UNIVERSITÀ DEGLI STUDI DI MILANO

DIPARTIMENTO DI MATEMATICA "FEDERIGO ENRIQUES"

Corso di Laurea Magistrale in Matematica

Stone duality above dimension zero

Relatore:
Prof. Vincenzo Marra

Laureando:
Luca Reggio

Anno Accademico 2013–2014

Abstract

The ordinary algebraic structures usually constitute *finitary varieties*: that is, they are axiomatisable by means of equations and finitary operations, as in the case of groups and rings. It was only in the sixties that the algebraic theory of the structures equipped with infinitary operations — the so-called *infinitary varieties* — has been developed [64, 48, 49], after the pioneering works of G. Birkhoff dating back to the thirties. Still in the thirties, M. H. Stone showed in the fundamental work [65] that the dual of the category of zero-dimensional compact Hausdorff spaces and continuous maps is equivalent to the finitary variety of Boolean algebras and their homomorphisms. This is the celebrated *Stone duality*. If we now lift the zero-dimensionality assumption on spaces, we are left with the category KHaus of compact Hausdorff spaces. The question arises, is there a (finitary or infinitary) variety of algebras, providing a generalisation of Boolean algebras, that is equivalent to the dual category $\mathsf{KHaus}^{\mathsf{op}}$. The answer is positive, as proved by J. Duskin in 1969 [27, 5.15.3]. However, subsequent results by B. Banaschewski [9, p. 1116] entail that every variety that is equivalent to $\mathsf{KHaus}^{\mathsf{op}}$ must use an infinitary operation. On the other hand, J. Isbell had already shown [42] the existence of an infinitary variety equivalent to $\mathsf{KHaus}^{\mathsf{op}}$ in which a finite number of finitary operations, together with a single infinitary operation of countable arity, suffice. Semantically, Isbell's operation is the uniformly convergent series

$$\sum_{i=1}^{\infty} \frac{f_i}{2^i}.$$

The problem of providing an explicit axiomatisation of a variety equivalent to $\mathsf{KHaus}^{\mathsf{op}}$ has remained open. The main result of the thesis consists in a finite axiomatisation of such a variety.

Contents

Symbols

$\mathbb{N}$	Set of natural numbers $1, 2, 3, \ldots$
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	Set of integer, rational, real and complex numbers
$\aleph_0$	Cardinality of the set $\mathbb{N}$
$\aleph_1$	Cardinality of the set $\mathbb{R}$
ω	First infinite ordinal
ω_1	First uncountable ordinal
$\cong$	Isomorphism (in the appropriate category)
$\simeq$	Equivalence of categories
$\alpha := \beta$	α is defined as β
$X \setminus Y$	Set-theoretic difference of the sets X and Y
$\times$	Cartesian product in the category of sets
X^n	Cartesian product of the set X with itself n times
$f_{\mid K}$	Restriction of the function f to the subset K of its domain
C^{op}	Dual of the category C
A^c	Completion of the semisimple MV-algebra, or archimedean ℓ-group, A with respect to the norm induced by the unit
A^d	Divisible hull of the MV-algebra, or ℓ-group, A
$\mathrm{C}(X, Y)$	Set of all the continuous functions from the space X to the space Y
$\operatorname{coz} f$	Cozero-set of the function f
$\operatorname{Coz}(X)$	Family of the cozero-sets of the space X
$d(x, y)$	Chang distance of the elements x, y of an MV-algebra
Free_κ	Free MV-algebra over a set of κ generators
g^+, g^-	Positive and negative parts, respectively, of the element g of an ℓ-group
$\lvert g \rvert$	Absolute value of the element g of an ℓ-group
G_A	Chang ℓ-group associated to the MV-algebra A
G_X	Gleason cover of the compact Hausdorff space X
H_A	ℓ-group of the self-adjoint elements of the C*-algebra A
$\operatorname{infinit} A$	Set of infinitesimal elements of the MV-algebra A

Λ	Gelfand transform
$\operatorname{Max} A$	Maximal spectrum of the MV-algebra, or ℓ-group, A
$\operatorname{Rad} A$	Radical of the MV-algebra, or ℓ-group, or Banach algebra, A
Σ_A	Maximal spectrum of the C*-algebra A
σ_x	Spectrum of the element x of a Banach algebra
Y	Yosida map
Ω_x	Resolvent set of the element x of a Banach algebra
$\Omega(X)$	Lattice of the open subsets of the topological space X
$\|\cdot\|_\infty$	Uniform norm on $\mathrm{C}(X,\mathbb{R})$ or $\mathrm{C}(X,\mathbb{C})$
$\|\cdot\|_u$	Seminorm induced on the unital ℓ-group (G,u) by the unit u

Categories

$\mathsf{Alex_c}$	Countably compact Alexandroff algebras and lattice homomorphisms preserving countable joins
Bool	Boolean algebras and homomorphisms of Boolean algebras
C^*	Commutative unital C*-algebras and *-homomorphisms
Δ	δ-algebras and δ-homomorphisms
KHaus	Compact Hausdorff spaces and continuous maps
$\ell\mathsf{Grp}$	ℓ-groups and ℓ-homomorphisms
$\ell\mathsf{Grp_u}$	Unital ℓ-groups and unital ℓ-homomorphisms
$\mathsf{Mod}\,\mathbb{T}$	Models of the theory $\mathbb{T}$ and homomorphisms
MV	MV-algebras and MV-homomorphisms
Set	Sets and functions
St	Stone spaces and continuous maps
$\mathsf{Str}\,\Sigma$	Σ-structures and homomorphisms
YAlg	Cauchy-complete, divisible, and archimedean unital ℓ-groups and unital ℓ-homomorphisms

Introduction

The main object of study of this thesis is the dual of the category KHaus of compact Hausdorff spaces and continuous maps. A celebrated result by Stone [65] shows that the full subcategory St of the category KHaus, whose objects are *Stone spaces* (=zero-dimensional compact Hausdorff spaces), is dually equivalent to the finitary variety of Boolean algebras. The question arises, is KHaus dually equivalent to a finitary variety. In view of results by Rosický and Banaschewski [61, 9] not only is the answer known to be negative, but the dual category $\mathsf{KHaus}^{\mathsf{op}}$ is not axiomatisable by a wide class of first-order theories. However, Duskin had already proved in 1969 [27, 5.15.3] that the category KHaus is dually equivalent to a variety of *infinitary* algebras, i.e. structures with function symbols of infinite arity. Hence the problem of providing an explicit axiomatisation of an infinitary variety dually equivalent to KHaus arises. In 1982 Isbell proved [42] that $\mathsf{KHaus}^{\mathsf{op}}$ is equivalent to a variety in which every function symbol has arity at most countable. More precisely, the signature of the latter variety consists of finitely many finitary operations, along with exactly one operation of countably infinite arity. Indeed, Isbell defined an explicit set of operations and showed that it suffices to generate the algebraic theory of $\mathsf{KHaus}^{\mathsf{op}}$, in the sense of Słomińsky, Lawvere, and Linton [64, 48, 49]. The algebraic theory of $\mathsf{KHaus}^{\mathsf{op}}$ had been described by Negrepontis in [58], by means of Gelfand-Neumark duality between KHaus and the category of commutative unital C*-algebras. The problem of axiomatising by equations an infinitary variety dually equivalent to the category KHaus has remained open. The main contribution of the thesis is to offer a solution. Using as a key tool the theory of MV-algebras — a generalisation of Boolean algebras that provides the algebraic counterpart to Łukasiewicz' many-valued logic — along with Isbell's basic insight on the semantic nature of the infinitary operation, in Chapter 4 we provide a *finite* axiomatisation.

The thesis is organised as follows.

Chapter 1 gives a historical account of the problem of axiomatising the dual of the category KHaus.

The first two sections of Chapter 2 provide an introduction to the basic theory of lattice-ordered groups and MV-algebras. These two classes of algebraic structures are tightly related via the equivalence Γ. This connection is exploited in the third section of the chapter. The content of Chapter 2, and its exposition, are standard in the literature.

Historically, an important characterisation of the dual algebra of a compact Hausdorff space has been provided by Yosida in the language of lattice-ordered vector spaces. Chapter 3 is devoted to the exposition of the related categorical duality. Here, all the results are known. However, a detailed account of Yosida duality in the case of ℓ-groups with a strong order unit cannot be found in the literature.

Chapter 4 is the core of the thesis. Here we present a finite axiomatisation of a variety of infinitary algebras, and prove that this variety forms a category that is dually equivalent to the category KHaus. The whole chapter is original, however it relies on the theory of MV-algebras introduced in Chapter 2. All the MV-algebraic results which are employed in Chapter 4 are recalled in the first section, so that the latter chapter is self-contained.

In Chapter 5 we study the algebraic theory (in the sense of Słomiński, Lawvere, and Linton) of the variety introduced in Chapter 4, and show that this variety constitutes a full reflective subcategory of the category of MV-algebras. Further, some elementary universal-algebraic properties of the latter variety are proved.

Chapter 6 deals with the basic theory of Banach algebras and C*-algebras, as can be found in the literature. The well-known Gelfand-Neumark duality for commutative C*-algebras is proved in detail. In the last section of the chapter we draw the connection between commutative C*-algebras, lattice-ordered groups, and the infinitary algebras identified in Chapter 4. This viewpoint is not standard, and is not present in the literature. Moreover, we give a direct proof of the monadicity of the category of commutative C*-algebras with respect to the positive unit ball functor.

Finally, in Chapter 7 we turn back to the topic of the axiomatisability of the dual category $\mathsf{KHaus}^{\mathsf{op}}$ discussed in Chapter 1. On the one hand, we show that the category $\mathsf{KHaus}^{\mathsf{op}}$ cannot be axiomatised by a geometric theory of presheaf type. On the other hand, we give an explicit axiomatisation of the category $\mathsf{KHaus}^{\mathsf{op}}$ in an extension of first-order logic by means of Alexandroff duality. The former result is original, while the latter result is an observation — not to be found in the literature — relying on a known duality for compact Hausdorff spaces.

Chapter 1

Prologue: which language suffices to capture $\mathsf{KHaus}^{\mathsf{op}}$?

In 1969 Duskin proved that the category $\mathsf{KHaus}^{\mathsf{op}}$ is monadic over Set [27, 5.15.3]. This result, from a logical point of view, has two different consequences. On the one hand, it tells us that the dual category $\mathsf{KHaus}^{\mathsf{op}}$ is axiomatisable in a (possibly infinitary) algebraic language. On the other hand, that $\mathsf{KHaus}^{\mathsf{op}}$ is axiomatisable in some extension of ordinary first-order logic. In the following sections we explore these two directions.

1.1 Algebraic

Recall that a *one-sorted signature* consists in a class $\mathcal{F}$ of function symbols and in a class $\mathcal{R}$ of relation symbols. For every function symbol $f \in \mathcal{F}$ and for every relation symbol $R \in \mathcal{R}$ we assume that cardinal numbers λ_f and λ_R are given. The numbers λ_f and λ_R are the *arity* of f and R, respectively. Those function symbols whose arity is 0 are called *constant symbols*. For every cardinal number λ, we denote by $\mathcal{F}_\lambda$ (respectively $\mathcal{R}_\lambda$) the class of function symbols (respectively relation symbols) of arity λ. Throughout the thesis we assume that, for each cardinal λ, the classes $\mathcal{F}_\lambda$ and $\mathcal{R}_\lambda$ are not proper classes.

Notation 1.1.1. Many-sorted signatures will not be considered. Therefore, we shall omit the adjective *one-sorted* when dealing with signatures. An arbitrary signature is usually denoted by the symbol Σ. The equality symbol is considered as a logical symbol, as the propositional connectives and the quantifiers $\exists, \forall$.

By an *algebraic* signature we mean a signature with no relation symbols, i.e. such that $\mathcal{R}_\lambda = \varnothing$ for all cardinal numbers λ. Recall that a cardinal number λ is *regular* if there is no set of cardinality λ that is the union of μ sets of cardinality ν, with $\mu, \nu < \lambda$ cardinal numbers. For instance, 2 is a regular cardinal. Amongst the infinite regular cardinals are $\aleph_0$ and $\aleph_1$. We agree to say that an algebraic signature is a λ-*signature* if there exists an infinite regular cardinal λ such that $\mathcal{F}_\mu = \varnothing$ for every cardinal $\mu \geqslant \lambda$.

Let us consider a λ-signature Σ, together with a set of *variables*

$$\mathbf{Var} := \{x_\mu\}_{\mu<\lambda}.$$

The set **Term** of *terms* for the signature Σ is inductively defined in the following way: every variable is in **Term**; if $f \in \mathcal{F}_\mu$ and $\{t_\nu\}_{\nu<\mu} \subseteq \mathbf{Term}$, then $f(t_1, \ldots, t_\nu, \ldots) \in \mathbf{Term}$. Nothing else is in **Term**. A Σ-*structure* is a set U together with an operation $\widehat{f}\colon U^\mu \to U$ for each function symbol $f \in \mathcal{F}_\mu$. Observe that any function $\varphi\colon \mathbf{Var} \to U$ can be extended to a function $\overline{\varphi}\colon \mathbf{Term} \to U$. Indeed, suppose that the map $\overline{\varphi}$ is defined on the terms $\{t_\nu\}_{\nu<\mu} \subseteq \mathbf{Term}$, and let $f \in \mathcal{F}_\mu$ be a function symbol of arity μ. Then, we define

$$\overline{\varphi}(f(t_1, \ldots, t_\nu, \ldots)) := \widehat{f}(\overline{\varphi}(t_1), \ldots, \overline{\varphi}(t_\nu), \ldots).$$

A *homomorphism* between Σ-structures is a map preserving the operations. Given a λ-signature Σ, we denote by $\mathsf{Str}\,\Sigma$ the category that has Σ-structures as objects and homomorphisms as morphisms.

By an *equational theory* $\mathbb{T}$ over the λ-signature Σ we understand a set of *axioms*, i.e. pairs of terms (s,t), $s,t \in \mathbf{Term}$, where each such pair can informally be thought of as the equation $s = t$. In the following, we use the latter notation whenever convenient.

Definition 1.1.2. A *model* for an equational theory $\mathbb{T}$ is a Σ-structure U such that, for every function $\varphi\colon \mathbf{Var} \to U$ and for every pair $(s,t) \in \mathbb{T}$, the condition $\overline{\varphi}(s) = \overline{\varphi}(t)$ is satisfied.

