Y')) = \gamma(Y \cdot Y'),$$

  due to the fact that γ is a nucleus. Since $\mathbf{N}$ is closed under $\cdot$, $Y \cdot Y' \subseteq N$ and $\gamma(Y) \cdot_\gamma \gamma(Y') \in N^*$.

Furthermore, $\mathbf{N}^*$ is closed under arbitrary joins. Indeed let Y_i be subsets of N for all $i \in I$. Then

$$\bigvee_{i\in I} \gamma(Y_i) = \gamma(\bigcup_{i\in I} \gamma(Y_i)) = \gamma(\bigcup_{i\in I} Y_i).$$

We prove the second equality: since γ is a closure operator, $Y_i \subseteq \gamma(Y_i)$ $\forall i \in I$. So $\bigcup Y_i \subseteq \bigcup \gamma(Y_i)$. Applying γ to both these members, $\gamma(\bigcup_{i\in I} Y_i) \subseteq \gamma(\bigcup_{i\in I} \gamma(Y_i))$. Vice versa, we have $Y_i \subseteq \bigcup Y_i$ $\forall i \in I$; so, applying γ, $\gamma(Y_i) \subseteq \gamma(\bigcup Y_i)$ $\forall i \in I$. Then $\bigcup \gamma(Y_i) \subseteq \gamma(\bigcup Y_i)$ and consequently $\gamma(\bigcup_{i\in I} \gamma(Y_i)) \subseteq \gamma(\gamma(\bigcup_{i\in I} Y_i)) = \gamma(\bigcup_{i\in I} Y_i)$.

It follows that $\mathbf{N}^*$ is a complete lattice. Therefore $\mathbf{N}^*$ satisfies the hypothesis of Lemma 1.13 since for all $X \in M^*$ the set $\{Z \in N^* : Z \subseteq X\}$ has maximum. Indeed this set has sup and this sup belongs to the set; so the sup is a maximum. In the end, we can define a conucleus $\sigma : \mathbf{M}^* \to \mathbf{M}^*$ such that for all $X \in M^*$

$$\sigma(X) = \max\{Z \in N^* : Z \subseteq X\}$$

and $\sigma(M^*) = N^*$.

Therefore, in conclusion, we have provided a method to build, starting from a residuated lattice $\mathbf{M}$ and a submonoid of it, another residuated lattice $\mathbf{M}^*$ and a conucleus σ on $\mathbf{M}^*$.

Chapter 2

Substructural logics

In this chapter, we give a brief and general presentation of substructural logics. Substructural logics are logics lacking some or all the structural rules (exchange, weakening and contraction) when they are formalized in sequent systems. They encompass many famous logics, such as relevance logics (lacking the weakening rule), Łukasiewicz's many-valued logic and BCK-logic (lacking the contraction rule), linear logic (lacking contraction and weakening rule), etc... It is important to outline that at the beginning these types of logics were studied independently, mainly because of the different motivations that had led to their creation. Only later, a study on how structural rules affect logical properties, allowed us to consider these logics as special cases of the same concept. Indeed the purpose of substructural logics is to provide a uniform framework in which various kinds of non-classical logics, originated from different reasons, can be dealt with together, finding common features.

2.1 The sequent calculus FL

In this section we describe the sequent calculus **FL** (*Full-Lambek Calculus*), obtained by removing the structural rules from the sequent calculus of intuitionistic logic. It represents the base for all substructural logics.

The language of **FL** consists of propositional variables, constants 0 and 1, and binary connectives $\wedge$, $\vee$, $\cdot$, $\backslash$, $/$. Constant 0 allows us to define two connectives of negation: $\sim a = a\backslash 0$ and $-a = 0/a$.

A sequent of **FL** is an expression of the form $\alpha_1, ..., \alpha_n \Rightarrow \beta$, where $\alpha_1, ..., \alpha_n$ are formulas, $n \geq 0$ and β is a formula or the empty sequence.

The system **FL** consists of initial sequents (in particular two initial sequents for constants 0 and 1), cut rule and rules for logical connectives. In the following we adopt the convention that upper case letters are used for sequences of formulas, while lower case letters denote formulas.

Initial sequents:

$$\Rightarrow 1 \qquad 0 \Rightarrow \qquad \alpha \Rightarrow \alpha$$

Cut rule:

$$\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \alpha, \Xi \Rightarrow \varphi}{\Sigma, \Gamma, \Xi \Rightarrow \varphi} \text{ (cut)}$$

Rules for logical connectives:

$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, 1, \Delta \Rightarrow \varphi} \text{ (1w)} \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0} \text{ (0w)}$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \varphi \quad \Gamma, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \vee \beta, \Delta \Rightarrow \varphi} \text{ }(\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \text{ }(\Rightarrow \vee) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} \text{ }(\Rightarrow \vee)$$

$$\frac{\Gamma, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \varphi} \text{ }(\wedge \Rightarrow) \qquad \frac{\Gamma, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \wedge \beta, \Delta \Rightarrow \varphi} \text{ }(\wedge \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \text{ }(\Rightarrow \wedge)$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \alpha \cdot \beta, \Delta \Rightarrow \varphi} \text{ }(\cdot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta} \text{ }(\Rightarrow \cdot)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Xi, \beta, \Delta \Rightarrow \varphi}{\Xi, \Gamma, \alpha\backslash\beta, \Delta \Rightarrow \varphi}\ (\backslash \Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\backslash\beta}\ (\Rightarrow \backslash)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Xi, \beta, \Delta \Rightarrow \varphi}{\Xi, \beta/\alpha, \Gamma, \Delta \Rightarrow \varphi}\ (/ \Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta/\alpha}\ (\Rightarrow /)$$

We say that a sequent $\Gamma \Rightarrow \varphi$ is provable in **FL** (and write $\vdash_{FL} \Gamma \Rightarrow \varphi$) if $\Gamma \Rightarrow \varphi$ can be obtained from the initial sequents by repeated applications of the rules of **FL**. Hence, a formula α is provable in **FL**, if the sequent $\Rightarrow \alpha$ is provable in **FL**. Moreover, given Δ a set of formulas, we say that $\Gamma \Rightarrow \varphi$ is provable from Δ (and write $\Delta \vdash_{FL} \Gamma \Rightarrow \varphi$) if the sequent $\Gamma \Rightarrow \varphi$ is derivable in the sequent calculus of **FL** extended by initial sequents $\Rightarrow \delta$ for each $\delta \in \Delta$.

Logical connectives of **FL** are divided into two groups, according to the form of the rules involving the connectives. If the lower sequent of any of the corresponding rules has always the same environmental or context (namely the same side formulas) as the upper sequent(s), the connective is called *additive*; $\cdot$, $\backslash$ and $/$ are examples of connectives which belong to this group. The remaining connectives are called *multiplicative*.

Usually substructural logics are defined to be axiomatic extensions of **FL**. Let Φ be a set of formulas closed under substitutions. The axiomatic extension of **FL** by Φ is the calculus obtained from **FL** by adding new initial sequents $\Rightarrow \varphi$ for all formulas $\varphi \in \Phi$.

Sometimes, it is convenient to consider substructural logics as rule extensions of **FL**. An inference rule is an expression of the form

$$\frac{\Gamma_1 \Rightarrow \varphi_1 \quad \cdots \quad \Gamma_n \Rightarrow \varphi_n}{\Gamma_0 \Rightarrow \varphi_0}$$

The rule extension of **FL** is obtained adding to **FL** a set ϕ of inference rules closed under substitutions.

Some extensions of **FL** can be defined by adding combinations of structural rules (exchange, contraction, left and right weakening) to the set of rules of **FL**, as we will see in the next section.

2.2 Structural rules

In order to understand the roles of structural rules in a sequent calculus, we compare the sequent calculi **LK** and **LJ** of classical and intuitionistic logic respectively, with the sequent calculus **FL**.

A sequent of **LK** is an expression of the form $\alpha_1, ..., \alpha_m \Rightarrow \beta_1, ..., \beta_n$ with $n, m \geq 0$, which is interpreted as: $\beta_1 \vee ... \vee \beta_n$ *follows from assumptions* $\alpha_1, ..., \alpha_m$. In this sequent $\alpha_1, ..., \alpha_m$ are called antecedents, while $\beta_1, ..., \beta_n$ are called succedents. In the sequent calculus of **LK**, as well as cut rule and rules for logical connectives $\wedge, \vee, \rightarrow, \neg$, there is another kind of rules: structural rules.

Structural rules:

Weakening rules:

$$\frac{\Gamma, \Sigma \Rightarrow \Delta}{\Gamma, \alpha, \Sigma \Rightarrow \Delta}\ (w \Rightarrow) \qquad\qquad \frac{\Gamma \Rightarrow \Lambda, \Xi}{\Gamma \Rightarrow \Lambda, \alpha, \Xi}\ (\Rightarrow w)$$

Contraction rules:

$$\frac{\Gamma, \alpha, \alpha, \Sigma \Rightarrow \Delta}{\Gamma, \alpha, \Sigma \Rightarrow \Delta}\ (c \Rightarrow) \qquad\qquad \frac{\Gamma \Rightarrow \Lambda, \alpha, \alpha, \Xi}{\Gamma \Rightarrow \Lambda, \alpha, \Xi}\ (\Rightarrow c)$$

Exchange rules:

$$\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \Delta}{\Gamma, \beta, \alpha, \Sigma \Rightarrow \Delta}\ (e \Rightarrow) \qquad\qquad \frac{\Gamma \Rightarrow \Lambda, \alpha, \beta, \Xi}{\Gamma \Rightarrow \Lambda, \beta, \alpha, \Xi}\ (\Rightarrow e)$$

A sequent of **LJ** is an expression of the form $\alpha_1, ..., \alpha_m \Rightarrow \beta$, where $m \geq 0$ and β may be empty. The inference rules in **LJ** are the same as in **LK**, but we delete the structural rules ($\Rightarrow$ c) and ($\Rightarrow$ e) and consider that succedents consist of one formula or they are empty.

Analysing proofs in **LK** and **LJ**, it is easy to see that sequents of the form $\delta, \varphi \Rightarrow \delta \wedge \varphi$ can be proved using weakening rules, while sequents of the form

$\delta \wedge \varphi \Rightarrow \phi$ can be derived from the sequent $\delta, \varphi \Rightarrow \phi$ using contraction rule. Therefore, a sequent $\delta, \varphi \Rightarrow \phi$ is provable iff a sequent $\delta \wedge \varphi \Rightarrow \phi$ is provable. Generalizing this argument, we can conclude that in **LK** and **LJ** comma in the left-hand side of a sequent stands for conjunction, whereas in **LK** comma in the right-hand side stands for disjunction. Hence:

Proposition 2.1. *A sequent* $\alpha_1, ..., \alpha_m \Rightarrow \beta_1, ..., \beta_n$ *is provable in* **LK** *iff the sequent* $\alpha_1 \wedge ... \wedge \alpha_m \Rightarrow \beta_1 \vee ... \vee \beta_n$ *is provable in* **LK**. *This holds also for* **LJ**, *but in this case* $n \leq 1$.

In the end of this section we will explain the meaning of comma when the sequent calculus lacks some of the structural rules.

First, we analyse the roles of the left structural rules.

- **Exchange rule (e)**

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi}$$

  If the sequent calculus has the exchange rule, we can use antecedents in an arbitrary order.

- **Contraction rule (c)**

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$$

  If the sequent calculus has the contraction rule, we can use each antecedent multiple times. Instead, in a calculus without contraction, when a sequent $\Gamma \Rightarrow \varphi$ is proved, each antecedent in Γ is used at most once in the proof.