The full subcategory of $\mathsf{Str}\,\Sigma$ whose objects are the models of $\mathbb{T}$ is denoted by $\mathsf{Mod}\,\mathbb{T}$.

Definition 1.1.3. If λ is a regular infinite cardinal, a λ-*variety* is the class of models for an equational theory over a λ-signature.

We remark that the notion of Σ-structure can be defined more generally for an arbitrary signature Σ. Likewise, one can consider not only equational theories but also arbitrary first-order theories over an arbitrary signature, whose axioms are constructed by using propositional connectives and quantifiers in an appropriate way (see [1, p. 221–222]). If Σ is an arbitrary signature, and $\mathbb{T}$ is an arbitrary first-order theory over the signature Σ, we continue to denote by $\mathsf{Str}\,\Sigma$ and $\mathsf{Mod}\,\mathbb{T}$ the associated categories.

Remark 1.1.4. Let Σ be a λ-signature, and let $\mathbb{T}$ be an equational theory over the signature Σ. For $\lambda = \aleph_0$, we have that every function symbol in Σ has finite arity, and $\mathsf{Mod}\,\mathbb{T}$ is an equationally defined class of finitary algebras, as studied in classical universal algebra. In this context, a *variety* of algebras is defined as a class of algebras which is closed under homomorphic images, subalgebras and products. Birkhoff's theorem [18, Theorem 11.9] then states that the notions of equational class of (finitary) algebras and of variety of (finitary) algebras coincide. This shows that referring to $\mathsf{Mod}\,\mathbb{T}$ as a λ-variety makes sense if $\lambda = \aleph_0$. However, Słomiński showed in [64, 9.6] that the natural extension of Birkhoff's theorem to infinitary algebras also holds, so that the terminology becomes consistent for any infinite regular cardinal λ.

It is a classical result in category theory, that categories which are monadic over Set coincide, up to equivalence, with λ-varieties (see [49], or [54, Theorem 5.40 p. 66, Theorem 5.45 p. 68]). Then, Duskin's result about the monadicity of the dual category $\mathsf{KHaus}^{\mathsf{op}}$ [27, 5.15.3] entails that $\mathsf{KHaus}^{\mathsf{op}}$ is equivalent to a λ-variety for some infinite regular cardinal λ. The question arises, is $\mathsf{KHaus}^{\mathsf{op}}$ equivalent to a finitary variety of algebras. The answer is no, in view of a much stronger result of Banaschewski (see Theorem 1.2.11 below).

In 1971 Negrepontis described [58] the algebraic theory of $\mathsf{KHaus}^{\mathsf{op}}$ in the sense of Słomińsky, Lawvere, and Linton [64, 48, 49]. The dual category of KHaus is known to be equivalent to the category C^* of commutative unital C*-algebras by Gelfand-Neumark duality (see Section 6.2). Negrepontis showed that the unit ball functor from C^* to Set is monadic. In particular, if

$$I := \{z \in \mathbb{C} \mid \|z\| \leqslant 1\}$$

denotes the complex unit disc, a left adjoint to the unit ball functor maps a set X to the C*-algebra $\mathrm{C}(I^X, \mathbb{C})$ of all the continuous $\mathbb{C}$-valued functions on the space I^X. Hence, the algebraic theory of $\mathsf{KHaus}^{\mathsf{op}}$ has powers of the space I as objects, and continuous maps as morphisms (see Chapter 5 for some information about algebraic theories). It is known that the category C^* is monadic over Set also with respect to the Hermitian unit ball functor sending a C*-algebra to the set of its self-adjoint elements whose norm does not exceed 1 (see Section 6.3 for details). A left adjoint to the latter functor induces a monad over Set which maps a set X to the set $\mathrm{C}([-1,1]^X, \mathbb{C})$. The associated algebraic theory has the cubes $[-1,1]^\lambda$ as objects, with λ an arbitrary cardinal number, and continuous maps between the cubes as morphisms. In [42] Isbell gave an explicit set of operations which suffice to generate the latter algebraic theory. In the intended model $\mathrm{C}(X, [-1,1])$, for X a compact Hausdorff space, the identified operations are interpreted as follows. There are three finitary operations: the constant function of value 1 on X, a unary operation mapping a continuous function f to the function $-f$, and the truncated multiplication by 2 sending f to $\min(1, \max(-1, 2f))$. Further, there is an infinitary operation of countably infinite arity which maps a sequence of continuous functions $\{f_i\}_{i\in\mathbb{N}}$ to the uniformly convergent series

$$\sum_{i=1}^{\infty} \frac{f_i}{2^i}.$$

In particular, this means that the category $\mathsf{KHaus}^{\mathsf{op}}$ is equivalent to an $\aleph_1$-variety. Negrepontis' result was then generalised by Van Osdol [66] who proved that the category of (possibly non-commutative and non-unital) C*-algebras is monadic over Set with respect to the unit ball functor. In [59, 60] Pelletier and Rosický gave explicit sets of operations generating the algebraic theory of all unital C*-algebras with respect to the unit ball functor, and of related categories. However, the problem of providing a tractable (i.e. finite or recursive) set of identities for these theories has remained open. In Chapter 4 we give a finite axiomatisation of an $\aleph_1$-variety that is dually equivalent to the category KHaus. This can be regarded as an axiomatisation of the class of positive unit balls of commutative unital C*-algebras.

1.2 First-order and extensions

In this section we shall see that every category that is monadic over Set, or equivalently every λ-variety, can be axiomatised in an appropriate extension of first-order logic.

Let κ, λ be infinite cardinal numbers, and let us consider a set of variables $\mathbf{Var} = \{x_\mu\}_{\mu<\gamma}$ of cardinality $\gamma = \max(\kappa, \lambda)$. The infinitary language $\mathrm{L}_{\kappa,\lambda}$ is described as follows. On the one hand we consider logical symbols, i.e. the propositional connectives $\wedge, \vee, \neg, \Rightarrow$, the quantifiers $\exists, \forall$, and the equality symbol $=$. On the other hand, the non-logical symbols are provided by a signature Σ containing only finitary function symbols and finitary relation symbols. In other words, $\mathcal{F}_\mu = \varnothing = \mathcal{R}_\mu$ for each infinite cardinal μ. The class **Term** of terms for the signature Σ is defined as usual: every variable is a term; if $t_1, \dots, t_n \in \mathbf{Term}$ and $f \in \mathcal{F}_n$, then $f(t_1, \dots, t_n) \in \mathbf{Term}$. Nothing else is in **Term**.

Remark 1.2.1. In the context of infinitary languages, infinitary function symbols and infinitary relation symbols are not allowed. Therefore, when dealing with a language $\mathrm{L}_{\kappa,\lambda}$, we implicitly assume that a signature Σ is given, which satisfies $\mathcal{F}_\mu = \varnothing = \mathcal{R}_\mu$ for each infinite cardinal μ.

Now, we can define inductively the notion of *expression* of $\mathrm{L}_{\kappa,\lambda}$.

1. If t_1, t_2 are terms, then $t_1 = t_2$ is an expression.

2. If $R \in \mathcal{R}_n$ and $t_1, \dots, t_n \in \mathbf{Term}$, then $R(t_1, \dots, t_n)$ is an expression.

3. If ε is an expression, then $\neg\varepsilon$ is an expression.

4. If $\varepsilon_1, \varepsilon_2$ are expressions, then $\varepsilon_1 \Rightarrow \varepsilon_2$ is an expression.

5. If $\rho < \kappa$ is a cardinal number and $\{\varepsilon_\mu\}_{\mu<\rho}$ is a set of expressions, then
$$\bigwedge \varepsilon_1 \varepsilon_2 \cdots \varepsilon_\mu \cdots,$$
and
$$\bigvee \varepsilon_1 \varepsilon_2 \cdots \varepsilon_\mu \cdots$$
are expressions.

6. If ε is an expression, $\delta < \lambda$ is a cardinal number and $\{x_{\mu_\xi}\}_{\xi<\delta} \subseteq \mathbf{Var}$ (where $\mu_\xi < \max(\kappa, \lambda)$ for each $\xi < \delta$), then
$$\exists x_{\mu_1} x_{\mu_1} \cdots x_{\mu_\xi} \cdots \varepsilon,$$
and
$$\forall x_{\mu_1} x_{\mu_1} \cdots x_{\mu_\xi} \cdots \varepsilon$$
are expressions.

7. Nothing else is an expression.

In other words, the infinitary language $\mathrm{L}_{\kappa,\lambda}$ extends ordinary first-order logic by allowing conjunctions and disjunctions of sets of formulæ of cardinality strictly smaller than κ, and quantification over sets of variables of cardinality strictly smaller than λ. If, moreover, we introduce conjunctions and disjunctions of sets of formulæ of arbitrary cardinality, we obtain the language usually denoted by $\mathrm{L}_{\infty,\lambda}$ (and similarly one could define the language $\mathrm{L}_{\kappa,\infty}$). Finally, the infinitary language $\mathrm{L}_{\infty,\infty}$ allow us to consider arbitrary conjunctions and disjunctions, and arbitrary quantifications. Observe that the language $\mathrm{L}_{\aleph_0,\aleph_0}$ is the ordinary first-order language.

Notation 1.2.2. When κ, λ are amongst the cardinals $\aleph_0, \aleph_1$, we write the ordinal ω (respectively ω_1), in place of the cardinal $\aleph_0$ (respectively $\aleph_1$). For instance, the usual first-order logic is denoted by $\mathrm{L}_{\omega,\omega}$.

Definition 1.2.3. Let κ, λ be infinite cardinals. A *sentence* of the language $\mathrm{L}_{\kappa,\lambda}$ is an expression with no free variables. A *theory* $\mathbb{T}$ in the infinitary language $\mathrm{L}_{\kappa,\lambda}$ is a set of sentences of $\mathrm{L}_{\kappa,\lambda}$.

A straightforward generalisation of Tarski's truth definition employed in ordinary model theory allows one to define a *model* for a theory $\mathbb{T}$ in an infinitary language $\mathrm{L}_{\kappa,\lambda}$ (see [25, p. 68] for details). A homomorphisms between models is a map preserving the operations and the relations, and the category of models for a theory $\mathbb{T}$, with their homomorphisms, is denoted by $\mathsf{Mod}\,\mathbb{T}$.

Definition 1.2.4. Let κ, λ be infinite cardinals. A category C is *axiomatisable* in the language $\mathrm{L}_{\kappa,\lambda}$ if there exists a theory $\mathbb{T}$ in the language $\mathrm{L}_{\kappa,\lambda}$ such that $\mathsf{C} \simeq \mathsf{Mod}\,\mathbb{T}$.

For a thorough treatment of infinitary languages the interested reader is referred to [25]. Now we turn to the connection between infinitary languages and category theory. Recall that a non-empty partially ordered set is *directed* provided that each pair of elements has an upper bound. If λ is an infinite regular cardinal, we can generalise the latter definition by saying that a partially ordered set is λ-*directed* if every subset of cardinality strictly smaller than λ has an upper bound. If $D\colon (I, \leqslant) \to \mathsf{C}$ is a diagram in the category C, and $(I, \leqslant)$ is a λ-directed partially ordered set (regarded as a category), then D is a λ-*directed diagram* and its colimit is a λ-*directed colimit*.

We remark that there is another construction related to that of directed colimits, namely that of filtered colimits. Recall that a non-empty category C is *filtered* if the following properties are satisfied.

1. For each pair of objects C_1, C_2 of C there is an object D and morphisms $f_1\colon C_1 \to D$, $f_2\colon C_2 \to D$ in C.
2. For each pair of parallel arrows $g_1, g_2\colon C_1 \to C_2$ in C there is a morphism $f\colon C_2 \to D$ in C such that $f \circ g_1 = f \circ g_2$.

It is elementary that every directed partially ordered set, regarded as a category, is a filtered category. A *filtered diagram* in a category D is a functor $D\colon \mathsf{C} \to \mathsf{D}$ where C is a filtered category. The colimit of a filtered diagram is called a *filtered colimit*. Directed colimits and filtered colimits are equivalent constructions, in the following sense.

Lemma 1.2.5. *A category* C *has directed colimits if, and only if, it has filtered colimits. If the category* C *satisfies one of the latter equivalent conditions, and* D *is any category, then a functor* $F\colon \mathsf{C} \to \mathsf{D}$ *preserves directed colimits if, and only if, it preserves filtered colimits.*

Proof. See [1, Corollary p. 15]. □

The foregoing lemma admits a generalisation to λ-directed colimits. We agree to say that a non-empty category C is λ-*filtered*, for λ an infinite regular cardinal, if the following hold.

1. For each set $\{C_i\}_{i\in I}$ of objects of C of cardinality strictly smaller than λ there exists an object D in C and morphisms $f_i\colon C_i \to D$ for all $i \in I$.
2. For each collection $\{g_i\}_{i\in I}$ of morphisms $g_i\colon C_1 \to C_2$ in C of cardinality strictly smaller then λ there exists a morphism $f\colon C_2 \to D$ in C such that $f \circ g_i = f \circ g_j$ for all $i, j \in I$.

A λ-*filtered diagram* in a category D is a functor $D\colon \mathsf{C} \to \mathsf{D}$ where C is a λ-filtered category. The colimit of a λ-filtered diagram is called a λ-*filtered colimit*. Again, a category has λ-directed colimits if, and only if, it has λ-filtered colimits (see [1, Remark 1.21 p. 22]). Therefore, the difference between directed colimits and filtered colimits is immaterial. Depending on the specific situation, we shall use whichever is more convenient.

Notation 1.2.6. If C is a category and A is an object of C, we agree to denote by

$$\mathsf{C}(A, -)\colon \mathsf{C} \to \mathsf{Set}$$

the functor mapping an object B of C to the set $\mathsf{C}(A, B)$ of morphisms $A \to B$ in C.

Now, let us fix an infinite regular cardinal λ.

Definition 1.2.7. An object A of a category C is λ-*presentable* if the functor $\mathsf{C}(A, -)\colon \mathsf{C} \to \mathsf{Set}$ preserves λ-filtered colimits.