- **Left weakening rule (i)**

$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi}$$

  With the left weakening rule, we can add any redundant formula as an antecedent. Instead, without left weakening rule, each antecedent is used at least once in the proof.

- **Right weakening rule (o)**

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}$$

These explanations of structural rules allow us to define substructural logics as *resource-sensitive* logics, since they are sensitive to the number and order of the assumptions. For instance, in linear logic ([9]), a sequent calculus lacking both contraction and weakening rule, and having exchange as its only structural rule, every assumption must be used exactly once to derive the conclusion.

The above argument suggests that the role of comma in the left-hand side of sequents when there are not some of structural rules, is very different from the role of comma in **LK** and **LJ**, since this time comma cannot be interpreted as the conjunction $\wedge$. Indeed in substructural logics, comma is represented by the logical connective $\cdot$, whose behaviour is described by the rules $(\cdot \Rightarrow)$ and $(\Rightarrow \cdot)$ that we have already displayed in the sequent calculus **FL**. Therefore the meaning of a sequent is very different in **FL** from **LK** or **LJ**. We can state the following theorem:

Theorem 2.2. *In the sequent calculus* **FL***, a sequent* $\alpha_1, ..., \alpha_n \Rightarrow \beta$ *is provable if and only if the sequent* $\alpha_1 \cdot ... \cdot \alpha_n \Rightarrow \beta$ *is provable.*

In addition, we have the following lemma which relates the connective $\cdot$ to the two connectives $\backslash$ and $/$:

Lemma 2.3. *In* **FL***, the following conditions are mutually equivalent. For all formulas* α, β *and* γ*:*

1. $\alpha \cdot \beta \Rightarrow \gamma$ *is provable;*

2. $\alpha \Rightarrow \gamma/\beta$ *is provable;*

3. $\beta \Rightarrow \alpha\backslash\gamma$ *is provable.*

Some substructural logics are obtained adding some of the structural rules to the sequent calculus **FL**.

Let S be a subset of $\{$e,c,i,o$\}$. Then $\mathbf{FL}_S$ denotes the extension of **FL** obtained by adding the structural rules from S. For example, $\mathbf{FL}_e$ denotes the sequent calculus **FL** endowed with the exchange rule. The combination of (i) and (o) is abbreviated by (w). Moreover, $\mathbf{FL}_S$ can be viewed as an axiomatic extension of **FL**. The axioms which correspond to (e), (c), (i) and (o) are respectively:

$$\alpha \cdot \beta \to \beta \cdot \alpha,$$

$$\alpha \to \alpha \cdot \alpha,$$

$$\alpha \to 1,$$

$$0 \to \alpha.$$

Naturally, $\mathbf{FL}_{ewc}$ is intuitionistic logic.

2.3 Expressive power of substructural logics

As we know, classical logic is not able to express many concepts coming from the natural language. For example, the natural language has at least two conjunctions, one additive, whose algebraic counterpart is the lattice operation of "meet" $\wedge$, and one multiplicative, whose algebraic counterpart is a monoidal operation $\cdot$. These two conjunctions can be interpreted in the following way in terms of resources: $A \wedge B$ means that I can choose A or B but not both of them at the same time, while $A \cdot B$ means that I can have both A and B simultaneously. Classical logic has only one kind of conjunction, which is the idempotent conjunction $\wedge$.

Moreover, in classical logic the implication $A \to B$ is viewed as $\neg A \vee B$, and this interpretation does not respect at all the relationship cause-effect between antecedent and succedent typical of the implication in the natural language.

In this way, substructural logics provide a better interpretation of the natural language. Indeed the lack of expressivity of classical logic about many situations involving natural language is due to the presence of structural rules.

For example, probabilistic reasonings typically do not obey weakening. If Giulia studies in Siena, she is probably Tuscan. But if Giulia studies in Siena, she was born in Orvieto and she lives in Orvieto, she is probably not Tuscan.

Situations involving finitary resources do not respect contraction: if I am in front of the coffee machine and I have 50 cent., I can buy a coffee. But I cannot use the same 50 cent. twice to buy also a tea. Having twice 50 cent. is not the same as having 50 cent. once.

Exchange cannot be valid in the natural language: indeed the sentence “I opened the door and I entered the room” is not the same as the sentence “I entered the room and I opened the door”.

Therefore, we take advantages of removing some of the structural rules; we gain more expressivity as regards to the natural language. For example, in a logic without contraction, we can define two conjunctions (one of them is not idempotent) which are able to reflect the meanings of the two conjunctions of the natural language. Linear logic is an example of substructural logic obtained deleting both contraction and weakening rule. As another example we can cite relevant logic. In this logic, weakening rules are rejected. It has the advantage of having an implication more closely related to the implication of the natural language than the material implication of classical logic. In fact, in relevant logic, an implication $A \to B$ can be valid if and only if A and B have a particular logical relationship, exactly if they have at least one common variable.

2.4 Algebraic semantic for substructural logics

A famous result ([10]) is that **FL** is algebraizable and its algebraic counterpart is represented by the variety FL of FL-algebras, which we have described in the previous chapter. This is an important result because it allows us to investigate substructural logics from both a logical and algebraic point of view. Therefore, the analysis is carried out in a particular background called *algebraic logic*.

Moreover, it is known that the extension of **FL** by an axiomatic schema φ is equivalent to a subvariety of the variety FL defined by $1 \leq \varphi$. This induces a dual-isomorphism V from the lattice of axiomatic extensions of **FL** to the subvariety lattice of FL. In [10], the following completeness theorem is proved:

Theorem 2.4. *Let L be an axiomatic extension of* **FL** *and $V(L)$ the corresponding variety of* FL*-algebras. Then there are translations τ, ρ such that for any set Φ of formulas, any formula φ and any set $E \cup \{t = u\}$ of identities, we have:*

$$\Phi \vdash_L \varphi \textit{ iff } \tau(\Phi) \models_{V(L)} \tau(\varphi),$$

$$E \models_{V(L)} t = u \textit{ iff } \rho(E) \vdash_L \rho(t = u).$$

The translations τ and ρ are defined as follows:

$$\varphi \xrightarrow{\tau} 1 \leq \varphi$$

$$t = u \xrightarrow{\rho} (u \backslash t) \wedge (t \backslash u).$$

Therefore a formula φ is provable in **FL** if and only if the corresponding inequation $\varphi \geq 1$ is valid in the variety of FL-algebras.

This algebraization result can be generalized to a correspondence between rule extensions of **FL** and subquasivarieties of FL. Indeed, given a sequent $\alpha_1, ..., \alpha_n \Rightarrow \beta$, we can build the inequation $\alpha_1 \cdot .... \cdot \alpha_n \leq \beta$.

Moreover, given an inference rule

$$\frac{\Gamma_1 \Rightarrow \varphi_1 \quad \cdots \quad \Gamma_n \Rightarrow \varphi_n}{\Gamma_0 \Rightarrow \varphi_0}\ \text{(r)}$$

we can build a quasi-identity

$$\Gamma_1 \leq \varphi_1 \text{ and } \ldots \text{ and } \Gamma_n \leq \varphi_n \Longrightarrow \Gamma_0 \leq \varphi_0.$$

This association allows us to define a dual-isomorphism Q from the lattice of rule extensions of **FL** to the lattice of quasivarieties of FL-algebras. Hence we can state the following theorem:

Theorem 2.5. *Let L be a rule extension of* **FL** *and Q(L) the corresponding quasivariety of* FL*-algebras. Then for any set Φ of formulas, any formula φ and any set $E \cup \{t = u\}$ of identities, we have:*

$$\Phi \vdash_L \varphi \textit{ iff } \tau(\Phi) \models_{Q(L)} \tau(\varphi),$$

$$E \models_{Q(L)} t = u \textit{ iff } \rho(E) \vdash_L \rho(t = u),$$

where the translations τ and ρ are defined as in the previous theorem.

Therefore a rule extension L of **FL** is consistent if and only if the quasivariety $Q(\mathrm{L})$ is nontrivial, that is it contains at least one algebra different from the trivial one-element FL-algebra.

Due to the algebraization of substructural logics, we can see the structural rules from an algebraic point of view. Indeed the algebraic meanings of the structural rules are the following:

- the exchange rule is equivalent to the commutativity of monoidal operation, so it corresponds to the identity $x \cdot y = y \cdot x$;
- the contraction rule corresponds to the property for an FL-algebra to be contractive, i.e., $x \leq x \cdot x$;
- left weakening corresponds to the integrality of FL-algebra, i.e., $x \leq 1$;
- right weakening corresponds to the inequation $0 \leq x$, namely 0 is the minimum of the FL-algebra.

Therefore, $\mathbf{FL}_e$ corresponds to the variety CFL of commutative FL-algebras and a formula φ is valid in $\mathbf{FL}_e$ if and only if the inequation $\varphi \geq 1$ is valid in the variety CFL. Similarly, $\mathbf{FL}_c$ corresponds to the variety KFL of contractive FL-algebras, $\mathbf{FL}_i$ corresponds to the variety IFL of integral FL-algebras and $\mathbf{FL}_o$ corresponds to the variety of zero-bounded FL-algebras.

2.5 Examples of substructural logics

In this section we present some famous examples of substructural logics. In most cases, these logics were born independently and for different reasons. As limit examples, we can consider classical and intuitionistic logic. They are axiomatic extensions of **FL** since they are obtained by FL adding all the structural rules: contraction, weakening and exchange. We now deal with two kinds of non-classical logics.

Relevance logic. Relevance logic (or relevant logic) was born to avoid the paradoxes of material implication. Among them, we recall $p \to (q \to p)$, $\neg p \to (p \to q)$ and $(p \to q) \vee (q \to p)$. In order to clarify this concept, we take an example of [8]. Consider the following reasoning: 'if 2+2=4, then the fact that the Moon is made of Camembert implies that 2+2=4'. Therefore, since 2+2=4, by modus ponens, we have the fact that the Moon is made of Camembert implies 2+2=4. This is a classically valid reasoning but it is extremely counterintuitive. The problem is that antecedent is irrelevant to succedent; in fact they are on completely different topics. In order to give a precise mathematical definition to this concept, relevant logicians built various versions of the *variable sharing property*, also known as *relevance principle*, stating that an implication $\alpha \to \beta$ can be only a theorem if α and β have at least a variable in common. This creates a particular logical relationship between antecedent and succedent.

There are several kinds of relevance logics. A famous relevant logic is the system **E**, generally presented as Hilbert system. Among the axioms of **E**, we recall distributivity, contraction and double negation. In particular weakening is rejected. **E** is not algebraizable and often extensions of **E** are considered. For example, **R** is the extension of **E** obtained adding the constant 1 and axioms for 1. Moreover, **RM** (or *R-mingle*) is the extension of **R** with the formula

$$\text{(M) } \varphi \to (\varphi \to \varphi).$$

Algebrically, (M) is equivalent to the property of being *square decreasing*, i.e., the identity $x^2 \leq x$. The logic **RM** is algebraizable and its equivalent quasivariety semantic is precisely the variety generated by *Sugihara algebras*. A Sugihara algebra is an algebra whose universe is $S_n = \{a_{-n}, a_{-n+1}, ..., a_{-1}, a_0, a_1, ..., a_{n-1}, a_n\}$ for some natural number n, or $S_\infty = \{a_i : i \in \mathbb{Z}\}$. The lattice operations are determined by the natural total ordering of the indices and multiplication is defined by

$$a_i \cdot a_j = \begin{cases} a_i & \text{if } |i| > |j| \\ a_j & \text{if } |i| < |j| \\ a_i \wedge a_j & \text{if } |i| = |j| \,. \end{cases}$$

It can be proved that multiplication is residuated. Therefore each Sugihara algebra is a residuated lattice $\langle S_\alpha, \wedge, \vee, \cdot, \rightarrow, 1\rangle$ whose identity 1 is a_0. Studying Sugihara algebras, it can be observed that the algebraic semantic for **RM** is the subvariety of $InDFL_{ec}$ (involutive and distributive FL-algebras with exchange and contraction) satisfying $x^2 \leq x$.

Sometimes, also relevant logics without contraction are considered. The most famous example among them is *Abelian logic*. Its Hilbert system is obtained deleting from R the contraction axiom and adding the axiom

$$\text{(A) } ((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow \varphi.$$

This axiom is known as *relativization axiom* and it axiomatizes abelian lattice-ordered groups. Therefore, abelian logic is algebraizable and its algebraic counterpart is the variety CLG of abelian lattice-ordered groups.

Łukasiewicz logic. The three-valued system of Łukasiewicz was introduced in 1920. In order to prove the necessity of leave the two-valued classical logic, we cite the following example taken from [8]. Consider the proposition 'there will be a sea battle tomorrow'. That proposition, to be true, has to describe things in the way they really are, so a sea battle has to happen tomorrow. Nevertheless today no sea battle happened (yet), so our proposition is not true. On the other hand, to be false, the proposition has to describe things in the way they really are not, so there has to be no sea battle tomorrow. But the absence of sea battles today says nothing about sea battles tomorrow, so our proposition is not false. This is the motivation for introducing a third value, $\frac{1}{2}$. Logical connectives are extended in order to include also this third value. This argument can also be generalized leading to the introduction of values $\frac{1}{n}$ for any natural number n and also infinite-valued logic. MV-algebras, introduced in the previous chapter, are the algebraic counterpart of Łukasiewicz's infinite-valued logic. They are term-equivalent to the subvariety MV of $\mathbf{FL}_{eo}$ axiomatized by $(x \rightarrow y) \rightarrow y = x \vee y$. This axiom is known as *relativized law of double negation* since it is a generalization of the law of double negation $\neg\neg x = x$; indeed, if we take $y = 0$, we obtain

$(x \to 0) \to 0 = x$.

An example of MV-algebra is the algebra $\mathbf{C}_n$ where $C_n = \{c_{n-1}, ..., c_2, c_1, c_0 = 1\}$, $c_i \leq c_j$ iff $i \geq j$ and $c_i \cdot c_j = c_{min\{i+j,n\}}$. It is proved that $\mathbf{C}_n$ is an FL-algebra. Note that $\mathbf{C}_2$ is isomorphic to Boolean algebra $\mathbf{2}$, which consists of two elements.

2.6 Conuclear images of substructural logics

In this section we provide a particular construction on substructural logics, that allows us to pass from a substructural logic to another substructural logic, which is weaker than the initial substructural logic. Our aim for the rest of the paper is to investigate this construction, both from a logical and algebraic point of view, and to analyse the substructural logics obtained through this method.

In the previous chapter, we have seen that, given a residuated lattice and a conucleus on it, the conuclear image of a residuated lattice is a residuated lattice. We have also extended this concept for varieties of residuated lattices. We recall that, given a variety $\mathcal{V}$ of residuated lattices, we denote by $\mathcal{V}_\sigma$ the variety which consists of algebras $\langle \mathbf{A}, \sigma \rangle$, where $\mathbf{A} \in \mathcal{V}$ and σ is a conucleus on $\mathbf{A}$. In addition, we indicate with $\sigma(\mathcal{V})$ the variety generated by the conuclear images $\sigma(\mathbf{A})$, where $\langle \mathbf{A}, \sigma \rangle \in \mathcal{V}_\sigma$. Now we present the same concept but from a logical point of view.

Given a substructural logic L, we denote by L_σ the logic L endowed with an unary operator σ which satisfies the following axioms[1]:

1. $\sigma(A) \to A$;
2. $\sigma(A) \to \sigma(\sigma(A))$;
3. $(\sigma(A) \cdot \sigma(B)) \to \sigma(A \cdot B)$;

and the necessitation rule:

$$\frac{A}{\sigma(A)}.$$

We can easily verify that Axiom 3 implies the axiom

[1]If L is commutative, the left and the right implications coincide and we denote both by $\to$.

4. $\sigma(A \to B) \to (\sigma(A) \to \sigma(B))$.

In fact, if we work in a residuated lattice $\mathbf{R}$, the logical axiom $(\sigma(A) \cdot \sigma(B)) \to \sigma(A \cdot B)$ corresponds to the property $\sigma(x) \cdot \sigma(y) \leq \sigma(x \cdot y)$. Thus, given $a, b \in R$, using the above axiom, $\sigma(a) \cdot \sigma(a \to b) \leq \sigma(a \cdot (a \to b)) \leq \sigma(b)$. Therefore, by the residuation law, we obtain $\sigma(a \to b) \leq \sigma(a) \to \sigma(b)$, whose corresponding logical axiom is $\sigma(A \to B) \to (\sigma(A) \to \sigma(B))$, and the claim is settled.

So we observe that σ satisfies the S4 axioms (axioms of the modal logic S4) and the further axiom $(\sigma(A) \cdot \sigma(B)) \to \sigma(A \cdot B)$.

In other words, we have translated in logical terms the algebraic properties of conuclei. It is well-known that the algebraic counterpart of the modal logic S4 are Boolean algebras endowed with an interior operator. Indeed, it is easy to see that, from Axioms 1, 2, 4, we can conclude that σ is an interior operator. Furthermore, the property of conucleus, $\sigma(x) \cdot \sigma(y) \leq \sigma(x \cdot y)$, is directly translated through Axiom 3 and the necessitation rule allows us to prove that σ satisfies also the property $\sigma(1) = 1$. In fact, the implication $\sigma(1) \to 1$ follows directly from Axiom 1. Furthermore, the necessitation rule says that if A is a theorem, then $\sigma(A)$ is a theorem. Thus, since 1 is a theorem, $\sigma(1)$ is a theorem. Moreover, in any substructural logic, the formula $1 \to \varphi$ is equivalent to the formula φ. Indeed, in all residuated lattices

$$1 \to x = max\,\{y : 1 \cdot y \leq x\} = max\,\{y : y \leq x\} = x.$$

Hence, since $\sigma(1)$ is a theorem, also $1 \to \sigma(1)$ is a theorem. In conclusion, since $1 \to \sigma(1)$ and $\sigma(1) \to 1$, $\sigma(1) = 1$.

Therefore L_σ is the logic L endowed with a conucleus σ. We call L_σ the *conuclear extension* of L.

We now define an interpretation ${}^\sigma$ of L into L_σ in the following way:

- $p^\sigma = \sigma(p)$ if p is a propositional variable,
- $0^\sigma = \sigma(0)$,
- $1^\sigma = 1$,

- $(A \circ B)^\sigma = A^\sigma \circ B^\sigma$, for $\circ \in \{\vee, \cdot\}$,
- $(A \circ B)^\sigma = \sigma(A^\sigma \circ B^\sigma)$, for $\circ \in \{\backslash, /, \wedge\}$.

Thus σ(L) denotes the logic whose theorems are those formulas A such that A^σ is a theorem of L_σ. σ(L) is called the *conuclear image* of the substructural logic L.

We will see in the next chapter that σ(L) is a weaker logic than L, namely, each theorem of σ(L) is also a theorem of L. Hence, from an algebraic point of view, through this construction, we obtain a wider variety of residuated lattices than the initial variety of residuated lattices.

We have some famous examples of this construction. For instance, by a result of McKinsey and Tarski ([16] and [17]), already explained in Chapter 1, if L is classical logic, then σ(L) is intuitionistic logic, a logic which is weaker than classical logic.

Moreover, in the previous chapter, we have seen another important example due to Montagna and Tsinakis: if L is the logic of abelian ℓ-groups, then σ(L) is the logic of commutative and cancellative residuated lattices.

Although we have some specific examples of conuclear images of substructural logics, as far as we know, this construction has not been studied yet from a general point of view. Our aim is to investigate conuclear images of generic substructural logics; in other words, starting from a generic substructural logic L, we want to analyse the substructural logic which represents the conuclear image σ(L) of L, and the relationship between L and σ(L).

Chapter 3

Properties excluded to hold in a conuclear image

This and the following chapters are devoted to the original part of the thesis. In the previous chapter we have seen some specific examples of substructural logics and of their conuclear images. Now we want to face up to the topic of conuclear images from a general point of view. Indeed the aim of the thesis is to investigate the relationship between a substructural logic L and its conuclear image $\sigma(\mathrm{L})$, whichever the substructural logic L is.

We have carried out our analysis dealing with the following problems:

1. Which properties are excluded to hold in $\sigma(\mathrm{L})$, whatever L is?
2. Which properties may be valid in $\sigma(\mathrm{L})$ for some particular logic L but are not necessarily preserved under conuclear images?
3. Which theorems of L are preserved by the map $\mathrm{L} \mapsto \sigma(\mathrm{L})$?

We start talking about those properties which never hold in a conuclear image. We will discover that, in order to answer to this question, the disjunction property plays a fundamental role.

3.1 Disjunction property

A variety of residuated lattices $\mathcal{V}$ has the *disjunction property* if whenever $t_1 \vee t_2 \geq 1$ holds in $\mathcal{V}$, then $t_1 \geq 1$ or $t_2 \geq 1$ holds in $\mathcal{V}$, where t_1 and t_2 are terms of the variety $\mathcal{V}$. If we interpret this concept in logic, a logic L has the disjunction property when for any formulas A and B, if $A \vee B$ is provable in L, then either A or B is provable in it. The disjunction property is a constructive property: it says that a disjunction $A \vee B$ is only provable if one of the disjuncts A or B is provable, in accordance with Heyting semantics of proofs, according to which a proof of $A \vee B$ is either a proof of A or a proof of B. For example, classical logic does not have the disjunction property, since $p \vee \neg p$ is provable but neither of p and $\neg p$ are provable. On the contrary, the following result for intuitionistic logic follows as a consequence of cut elimination of its sequent calculus:

Theorem 3.1. *[8] Intuitionistic logic has the disjunction property.*

We prove that the conuclear image of any variety of residuated lattices (and, hence, the conuclear image of any substructural logic), has the disjunction property.

The following lemma will be useful for the construction that we are going to present.

Lemma 3.2. *[12] Let* $\mathbf{B}$ *be a nontrivial residuated lattice. There exists an element* $a \in B$ *such that* $a < 1$*.*

Proof. Since $\mathbf{B}$ is nontrivial, there exists an element $b \in B$ such that $b \neq 1$. Then we have two possibilities. If $1 \not\leq b$, we take $a = b \wedge 1 < 1$. Instead, if $1 < b$, then we take $a = b\backslash 1$. Clearly we have $a \leq 1\backslash 1 = 1$. Moreover, we can prove that $a < 1$; in fact, if $a = 1$, then $b = b \cdot a = b \cdot (b\backslash 1) \leq 1$, against the hypothesis. □

Let $\mathcal{V}$ be a variety of residuated lattices and let $\mathbf{C}$ be a nontrivial algebra in $\mathcal{V}$. By the previous lemma, we can fix an element $c_0 \in \mathbf{C}$ such that $c_0 < 1$.