Definition 1.2.8. A category C is λ-*accessible* provided that it has λ-directed colimits and a *dense* subset $\mathcal{A}$ of λ-presentable objects, i.e. every object of C is a λ-directed colimit of objects of $\mathcal{A}$. The category C is *locally* λ-*presentable* if it is λ-accessible and cocomplete.

A great number of examples of locally λ-presentable categories is provided by the following

Theorem 1.2.9. *Every* λ*-variety is a locally* λ*-presentable category. Equivalently, every category that is monadic over* Set *is locally* λ*-presentable, for some infinite regular cardinal* λ*.*

Proof. See [1, Theorem 3.28]. □

If $\lambda = \aleph_0$ we speak of *finitely presentable* objects and of *finitely accessible* and *locally finitely presentable* categories, respectively. A category is called *accessible* (respectively *locally presentable*) if it is λ-accessible (respectively locally λ-presentable) for some infinite regular cardinal λ. The notion of accessible category originates from the work of Grothendieck [7], and the related theory was further developed by Gabriel and Ulmer in [30], where locally presentable categories are studied for the first time. Moreover, accessible categories have been intensively studied by Makkai and Paré [53] in connection with model theory. In fact, here logic enters the picture: accessible and locally presentable categories can be characterised, up to equivalence, precisely as categories of models for certain theories in the infinitary language $\mathrm{L}_{\infty,\infty}$. We shall be concerned only with the case of locally presentable categories, for a characterisation of accessible categories please see [1, pp. 227–229].

Let us fix a signature Σ and an infinite regular cardinal λ. A *limit theory* in the language $\mathrm{L}_{\lambda,\lambda}$ is a set of sentences, each of them being of the form

$$\forall \{x_i\}_{i\in I} \left(\varphi\left(\{x_i\}_{i\in I}\right) \Rightarrow \exists! \{y_j\}_{j\in J} \psi\left(\{x_i\}_{i\in I}, \{y_j\}_{j\in J}\right)\right),$$

where $\{x_i\}_{i\in I}, \{y_j\}_{j\in J}$ are sets of variables of cardinality strictly smaller than λ, and φ, ψ are conjunctions of less than λ *atomic* formulæ (=formulæ that do not contain propositional connectives or quantifiers). Limit theories completely characterise locally presentable categories:

Theorem 1.2.10. *Let λ be an infinite regular cardinal. A category is locally λ-presentable if, and only if, it is equivalent to* $\mathsf{Mod}\,\mathbb{T}$ *for a limit theory* $\mathbb{T}$ *in the language* $\mathrm{L}_{\lambda,\lambda}$.

Proof. See [1, Theorem 5.30]. □

It follows that $\mathsf{KHaus}^{\mathsf{op}}$, being equivalent to an $\aleph_1$-variety (in view of Isbell's result), is axiomatisable in the language $\mathrm{L}_{\omega_1,\omega_1}$. However, this does not mean that $\mathsf{KHaus}^{\mathsf{op}}$ cannot be axiomatised in a smaller fragment of language, e.g. in ordinary first-order logic $\mathrm{L}_{\omega,\omega}$. In the eighties Bankston asked [10] whether KHaus is dually equivalent to any elementary *P-class* of finitary algebras. Recall that a subcategory D of a category C is said to be *closed* in C *under product* if the following property is satisfied: the product of objects of D, computed in C is, in fact, in D. Then a P-class of finitary algebras is a category of the form $\mathsf{Mod}\,\mathbb{T}$, for $\mathbb{T}$ a theory in the language $\mathrm{L}_{\omega,\omega}$ over an algebraic signature Σ, such that $\mathsf{Mod}\,\mathbb{T}$ is closed under products in $\mathsf{Str}\,\Sigma$. A negative answer was given, independently, by Rosický [61] and Banaschewski [9]. In fact, the latter proved the following stronger result, where we recall that St denotes the category of Stone spaces (=zero-dimensional compact Hausdorff spaces).

Theorem 1.2.11. *The only full subcategory of* KHaus *extending* St*, which is dually equivalent to an elementary P-class of finitary algebras, is* St.

Proof. See [9, p. 1116]. □

By a celebrated result of Stone [65], the category St of Stone spaces and continuous maps is dually equivalent to the category Bool of Boolean algebras an their homomorphisms. Hence, Banaschewski's result shows that Stone duality cannot be extended further retaining the finitary algebraic nature of the dual category.

Chapter 2

Lattice-ordered groups and MV-algebras

2.1 Lattice-ordered groups

2.1.1 The variety of ℓ-groups

Recall that a *lattice* is a partially ordered set $(G, \leqslant)$ in which every pair of elements $x, y \in G$ has a greatest lower bound and a least upper bound, denoted by $x \wedge y$ and $x \vee y$, respectively. Equivalently, it can be described as an (equationally defined) algebra $(G, \wedge, \vee)$ satisfying the commutative, associative, idempotent and absorption laws [18, Definition 1.1]. Whenever we are given a lattice in the form $(G, \wedge, \vee)$, we shall denote by $\leqslant$ the canonically associated ordering, defined by $x \leqslant y$ if, and only if, $x \wedge y = x$.

The language of ℓ-groups is given by $\mathcal{L}_{\ell\mathsf{Grp}} := \{0, +, \wedge, \vee\}$ where 0 is a function symbol of arity 0, and $+, \wedge, \vee$ are binary function symbols.

Definition 2.1.1. A *lattice-ordered abelian group* (*abelian ℓ-group* for short) is an algebra $(G, +, 0, \wedge, \vee)$ satisfying the following conditions.

1. $(G, +, 0)$ is an abelian group.
2. $(G, \wedge, \vee)$ is a lattice.
3. For all $x, y, t \in G$, if $x \leqslant y$, then $x + t \leqslant y + t$.

It is clearly possible to generalise this definition to (possibly non-commutative) ℓ-groups. However, we will be concerned with commutative ℓ-groups only. For this reason, henceforth by an ℓ-group we understand an abelian ℓ-group.

Remark 2.1.2. In item 2 of Definition 2.1.1 we do not require that the lattice $(G, \wedge, \vee)$ is either distributive or bounded. In particular, we do not require that every finite subset $F \subseteq G$ admits a greatest lower bound $\bigwedge F$ and a least upper bound $\bigvee F$, for otherwise the lattice G would be bounded by $\top := \bigwedge \varnothing$ and $\bot := \bigvee \varnothing$ (this definition is adopted by some authors, see e.g. [44, 1.2, 1.4]). It will soon transpire that the underlying lattice $(G, \wedge, \vee)$ of any ℓ-group is automatically distributive, and that it is bounded if, and only if, G is the trivial (singleton) group.

It is not evident by Definition 2.1.1 that the class of ℓ-groups can be defined by equations. We show that, in fact, this is possible.

Lemma 2.1.3. *Let $(G, +, 0, \wedge, \vee)$ be an algebra such that conditions 1 and 2 in Definition 2.1.1 are satisfied. Then $(G, +, 0, \wedge, \vee)$ is an ℓ-group if, and only if, the following hold for all $x, y, t \in G$.*

$$t + (x \wedge y) = (t + x) \wedge (t + y).$$

$$t + (x \vee y) = (t + x) \vee (t + y).$$

Proof. Observe that, if the condition $x \leqslant y \;\Rightarrow\; x + t \leqslant y + t$ holds, then $x + t \leqslant y + t$ entails $x = x + t + (-t) \leqslant y + t + (-t) = y$. Thus $x \leqslant y$ if, and only if, $x + t \leqslant y + t$. Now, assume that $x \leqslant y \;\Rightarrow\; x + t \leqslant y + t$. We have

$$t + (x \wedge y) \leqslant t + x \;\Leftrightarrow\; x \wedge y \leqslant x,$$

$$t + (x \wedge y) \leqslant t + y \;\Leftrightarrow\; x \wedge y \leqslant y.$$

Hence $t + (x \wedge y) \leqslant (t + x) \wedge (t + y)$. If $z \in G$ is such that $z \leqslant t + x$ and $z \leqslant t + y$, then

$$z \leqslant t + x \;\Leftrightarrow\; z - t \leqslant x,$$

$$z \leqslant t + y \;\Leftrightarrow\; z - t \leqslant y.$$

We conclude that

$$z - t \leqslant x \wedge y \;\Leftrightarrow\; z \leqslant t + (x \wedge y).$$

In other words $t + (x \wedge y) = (t + x) \wedge (t + y)$. Similarly, it is possible to show that $t + (x \vee y) = (t + x) \vee (t + y)$. In the other direction,

$$x \leqslant y \;\Leftrightarrow\; x \wedge y = x \;\Leftrightarrow\; (x \wedge y) + t = x + t \;\Leftrightarrow$$

$$(x + t) \wedge (y + t) = x + t \;\Leftrightarrow\; x + t \leqslant y + t.$$

□

By Birkhoff's theorem [18, Theorem 11.9], together with Lemma 2.1.3, the class of ℓ-groups is a variety of (finitary) algebras, meaning that it is closed under the operators $\mathbb{H}$ (homomorphic images), $\mathbb{S}$ (subalgebras) and $\mathbb{P}$ (products).

The following fact is a key property.

Lemma 2.1.4. *If G is an ℓ-group and $x, y \in G$, then*

$$(x - (x \wedge y)) \wedge (y - (x \wedge y)) = 0.$$

Proof. A straightforward computation shows that

$$\begin{aligned}(x-(x\wedge y))\wedge(y-(x\wedge y)) &= (x\wedge y)-(x\wedge y) && \text{(Lemma 2.1.3)}\\ &= 0.\end{aligned}$$

□

As anticipated above, the lattice reduct of any ℓ-group is distributive:

Proposition 2.1.5. *If $(G,+,0,\wedge,\vee)$ is an ℓ-group, then $(G,\wedge,\vee)$ is a distributive lattice.*

Proof. See [13, Proposition 1.2.14]. □

Definition 2.1.6. Let G,H be ℓ-groups. A function $h\colon G\to H$ is called a *homomorphism* of ℓ-groups (*ℓ-homomorphism* for short) if it is both a group homomorphism and a lattice homomorphism.

Remark 2.1.7. An ℓ-homomorphism, being a homomorphism of abelian groups, is linear with respect to the $\mathbb{Z}$-module structure of the underlying group.

Definition 2.1.8. The *positive cone* of an ℓ-group G is the set

$$G^+ := \{x\in G \mid 0\leqslant x\}.$$

Remark 2.1.9. The term *positive* in the preceding definition, instead of the more appropriate *non-negative*, is standard in the literature.

The name *cone* suggests that the following property holds: if $x,y\in G^+$, then $x+y\in G^+$. This is true, indeed

$$0\leqslant x \;\Rightarrow\; y\leqslant x+y \;\Rightarrow\; 0\leqslant y\leqslant x+y.$$

Remark 2.1.10. Notice that, if we know the positive cone, we can recover the partial order of the ℓ-group. Indeed,

$$x\leqslant y \;\Leftrightarrow\; 0\leqslant y+(-x) \;\Leftrightarrow\; y+(-x)\in G^+.$$

The same observations apply to the *negative cone* of G, defined by

$$G^- := \{x\in G \mid x\leqslant 0\}.$$

Lemma 2.1.11. *If $h\colon G\to H$ is an ℓ-homomorphism, then*

1. *h is order-preserving.*
2. *$h(G^+)\subseteq H^+$.*

Proof. For item 1, assume that $x \leqslant y$, i.e. $x \wedge y = x$. We find $h(x \wedge y) = h(x)$ if, and only if, $h(x) \wedge h(y) = h(x)$, that is $h(x) \leqslant h(y)$. With respect to item 2, pick $x \in G^+$, so that $0 \wedge x = 0$. Then

$$h(0 \wedge x) = h(0) \;\Leftrightarrow\; h(0) \wedge h(x) = h(0) \;\Leftrightarrow\; 0 \wedge h(x) = 0 \;\Leftrightarrow\; h(x) \in H^+.$$

□

Denote by $\ell\mathsf{Grp}$ the category whose objects are ℓ-groups and whose morphisms are ℓ-homomorphisms. It is a consequence of general results about varieties of finitary algebras, viewed as categories, that the category $\ell\mathsf{Grp}$ has the following properties.

1. The category $\ell\mathsf{Grp}$ is complete and cocomplete (further, $\ell\mathsf{Grp}$ is locally finitely presentable [1, Corollary 3.7]). In particular, it has an initial and a terminal object.
2. The category $\ell\mathsf{Grp}$ has free objects on generating sets of any cardinality: in other words, there is a functor $F\colon \mathsf{Set} \to \ell\mathsf{Grp}$ that is left adjoint to the underlying-set functor $U\colon \ell\mathsf{Grp} \to \mathsf{Set}$ (see [18, Theorem 10.12] or [54, Theorem 4.15 p. 37]).

Lemma 2.1.12. *If G is an ℓ-group and $x, y \in G$ satisfy $x \wedge y = 0$, then $nx \wedge ny = 0$ for all $n \in \mathbb{N}$.*

Proof. By [13, 1.2.24], for all $x, y, z \in G$, $x \wedge y = 0 = x \wedge z \;\Rightarrow\; x \wedge (y + z) = 0$. In particular, $x \wedge y = 0 \;\Rightarrow\; x \wedge (y + y) = 0$. Reasoning by induction on $n \in \mathbb{N}$, it is easy to see that $x \wedge ny = 0$ and, consequently, that $nx \wedge ny = 0$. □

Given an ℓ-group G and an element $g \in G$, define the *positive part* $g^+ := g \vee 0$ and the *negative part* $g^- := -g \vee 0$ of g.

Lemma 2.1.13. *Let G be an ℓ-group. For an arbitrary element $g \in G$, the following hold.*

1. $g = g^+ - g^-$.
2. $g^+ \wedge g^- = 0$.
3. *If $a, b \in G$ satisfy $g = a - b$ and $a \wedge b = 0$, then $a = g^+$ and $b = g^-$.*

Proof. Clearly $g + g^- = g + (-g \vee 0) = (g - g) \vee (g + 0) = 0 \vee g = g^+$, that is $g = g^+ - g^-$. For the proof of items 2 and 3 the interested reader is referred to [13, Proposition 1.3.4]. □

Corollary 2.1.14. *Let G be an ℓ-group. For every pair of elements $g, h \in G$, $g \geqslant h$ if, and only if, $g^+ \geqslant h^+$ and $h^- \geqslant g^-$.*

Proof. Assume that $g \geqslant h$. Then $g^+ = g \vee 0 \geqslant h \vee 0 = h^+$ and $g^- = -g \vee 0 \leqslant -h \vee 0 = h^-$. Conversely, if $g^+ \geqslant h^+$ and $g^- \leqslant h^-$, then $-g^- \geqslant -h^-$ and $g = g^+ - g^- \geqslant h^+ - h^- = h$ by Lemma 2.1.13. □

Definition 2.1.15. An ℓ-group G is *totally-ordered* (or *linearly ordered*) if, for all $a, b \in G$, either $a \leqslant b$ or $b \leqslant a$.