Let $\mathbf{B}_1 = \langle \mathbf{A}_1, \sigma_1 \rangle$ and $\mathbf{B}_2 = \langle \mathbf{A}_2, \sigma_2 \rangle \in \mathcal{V}_\sigma$. Then $\sum(\mathbf{B}_1, \mathbf{B}_2, \mathbf{C})$ denotes the algebra $\langle \mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{C}, \sigma \rangle$, where $\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{C}$ is the direct product of the three algebras and the operator $\sigma : A_1 \times A_2 \times C \to A_1 \times A_2 \times C$ is defined as follows:

$$\sigma(a_1, a_2, c) = \begin{cases} (\sigma_1(a_1), \sigma_2(a_2), c \wedge 1) & \text{if } a_1, a_2 \geq 1 \\ (\sigma_1(a_1), \sigma_2(a_2), c \wedge c_0) & \text{otherwise} \end{cases}$$

Theorem 3.3. *σ is a conucleus on $\mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{C}$; therefore $\sigma(\sum(\mathbf{B}_1, \mathbf{B}_2, \mathbf{C}))$ is a residuated lattice which belongs to $\sigma(\mathcal{V})$.*

Proof. To begin with, we observe that σ is contracting, whatever the third component is. Indeed the claim for the first and the second component follows from the definition of conucleus for σ_1 and σ_2, and, as regards to the third component, $c \wedge 1 \leq c$ and $c \wedge c_0 \leq c$. To prove that σ is idempotent, we have to distinguish two cases: if $a_1, a_2 \geq 1$, then $\sigma(\sigma(a_1, a_2, c)) = \sigma(\sigma_1(a_1), \sigma_2(a_2), c \wedge 1)$. Since σ_1 and σ_2 are monotone, $\sigma_1(a_1) \geq 1$ and $\sigma_2(a_2) \geq 1$, thus $\sigma(\sigma_1(a_1), \sigma_2(a_2), c \wedge 1) = (\sigma_1(\sigma_1(a_1)), \sigma_2(\sigma_2(a_2)), c \wedge 1 \wedge 1) = (\sigma_1(a_1), \sigma_2(a_2), c \wedge 1) = \sigma(a_1, a_2, c)$. On the other hand, if at least one between a_1 and a_2 is not ≥ 1, then $\sigma(\sigma(a_1, a_2, c)) = \sigma(\sigma_1(a_1), \sigma_2(a_2), c \wedge c_0)$. Now, if we apply σ again, we obtain that the third component is either $c \wedge c_0 \wedge c_0$ or $c \wedge c_0 \wedge 1$ which, in both these cases, is equal to $c \wedge c_0$ (because $c_0 < 1$), so the thesis is proved. In order to prove that σ is monotone, suppose that $(a_1, a_2, c) \leq (a_1', a_2', c')$, that is $a_1 \leq a_1'$, $a_2 \leq a_2'$ and $c \leq c'$. If $a_1, a_2 \geq 1$, then $\sigma(a_1, a_2, c) = (\sigma_1(a_1), \sigma_2(a_2), c \wedge 1)$ but, since $a_1' \geq a_1 \geq 1$ and $a_2' \geq a_2 \geq 1$, $\sigma(a_1', a_2', c') = (\sigma_1(a_1'), \sigma_2(a_2'), c' \wedge 1)$ and the thesis is proved due to the fact that $c \wedge 1 \leq c' \wedge 1$. Instead, if one of a_1 and a_2 is not ≥ 1, then $\sigma(a_1, a_2, c) = (\sigma_1(a_1), \sigma_2(a_2), c \wedge c_0)$ and the third component of $\sigma(a_1', a_2', c')$ can be either $c' \wedge c_0$ or $c' \wedge 1$. Since $c \wedge c_0$ is smaller than or equal to both these quantities, the claim is settled. Moreover, $\sigma(1, 1, 1) = (\sigma_1(1), \sigma_2(1), 1 \wedge 1) = (1, 1, 1)$. As regards to the property $\sigma(x) \cdot \sigma(y) \leq \sigma(x \cdot y)$, it obviously holds for the first and the second component due to the fact that σ_1 and σ_2 are conuclei. It remains to prove that also the third component satisfies it. Let (a_1, a_2, c) and $(a_1', a_2', c') \in \mathbf{A}_1 \times \mathbf{A}_2 \times \mathbf{C}$; if $a_1, a_2, a_1', a_2' \geq 1$, then the third components of $\sigma(a_1, a_2, c)$ and of $\sigma(a_1', a_2', c')$ respectively, are $c \wedge 1$ and $c' \wedge 1$ respectively. Moreover, since $a_1 \cdot a_1', a_2 \cdot a_2' \geq 1$, the third component of $\sigma(a_1 \cdot a_1', a_2 \cdot a_2', c \cdot c')$ is $(c \cdot c') \wedge 1$. Since $(c \wedge 1) \cdot (c' \wedge 1) \leq (c \cdot c') \wedge 1$, the claim is settled. If $a_1, a_2 \geq 1$ but at least one between a_1' and a_2' is not ≥ 1, then the third component of $\sigma(a_1, a_2, c)$ is $c \wedge 1$ and the third component of $\sigma(a_1', a_2', c')$ is $c' \wedge c_0$. In this case the third component of $\sigma(a_1 \cdot a_1', a_2 \cdot a_2', c \cdot c')$ can be either $(c \cdot c') \wedge 1$ or $(c \cdot c') \wedge c_0$. Since $(c \wedge 1) \cdot (c' \wedge c_0) \leq (c \cdot c') \wedge c_0 \leq (c \cdot c') \wedge 1$, the claim is settled in both these cases. Similarly the case in which $a_1', a_2' \geq 1$ but at least one between a_1 and a_2 is not ≥ 1. In the end, we assume that at least one between a_1 and a_2 is not ≥ 1 and at least one between a_1' and a_2' is not ≥ 1. In this case the third components of $\sigma(a_1, a_2, c)$ and of $\sigma(a_1', a_2', c')$ respectively, are $c \wedge c_0$ and $c' \wedge c_0$ respectively, while the third component of $\sigma(a_1 \cdot a_1', a_2 \cdot a_2', c \cdot c')$ can be either $(c \cdot c') \wedge 1$ or $(c \cdot c') \wedge c_0$. Since $(c \wedge c_0) \cdot (c' \wedge c_0) \leq (c \cdot c') \wedge c_0 \leq (c \cdot c') \wedge 1$, the proof is finished. □

Lemma 3.4. *Let t_1 and t_2 be terms in the language of residuated lattices. If $\sigma_i(\mathbf{B}_i)$ does not satisfy $t_i \geq 1$ ($i = 1, 2$), then $\sigma(\sum(\mathbf{B}_1, \mathbf{B}_2, \mathbf{C}))$ does not satisfy $t_1 \vee t_2 \geq 1$.*

Proof. If $\sigma_1(\mathbf{B}_1)$ does not satisfy $t_1 \geq 1$, there exists an interpretation of t_1 into $\sigma_1(\mathbf{B}_1)$ (which we indicate with $t_1^{\sigma_1(B_1)}$) such that $t_1^{\sigma_1(B_1)}$ is not ≥ 1. Similarly, if $\sigma_2(\mathbf{B}_2)$ does not satisfy $t_2 \geq 1$, there exists an interpretation of t_2 into $\sigma_2(\mathbf{B}_2)$ (which we indicate with $t_2^{\sigma_2(B_2)}$) such that $t_2^{\sigma_2(B_2)}$ is not ≥ 1. We have to prove that there exists an interpretation of $t_1 \vee t_2$ into $\sigma(\sum(\mathbf{B}_1, \mathbf{B}_2, \mathbf{C}))$ such that $(t_1 \vee t_2)^{\sigma(\sum(B_1, B_2, C))}$ is not ≥ 1. The interpretations of t_1 and of t_2 respectively into $\sigma(\sum(\mathbf{B}_1, \mathbf{B}_2, \mathbf{C}))$, have the forms $(t_1^{\sigma_1(B_1)}, k, c \wedge c_0)$ and $(k', t_2^{\sigma_2(B_2)}, c' \wedge c_0)$ respectively. Thus, if we interpret $t_1 \vee t_2$ into $\sigma(\sum(\mathbf{B}_1, \mathbf{B}_2, \mathbf{C}))$, it has the last component equal to $(c \wedge c_0) \vee (c' \wedge c_0) \leq c_0 < 1$, so it cannot be $\geq (1, 1, 1)$. Therefore $t_1 \vee t_2 \geq 1$ is not valid in $\sigma(\sum(\mathbf{B}_1, \mathbf{B}_2, \mathbf{C}))$. □

The following theorem proves that the conuclear image of *any* variety of residuated lattices, and hence the conuclear image of *any* substructural logic, has the disjunction property.

Theorem 3.5. *For each variety $\mathcal{V}$ of residuated lattices, $\sigma(\mathcal{V})$ has the disjunction property, i.e., if $\sigma(\mathcal{V})$ satisfies $t_1 \vee t_2 \geq 1$, then there exists $i \in \{1, 2\}$ such that $\sigma(\mathcal{V})$ satisfies $t_i \geq 1$.*

Proof. We argue contrapositively. If $t_i \geq 1$ does not hold in $\sigma(\mathcal{V})$ for $i \in \{1, 2\}$, then there exist $\mathbf{B}_i = \langle \mathbf{A}_i, \sigma_i \rangle \in \mathcal{V}_\sigma$ for $i = 1, 2$, such that $t_i \geq 1$ does not hold in $\sigma_i(\mathbf{B}_i)$. Thus, by the previous lemma, $t_1 \vee t_2 \geq 1$ does not hold in $\sigma(\sum(\mathbf{B}_1, \mathbf{B}_2, \mathbf{C}))$, where $\mathbf{C}$ is an arbitrary and nontrivial algebra in $\mathcal{V}$. In conclusion, since $\sigma(\sum(\mathbf{B}_1, \mathbf{B}_2, \mathbf{C})) \in \sigma(\mathcal{V})$, $t_1 \vee t_2 \geq 1$ does not hold in $\sigma(\mathcal{V})$. □

3.1.1 Applications to logic

In the previous chapter we have built, starting from a substructural logic L, its conuclear image σ(L). We recall that σ(L) is a substructural logic whose theorems are those formulas A such that A^σ is a theorem of L_σ. In order to begin to investigate the relationship between L and σ(L), we state the following theorem:

Theorem 3.6. *L extends σ(L). Moreover, σ(L) has the disjunction property.*

Proof. For the first part, we argue contrapositively. Indeed, if A is not a theorem of L, then, taking $\sigma = id$ where id is the identical function, $A^\sigma = A$ and A^σ is not a theorem of L_σ. Therefore each theorem of σ(L) is also a theorem of L. The second part of the theorem follows from Theorem 3.5. □

Furthermore, the disjunction property gives us interesting information about the complexity of the decision problem of conuclear images of substructural logics. In fact, a recent work of Horčík and Terui ([12]) deals with the disjunction property in substructural logics. In addition to proving that a wide class of substructural logics satisfies the disjunction property, the authors prove the following result, involving the problem of complexity for substructural logics:

Theorem 3.7. *[12] Let L be a consistent substructural logic. The decision problem for L is coNP-hard. If L further satisfies the disjunction property, then it is PSPACE-hard.*

Therefore we can conclude the following result for the conuclear image of any substructural logic:

Theorem 3.8. *If L is a substructural logic, then L_σ and σ(L) are PSPACE-hard.*

Proof. The claim for $\sigma(\mathrm{L})$ follows from Theorem 3.7. As regards to L_σ, we observe that the map $A \mapsto A^\sigma$ reduces in polynomial time $\sigma(\mathrm{L})$ to L_σ, so the claim is settled. □

3.2 Properties excluded to hold in a conuclear image

To sum up, in the previous section we have seen that all conuclear images of any variety of residuated lattices have the disjunction property, even if the variety of residuated lattices does not have the disjunction property. This means that we have found a way, starting from a (possibly non-constructive) substructural logic, to obtain a constructive one. This result allows us to outline some properties which a conuclear image never satisfies.

It is well-known that the excluded middle ($x \vee \neg x \geq 1$) and the prelinearity axiom ($x\backslash y \vee y\backslash x \geq 1$) hold in classical logic but not in intuitionistic logic. Since intuitionistic logic is the conuclear image of classical logic, we can conclude that these properties are not preserved under conuclear images. Actually, using the previous results, we can conclude something stronger, namely, these properties never hold in a conuclear image, as they are in contrast with the disjunction property.

Since all conuclear images have the disjunction property, one might conjecture that every logic with the disjunction property is the conuclear image of some substructural logic. This conjecture is false and the counterexample is provided by the double negation axiom DN: $\neg\neg x = x$. Indeed, in [21], it is proved that **FL** plus the double negation axiom has the disjunction property. On the other hand, we present the following theorem which states that DN is another property excluded to hold in a conuclear image:

Theorem 3.9. *For any substructural logic L, its conuclear image does not satisfy the double negation principle.*

Proof. Let $\mathcal{V}$ be a variety of residuated lattices and $\mathbf{A}$ and $\mathbf{C}$ two algebras of the variety. We consider the direct product $\mathbf{A}\times\mathbf{C}$ and we define for all $(a,x)\in\mathbf{A}\times\mathbf{C}$

$$\sigma(a,x)=\begin{cases}(a,x\wedge 1) & \text{if } a\geq 1\\ (a,x\wedge c_0) & \text{otherwise}\end{cases}$$

where $c_0\in\mathbf{C}$, $c_0<1$ and $c_0\leq 0$.

We verify that σ is a conucleus on $\mathbf{A}\times\mathbf{C}$. We observe that σ is contracting, whatever the second component of σ is. As regards to idempotence, if $a\geq 1$, then $\sigma(\sigma(a,x))=\sigma(a,x\wedge 1)=(a,x\wedge 1\wedge 1)=(a,x\wedge 1)=\sigma(a,x)$. Otherwise, $\sigma(\sigma(a,x))=\sigma(a,x\wedge c_0)=(a,x\wedge c_0\wedge c_0)=(a,x\wedge c_0)=\sigma(a,x)$. Now we suppose that $(a,x)\leq(a',x')$, namely $a\leq a'$ and $x\leq x'$. If $a\geq 1$, then $\sigma(a,x)=(a,x\wedge 1)$. Since $a'\geq a\geq 1$, $\sigma(a',x')=(a',x'\wedge 1)$. Therefore, since $x\wedge 1\leq x'\wedge 1$, $\sigma(a,x)\leq\sigma(a',x')$. Instead, if a is not ≥ 1, $\sigma(a,x)=(a,x\wedge c_0)$. Then $\sigma(a',x')$ can be either $(a',x'\wedge 1)$ or $(a',x'\wedge c_0)$. Since both $x'\wedge 1$ and $x'\wedge c_0$ are greater than or equal to $x\wedge c_0$, monotonicity is proved. In order to prove the property $\sigma(a,x)\cdot\sigma(a',x')\leq\sigma(a\cdot a',x\cdot x')$, we have to distinguish four cases. If $a\geq 1$ and $a'\geq 1$, then $\sigma(a,x)=(a,x\wedge 1)$ and $\sigma(a',x')=(a',x'\wedge 1)$. Thus $\sigma(a,x)\sigma(a',x')=(aa',(x\wedge 1)(x'\wedge 1))$. Since $aa'\geq 1$, $\sigma(aa',xx')=(aa',xx'\wedge 1)$ and, since $(x\wedge 1)\cdot(x'\wedge 1)\leq xx'\wedge 1$, the claim is settled. If $a\ngeq 1$ and $a'\geq 1$, then $\sigma(a,x)=(a,x\wedge c_0)$ and $\sigma(a',x')=(a',x'\wedge 1)$, therefore $\sigma(a,x)\sigma(a',x')=(aa',(x\wedge c_0)(x'\wedge 1))$. In this case $\sigma(aa',xx')$ has the first component equal to aa', while the second component can be either $xx'\wedge 1$ or $xx'\wedge c_0$. Since both these two last quantities are greater than or equal to $(x\wedge c_0)(x'\wedge 1)$, also this case is verified. The case $a\geq 1$ and $a'\ngeq 1$ is symmetrical to the previous one. Now suppose that $a\ngeq 1$ and $a'\ngeq 1$. Then $\sigma(a,x)=(a,x\wedge c_0)$ and $\sigma(a',x')=(a',x'\wedge c_0)$, therefore $\sigma(a,x)\sigma(a',x')=(aa',(x\wedge c_0)(x'\wedge c_0))$. In this case $\sigma(aa',xx')$ has the first component equal to aa', while the second component can be either $xx'\wedge 1$ or $xx'\wedge c_0$. Since both these two last quantities are greater than or equal to $(x\wedge c_0)(x'\wedge c_0)$, we conclude that σ is a conucleus.

Now we prove that

$$\neg\neg(1,c_0)=\sigma(\sigma((1,c_0)\backslash(0,0))\backslash(0,0))\neq(1,c_0)$$

in $\sigma(\mathbf{A} \times \mathbf{C})$.

First, we consider $\sigma((1, c_0)\backslash(0, 0)) = \sigma(1\backslash 0, c_0\backslash 0) = \sigma(0, c_0\backslash 0)$, where, by definition of $\backslash$, $c_0\backslash 0 \geq 1$. Thus, $\sigma(0, c_0\backslash 0) = (0, (c_0\backslash 0) \wedge c_0) = (0, c_0)$. Now we consider $\sigma((0, c_0)\backslash(0, 0)) = \sigma(0\backslash 0, c_0\backslash 0)$, where $0\backslash 0 \geq 1$. So, $\sigma(0\backslash 0, c_0\backslash 0) = (0\backslash 0, (c_0\backslash 0) \wedge 1)) = (0\backslash 0, 1)$. Therefore $(1, c_0) < (0\backslash 0, 1) = \neg\neg(1, c_0)$ and the claim is settled. $\square$

In conclusion, while all conuclear images of substructural logics have the disjunction property, not all substructural logics with the disjunction property are conuclear images of a substructural logic.

Chapter 4

Properties compatible with conuclear images but not preserved under them

In this chapter we analyse two examples of properties compatible with conuclear images but not preserved under them. The examples are distributivity and divisibility. Both these properties hold in intuitionistic logic, which is the conuclear image of classical logic, so they are compatible with conuclear images. Nevertheless they are not preserved under conuclear images. The counterexample is provided by Montagna and Tsinakis in [18], where they prove that each commutative cancellative residuated lattice is the conuclear image of an abelian ℓ-group.

As regards to abelian ℓ-groups, it is well-known that their lattice reduct satisfies the distributive law, which can be expressed in the two equivalent forms: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. We now give a direct proof ([3]):

Theorem 4.1. *Each ℓ-group has a distributive lattice reduct.*

Proof. We prove that, given an ℓ-group $\mathbf{G}$ and $x, y, t \in G$, $x \vee (y \wedge t) = (x \vee y) \wedge (x \vee t)$. First, we have that $x \vee (y \wedge t) \leq x \vee y$ and $x \vee (y \wedge t) \leq x \vee t$. Thus $x \vee (y \wedge t) \leq (x \vee y) \wedge (x \vee t)$. In order to prove the reverse, set $z = y \wedge t$. Then, since $z \leq y$, $1 \leq yz^{-1}$. Hence, $x \leq yz^{-1}x \leq yz^{-1}(z \vee x)$. Since $z \leq z \vee x$, then

$1 \leq z^{-1}(z \vee x)$. Therefore $y \leq yz^{-1}(z \vee x)$. In conclusion, $x \vee y \leq yz^{-1}(z \vee x)$. Similarly we can prove that $x \vee t \leq tz^{-1}(z \vee x)$. Therefore

$$\begin{aligned}(x \vee y) \wedge (x \vee t) &\leq yz^{-1}(z \vee x) \wedge tz^{-1}(z \vee x)\\ &= (y \wedge t)z^{-1}(z \vee x)\\ &= zz^{-1}(z \vee x)\\ &= z \vee x\\ &= x \vee (y \wedge t).\end{aligned}$$

□

In the previous proof we have used a particular property that holds for ℓ-groups but not in general for residuated lattices, namely the operation of multiplication distributes over meets. Indeed we can state the following lemma:

Lemma 4.2. *([1]) Let h_i be an element of an ℓ-group $\mathbf{G}$ for all $i \in I$ and let $g \in G$. Then*

$$g \cdot (\bigwedge h_i) = \bigwedge (g \cdot h_i) \text{ and dually, } g \cdot (\bigvee h_i) = \bigvee (g \cdot h_i).$$

Proof. It is evident that $g \cdot (\bigwedge h_i) \leq g \cdot h_i$ for all $i \in I$. Suppose that $k \leq g \cdot h_i$ for all $i \in I$, namely k is a lower bound of $g \cdot h_i$ for all $i \in I$. Then $g^{-1}k \leq h_i$ and thus $g^{-1}k \leq \bigwedge h_i$. Therefore $k \leq g \cdot \bigwedge h_i$. Consequently $g \cdot \bigwedge h_i$ is the greatest lower bound of $g \cdot h_i$ for all $i \in I$ and $g \cdot \bigwedge h_i = \bigwedge (g \cdot h_i)$. □

As well as distributivity, ℓ-groups satisfy also the property of divisibility. Indeed, as we know, the two residuals in ℓ-groups are defined as $x \backslash y = x^{-1}y$ and $x/y = xy^{-1}$. Therefore, if we take $x \leq y$, we can define $u = y \backslash x$ and $z = x/y$ and we have

$$yu = y(y \backslash x) = y(y^{-1}x) = yy^{-1}x = x$$

and

$$zy = (x/y)y = xy^{-1}y = x.$$

Therefore, also divisibility holds in ℓ-groups.

On the contrary, as the below examples show, there exist commutative cancellative residuated lattices which are not distributive or divisible. The following examples are taken from [2] (Example 3.4 and Example 4.5).

Example 4.1: Let **F** be the universe of the free 2-generated commutative monoid on a and b. So we can define **F** as $\{a^m b^n : n, m \in \omega\}$. The length of the word $a^m b^n$ is the sum of the number of occurrences of a and of occurrences of b, namely $m+n$. We denote the length of a word u by $|u|$. We denote the empty word by e and order on **F** by dual shortex order, i.e., for words $u, v \in F$, we have $u \leq v$ iff $|u| > |v|$ or $|u| = |v|$ and $u <_{lex} v$, where $<_{lex}$ is the lexicographic order generated by $b < a$. For example

$$e > a > b > a^2 > ab > b^2 > a^3 > a^2b > ab^2 > b^3 > ...$$

The operation of multiplication is defined as follows:

$$a^m b^n \cdot a^k b^h = a^{m+k} b^{n+h}.$$

Therefore the multiplication is commutative, as well as associative.

Hence $\langle F, \cdot, e\rangle$ is a commutative monoid; indeed e is the unit of the operation of multiplication, since $u \cdot e = u$ for all $u \in F$.

Furthermore, we can observe that e is the maximum of **F**, since $u \leq e$ for all $u \in F$.