Although the variety of ℓ-groups arises from the abstraction of the integers $(\mathbb{Z}, +, 0, \leqslant)$, it captures a wider class of structures. We illustrate this comment by giving some examples of models for the theory of ℓ-groups.

Example 2.1.16. $(\mathbb{Z}, +, 0, \leqslant)$, $(\mathbb{Q}, +, 0, \leqslant)$ and $(\mathbb{R}, +, 0, \leqslant)$ are ℓ-groups. Further, they are totally-ordered.

Example 2.1.17. $(\mathbb{Z} \times \mathbb{Z}, +, (0,0), \leqslant)$ is an ℓ-group, where sum and order are defined componentwise, in other words

$$(a, b) + (a', b') := (a + a', b + b'),$$

$$(a, b) \leqslant (a', b') \ \Leftrightarrow \ a \leqslant a' \ \text{ and } \ b \leqslant b'.$$

This is a partial order, and not a total one. Thus the class of totally-ordered ℓ-groups cannot be a variety of algebras because it is not closed under products. In fact, the axiom for a total order fails to be equationally definable.

Example 2.1.18. We consider a different order on the free abelian group of rank 2, $\mathbb{Z} \times \mathbb{Z}$. Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ be any irrational number, and consider the group

$$G := \{z_1 + \xi z_2 \mid z_1, z_2 \in \mathbb{Z}\}.$$

It is easy to see that the map $(z_1, z_2) \mapsto z_1 + \xi z_2$ is an isomorphism between the groups $\mathbb{Z}^2$ and G. Notice, in particular, that the injectivity is due to the irrationality of ξ:

$$z_1 + \xi z_2 = z_3 + \xi z_4 \ \Leftrightarrow \ 1(z_1 - z_3) + \xi(z_2 - z_4) = 0 \ \Leftrightarrow \ z_1 = z_3 \ \text{ and } \ z_2 = z_4$$

for, otherwise, there would exist integers m $(= z_1 - z_3)$ and n $(= z_2 - z_4)$ such that

$$1 \cdot m + \xi n = 0 \ \Leftrightarrow \ \xi n = -m \ \Leftrightarrow \ \xi = -\frac{m}{n},$$

which is impossible. The group G, equipped with the order induced by $\mathbb{R}$, is a totally-ordered group. Moreover,

$$0 \leqslant z_1 + \xi z_2 \ \Leftrightarrow \ 0 \leqslant \langle (z_1, z_2), (1, \xi) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product of $\mathbb{R}^2$. That is,

$$G^+ = \{z_1 + \xi z_2 \mid \langle (z_1, z_2), (1, \xi) \rangle \geqslant 0\}.$$

Geometrically, the positive cone of G, regarded as a subset of $\mathbb{Z}^2 \cong G$, is the halfspace containing $(1, \xi)$ determined by the line

$$l := \{(x, y) \in \mathbb{R}^2 \mid \langle (x, y), (1, \xi) \rangle = 0\}.$$

The key property of l is that its intersection with $\mathbb{Z}^2$ is $\{(0, 0)\}$.

Henceforth, we shall write $a < b$ meaning that the two conditions $a \leqslant b$ and $a \neq b$ are satisfied.

Definition 2.1.19. An ℓ-group G is *archimedean* if, for all $a, b \in G$, the following condition holds.

$$\text{If } \ 0 < a \leqslant b, \ \text{ then there exists } n \in \mathbb{N} \ \text{ such that } \ na \nleqslant b.$$

The examples for the theory of ℓ-groups that we have introduced so far are all archimedean. In the next example we define an order on the group $\mathbb{Z} \times \mathbb{Z}$, such that the resulting ℓ-group is not archimedean.

Example 2.1.20. The *lexicographic product* of $\mathbb{Z}$ with itself is the ordered group $\mathbb{Z} \overrightarrow{\times} \mathbb{Z} = (\mathbb{Z} \times \mathbb{Z}, +, (0, 0), \leqslant)$ where the sum is componentwise and the order, called *lexicographic order*, is given by

$$(a', b') \leqslant (a, b) \ \Leftrightarrow \ (a' < a) \ \text{ or } \ (a' = a \ \text{ and } \ b' \leqslant b).$$

It is elementary that $\mathbb{Z} \overrightarrow{\times} \mathbb{Z}$ is a totally-ordered ℓ-group, and its positive cone is

$$(\mathbb{Z} \overrightarrow{\times} \mathbb{Z})^+ = \{(a, b) \in \mathbb{Z}^2 \mid (0 < a) \ \text{ or } \ (a = 0 \ \text{ and } \ 0 \leqslant b)\}.$$

However, $\mathbb{Z} \overrightarrow{\times} \mathbb{Z}$ is not archimedean: there exist two elements $0 < a, b$ such that $na \leqslant b$ holds for all $n \in \mathbb{N}$. For example, take $a := (0, 1)$ and $b := (1, 0)$. We have $a \leqslant b$ but, for all $n \in \mathbb{N}$, $n(0, 1) = (0, n) \leqslant (1, 0)$. We say that the element $(0, 1)$ is *infinitesimal* with respect to $(1, 0)$, and write $(0, 1) \ll (1, 0)$. Similarly, the ordered group $\mathbb{Z} \overrightarrow{\times} \mathbb{Z} \overrightarrow{\times} \mathbb{Z}$ is seen to be a non-archimedean ℓ-group. In this case there are infinitesimal elements of two different ranks:

$$(0, 0, 1) \ll (0, 1, 0) \ll (1, 0, 0).$$

We conclude with one more example of ℓ-group, which will be central in Chapter 3. Given a topological space X, the family $\mathrm{C}(X)$ of all the continuous functions on X with values in $\mathbb{R}$ (or, more generally, in $\mathbb{C}$) has been extensively studied under different points of view, depending on which structure one chooses to equip $\mathbb{R}$ with. For instance, the latter could be regarded as a group, a ring, or even a Banach algebra, and this choice determines a corresponding structure on $\mathrm{C}(X)$ (see [62] for a thorough treatment of the subject).

Example 2.1.21. Let X be a topological space, and consider the euclidean topology on $\mathbb{R}$. We denote by $\mathrm{C}(X, \mathbb{R})$ the set $\{f \colon X \to \mathbb{R} \mid f \ \text{ is continuous}\}$. Upon considering

$\mathbb{R}$ as an ℓ-group, we can define a structure $(\mathrm{C}(X,\mathbb{R}),+,0,\wedge,\vee)$ where the sum is taken to be pointwise, and for every $f,g\in\mathrm{C}(X,\mathbb{R})$ and for every $x\in X$,

$$(f\wedge g)(x):=\min\left(f(x),f(y)\right),$$

and

$$(f\vee g)(x):=\max\left(f(x),g(x)\right).$$

The element 0 of $\mathrm{C}(X,\mathbb{R})$ is the constant function of value 0 on X. It is easy to verify that $\mathrm{C}(X,\mathbb{R})$ is an ℓ-group; in fact, it is also archimedean, as we shall now prove.

Lemma 2.1.22. *If X is a topological space, then the ℓ-group $\mathrm{C}(X,\mathbb{R})$ is archimedean.*

Proof. Let $f,g\in\mathrm{C}(X,\mathbb{R})$ be such that $0<f\leqslant g$. Since $0<f$, there exists a point $x\in X$ such that $0<f(x)$; then $0<f(x)\leqslant g(x)$ because $f\leqslant g$. The ℓ-group $\mathbb{R}$ is archimedean, hence there exists $n\in\mathbb{N}$ satisfying $nf(x)\nleqslant g(x)$, i.e. $nf\nleqslant g$. □

2.1.2 Hölder's theorem

The following result is fundamental. It was first proved in 1901 by Hölder [39] (for an English translation please see [40, 41]).

Theorem 2.1.23 (Hölder)**.** *Every totally-ordered archimedean ℓ-group G can be embedded in $\mathbb{R}$, i.e. there exists an injective ℓ-homomorphism $G\to\mathbb{R}$.*

Remark 2.1.24 (Assumes knowledge in logic)*.* The archimedean property, as stated in Definition 2.1.19, is not elementary, meaning that it is not expressible in a first-order language. Indeed, the construction of a non-archimedean structure as an ultrapower of $\mathbb{R}$, fundamental in non-standard analysis, shows that such a first-order formulation cannot exist, for otherwise it would contradict Łoś's theorem. Consistently with the fact that the hypotheses of the upward Löwenheim-Skolem theorem are not satisfied, Hölder's theorem gives an upper bound on the cardinality of any model for the theory of totally-ordered archimedean ℓ-groups: the cardinality of an infinite model for this theory is either $\aleph_0$ or $\aleph_1$.

The proof of Hölder's theorem requires some preliminary results that hold more generally for partially ordered groups. A *partially ordered group* is a structure $(G,+,0,\leqslant)$ such that $(G,+,0)$ is an abelian group, $(G,\leqslant)$ is a partially ordered set, and for all $x,y,t\in G$, if $x\leqslant y$ then $x+t\leqslant y+t$. Clearly, a partially ordered group is an ℓ-group if, and only if, it is lattice-ordered as a partially ordered set. The notion of positive and negative cone can be extended from ℓ-groups to partially ordered groups in the obvious way. We continue using the notation G^+,G^-.

Definition 2.1.25. A partially ordered set is *2-directed* if every pair of elements has an upper bound and a lower bound.

Remark 2.1.26. We reserve the term *directed* to mean that each pair of elements has an upper bound. The non-standard terminology *2-directed* is only used in this section for the sake of clarity.

Lemma 2.1.27. *A partially ordered group G is 2-directed if, and only if,*

$$G = G^+ + G^- := \{g_1 + g_2 \mid g_1 \in G^+ \ \text{and}\ g_2 \in G^-\}.$$

Proof. Suppose that G is 2-directed and pick $g \in G$. Upon considering the pair $\{0, g\}$, there exists $y \in G$ such that $g \leqslant y$ and $0 \leqslant y$. Thus $y \in G^+$ and $g \leqslant y \ \Leftrightarrow\ g - y \leqslant 0 \ \Leftrightarrow\ g - y \in G^-$. We conclude that $g = y + (g - y)$, so that $g \in G^+ + G^-$. Conversely, let $g_1, g_2 \in G$ and assume that there exist $x, y, u, v \in G^+$ satisfying $g_1 = x + (-y)$ and $g_2 = u + (-v)$. It follows

$$g_1 = x + (-y) \ \Leftrightarrow\ x - g_1 \in G^+ \ \Leftrightarrow\ g_1 \leqslant x,$$

$$g_2 = u + (-v) \ \Leftrightarrow\ u - g_2 \in G^+ \ \Leftrightarrow\ g_2 \leqslant u.$$

Hence $g_1 \leqslant x \leqslant x + u$ and $g_2 \leqslant u \leqslant x + u$, because $x, u \in G^+$. This shows that $x + u$ is an upper bound for the pair g_1, g_2. Similarly, one can prove that $-y - v$ is a lower bound for the pair. $\square$

Lemma 2.1.28. *Let G and H be partially ordered set, and let $f\colon G^+ \to H^+$ be a function satisfying $f(a+b) = f(a) + f(b)$ for all $a, b \in G^+$. If G is 2-directed, then there exists a unique monotonic group homomorphism $\bar{f}\colon G \to H$ extending f.*

Proof. Consider an element $g \in G$. By Lemma 2.1.27 there exist $a, b \in G^+$ such that $g = a + (-b)$. Define the function $\bar{f}\colon G \to H$ as

$$\bar{f}(g) := f(a) + (-f(b)).$$

This function is well-defined: if $c, d \in G^+$ satisfy $g = c + (-d)$, then

$$a + (-b) = c + (-d) \ \Leftrightarrow\ a + d = c + b \ \Rightarrow\ f(a + d) = f(c + b) \ \Leftrightarrow$$

$$f(a) + f(d) = f(c) + f(b) \ \Leftrightarrow\ f(a) + (-f(b)) = f(c) + (-f(d)).$$

We show that $\bar{f}$ is a group homomorphism. Let $g_1, g_2 \in G$ and $a, b, c, d \in G^+$ be such that $g_1 = a + (-b)$ and $g_2 = c + (-d)$. Since

$$g_1 + g_2 = a + (-b) + c + (-d) = (a + c) + (-(b + d)),$$

we have

$$\begin{aligned}
\bar{f}(g_1 + g_2) &= f(a + c) + (-f(b + d)) \\
&= f(a) + f(c) + (-f(b)) + (-(f(d)) \\
&= f(a) + (-f(b)) + f(c) + (-f(d)) \\
&= \bar{f}(g_1) + \bar{f}(g_2).
\end{aligned}$$

Moreover,

$$\bar{f}(-g_1) = \bar{f}(b + (-a)) = f(b) - f(a) = -(f(a) - f(b)) = -\bar{f}(g_1).$$

Now, suppose that $g_1 \leqslant g_2$, i.e. $g_2 - g_1 \in G^+$. Then

$$\overline{f}(g_2) - \overline{f}(g_1) = \overline{f}(g_2 - g_1) = f(g_2 - g_1) \geqslant 0,$$

that is $\overline{f}(g_1) \leqslant \overline{f}(g_2)$. Finally, suppose that f' is another group homomorphism extending f. Upon considering an arbitrary element $g \in G$ and elements $a, b \in G^+$ such that $g = a + (-b)$ (their existence is assured by Lemma 2.1.27), the map f' will satisfy

$$f'(g) = f'(a + (-b)) = f'(a) + (-f'(b)) = f(a) + (-f(b)) = \bar{f}(g).$$

□

Corollary 2.1.29. *If G, H are ℓ-groups and $f\colon G^+ \to H^+$ is a function preserving sums and joins, then there exists a unique ℓ-homomorphism $\bar{f}\colon G \to H$ extending f.*

Proof. Clearly, every lattice is a 2-directed partially ordered set, so that every ℓ-group is 2-directed. By Lemma 2.1.28 there exists a unique monotonic group homomorphism $\bar{f}$ which extends f. We prove that $\bar{f}$ preserves the lattice structure. If $g_1, g_2 \in G$, then

$$\begin{aligned}
\bar{f}(g_1 \vee g_2) - \bar{f}(g_1 \wedge g_2) &= f((g_1 \vee g_2) - (g_1 \wedge g_2)) \\
&= f((g_1 - (g_1 \wedge g_2)) \vee (g_2 - (g_1 \wedge g_2))) \\
&= f(g_1 - (g_1 \wedge g_2)) \vee f(g_2 - (g_1 \wedge g_2)) \\
&= \bar{f}(g_1 - (g_1 \wedge g_2)) \vee \bar{f}(g_2 - (g_1 \wedge g_2)) \\
&= (\bar{f}(g_1) - \bar{f}(g_1 \wedge g_2)) \vee (\bar{f}(g_2) - \bar{f}(g_1 \wedge g_2)) \\
&= (\bar{f}(g_1) \vee \bar{f}(g_2)) - \bar{f}(g_1 \wedge g_2).
\end{aligned}$$

We conclude that $\bar{f}(g_1 \vee g_2) = \bar{f}(g_1) \vee \bar{f}(g_2)$. Furthermore,

$$\bar{f}(g_1 \wedge g_2) = \bar{f}(-(-g_1 \vee -g_2)) = -(\bar{f}(-g_1) \vee \bar{f}(-g_2)) = \bar{f}(g_1) \wedge \bar{f}(g_2).$$

□

We can finally prove Hölder's theorem:

Proof of Theorem 2.1.23. If $G = \{0\}$ the proof is trivial. Let us fix an arbitrary element $a \in G$ such that $a > 0$. For each $b \in G^+$ define the set

$$\mathcal{I}(b) := \left\{ \tfrac{m}{n} \in \mathbb{Q} \mid m \geqslant 0,\ n > 0 \ \text{ and } \ ma \leqslant nb \right\},$$

where m, n are integers and the notation ma, nb refers to the $\mathbb{Z}$-module structure of the group (e.g. ma represents the iterated sum of a with itself m times). The set $\mathcal{I}(b)$ is non-empty because $0 \in \mathcal{I}(b)$, indeed for all $n > 0$ the condition $0 \leqslant nb$ holds. Observe that, if $\frac{r}{s} \leqslant \frac{m}{n}$ and $\frac{m}{n} \in \mathcal{I}(b)$, then $\frac{r}{s} \in \mathcal{I}(b)$. Assuming by contradiction that $\frac{r}{s} \notin \mathcal{I}(b)$, we have $ra > sb$ because G is totally-ordered. It follows

$$s(ma) \leqslant s(nb) = n(sb) < n(ra) = (nr)a \leqslant (ms)a = s(ma),$$

a contradiction. Further, the set $\mathcal{I}(b) \subseteq \mathbb{R}$ is bounded: G is archimedean, thus there exists $k \in \mathbb{N}$ such that $ka \nleqslant b$, hence $ka > b$. This means that $k = \frac{k}{1} \notin \mathcal{I}(b)$ and $\frac{k}{1} > \frac{m}{n}$ for every $\frac{m}{n} \in \mathcal{I}(b)$, i.e. k is an upper bound for $\mathcal{I}(b)$. Clearly, a lower bound for $\mathcal{I}(b)$ is given by 0. Define the function $f\colon G^+ \to \mathbb{R}^+$ which maps $b \in G^+$ to

$$f(b) := \sup \mathcal{I}(b).$$

The map f is well-defined because $\mathbb{R}$ is totally-ordered, hence lattice ordered. We show that $f(b_1 + b_2) = f(b_1) + f(b_2)$ for all $b_1, b_2 \in G^+$. Consider $\frac{m}{n} \in \mathcal{I}(b_1)$ and $\frac{r}{s} \in \mathcal{I}(b_2)$. We can assume without loss of generality that $n = s$. By the inequalities $ma \leqslant nb_1$ and $ra \leqslant nb_2$ we obtain

$$(m + r)a = ma + ra \leqslant nb_1 + nb_2 = n(b_1 + b_2),$$

that is $\frac{m+r}{n} \in \mathcal{I}(b_1 + b_2)$. In other words, $f(b_1) + f(b_2) \leqslant f(b_1 + b_2)$. Conversely, if $\frac{m}{n} > \sup \mathcal{I}(b_1)$ and $\frac{r}{n} > \sup \mathcal{I}(b_2)$, then $ma > nb_1$ and $ra > nb_2$, so that $(m + r)a > n(b_1 + b_2)$. This shows that $\frac{m+r}{n} > \sup \mathcal{I}(b_1 + b_2)$, that is

$$f(b_1) + f(b_2) \geqslant f(b_1 + b_2).$$

If $b_1 \leqslant b_2$, then $\mathcal{I}(b_1) \subseteq \mathcal{I}(b_2)$ by definition. Hence, since G is totally-ordered, $f(b_1 \vee b_2) = f(b_1) \vee f(b_2)$. By Corollary 2.1.29 there exists a unique ℓ-homomorphism $\bar{f}\colon G \to \mathbb{R}$ extending f. We shall prove that $\bar{f}$ is an embedding. On the one hand, if $b > 0$ there exists $k \in \mathbb{N}$ such that $a < kb$, since G is archimedean and totally-ordered. In this case $\frac{1}{k} \in \mathcal{I}(b)$, whence $\bar{f}(b) = f(b) = \sup \mathcal{I}(b) \geqslant \frac{1}{k} > 0$. On the other hand, if $b < 0$ we have $\bar{f}(b) = -\bar{f}(-b) = -f(-b) < 0$. This means that $\bar{f}(b) = 0 \Rightarrow b = 0$, i.e. $\bar{f}$ is an injective ℓ-homomorphism. □

Remark 2.1.30. We will see in Theorem 2.1.57, after introducing strong order units, that Hölder's theorem holds in a stronger form for unital ℓ-groups.

2.1.3 Subobjects and Quotients

It is elementary that in every variety of (finitary) algebras, the monomorphisms coincide with the injective homomorphisms. In the case of ℓ-groups, the following definition describes precisely the subobjects in the category $\ell\mathsf{Grp}$.

Definition 2.1.31. An ℓ-group $(H, +, 0, \wedge, \vee)$ is said to be an *ℓ-subgroup* of $(G, +, 0, \wedge, \vee)$ if $(H, +, 0)$ is a subgroup of $(G, +, 0)$, and $(H, \wedge, \vee)$ is a sublattice of $(G, \wedge, \vee)$.

It is easy to see that (as in any variety of algebras), given an ℓ-group G, a subset $H \subseteq G$ is an ℓ-subgroup if, and only if, H is (non-empty and) closed under the operations $+, -, \wedge, \vee$.

Example 2.1.32. $\mathbb{Z}$ is an ℓ-subgroup of $\mathbb{R}$ and $\mathbb{Z} \times \mathbb{Z}$ is an ℓ-subgroup of $\mathbb{R} \times \mathbb{R}$, where sum and order are defined componentwise (see Example 2.1.17).

However, while monomorphisms coincide with injective homomorphisms in a variety of algebras, epimorphisms do not correspond to surjective homomorphisms in general. For this reason, the notion of epimorphism is not suitable to describe the quotient objects in the variety. The following standard example shows that the category of commutative unital rings admits morphisms which are both mono and epi, but not surjective.

Example 2.1.33. For an arbitrary commutative unital ring R, let us consider homomorphisms of (commutative unital) rings

$$\mathbb{Z} \overset{h}{\hookrightarrow} \mathbb{Q} \underset{\psi}{\overset{\varphi}{\rightrightarrows}} R$$

satisfying $\varphi \circ h = \psi \circ h$, where h is the inclusion of the ring of integers in the ring of rationals. For all $\frac{m}{n} \in \mathbb{Q}$, with $m, n \in \mathbb{Z}$, we have

$$\begin{aligned}\varphi\left(\tfrac{m}{n}\right) &= \varphi(m \cdot n^{-1}) = \varphi(m) \cdot \varphi(n^{-1}) = \varphi(m) \cdot \varphi(n)^{-1} \\ &= \psi(m) \cdot \psi(n)^{-1} = \psi(m) \cdot \psi(n^{-1}) = \psi(m \cdot n^{-1}) = \psi\left(\tfrac{m}{n}\right).\end{aligned}$$

The morphism h is an epi, however it is clearly not surjective.

The categorical notion which allows to define quotient objects is that of *regular epimorphism*, that is, a coequalizer of a pair of parallel arrows. For a variety of algebras, quotients are in bijective correspondence with congruences which, in turn, correspond to kernels of homomorphisms, defined as the sets of pairs of elements identified by the homomorphism. In many notable instances, kernels can be defined more simply as subsets of the domain. For example, in the theory of groups, the quotient groups are defined by means of normal subgroups, that are precisely the kernels of group homomorphisms in the usual sense.

If $h\colon G \to H$ is an ℓ-homomorphism, we shall find those properties which characterise the group-theoretic kernel $h^{-1}(0) = \{g \in G \mid h(g) = 0\}$. Notice that $h^{-1}(0)$ is a subgroup of G, automatically normal, due to commutativity, since h is a group homomorphism. Also, h is a lattice homomorphism, whence $h^{-1}(0)$ is a sublattice of G. Furthermore, if $g_1 \geqslant g_2 \geqslant g_3$ are elements of G, then

$$h(g_1) = h(g_3) = 0 \;\Rightarrow\; h(g_2) = 0$$

because h is order-preserving by Lemma 2.1.11. In other words, if $g_1, g_3 \in h^{-1}(0)$ and $g_1 \geqslant g_2 \geqslant g_3$, then $g_2 \in h^{-1}(0)$. Given a partially ordered set G and a subset $I \subseteq G$, we say that I is *order-convex* (*convex*, for short) if, given arbitrary elements $x, z \in I$ and $y \in G$, $y \in I$ holds whenever $x \leqslant y \leqslant z$.

Definition 2.1.34. Let G be an ℓ-group. A subset $I \subseteq G$ is called an *ℓ-ideal* of G if it is a convex ℓ-subgroup.

If no confusion arises, we write *ideal* meaning ℓ-ideal. Every ℓ-group $G \neq \{0\}$ contains at least two distinct ideals: namely, the *trivial* ideal $\{0\}$ and the *improper* ideal G. It is elementary that arbitrary intersections of ideals are ideals. One therefore defines the ideal $\langle S \rangle$ *generated* by a subset $S \subseteq G$ as the intersection of all the ideals extending S. If $g \in G$ is an element of the ℓ-group, we denote by $\langle g \rangle$ the *principal* ideal generated by g, that is the smallest ideal of G containing g. One can prove that such an ideal can be explicitly described as

$$\langle g \rangle = \{x \in G \mid \exists n \in \mathbb{N} \text{ such that } -n|g| \leqslant x \leqslant n|g|\},$$

where $|g| := g \vee (-g)$ is the *absolute value* of g [34, Lemma 1.1.6].

Example 2.1.35. Consider the ℓ-group $\mathbb{Z} \times \mathbb{Z}$, where sum and order are defined componentwise as in Example 2.1.17. The set $G := \{(2x, 2y) \in \mathbb{Z} \times \mathbb{Z} \mid x, y \in \mathbb{Z}\}$ is both a subgroup and a sublattice of $\mathbb{Z} \times \mathbb{Z}$. However, G is not convex since $(2,0), (4,2) \in G$ and $(2,0) \leqslant (4,1) \leqslant (4,2)$, but $(4,1) \notin G$. A similar argument shows that the subset $H := \{(x,x) \in \mathbb{Z} \times \mathbb{Z} \mid x \in \mathbb{Z}\}$ is a non-convex ℓ-subgroup of $\mathbb{Z} \times \mathbb{Z}$. The ideal generated by H, i.e. the smallest convex ℓ-subgroup of $\mathbb{Z} \times \mathbb{Z}$ containing H, is the whole $\mathbb{Z} \times \mathbb{Z}$. The only non-trivial proper ideals of $\mathbb{Z} \times \mathbb{Z}$ are

$$I_1 := \{(x,0) \in \mathbb{Z} \times \mathbb{Z} \mid x \in \mathbb{Z}\} \text{ and } I_2 := \{(0,y) \in \mathbb{Z} \times \mathbb{Z} \mid y \in \mathbb{Z}\}.$$

Definition 2.1.36. If $I \subseteq G$ is an ideal of the ℓ-group G, we define an equivalence relation $\equiv_I$ on G in the following way: for all $x, y \in G$,

$$x \equiv_I y \text{ if, and only if, } x - y \in I.$$

Lemma 2.1.37. *The equivalence relation $\equiv_I$ is a congruence on G.*

Proof. An elementary verification. □

As a consequence of the previous lemma, the set of equivalence classes

$$\frac{G}{\equiv_I} := \{[g]_{\equiv_I} \mid g \in G\}$$

is naturally endowed with the structure of an ℓ-group. By abuse of notation, we will denote this ℓ-group by $\frac{G}{I}$. There is a surjective ℓ-homomorphism which is naturally associated to the congruence $\equiv_I$: this is the map $q_I \colon G \twoheadrightarrow \frac{G}{I}$ sending g to $[g]_{\equiv_I}$.

Starting from an ideal in G, we have constructed a surjective homomorphism q_I with domain G. The following proposition says that the converse is possible as well, and that $I \subseteq G$ is an ideal if, and only if, $I = h^{-1}(0)$ for some ℓ-group H and some ℓ-homomorphism $h \colon G \to H$.