Moreover, multiplication is residuated. In fact, given two words $u, v \in F$, $u \to v = max\,\{k : k \cdot u \leq v\}$ and this max always exists. Indeed, if $u \leq v$, then $u \to v = e$, since $u \cdot e = u \leq v$ and e is the maximum of **F**. Otherwise, if $u > v$, then either u is shorter that v or u and v have the same length but u has more occurrences of a than v. In the first case, we search all the words k such that the length of $u \cdot k$ is the same as v and we take the biggest of these words k such that $k \cdot u \leq v$. If none of these words k are such that $k \cdot u \leq v$, then we take the biggest word k such that the length of $k \cdot u$ is greater than the length of v (in this case k has the form a^z for some $z \in \omega$). Therefore, this max exists since we have to check a finite number of words, due to the fact that the number of words of fixed length is finite. In the latter case, $u \to v = a$.

Therefore $\mathbf{F} = \langle F, \wedge, \vee, \cdot, \to, e\rangle$ is a commutative, integral residuated chain.

As regards to cancellativity, we have to prove that if $u \cdot w = v \cdot w$, then $u = v$, where $u = a^m b^n$, $v = a^h b^k$ and $w = a^g b^j$. If $u \cdot w = v \cdot w$, then $a^{m+g} b^{n+j} = a^{h+g} b^{k+j}$. Consequently, $m+g = h+g$ and $n+j = k+j$. Therefore, due to the cancellativity of $\mathbb{N}$, $m = h$ and $n = k$, hence $u = v$.

In conclusion, $\mathbf{F} = \langle F, \wedge, \vee, \cdot, \to, e\rangle$ is a cancellative, commutative, integral residuated chain.

Since $\mathbf{F}$ is integral, we have seen in Chapter 1 that the condition of divisibility

$$\text{if } x \leq y \text{ then there exist } u, z \text{ such that } yu = zy = x,$$

is equivalent to the condition

$$y(y \backslash x) = x \wedge y = (x/y)y.$$

Now we prove that this last condition does not hold in $\mathbf{F}$; in fact $a(a \to b) = a^2$, since $a \to b = max\,\{c \in F : a \cdot c \leq b\} = a$. On the other hand, $a \wedge b = b$ and $a^2 \neq b$.

Therefore $\mathbf{F}$ is not divisible and this example proves that divisibility is not preserved under conuclear images.

Example 4.2: Let $\mathbf{F} = \mathbf{F}_{CM}(a, b, c) = \left\{a^i b^j c^k : i, j, k \in \omega\right\}$ be the 3-generated free commutative monoid. For a word $\alpha \in F$, we denote the length of α by $|\alpha|$, and for $x \in \{a, b, c\}$, we define $|\alpha|_x$ to be the number of occurrences of x in α. The order on $\mathbf{F}$ is defined by $\alpha \leq \beta$ if $|\alpha| > |\beta|$, or $|\alpha| = |\beta|, |\alpha|_b \geq |\beta|_b$ and $|\alpha|_c \geq |\beta|_c$.

It is possible to observe that each block of words of the same length is a finite join-subsemilattice of a product of two chains, hence $\mathbf{F}$ is a lattice in which every join of elements is attained by a finite subjoin.

Recall that a binary operation on a join-complete lattice is residuated iff it distributes over arbitrary joins. In order to see that the monoid operation of $\mathbf{F}$

is residuated, it therefore suffices to show that it distributes over finite joins. For two words α, β of the same length, we have $|\alpha \vee \beta|_b = min(|\alpha|_b, |\beta|_b)$, so

$$\begin{aligned} |(\alpha \vee \beta)\gamma|_b &= min(|\alpha|_b, |\beta|_b) + |\gamma|_b \\ &= min(|\alpha|_b + |\gamma|_b, |\beta|_b + |\gamma|_b) \\ &= |\alpha\gamma \vee \beta\gamma|_b \end{aligned}$$

and similarly for $|\ |_c$.

Therefore **F** is a residuated lattice and, since the underlying commutative monoid is freely generated, **F** is cancellative.

Finally, we can prove that it is not distributive since $bb \vee (ab \wedge cc) = bb \vee aaa = bb$, while $(bb \vee ab) \wedge (bb \vee cc) = ab \wedge aa = ab$.

In conclusion, this example shows that distributivity is not preserved under conuclear images.

Chapter 5

Preservation under conuclear images

In this chapter we face up to the problem of preservation under conuclear images. In other words, we want to discover which properties are always preserved from L to σ(L), whichever the substructural logic L is.

Given f and g terms in the language of residuated lattices, we say that an inequation $f \leq g$ is preserved under conuclear images when, given a residuated lattice $\mathbf{A}$, if $f \leq g$ holds in $\mathbf{A}$, then $f^\sigma \leq g^\sigma$ holds in $\sigma(\mathbf{A})$.

The map $^\sigma$, defined in the previous chapter from L to L_σ, can be also applied to terms in the language of residuated lattices (possibly endowed with a constant 0) in the following way:

$x^\sigma = \sigma(x)$, where x is a variable;

$1^\sigma = \sigma(1) = 1$;

$0^\sigma = \sigma(0)$;

$(f \circ g)^\sigma = f^\sigma \circ g^\sigma$ for $\circ \in \{\vee, \cdot\}$;

$(f \circ g)^\sigma = \sigma(f^\sigma \circ g^\sigma)$ for $\circ \in \{\wedge, \backslash, /\}$;

The aim of this investigation is to characterize the substructural logics which are invariant under conuclear images and therefore coincide with their conuclear image. We start our analysis considering particular properties, which are familiar to substructural logics and residuated lattices, and then we try to deal with the general case. In the end, we provide a sufficient condition in order that an inequation is preserved under conuclear images.

5.1 Properties preserved under conuclear images

We start considering the property of integrality. We recall that a residuated lattice $\mathbf{A}$ is integral if $x \leq 1$ for all $x \in A$. We observe that integrality is preserved under conuclear images. In fact, in order to prove that $\sigma(\mathbf{A})$ satisfies $\sigma(x) \leq \sigma(1)$ for all $x \in A$, note that $\sigma(1) = 1$ and $\sigma(A) \subseteq A$. Therefore $\sigma(x) \in A$ and, by the integrality of $\mathbf{A}$, $\sigma(x) \leq 1$.

Now we suppose that $\mathbf{A}$ is a contractive residuated lattice, namely $x \leq x \cdot x$ for all $x \in A$. We want to prove that also $\sigma(\mathbf{A})$ is contractive, i.e., $\sigma(x) \leq \sigma(x)\sigma(x)$ for all $x \in A$. Since $\sigma(A) \subseteq A$ and the multiplication of $\sigma(\mathbf{A})$ is the same as in $\mathbf{A}$, the claim follows from the property of contraction of $\mathbf{A}$.

Let $\mathbf{A}$ be a commutative residuated lattice, i.e., $x \cdot y = y \cdot x$ for all $x, y \in A$. Since $\sigma(A) \subseteq A$, $\sigma(x), \sigma(y) \in A$ for all $x, y \in A$. Therefore, since the multiplication in $\mathbf{A}$ coincides with the multiplication in $\sigma(\mathbf{A})$, $\sigma(x) \cdot \sigma(y) = \sigma(y) \cdot \sigma(x)$ for all $x, y \in A$, because of the commutativity of $\mathbf{A}$. Hence, also $\sigma(\mathbf{A})$ is a commutative residuated lattice and commutativity is preserved under conuclear images.

Now we consider an idempotent residuated lattice. We recall that a residuated lattice $\mathbf{A}$ is *idempotent* if $x \cdot x = x$ for all $x \in A$. As we have proved for commutativity and contraction, idempotence is preserved under conuclear images due to the fact that $\sigma(A) \subseteq A$ and the multiplication in $\mathbf{A}$ is the same as in $\sigma(\mathbf{A})$.

As we know, by Lemma 1.5, a residuated lattice is cancellative if and only if it satisfies the identity $xy/y = x = y\backslash yx$. Let $\mathbf{A}$ be a cancellative residuated lattice. In order to prove that $\sigma(\mathbf{A})$ is cancellative, we have to check that $\sigma(\mathbf{A})$ satisfies the identity

$$\sigma(x)\sigma(y)/_{\sigma}\sigma(y) = \sigma(x) = \sigma(y)\backslash_{\sigma}\sigma(y)\sigma(x).$$

We verify the first identity; in fact, by definition of $/_\sigma$,

$$\sigma(x)\sigma(y)/_\sigma\sigma(y) = \sigma(\sigma(x)\sigma(y)/\sigma(y)).$$

Since $\mathbf{A}$ is cancellative and $\sigma(x), \sigma(y) \in A$, we have that

$$\sigma(x)\sigma(y)/\sigma(y) = \sigma(x).$$

Substituting in the previous expression,

$$\sigma(\sigma(x)\sigma(y)/\sigma(y)) = \sigma(\sigma(x)) = \sigma(x),$$

where the last equality follows from the idempotence of the conucleus σ.

Therefore also $\sigma(\mathbf{A})$ is cancellative and cancellativity is preserved under conuclear images.

Also weak contraction ($x \wedge \neg x \leq 0$) is preserved under conuclear images. Indeed, let $\mathbf{A}$ be a residuated lattice which satisfies the inequation $x \wedge x\backslash 0 \leq 0$. Thus, we have to verify that $\sigma(\mathbf{A})$ satisfies

$$\sigma(x) \wedge_\sigma \sigma(x)\backslash_\sigma\sigma(0) = \sigma(\sigma(x) \wedge \sigma(\sigma(x)\backslash\sigma(0))) \leq \sigma(0).$$

(In this case we are considering a residuated lattice $\mathbf{A}$ with an additional constant 0 which corresponds in $\sigma(\mathbf{A})$ to the constant $\sigma(0)$). In fact

$$\sigma(x) \wedge \sigma(\sigma(x)\backslash\sigma(0)) \leq \sigma(x) \wedge \sigma(x)\backslash\sigma(0) \leq \sigma(x) \wedge \sigma(x)\backslash 0 \leq 0,$$

where the first two inequalities follow from the fact that the conucleus σ is contracting and that the residuals are order-preserving in the numerators; the last inequality instead is due to the fact that $\sigma(A) \subseteq A$ and $\mathbf{A}$ satisfies weak contraction. At this point, applying σ to the first and the last element, we obtain the statement.

5.2 General case

Now we try to find general conditions in order that an inequation is preserved under conuclear images.

To begin with, we state the following lemmas:

Lemma 5.1. *Let f and g be terms in the language of residuated lattices. If f and g contain only $\vee$, $\cdot$ and 1, then the inequation $f \leq g$ is preserved under conuclear images.*

Proof. Since $\vee$, $\cdot$ and 1 are the same in $\mathbf{A}$ and $\sigma(\mathbf{A})$, $f^\sigma(\sigma(x_1), ..., \sigma(x_n)) = f(\sigma(x_1), ..., \sigma(x_n))$ and $g^\sigma(\sigma(x_1), ..., \sigma(x_n)) = g(\sigma(x_1), ..., \sigma(x_n))$. Moreover, since $\sigma(A) \subseteq A$ and we are supposing that $f \leq g$ holds in $\mathbf{A}$, $f(\sigma(x_1), ..., \sigma(x_n)) \leq g(\sigma(x_1), ..., \sigma(x_n))$ and the claim is settled. □

Lemma 5.2. *Let f be a term in the language of residuated lattices. If f is a term such that any occurrence of $/$ or $\backslash$ is of the form f_1/f_2 or $f_2\backslash f_1$ where f_2 only contains $1, \vee$ and $\cdot$, then*

$$f^\sigma(\sigma(x_1), ..., \sigma(x_n)) \leq f(\sigma(x_1), ..., \sigma(x_n)).$$

Proof. We will prove the lemma by induction on complexity of f.
Base case:

- If f is a variable: $f^\sigma(\sigma(x_i)) = \sigma(x_i) \leq \sigma(x_i) = f(\sigma(x_i))$.
- If $f = 1$, then $f^\sigma = \sigma(1) \leq 1 = f$.
- If $f = 0$, then $f^\sigma = \sigma(0) \leq 0 = f$.