Proposition 2.1.38. *If $h \colon G \to H$ is an ℓ-homomorphism, then $h^{-1}(0)$ is an ideal of G. If, in addition, h is surjective, then*

$$G \xrightarrow{h} H \cong \frac{G}{h^{-1}(0)} \xleftarrow{q} G.$$

Further, if $I \subseteq G$ is an ideal, and $q_I \colon G \twoheadrightarrow \frac{G}{I}$ is the natural quotient map, then

$$q_I^{-1}([0]_{\equiv_I}) = I.$$

Proof. It is elementary that $h^{-1}(0)$ is an ideal of G. The second part of the statement is a particular case of the first isomorphism theorem [18, Theorem 6.12]. Finally

$$q_I^{-1}([0]_{\equiv_I}) = \{g \in G \mid [g]_{\equiv_I} = [0]_{\equiv_I}\} = \{g \in G \mid g - 0 \in I\} = I.$$

□

From what we have seen so far, given an ideal I of an ℓ-group G, it makes sense to ask what the associated quotient map is. For example, if $I := \{0\}$ is the trivial ideal, then the associated map is essentially the identity $G \to G$. If $I := G$ is the improper ideal, the associated map is the unique ℓ-homomorphism onto the terminal object of the variety, that is $G \xrightarrow{!} \{0\}$. Observe that the category $\ell\mathsf{Grp}$ of ℓ-groups, as the category of groups, has a zero object, namely $\{0\}$, which is both initial and terminal.

Example 2.1.39. Let I_1, I_2 be the two ideals of $\mathbb{Z} \times \mathbb{Z}$ defined in Example 2.1.35. The quotient map associated to the ideal I_1 sends a pair $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ to $[(x, y)]_{\equiv_{I_1}}$, where

$$\begin{aligned}
[(x, y)]_{\equiv_{I_1}} &= \{(z_1, z_2) \in \mathbb{Z} \times \mathbb{Z} \mid (x - z_1, y - z_2) \in I_1\} \\
&= \{(z_1, z_2) \in \mathbb{Z} \times \mathbb{Z} \mid y - z_2 = 0\} \\
&= \{(z_1, y) \in \mathbb{Z} \times \mathbb{Z} \mid z_1 \in \mathbb{Z}\} \\
&= [(0, y)]_{\equiv_{I_1}}.
\end{aligned}$$

Upon identifying $[(0, y)]_{\equiv_{I_1}}$ with its canonical representative $(0, y)$, we see that

$$q_1 : \mathbb{Z} \times \mathbb{Z} \twoheadrightarrow \frac{\mathbb{Z} \times \mathbb{Z}}{I_1} \cong \mathbb{Z}.$$

Similarly, for the ideal I_2, $q_2 : (x, y) \mapsto [(x, 0)]_{\equiv_{I_2}}$ and $\dfrac{\mathbb{Z} \times \mathbb{Z}}{I_2} \cong \mathbb{Z}$.

Example 2.1.40. The lexicographic product $\mathbb{Z} \overrightarrow{\times} \mathbb{Z}$ (see Example 2.1.20) has only one non-trivial proper ideal, hence it admits only one non-trivial quotient. Assume that $J \subseteq \mathbb{Z} \overrightarrow{\times} \mathbb{Z}$ is an ideal. Observe that, if J contains a non-zero element z, then it contains all the elements between z and $(0, 0)$ by convexity. For example, if $(1, 0) \in J$, then J contains all the infinitesimal elements of the form $(0, z_2)$. On the other hand J is a group, so it is closed under sums and inverses, so that $\{(z_1, 0) \in \mathbb{Z} \times \mathbb{Z} \mid z_1 \in \mathbb{Z}\} \subseteq J$. It follows that $J = \mathbb{Z} \overrightarrow{\times} \mathbb{Z}$. Therefore, the only non-trivial proper ideal is the ideal of infinitesimal elements $I := \{(0, z_2) \in \mathbb{Z} \times \mathbb{Z} \mid z_2 \in \mathbb{Z}\}$ corresponding to the quotient map

$$q \colon \mathbb{Z} \overrightarrow{\times} \mathbb{Z} \twoheadrightarrow \frac{\mathbb{Z} \overrightarrow{\times} \mathbb{Z}}{I} \cong \mathbb{Z}$$

which sends every infinitesimal element to 0.

2.1.4 Maximal ideals

Definition 2.1.41. Let G be an ℓ-group. An ideal $I \subseteq G$ is said to be *maximal* if it is proper and for every ideal $J \subseteq G$, if $J \neq G$ and $I \subseteq J$, then $I = J$.

Notation 2.1.42. A maximal ideal of an ℓ-group is usually denoted by $\mathfrak{m}$.

Remark 2.1.43 (For logicians). In the variety of Boolean algebras a congruence can be represented by an ideal, or by its dual concept: a *filter*. Maximal filters are usually called *ultrafilters*. The reason why the language of filters is used in logic, rather than that of ideals, is due to the study of the Lindenbaum algebra of classical propositional logic. In this context, a filter F in the algebra corresponds to a deductively closed theory, and the property $x \in F,\ x \leqslant y \ \Rightarrow\ y \in F$ coincides with the rule of inference known as *modus ponens*. Furthermore, ultrafilters represent those deductively closed theories which are consistent and maximal. The latter are central in logic, since they are the syntactic counterpart of the semantic notion of logical evaluation, in the following sense. Denote by FORM the set of all well-formed formulæ over a (countable) set of propositional variables. If $\mu\colon \mathrm{FORM} \to \{0,1\}$ is an evaluation, then

$$\Theta_\mu := \{\alpha \in \mathrm{FORM} \mid \mu \models \alpha\}$$

is a consistent maximal (deductively closed) theory. Conversely, if Θ is a (deductively closed) consistent and maximal theory, the map $\mu_\Theta\colon \mathrm{FORM} \to \{0,1\}$ defined by

$$\mu_\Theta(\alpha) = 1 \ \Leftrightarrow\ \alpha \in \Theta$$

is an evaluation. Therefore, we can identify the ultrafilters of the Lindenbaum algebra (the consistent maximal theories) with the models of the theory (the evaluations). This correspondence between maximal consistent deductively closed theories and logical evaluations generalises: since every Boolean algebra is the Lindenbaum algebra of some classical propositional theory, we can think of the dual Stone space of a Boolean algebra as the space of models for the associated theory.

The study of quotients of ℓ-groups by maximal ideals is of particular interest. Specifically, the next result will be fundamental in the development of Yosida duality in Chapter 3.

Lemma 2.1.44. *Let G be an ℓ-group. The following are equivalent, for any ideal $\mathfrak{m} \subseteq G$.*

1. *The ideal $\mathfrak{m}$ is either maximal or improper.*
2. *There exists an ℓ-embedding $\frac{G}{\mathfrak{m}} \to \mathbb{R}$.*
3. *The ℓ-group $\frac{G}{\mathfrak{m}}$ is totally-ordered and archimedean.*

Before proving the previous result, we shall introduce the concept of simple ℓ-group. In a variety of algebras, if $h\colon A \twoheadrightarrow B$ is a surjective homomorphism, the congruences on B are in bijective correspondence, via h, with the congruences on A that extend $\ker h = \{(a_1, a_2) \in A \times A \mid h(a_1) = h(a_2)\}$. Furthermore, this bijection is order-preserving with respect to set-theoretic inclusion. We say that the algebra B is *simple* if there are no non-trivial congruences on B.

Corollary 2.1.45. *If $h\colon A \twoheadrightarrow B$ is a surjective homomorphism of algebras, then* $\ker h$ *is maximal (with respect to inclusion of congruences) if, and only if, B is simple.* $\square$

In the case of the variety of ℓ-groups, the notion of simple algebra can be rephrased in the following way.

Definition 2.1.46. An ℓ-group G is *simple* if its only ideals are $\{0\}$ and G, or equivalently if its only quotients are the identity $G \to G$ and $G \xrightarrow{!} \{0\}$.

Clearly, simple ℓ-groups can be characterised as those ℓ-groups for which the trivial ideal $\{0\}$ is either maximal or improper.

Remark 2.1.47. For the variety of ℓ-groups, Corollary 2.1.45 states that, given a surjective ℓ-homomorphism $h\colon G \twoheadrightarrow H$, the ideal $h^{-1}(0)$ is maximal if, and only if, H is simple and non-trivial.

Example 2.1.48. It is easy to show that the ℓ-group $\mathbb{Z}$ is simple, since the convex closure of an arbitrary non-trivial ℓ-subgroup is the whole $\mathbb{Z}$.

Corollary 2.1.49. *Let G be a non-trivial ℓ-group, and let $\mathfrak{m} \subseteq G$ be an ideal of G. Then $\mathfrak{m}$ is maximal if, and only if, $\frac{G}{\mathfrak{m}}$ is a simple non-trivial ℓ-group.*

Proof. Denote by $q\colon G \twoheadrightarrow \frac{G}{\mathfrak{m}}$ the quotient map sending g to $[g]_{\equiv_\mathfrak{m}}$. By Proposition 2.1.38 we have $q^{-1}([0]_{\equiv\mathfrak{m}}) = \mathfrak{m}$, hence by Remark 2.1.47 $\mathfrak{m}$ is a maximal ideal if, and only if, $\frac{G}{\mathfrak{m}}$ is simple and non-trivial. $\square$

Lemma 2.1.50. *Every simple ℓ-group is totally-ordered and archimedean.*

Proof. Let G be a simple ℓ-group and assume, by contradiction, that it is not archimedean (hence, in particular, non-trivial). Then there exist $g, g' \in G$ such that g is non-zero and infinitesimal with respect to g', i.e. $0 < ng \leqslant g'$ for all $n \in \mathbb{N}$. If $\langle g\rangle$ is the ideal generated by g, then $g' \notin \langle g\rangle$, i.e. $\langle g\rangle$ is a proper non-trivial ideal of G. However, this is a contradiction because G is simple. Suppose now that G is simple, but not totally-ordered: there exist $x, y \in G$ satisfying $x \nleqslant y, y \nleqslant x$. We can assume, without loss of generality, that $x, y > 0$. Then $x \neq x \wedge y \neq y$, and

$$(x - (x \wedge y)) \wedge (y - (x \wedge y)) = 0$$

by Lemma 2.1.4. Set $\bar{x} := x - (x \wedge y)$ and $\bar{y} := y - (x \wedge y)$, so that $\bar{x}, \bar{y} > 0$ and $\bar{x} \wedge \bar{y} = 0$. Consider the ideal $\langle \bar{x}\rangle$ generated by $\bar{x}$, and suppose that $\bar{y} \in \langle \bar{x}\rangle$. This means that there exists $n \in \mathbb{N}$ such that $n\bar{x} \geqslant \bar{y}$. However $n\bar{y} \geqslant \bar{y}$ implies $n\bar{x} \wedge n\bar{y} \geqslant \bar{y} > 0$ and, by Lemma 2.1.12, it follows that $\bar{x} \wedge \bar{y} \neq 0$ which is not the case. Whence $\bar{y} \notin \langle \bar{x}\rangle$ and $\langle \bar{x}\rangle$ is a non-trivial proper ideal of G, that is a contradiction. $\square$

The converse of Lemma 2.1.50 holds:

Corollary 2.1.51. *Every totally-ordered archimedean ℓ-group is simple.*

Proof. Suppose that G is a totally-ordered archimedean ℓ-group, and let $I \subseteq G$ be a non-trivial ideal of G. In particular, there exists a non-zero positive element $y \in I$. Since G is archimedean, for every element $x \in G^+$ with $0 < y \leqslant x$ there exists $n \in \mathbb{N}$ such that $ny \nleqslant x$, which is equivalent to $x < ny$ because G is totally-ordered. Now, $ny \in I$ since the latter is a group, and from the convexity of I we conclude that $x \in I$. It follows that $-x \in I$ as well, so that $G^+ \cup G^- \subseteq I$. However $G = G^+ \cup G^-$ for totally-ordered groups, whence I must be the improper ideal. □

Proof of Lemma 2.1.44. The equivalence between items 2 and 3 is exactly Hölder's Theorem 2.1.23. We prove that item 1 is equivalent to item 2. If we assume that $\mathfrak{m} \subseteq G$ is either a maximal ideal or the improper ideal, then $\frac{G}{\mathfrak{m}}$ is a simple ℓ-group by Corollary 2.1.49, hence it is totally-ordered and archimedean by Lemma 2.1.50. The existence of an ℓ-embedding $\frac{G}{\mathfrak{m}} \to \mathbb{R}$ is deduced by Hölder's theorem. Conversely, suppose that there exists an ℓ-embedding $\iota\colon \frac{G}{\mathfrak{m}} \to \mathbb{R}$. If $\frac{G}{\mathfrak{m}}$ is trivial there is nothing to prove. By Corollary 2.1.49 it therefore suffices to show that $\frac{G}{\mathfrak{m}}$ is simple. If $I \subseteq \frac{G}{\mathfrak{m}}$ is a non-trivial ideal, then $\iota(I)$ is a subgroup and sublattice of $\mathbb{R}$, but it is not necessarily convex. Consider the ideal $\langle \iota(I) \rangle$ obtained as the convex closure of $\iota(I)$; this is a non-trivial ideal since it contains I, whence $\langle \iota(I) \rangle = \mathbb{R}$ by Corollary 2.1.4. We shall prove, by contradiction, that $I = \frac{G}{\mathfrak{m}}$. Pick an element $x \in \frac{G}{\mathfrak{m}} \setminus I$; we can suppose, without loss of generality, that $x > 0$. We have $\iota(x) \notin \iota(I)$, but $\iota(x) \in \langle \iota(I) \rangle = \mathbb{R}$, i.e. there exist elements $\iota(a), \iota(b) \in \iota(I)$ such that $\iota(a) \leqslant \iota(x) \leqslant \iota(b)$. The injectivity of ι implies $a \leqslant x \leqslant b$, indeed e.g. $\iota(a) \leqslant \iota(x) \Leftrightarrow \iota(a) \wedge \iota(x) = \iota(a) \Leftrightarrow \iota(a \wedge x) = \iota(a) \Rightarrow a \wedge x = a \Leftrightarrow a \leqslant x$. The ideal I is convex in $\frac{G}{\mathfrak{m}}$ and $a, b \in I$, whence $x \in I$ which is a contradiction. Therefore $\frac{G}{\mathfrak{m}} \setminus I = \varnothing$ and I is the improper ideal. □

2.1.5 Strong order units

Definition 2.1.52. Let G be an ℓ-group. An element $u \in G^+$ is a *strong order unit* for G if, for each $g \in G$, there exists $n \in \mathbb{N}$ such that $g \leqslant nu$.