Inductive steps:

- if $f = f_1 \vee f_2$, then
$$\begin{aligned} f^\sigma(\sigma(x_1), ..., \sigma(x_n)) &= (f_1 \vee f_2)^\sigma(\sigma(x_1), ..., \sigma(x_n)) \\ &= f_1^\sigma \vee f_2^\sigma(\sigma(x_1), ..., \sigma(x_n)) \\ &\leq f_1(\sigma(x_1), ..., \sigma(x_n)) \vee f_2(\sigma(x_1), ..., \sigma(x_n)), \end{aligned}$$
where the last inequality follows from the induction hypothesis.
- if $f = f_1 \cdot f_2$ then
$$\begin{aligned} f^\sigma(\sigma(x_1), ..., \sigma(x_n)) &= (f_1 \cdot f_2)^\sigma(\sigma(x_1), ..., \sigma(x_n)) \\ &= f_1^\sigma \cdot f_2^\sigma(\sigma(x_1), ..., \sigma(x_n)) \\ &\leq f_1(\sigma(x_1), ..., \sigma(x_n)) \cdot f_2(\sigma(x_1), ..., \sigma(x_n)). \end{aligned}$$

- if $f = f_1 \wedge f_2$, then

$$\begin{aligned} f^\sigma(\sigma(x_1),...,\sigma(x_n)) &= (f_1 \wedge f_2)^\sigma(\sigma(x_1),...,\sigma(x_n)) \\ &= \sigma(f_1^\sigma \wedge f_2^\sigma)(\sigma(x_1),...,\sigma(x_n)) \\ &\leq (f_1^\sigma \wedge f_2^\sigma)(\sigma(x_1),...,\sigma(x_n)) \\ &\leq f_1(\sigma(x_1),...,\sigma(x_n)) \wedge f_2(\sigma(x_1),...,\sigma(x_n)), \end{aligned}$$

- if $f = f_1/f_2$ where f_2 only contains $\vee$, $\cdot$ and 1, then

$$\begin{aligned} f^\sigma(\sigma(x_1),...,\sigma(x_n)) &= (f_1/f_2)^\sigma(\sigma(x_1),...,\sigma(x_n)) \\ &= \sigma(f_1^\sigma/f_2^\sigma)(\sigma(x_1),...,\sigma(x_n)) \\ &\leq (f_1^\sigma/f_2^\sigma)(\sigma(x_1),...,\sigma(x_n)) \\ &\leq f_1(\sigma(x_1),...,\sigma(x_n))/f_2(\sigma(x_1),...,\sigma(x_n)), \end{aligned}$$

  due to the fact that $f_1^\sigma(\sigma(x_1),...,\sigma(x_n)) \leq f_1(\sigma(x_1),...,\sigma(x_n))$ by the induction hypothesis and, since f_2 only contains $\vee$, $\cdot$ and 1,
  $f_2^\sigma(\sigma(x_1),...,\sigma(x_n)) = f_2(\sigma(x_1),...,\sigma(x_n))$.

- The case $f = f_2 \backslash f_1$ is similar to the previous one.

□

Corollary 5.3. *Let f and g be terms in the language of residuated lattices. If f is a term such that any occurrence of $/$ or $\backslash$ is of the form f_1/f_2 or $f_2 \backslash f_1$ where f_2 only contains $1, \vee$ and $\cdot$, and g only contains $\vee$, $\cdot$ and 1, then $f \leq g$ is preserved under conuclear images.*

Proof. Suppose that $f \leq g$ holds in $\mathbf{A}$. Then we have:
$f^\sigma(\sigma(x_1),...,\sigma(x_n)) \leq f(\sigma(x_1),...,\sigma(x_n))$ by the previous lemma,
$f(\sigma(x_1),...,\sigma(x_n)) \leq g(\sigma(x_1),...,\sigma(x_n))$ since $\sigma(A) \subseteq A$, and
$g(\sigma(x_1),...,\sigma(x_n)) = g^\sigma(\sigma(x_1),...,\sigma(x_n))$ due to the fact that g only contains $\vee$, $\cdot$ and 1.
Thus $f^\sigma(\sigma(x_1),...,\sigma(x_n)) \leq g^\sigma(\sigma(x_1),...,\sigma(x_n))$ and therefore the inequation $f \leq g$ is preserved under conuclear images. □

Now we consider a particular classification of terms in the language of residuated lattices: it is called *substructural hierarchy* and it is introduced in [4]. This hierarchy classifies logical formulas according to their syntactic complexity, namely how difficult they are to deal with. It has been created with the aim to find an algebraic characterization to cut elimination. In particular, in [4], it is shown that a stronger form of cut elimination and Dedekind-McNeille completion coincide up to level N_2 in the substructural hierarchy. In the thesis we use the same classes to find a condition in order that an inequation is preserved under conuclear images. It is surprising that the same classes that provide an algebraic characterization of cut elimination, are useful for investigating properties preserved under conuclear images. Actually, we use a slight different version of classes P_n and N_n, due to the fact that we are not considering logics with $\bot$ and $\top$.

Definition 5.4. *For each $n \geq 0$, the sets P_n, N_n of terms are defined as follows:*
(0) $P_0 = N_0 =$the set of variables.
(P1) 1 and all terms of N_n belong to P_{n+1}.
(P2) If $t, u \in P_{n+1}$, then $t \vee u, t \cdot u \in P_{n+1}$.
(N1) 0 and all terms of P_n belong to N_{n+1}.
(N2) If $t, u \in N_{n+1}$, then $t \wedge u \in N_{n+1}$.
(N3) If $t \in P_{n+1}$ and $u \in N_{n+1}$, then $t\backslash u$, $u/t \in N_{n+1}$.

The next two lemmas are taken from [4] (Proposition 3.2 and Lemma 3.3).

Lemma 5.5. *Every term belongs to some P_n and N_n. Furthermore, $P_n \subseteq P_{n+1}$ and $N_n \subseteq N_{n+1}$ for every n.*

Lemma 5.6. *(P) If $t \in P_{n+1}$, then t is equivalent to $u_1 \vee ... \vee u_m$, where each u_i is a product of terms in N_n.*
(N) If $t \in N_{n+1}$, then t is equivalent to $\bigwedge_{1\leq i\leq m} l_i\backslash u_i/r_i$, where each u_i is either 0 or a term in P_n and each l_i and r_i are products of terms in N_n.

Proof. The lemma is proved by simultaneous induction of the two statements: (P) and (N). As regards to statement (P), the case $t = 1$ is a special case for

$m = 1$ and u_1 the empty product. If (P) holds for $t, u \in P_{n+1}$, then it clearly holds for $t \vee u$. For $t \cdot u$, we use the fact that multiplication distributes over joins.

As regards to statement (N), if $t = 0$, then we take $m = 1$, $l_1 = r_1 = 1$ and $u_1 = 0$. If (N) holds for $t, u \in N_{n+1}$, then it clearly holds for $t \wedge u$. If $t \in P_{n+1}$ and $u \in N_{n+1}$, we know that $t = t_1 \vee ... \vee t_m$ for t_i product of terms in N_n. We have $t \backslash u = (t_1 \vee ... \vee t_m) \backslash u = (t_1 \backslash u) \wedge ... \wedge (t_m \backslash u)$. Moreover, by the induction hypothesis, for all $j \in \{1, ..., m\}$, $t_j \backslash u = t_j \backslash (\bigwedge_{1 \leq i \leq k} l_i \backslash u_i / r_i) = \bigwedge_{1 \leq i \leq k} t_j \backslash (l_i \backslash u_i / r_i) = \bigwedge_{1 \leq i \leq k} (l_i t_j) \backslash u_i / r_i$ and the claim is settled. □

We can state the following theorem, which provides a sufficient condition in order that an inequation is preserved under conuclear images.

Theorem 5.7. *Let f and g be terms in the language of residuated lattices. If $f \in P_2$ and $g \in N_2$, then $f \leq g$ is preserved under conuclear images.*

Proof. If $f \in P_2$, then, by Lemma 5.6, $f = u_1 \vee ... \vee u_m$, where for all i, u_i is a product of terms in N_1. Instead, if $g \in N_2$, then, using Lemma 5.6 again, $g = \bigwedge_{1 \leq i \leq k} l_i \backslash v_i / r_i$, where $v_i = 0$ or $v_i \in P_1$ and l_i and r_i are products of terms in N_1. Thus we consider the inequation

$$u_1 \vee ... \vee u_m \leq \bigwedge_{1 \leq i \leq k} l_i \backslash v_i / r_i$$

which is equivalent to

$$u_1 \leq \bigwedge_{1 \leq i \leq k} l_i \backslash v_i / r_i \text{ and...and } u_m \leq \bigwedge_{1 \leq i \leq k} l_i \backslash v_i / r_i$$

which is equivalent to

$$u_1 \leq l_1 \backslash v_1 / r_1 \text{ and...and } u_1 \leq l_k \backslash v_k / r_k \text{ and...and } u_m \leq l_1 \backslash v_1 / r_1 \text{ and...and } u_m \leq l_k \backslash v_k / r_k.$$

Therefore we can consider an inequation $u_i \leq l_i \backslash v_i / r_i$, which is equivalent, by the residuation law, to

$$l_i \cdot u_i \cdot r_i \leq v_i.$$

$l_i \cdot u_i \cdot r_i$ is a product of terms in N_1, namely, by Lemma 5.6, it is a product of $\bigwedge_{1 \leq i \leq s} l'_i \backslash v'_i / r'_i$, where v'_i is 0 or a variable and l'_i and r'_i are products of variables.

Therefore on the left of $\leq$, we have only occurrences of $\cdot, \wedge, 0, \backslash, /$ and in any occurrence of $\backslash$ and of $/$, the denominators are products of variables. Thus, if $v_i \in P_1$, using Corollary 5.3, we can conclude that this inequation is preserved under conuclear images. Instead, if $v_i = 0$, the inequation becomes

$$l_i \cdot u_i \cdot r_i \leq 0.$$

In this case, we have to prove that $l_i^\sigma \cdot u_i^\sigma \cdot r_i^\sigma \leq \sigma(0)$. Since $l_i \cdot u_i \cdot r_i$ fulfills the hypothesis of Lemma 5.2,

$$l_i^\sigma \cdot u_i^\sigma \cdot r_i^\sigma(\sigma(x_1), ..., \sigma(x_n)) \leq l_i(\sigma(x_1), ..., \sigma(x_n)) \cdot u_i(\sigma(x_1), ..., \sigma(x_n)) \cdot r_i(\sigma(x_1), ..., \sigma(x_n)) \leq 0.$$

Applying σ to the first and the last element of the above chain,

$$\sigma(l_i^\sigma \cdot u_i^\sigma \cdot r_i^\sigma) \leq \sigma(0).$$

As we know, $l_i^\sigma \cdot u_i^\sigma \cdot r_i^\sigma$ is an element of $\sigma(A)$, so it is a fixed point of σ, i.e., $\sigma(l_i^\sigma \cdot u_i^\sigma \cdot r_i^\sigma) = l_i^\sigma \cdot u_i^\sigma \cdot r_i^\sigma$. In conclusion, $l_i^\sigma \cdot u_i^\sigma \cdot r_i^\sigma \leq \sigma(0)$ and the inequation is preserved also in this case. □

Commutativity, integrality, contraction, weak contraction and idempotence satisfy the hypothesis of Theorem 5.7. Therefore substructural logics axiomatized by these identities are invariant under conuclear images and they coincide with their conuclear image.