If u is a strong order unit for the ℓ-group G, we say that (G, u) is a *unital ℓ-group*. An ℓ-homomorphism $h\colon (G, u) \to (H, v)$ between unital ℓ-groups is a *unital ℓ-homomorphism* if $h(u) = v$. We denote by $\ell\mathsf{Grp_u}$ the category whose objects are unital ℓ-groups and whose morphisms are unital ℓ-homomorphisms.

Remark 2.1.53. If G is an archimedean totally-ordered ℓ-group, then any strictly positive element of G is a strong order unit. Moreover, unless G is the trivial ℓ-group, each strong order unit is a strictly positive element. This is the case, for example, for the ℓ-group $\mathbb{R}$. We agree to fix the real number 1 as strong order unit: when referring to the unital ℓ-group of the reals, we understand the pair $(\mathbb{R}, 1)$.

Lemma 2.1.54. *Let (G, u) be a unital ℓ-group and let $I \subseteq G$ be an ideal. Then I is proper if, and only if, $u \notin I$.*

Proof. Suppose that $u \in I$. For all $g \in G$ there exists $n \in \mathbb{N}$ such that $-nu \leqslant g \leqslant nu$. Since I is a convex group, $g \in I$, that is $I = G$. The other direction is trivial. □

Lemma 2.1.55. *Let (G, u) be a unital ℓ-group and let $\mathfrak{m} \subseteq G$ be a proper ideal of G. The following are equivalent.*

1. *The ideal $\mathfrak{m}$ is maximal.*
2. *For all $x \in G$, $x \notin \mathfrak{m}$ if, and only if, there exists $n \in \mathbb{N}$ such that $u - nx \in \mathfrak{m}$.*

Proof. This is a straightforward translation of the analogue statement for MV-algebras, proved in [21, Proposition 1.2.2]. □

Remark 2.1.56. With the same notation of the proof of Hölder's Theorem 2.1.23, we observe that there exists a unique element $x \in G$ such that $\bar{f}(x) = 1$. In fact, $x = a$ because

$$\mathcal{I}(a) = \left\{ \tfrac{m}{n} \in \mathbb{Q} \mid m \geqslant 0,\ n > 0 \text{ and } ma \leqslant na \right\} = \left\{ \tfrac{m}{n} \in \mathbb{Q} \mid m \geqslant 0,\ n > 0 \text{ and } \tfrac{m}{n} \leqslant 1 \right\}$$

which shows that $\bar{f}(a) = \sup \mathcal{I}(a) = 1$. The uniqueness is a consequence of the injectivity of the ℓ-homomorphism $\bar{f}$. Now, let (G, u) be a unital totally-ordered ℓ-group. By Remark 2.1.53, u is an arbitrary strictly positive element of G, unless $G = \{0\}$, in which case $u = 0$. But if $G = \{0\}$ and therefore $u = 0$, there is no unital ℓ-homomorphism $(G, u) \to (\mathbb{R}, 1)$. Hence, let us assume that $G \neq \{0\}$ and $u > 0$. By repeating the construction in the proof of Hölder's theorem, with $a = u$, we see that there exists a unital injective ℓ-homomorphism $(G, u) \to (\mathbb{R}, 1)$. It is possible to prove that this unital embedding is unique: the ℓ-homomorphism is completely determined, once a strong order unit has been fixed. This is the content of the following

Theorem 2.1.57 (Hölder, unital version)**.** *If (G, u) is a non-trivial totally-ordered archimedean unital ℓ-group, then there exists a unique unital ℓ-embedding*

$$(G, u) \to (\mathbb{R}, 1).$$

Proof. In light of Remark 2.1.56, it suffices to prove the uniqueness. Suppose that $\phi, \psi \colon (G, u) \to (\mathbb{R}, 1)$ are unital injective ℓ-homomorphisms. The map ϕ provides a unital ℓ-isomorphism between (G, u) and $(A, 1) := \phi(G, u)$, while ψ provides a unital ℓ-isomorphism between (G, u) and $(B, 1) := \psi(G, u)$. Then $f := \psi \circ \phi^{-1} \colon (A, 1) \to (B, 1)$ is a unital ℓ-isomorphism. We claim that f is the identity map. For an arbitrary element $x \in \mathbb{R}$, define

$$\mathcal{I}(x) := \left\{ \tfrac{m}{n} \in \mathbb{Q} \mid m \geqslant 0,\ n > 0 \text{ and } m \cdot 1 \leqslant nx \right\}.$$

If $x \in A$ and $\frac{m}{n} \in \mathcal{I}(x)$, we have $mf(1) = m \cdot 1 \leqslant nf(x)$, that is $\frac{m}{n} \in \mathcal{I}(f(x))$. The converse is analogous, hence $\mathcal{I}(x) = \mathcal{I}(f(x))$ and $\sup \mathcal{I}(x) = \sup \mathcal{I}(f(x))$. It is elementary that, for all $x \in \mathbb{R}$, $x = \sup \mathcal{I}(x)$. We conclude that $f(x) = x$ for all $x \in A$, i.e. $\psi \circ \phi^{-1}$ is the identity map. Upon considering the inverse map f^{-1}, it is clear that also $\phi \circ \psi^{-1}$ is the identity map, therefore $\phi = \psi$. □

Example 2.1.58. Recall by Lemma 2.1.22 that $\mathrm{C}(X, \mathbb{R})$ is an archimedean ℓ-group, for any topological space X. Assume now that X is a compact Hausdorff space. The compactness of the space implies that every function $f \in \mathrm{C}(X, \mathbb{R})$ is bounded, in particular there exists $\alpha \in \mathbb{R}$ satisfying $f(x) \leqslant \alpha$ for every $x \in X$. Since 1 is a strong order unit for the ℓ-group $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $\alpha \leqslant n \cdot 1$; this shows that the constant function of value 1 on X, denoted by $1_X \in \mathrm{C}(X, \mathbb{R})$, is a strong order unit for $\mathrm{C}(X, \mathbb{R})$.

Henceforth, when dealing with the ℓ-group $\mathrm{C}(X, \mathbb{R})$, we assume that X is a compact Hausdorff space. Although every ℓ-group of the kind $\mathrm{C}(X, \mathbb{R})$ is unital and archimedean, we shall see in the next example that not every archimedean unital ℓ-group is of this kind. A characterisation of those unital ℓ-groups of the form $\mathrm{C}(X, \mathbb{R})$, for some compact Hausdorff space X, is given by Yosida's representation Theorem 3.2.18.

Example 2.1.59. Consider the unital ℓ-group $(\mathbb{Z}, 1)$, and suppose that there exists a compact Hausdorff space X such that $(\mathbb{Z}, 1) \cong (\mathrm{C}(X, \mathbb{R}), 1_X)$. If $X = \varnothing$, then $\mathrm{C}(X, \mathbb{R}) = \mathrm{C}(\varnothing, \mathbb{R}) = \{\star\} = \{0 = u\}$ is the terminal object of the category $\ell\mathsf{Grp}_{\mathsf{u}}$. In the ℓ-group $\{0 = u\}$ the strong order unit coincides with the identity element of the group. However $\mathbb{Z} \ncong \{\star\}$, thus $X \neq \varnothing$. If the underlying set of the space X has cardinality at least 2, then there exist distinct points $p_1, p_2 \in X$. Observe that $\{p_1\}, \{p_2\}$ are closed sets since X is a T_1-space. Every compact Hausdorff space is normal [28, Theorem 3.1.9], so that Urysohn's lemma [28, Theorem 1.5.11] applies: there exist two functions $f, g \in \mathrm{C}(X, \mathbb{R})$ satisfying $f(p_1) = 0, f(p_2) = 1$ and $g(p_1) = 1, g(p_2) = 0$. The elements f, g do not satisfy either $f \leqslant g$ or $g \leqslant f$, therefore $\mathrm{C}(X, \mathbb{R})$ is not totally-ordered, while $\mathbb{Z}$ is. The two ℓ-groups cannot be isomorphic, for an arbitrary ℓ-isomorphism is both order-preserving and order-reflecting. We are left with one possibility, that is $X = \{p\}$. But this cannot be the case because $\mathrm{C}(\{p\}, \mathbb{R}) \cong \mathbb{R}$ and $\mathbb{Z} \ncong \mathbb{R}$. In conclusion, the unital ℓ-group $(\mathbb{Z}, 1)$ is not of the form $(\mathrm{C}(X, \mathbb{R}), 1_X)$ for any compact Hausdorff space X.

2.2 MV-algebras

2.2.1 Basic theory

Let $\mathcal{L}_{\mathsf{MV}} := \{\oplus, \neg, 0\}$ be a language formed by a binary function symbol $\oplus$, a unary function symbol $\neg$ and a constant 0.

Definition 2.2.1. An *MV-algebra* is an algebra $(A, \oplus, \neg, 0)$ satisfying the following identities.

MV1 $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

MV2 $x \oplus y = y \oplus x$.

MV3 $x \oplus 0 = x$.

MV4 $\neg\neg x = x$.

MV5 $x \oplus \neg 0 = \neg 0$.

MV6 $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Observe that the first three axioms give to an arbitrary MV-algebra the structure of a commutative monoid. In other words, if $(A, \oplus, \neg, 0)$ is an MV-algebra, then $(A, \oplus, 0)$ is a commutative monoid.

Example 2.2.2. Note that $\{0\}$ is an MV-algebra, called the *trivial* MV-algebra. A class of examples is given by Boolean algebras: it is easy to see that, if $(A, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra, then $(A, \vee, \neg, 0)$ is an MV-algebra. It can be shown that the variety of Boolean algebras coincides with the subvariety of *idempotent* MV-algebras, i.e. MV-algebras satisfying the axiom $x \oplus x = x$ [21, Corollary 1.5.5].

Example 2.2.3. The real interval $[0, 1]$ is an MV-algebra with respect to the following operations: for all $x, y \in [0, 1]$, $x \oplus y := \min(1, x + y)$ and $\neg x := 1 - x$. Then $([0, 1], \oplus, \neg, 0)$ is an MV-algebra, called the *standard* MV-algebra.

Example 2.2.4. In Example 2.1.21, given a topological space X, we defined the ℓ-group of all the continuous functions on X with values in $\mathbb{R}$. Likewise, let $\mathrm{C}(X, [0, 1])$ denote the set

$$\{f : X \to [0, 1] \mid f \text{ is continuous}\}.$$

Equipping this set with the pointwise operations inherited from the standard MV-algebra $[0, 1]$, it is elementary that $\mathrm{C}(X, [0, 1])$ is an MV-algebra.

In an arbitrary MV-algebra we can define the following derived operations.

$$\begin{aligned} 1 &:= \neg 0, \\ x \odot y &:= \neg(\neg x \oplus \neg y), \\ x \ominus y &:= x \odot \neg y. \end{aligned}$$

In the standard MV-algebra $[0, 1]$, where $\oplus$ represents the truncated sum, the derived operation $\ominus$ represents the truncated difference given by $x \ominus y = \max(0, x - y)$.

Remark 2.2.5. If we set $y = 1$ in Axiom **MV6**, we see that

$$\neg(\neg x \oplus 1) \oplus 1 = \neg(0 \oplus x) \oplus x,$$

and therefore

$$\neg 1 \oplus 1 = \neg x \oplus x,$$

that is $x \oplus \neg x = 1$.

The monoidal operation $\oplus$ is not cancellative. However:

Lemma 2.2.6. *For arbitrary elements x, y, z in an MV-algebra A,*

$$\text{if } x \oplus z = y \oplus z \text{ and } x \odot z = 0 = y \odot z, \text{ then } x = y.$$

In particular, if $y = x \oplus y$ and $x \odot y = 0$, then $x = 0$.

Proof. See [57, p. 106]. □

Lemma 2.2.7. *Let A be an MV-algebra and let $x, y \in A$. The following are equivalent.*

1. $\neg x \oplus y = 1$.
2. $x \odot \neg y = 0$, *i.e.* $x \ominus y = 0$.
3. $y = x \oplus (y \ominus x)$.
4. *There exists* $z \in A$ *such that* $x \oplus z = y$.

Proof. See [21, Lemma 1.1.2]. □

The previous result allows us to introduce a natural partial order on every MV-algebra.

Definition 2.2.8. If A is an MV-algebra and $x, y \in A$, then $x \leqslant y$ if, and only if, x, y satisfy the equivalent conditions of Lemma 2.2.7. If the order is total, we say that A is an *MV-chain*.

Lemma 2.2.9. *The natural order of an arbitrary MV-algebra has the following properties.*

1. $x \leqslant y$ *if, and only if,* $\neg y \leqslant \neg x$.
2. *If* $x \leqslant y$, *then, for all* $z \in A$, $x \oplus z \leqslant y \oplus z$ *and* $x \odot z \leqslant y \odot z$.

Proof. See [21, Lemma 1.1.4]. □

Remark 2.2.10. The partial order of Definition 2.2.8 induces a lattice structure [21, Proposition 1.1.5]. Specifically, if $(A, \oplus, \neg, 0)$ is an MV-algebra, then $(A, \leqslant)$ is a lattice, where least upper bounds and greatest lower bounds are given by

$$\begin{aligned} x \vee y &:= (x \ominus y) \oplus y, \\ x \wedge y &:= \neg(\neg x \vee \neg y) = x \odot (\neg x \oplus y). \end{aligned}$$

Distributivity of $\odot$ (respectively $\oplus$) with respect to $\vee$ (respectively $\wedge$) holds:

Proposition 2.2.11. *The following identities hold in every MV-algebra.*

1. $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$.
2. $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$.

Proof. See [21, Proposition 1.1.6]. □

Notation 2.2.12. If x is an element of an MV-algebra and $n \in \mathbb{N}$, by nx we understand $\underbrace{x \oplus \cdots \oplus x}_{n \text{ times}}$.

Lemma 2.2.13. *Let A be an MV-algebra and let $x, y \in A$. If $x \wedge y = 0$, then $nx \wedge ny = 0$ for all $n \in \mathbb{N}$.*

Proof. See [21, Lemma 1.1.8]. □

We conclude with some properties enjoyed by totally-ordered MV-algebras.