Nevertheless, cancellativity does not fall into the scope of Theorem 5.7, although it is clearly preserved under conuclear images, as we have shown in the previous section. In fact, if we consider the verse $xy/y \leq x$, we have that $g = x \in N_2$ because g is a variable, while $f = xy/y \in N_2$ since $xy \in N_2$ and $y \in P_2$, but $f \notin P_2$.

The next result extends Theorem 5.7 and includes cancellativity as a special case.

We define P_2^* as the smallest class such that:

- $P_2 \subseteq P_2^*$;
- if $t, u \in P_2^*$, then $t \wedge u, t \vee u, t \cdot u \in P_2^*$;

- if $f \in P_2^*$ and $g \in P_1$, then $g\backslash f, f/g \in P_2^*$.

In other words, P_2^* is the smallest class containing P_2 and closed under $\wedge, \vee, \cdot$ and divisions $\backslash$ and $/$ with denominators in P_1.

Theorem 5.8. *Let f and g be terms in the language of residuated lattices. If $f \in P_2^*$ and $g \in N_2$, then $f \leq g$ is preserved under conuclear images.*

Proof. By induction on the definition of P_2^*, we prove that if $f \in P_2^*$, then f fulfills the hypothesis of Lemma 5.2 (the base case follows from the previous theorem). At this point, the argument is very similar to the proof of Theorem 5.7. □

Although Theorem 5.8 is more general than Theorem 5.7 and includes cancellativity as a special case, it may be further generalized. In fact, it is possible to extend also the class N_2 for g.

Let us define N_2^* similarly to N_2 but with axiom (N3) replaced by (N3'):
(N1) 0 and all terms of P_1 belong to N_2^*.
(N2) If $t, u \in N_2^*$, then $t \wedge u \in N_2^*$.
(N3') If $t \in P_2^*$ and $u \in N_2^*$, then $u/t, t\backslash u \in N_2^*$.

Therefore, denominators in N_2^* can also be P_2^*-terms.

Theorem 5.9. *Let f and g be terms in the language of residuated lattices. If $f \in P_2^*$ and $g \in N_2^*$, then $f \leq g$ is preserved under conuclear images.*

Proof. Let $g = g_1/g_2$ or $g = g_2\backslash g_1$, where $g_1 \in N_2^*$ and $g_2 \in P_2^*$. So the inequation $f \leq g_1/g_2$ is equivalent to

$$f \cdot g_2 \leq g_1$$

and we can easily return to the hypothesis of Theorem 5.8. □

We may wonder whether the condition in Theorem 5.9 characterizes the varieties of pointed residuated lattices closed under conuclear images. The next examples prove that the generalizations cannot be so strong, in the sense that if we further relax the constraints in Theorem 5.9, we meet some counterexamples:

1. As we know, a conuclear image never satisfies $1 \leq x \vee \neg x$. In this case $f = 1$, so $f \in P_2^*$, while $g = x \vee \neg x = x \vee x\backslash 0$ and $g \in P_2$ but $g \notin N_2^*$.

2. The axiom of prelinearity, $1 \leq x/y \vee y/x$, never holds in a conuclear image. In this case $f = 1 \in P_2^*$ and $g = x/y \vee y/x \in P_2$ but $\notin N_2^*$.

3. Distributivity, $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$, is not preserved under conuclear images: in this case $f = x \wedge (y \vee z) \in P_2^*$ and $g = (x \wedge y) \vee (x \wedge z) \in P_2$ but $\notin N_2^*$.

4. A conuclear image never satisfies the double negation law: $\neg\neg x \leq x$. In this case $g = x \in N_2^*$, whereas $f = (x\backslash 0)\backslash 0 \in N_2$ but $\notin P_2^*$.

5. The property of divisibility, which can be expressed in the form $x(x\backslash(x \wedge y)) = x \wedge y$, is not preserved under conuclear image. In fact if we take $f = x \wedge y$, then $f \in P_2^*$, while if $g = x(x\backslash(x \wedge y))$, then $g \in P_2^*$ but $g \notin N_2^*$.

Conclusions

The aim of the thesis was to investigate the relationship between a substructural logic L and its conuclear image $\sigma(\mathrm{L})$. In particular we have analysed which properties:

- are preserved under conuclear images;
- never hold in a conuclear image;
- are not preserved under conuclear images but are compatible with conuclear images.

We sum up the results shown in the thesis in Table 5.1. In the last column, P stands for "preserved under conuclear images", N stands for "never holds in a conuclear image" and NPC stands for "not preserved but compatible with conuclear images".

In the thesis we have seen that, while all conuclear images have the disjunction property, not all substructural logics with the disjunction property are conuclear images of a substructural logic. Hence, being a conuclear image seems to be a stronger and more constructive property than the disjunction property. It would be interesting to find a classification of axioms according to their "constructive" character (concept not defined yet) and to prove that constructive axioms, like commutativity, integrality, idempotence, contraction, weak contraction, etc..., are always preserved under conuclear images, non-constructive axioms, such as excluded middle, prelinearity, double negation,..., never hold in a conuclear image, and neutral axioms, like distributivity and divisibility, may hold in a conuclear image but are not necessarily preserved.

Equation	**Name**	**Behaviour**
$xy \leq yx$	Commutativity	P
$x \leq 1$	Left weakening	P
$0 \leq x$	Right weakening	P
$x \leq xx$	Contraction	P
$x = xx$	Idempotence	P
$x^n \leq x^m$	Knotted $(n, m \geq 0)$	P
$x \wedge \neg x \leq 0$	Weak contraction	P
$xy/y = x = y\backslash yx$	Cancellativity	P
$1 \leq x \vee \neg x$	Excluded middle	N
$1 \leq (x\backslash y) \vee (y\backslash x)$	Prelinearity	N
$1 \leq \neg x \vee \neg\neg x$	Weak excluded middle	N
$\neg\neg x \leq x$	Double negation	N
$x(x\backslash(x \wedge y)) = x \wedge y = ((x \wedge y)/x)x$	Divisibility	NPC
$x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$	Distributivity	NPC

Table 5.1: Some properties

Moreover, it would be interesting to present the construction in terms of categorical equivalence, thus generalizing equivalence results introduced in [17] and [18].

Another task could be to investigate varieties of residuated lattices endowed with particular conuclei, namely conuclei which satisfy further interesting properties, as well as the typical properties of conuclei.

For instance, in Chapter 1, we have considered the unary operator ! of linear logic and we have seen that its algebraic counterpart is a conucleus. As well as the properties of conuclei, ! satisfies the property

$$!x{\cdot}!x =!x.$$

In other words, the image under the conucleus consists of idempotent elements. The above property for ! follows directly from one of the axioms of linear logic. It would be interesting to investigate the conuclear image of a variety of residuated lattices under a special conucleus, in particular conuclei which satisfy the above property. Moreover, we could inspect which properties we have to suppose on a conucleus in order that particular axioms, generally not preserved, are preserved under conuclear images and see if these properties are not only sufficient but also necessary for the preservation of these particular axioms.

Finally, we propose the following topic. We can observe that the conuclear image of any intermediate variety between the variety of Boolean algebras and the variety of Heyting algebras is the variety of Heyting algebras. Similarly, if we consider an intermediate variety between the variety of abelian ℓ-groups and the variety of commutative cancellative residuated lattices, its conuclear image is the variety of commutative cancellative residuated lattices. Thus, we can say that the variety of Boolean algebras is the smallest variety whose conuclear image is the variety of Heyting algebras, while the variety of abelian ℓ-groups is the smallest variety whose conuclear image is the variety of commutative cancellative residuated lattices. From a logical point of view, classical logic is the strongest logic whose conuclear image is intuitionistic logic, and a similar result can be stated for the logic of abelian ℓ-groups and the logic of commutative cancellative residuated lattices. Now we take a substructural logic L closed under conuclear images. It would be interesting to find, if it exists, the strongest logic L$'$ such that $\sigma(\mathrm{L}')=\mathrm{L}$.

Bibliography

[1] M.E.Anderson and T.H.Feil. *Lattice-ordered groups: an introduction*. Springer Science & Business Media, pp. 190, (1988).

[2] P.Bahls, J.Cole, N.Galatos, P.Jipsen and C.Tsinakis. *Cancellative residuated lattices*. Algebra Universalis 50 (1), 83-106, (2003).

[3] A.Bigard, K.Keimel and S.Wolfenstein. *Groupes et anneaux reticules (Lectures Notes in Mathematics)*. Springer, XIII, pp. 334, (1977).

[4] A.Ciabattoni, N.Galatos and K.Terui. *Algebraic proof theory for substructural logics: cut-elimination and completions*. Annals of Pure and Applied Logic 163 (3), 266-290, (2012).

[5] Micheal R.Darnel. *Theory of lattice-ordered groups*. Pure and applied mathematics, pp. 400, (1994).

[6] P.Dubreil. *Sur le problèmes d'immersion et la théorie des modules*. C. R. Acad. Sci. Paris 216, 625-627, (1943).

[7] N.Galatos and P.Jipsen. *Residuated frames with applications to decidability*. Transaction of the American Mathematical Society 365 (3), 1219-1249, (2013).

[8] N.Galatos, P.Jipsen, T.Kowalski and H.Ono. *Residuated lattices: an algebraic glimpse at substructural logics*. Studies in logic and the foundations of mathematics 151, (2007).

[9] J.Y.Girard. *Linear logic*. Theoretical Computer Science 50, 1-102, (1987).

[10] N.Galatos and H.Ono. *Algebraization, parametrized local deduction theorem and interpolation for substructural logics over FL*. Studia Logica 83 (1-3), 279-308, (2006).

[11] J.B.Hart, L.Rafter and C.Tsinakis. *The structure of commutative residuated lattices*. Int. J. Algebra Comput. 12 (4), 509-525, (2001).

[12] R.Horčík and K.Terui. *Disjunction property and complexity of substructural logics*. Theoretical Computer Science 412 (31), 3992-4006, (2011).

[13] P.Jipsen and F.Montagna. *Embedding theorems for classes of GBL-algebras*. Journal of Pure and Applied Algebra 214 (9), 1559-1575, (2010).

[14] P.Jipsen and F.Montagna. *The Block-Ferreirim for normal GBL-algebras and application*. Algebra Universalis 60, 381-404, (2009).

[15] P.Jipsen and C.Tsinakis. *A survey of residuated lattices*. Ordered algebraic structures (J.Martinez, editor), Kluwer Academic Publishers, Dordrecht, 19-56, (2002).

[16] J.C.C.McKinsey and A.Tarski. *The algebra of topology*. Annals of Mathematics 45 (1), 141-191, (1944).

[17] J.C.C.McKinsey and A.Tarski. *Some theorems about the sentential calculi of Lewis and Heyting*. The Journal of Symbolic Logic 13, 1-15, (1948).

[18] F.Montagna and C.Tsinakis. *Ordered groups with a conucleus*. Journal of Pure and Applied Algebra 214 (1), 71-88, (2010).

[19] H.Ono. *Substructural logics and residuated lattices - An introduction*. Trends in Logica 20, 177-212, (2003).

[20] F.Paoli. *Substructural logics: a primer*. Trends in Logic-Studia Logica Library, Kluwer Academic Publisher, pp. 304, (2002).

[21] D.Souma. *An algebraic approach to the disjunction property of substructural logics*. Notre Dame Journal of Formal Logic 48 (4), 489-495, (2007).

[22] A.Ursini. *Girales: the algebra of linear logic*. Technical Report n. 473, University of Siena, (2005).

[23] W.Young. *From interior algebras to unital ℓ-groups: a unifying treatment of modal residuated lattices*. Studia Logica, 1-22, (2014).