Proposition 2.2.14. *Suppose that A is an MV-chain. For all $x, y \in A$, the following hold.*

1. *If $x \oplus y < 1$, then $x \odot y = 0$.*
2. *$x \oplus y = x$ if, and only if, either $x = 1$ or $y = 0$.*
3. *$x \oplus y \oplus (x \odot y) = x \oplus y$.*
4. *$(x \ominus y) \oplus ((x \oplus \neg y) \odot y) = x$.*

Proof. See [21, Lemma 1.6.1, Proposition 1.6.2]. □

2.2.2 Ideals and congruences

Definition 2.2.15. Let $(A, \oplus_A, \neg_A, 0_A)$ and $(B, \oplus_B, \neg_B, 0_B)$ be MV-algebras. An *MV-homomorphism* is a function $h\colon A \to B$ such that, for all $x, y \in A$, the following conditions hold.

1. $h(0_A) = 0_B$.
2. $h(x \oplus_A y) = h(x) \oplus_B h(y)$.
3. $h(\neg_A x) = \neg_B h(x)$.

In the following we shall omit the subscripts when referring to operations belonging to different MV-algebras, as long as the context avoids confusion.

Remark 2.2.16. It is elementary that any MV-homomorphism preserves the derived operations $1, \odot, \ominus$, e.g. if $h\colon A \to B$ is an MV-homomorphism and $x, y \in A$, then

$$\begin{aligned} h(x \odot y) &= h(\neg(\neg x \oplus \neg y)) = \neg(h(\neg x \oplus \neg y)) \\ &= \neg(h(\neg x) \oplus h(\neg y)) = \neg(\neg h(x) \oplus \neg h(y)) \\ &= h(x) \odot h(y). \end{aligned}$$

Remark 2.2.17. Every MV-homomorphism $h\colon A \to B$ is order-preserving. Indeed, let $x, y \in A$ be such that $x \leqslant y$, or equivalently such that $x = x \wedge y$. Then $h(x) = h(x \wedge y) = h(x) \wedge h(y)$, that is $h(x) \leqslant h(y)$.

Given an MV-homomorphism $h\colon A \to B$, we define the *kernel* of h as

$$\ker h := h^{-1}(0) = \{x \in A \mid h(x) = 0\}.$$

The kernel of an MV-homomorphism is not only a set, it has more structure:

Definition 2.2.18. A subset I of an MV-algebra A is an *ideal* if it satisfies the following properties.

1. $0 \in I$.
2. If $x \in I$, and $y \in A$ is such that $y \leqslant x$, then $y \in I$.
3. If $x, y \in I$, then $x \oplus y \in I$.

For an arbitrary MV-homomorphism h, $\ker h$ is an ideal: condition 1 is satisfied by definition, while condition 2 holds because MV-homomorphisms are order-preserving. Lastly, condition 3 is easily proved since $x, y \in \ker h$ implies $h(x \oplus y) = h(x) \oplus h(y) = 0 \oplus 0 = 0$, which shows that $x \oplus y \in \ker h$.

The intersection of an arbitrary family of ideals is again an ideal, hence we can define the ideal $\langle V\rangle$ *generated* by a subset V as the intersection of all the ideals containing V. It is easy to show that (see [21, Lemma 1.2.1])

$$\langle V\rangle = \{x \in A \mid \exists v_1, \ldots, v_k \in V \text{ such that } x \leqslant v_1 \oplus \cdots \oplus v_k\}.$$

In the particular case in which $V = \{z\}$ for some $z \in A$, this reduces to

$$\langle \{z\}\rangle = \{x \in A \mid \exists n \in \mathbb{N} \text{ such that } x \leqslant nz\}.$$

The ideal $\langle\{z\}\rangle$ is denoted just with $\langle z\rangle$, and it is called the *principal* ideal generated by z. Further, if I is an ideal and $z \in A$, it is elementary that

$$\langle I \cup z\rangle = \{x \in A \mid \exists n \in \mathbb{N}, \exists a \in I \text{ such that } x \leqslant nz \oplus a\}.$$

Remark 2.2.19. Every non-trivial MV-algebra A contains two distinct ideals: the trivial ideal $\{0\}$ and the improper ideal A.

Definition 2.2.20. A proper ideal I of an MV-algebra A is said to be *prime* if, for all $x, y \in A$, either $x \ominus y \in I$ or $y \ominus x \in I$. Moreover, I is *maximal* if, for every ideal J of A, $I \subset J$ implies $J = A$.

Notation 2.2.21. A prime ideal of an MV-algebra is usually denoted by $\mathfrak{p}$, while a maximal ideal is denoted by $\mathfrak{m}$.

Prime ideals can be characterised as follows.

Lemma 2.2.22. *Let $\mathfrak{p}$ be a proper ideal of the MV-algebra A. The following are equivalent.*

1. *The ideal* $\mathfrak{p}$ *is prime.*

2. *For all* $x, y \in A$, $x \wedge y \in \mathfrak{p}$ *if, and only if, either* $x \in \mathfrak{p}$ *or* $y \in \mathfrak{p}$.

Proof. See [57, Proposition 4.13]. □

Proposition 2.2.23. *Every proper ideal of an MV-chain is a prime ideal.*

Proof. Let I be a proper ideal of an MV-chain A, and consider $x, y \in A$. Since A is totally-ordered, we have either $x \leqslant y$ or $y \leqslant x$, that is either $x \ominus y = 0$ or $y \ominus x = 0$. Every ideal, by definition, contains the element 0, thus we must have either $x \ominus y \in I$ or $y \ominus x \in I$. We conclude that I is a prime ideal. □

Lemma 2.2.24. *Let* $h \colon A \to B$ *be an MV-homomorphism. The following hold.*

1. *If* $J \subseteq B$ *is an ideal of* B, *then* $h^{-1}(J)$ *is an ideal of* A.

2. *For all* $x, y \in A$, $h(x) \leqslant h(y)$ *if, and only if,* $x \ominus y \in \ker h$.

3. h *is injective if, and only if,* $\ker h = \{0\}$.

4. $\ker h$ *is a prime ideal of* A *if, and only if,* B *is non-trivial and* $h(A)$ *is an MV-chain.*

Proof. 1. Let J be an ideal of B and let $x, y \in A$. We prove that $h^{-1}(J)$ contains the element 0, is downward closed, and closed under finite $\oplus$-sums. Clearly $0 \in h^{-1}(J)$ because $h(0) = 0 \in J$. If $x \in h^{-1}(J)$ and $y \in A$ satisfies $y \leqslant x$, then $h(y) \leqslant h(x) \in J$ because any MV-homomorphism is order-preserving. Since J is downward closed, we have $h(y) \in J$, i.e. $y \in h^{-1}(J)$. Finally, suppose that $x, y \in h^{-1}(J)$. Then $h(x \oplus y) = h(x) \oplus h(y) \in J$ because J is closed under sums, whence $x \oplus y \in h^{-1}(J)$.

2. Let $x, y \in A$; by Lemma 2.2.7, $h(x) \leqslant h(y)$ if, and only if, $h(x) \ominus h(y) = 0$. Moreover

$$h(x) \ominus h(y) = 0 \;\Leftrightarrow\; h(x \ominus y) = 0 \;\Leftrightarrow\; x \ominus y \in \ker h.$$

3. If h is injective, then $h(x) = 0 = h(0)$ implies $x = 0$, that is $\ker h = \{0\}$. On the other hand, assume that $\ker h = \{0\}$. If $h(x) = h(y)$, then $h(x) \ominus h(y) = 0 = h(y) \ominus h(x)$. It follows that $h(x \ominus y) = 0$, so that $x \ominus y = 0$. Therefore $x \leqslant y$. In a similar fashion, $y \leqslant x$ since $h(y \ominus x) = 0$. We conclude that $x = y$.

4. We remark that the hypothesis of B being non-trivial is necessary: if B is trivial, then the unique MV-homomorphism $A \to B$ is the trivial one, and its kernel is improper. Now, $\ker h$ is prime if, and only if, for every $x, y \in A$ we have either $x \ominus y \in \ker h$ or $y \ominus x \in \ker h$, if, and only if, $h(x) \leqslant h(y)$ or $h(y) \leqslant h(x)$, by item 2. In turn, this is equivalent to saying that $h(A)$ is totally-ordered. □

Lemma 2.2.25. *Given an arbitrary MV-homomorphism* $h \colon A \to B$, *if* $\mathfrak{m}$ *is a maximal ideal of* B, *then* $h^{-1}(\mathfrak{m})$ *is a maximal ideal of* A.

Proof. See [21, Proposition 1.2.16]. □

In the variety of MV-algebras congruences, i.e. quotient objects, can be studied by means of ideals, just like quotient objects in the variety of ℓ-groups are studied through ℓ-ideals. With the aim of proving the existence of a bijection between congruences on an MV-algebra A and ideals of A, we introduce an A-valued distance on the MV-algebra A.

Definition 2.2.26. Let A be an MV-algebra. The *Chang distance* on A is the function $d\colon A \times A \to A$ defined by

$$d(x, y) := (x \ominus y) \oplus (y \ominus x).$$

Example 2.2.27. In the standard MV-algebra $[0, 1]$ the Chang distance coincides with the absolute value function. Indeed, for every $x, y \in [0, 1]$,

$$\begin{aligned} d(x, y) &= \min(1, \max(0, x - y) + \max(0, y - x)) \\ &= \min(1, |x - y|) \\ &= |x - y|. \end{aligned}$$

Proposition 2.2.28. *In an arbitrary MV-algebra A, the following hold.*

1. $d(x, y) = 0$ *if, and only if,* $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leqslant d(x, y) \oplus d(y, z)$.
4. $d(x, y) = d(\neg x, \neg y)$.
5. $d(x \oplus s, y \oplus t) \leqslant d(x, y) \oplus d(s, t)$.

Proof. See [21, Proposition 1.2.5]. □

Definition 2.2.29. Let I be an ideal of the MV-algebra A. The relation $\equiv_I$ on A is defined by

$$x \equiv_I y \text{ if, and only if, } d(x, y) \in I.$$

Proposition 2.2.30. *For every ideal I of A, the relation $\equiv_I$ is a congruence on A and $I = \{x \in A \mid x \equiv_I 0\}$. Conversely, if $\equiv$ is a congruence on A, then $\{x \in A \mid x \equiv 0\}$ is an ideal, and $x \equiv y$ if, and only if, $d(x, y) \equiv 0$.*

Proof. See [21, Proposition 1.2.6]. □

This shows that the correspondence $I \mapsto \equiv_I$ is a bijection between the ideals of the MV-algebra A and the congruences on A. We shall denote the set $A/\equiv_I$ of equivalence classes by the symbol A/I; the equivalence class of an element $x \in A$ is x/I. Since $\equiv_I$ is a congruence, upon endowing $A/\equiv_I$ with operations

$$\neg(x/I) := (\neg x)/I, \quad x/I \oplus y/I := (x \oplus y)/I,$$

the algebra $(A/I, \oplus, \neg, 0/I)$ is an MV-algebra, called *quotient* MV-algebra. Clearly, the quotient map $x \mapsto x/I$ is a surjective MV-homomorphism whose kernel is precisely the ideal I.

Remark 2.2.31. We have shown that the kernel of an MV-homomorphism $A \to B$ is always an ideal of A. On the other hand, every ideal I of A arises as the kernel of an MV-homomorphism with domain A, namely the kernel of the quotient map $A \to A/I$. Now, let us assume that A is a non-trivial MV-algebra and $I \subseteq A$ is a proper ideal of A. The MV-homomorphism $q\colon A \to A/I$ satisfies $\ker q = I$. By Lemma 2.2.24.(4), I is a prime ideal if, and only if, A/I is totally-ordered. In other words, every prime ideal of A arises as the kernel of an MV-homomorphism with domain A and codomain a totally-ordered MV-algebra.

Proposition 2.2.32. *If A is an MV-chain, the set of all the ideals of A is totally-ordered (with respect to set-theoretic inclusion).*

Proof. Suppose, by contradiction, that $I, J \subseteq A$ are two ideals of A satisfying $I \nsubseteq J$ and $J \nsubseteq I$. Then there exist elements $x \in I \setminus J$ and $y \in J \setminus I$. Now, A is totally-ordered, so that x, y satisfy either $x \leqslant y$ or $y \leqslant x$. Assume without loss of generality that $x \leqslant y$; since J is an ideal and $x \leqslant y \in J$, x must belong to J, and this is a contradiction. In the case $y \leqslant x$, a similar argument applies. □

Lemma 2.2.33. *Every prime ideal of an MV-algebra is contained in a unique maximal ideal.*

Proof. Let A be an MV-algebra, and let $\mathfrak{p} \subseteq A$ be a prime ideal of A. Consider the family $\mathcal{A}$ of all proper ideals of A extending $\mathfrak{p}$. Then $\mathcal{A}$ is totally-ordered by [21, Theorem 1.2.11]. Hence $\mathfrak{m} := \bigcup_{J \in \mathcal{A}} J$ is an ideal of A. The element 1 does not belong to any ideal $J \in \mathcal{A}$, hence $\mathfrak{m}$ is proper because $1 \notin \mathfrak{m}$. We prove that $\mathfrak{m}$ is maximal. Suppose that N is a proper ideal such that $\mathfrak{m} \subseteq N$. In particular $\mathfrak{p} \subseteq \mathfrak{m} \subseteq N$, which implies that $N \in \mathcal{A}$, that is $\mathfrak{m} = N$. The uniqueness follows easily: if $\mathfrak{m}'$ is a maximal (hence proper) ideal extending $\mathfrak{p}$, then $\mathfrak{m}' \in \mathcal{A}$. This shows that $\mathfrak{m}' \subseteq \mathfrak{m}$; from the maximality of $\mathfrak{m}'$ we conclude that $\mathfrak{m}' = \mathfrak{m}$. □

We next prove an MV-algebraic version of Stone's Lemma.

Proposition 2.2.34. *If I is an ideal of the MV-algebra A, and $a \in A \setminus I$, there exists a prime ideal $\mathfrak{p}$ such that $I \subseteq \mathfrak{p}$ e $a \notin \mathfrak{p}$.*

Proof. Denote by $\mathcal{F}$ the family of all the ideals of A that contain I, but do not contain a. The latter family is non-empty because $I \in \mathcal{F}$; moreover $\mathcal{F}$ is partially ordered by set-theoretic inclusion. We show that the hypotheses of Zorn's Lemma are satisfied, i.e. that every totally-ordered subset $\{F_\iota\}_\iota$ of $\mathcal{F}$ has an upper bound in $\mathcal{F}$. The upper bound is given by $\bigcup_\iota F_\iota$. To prove this claim, observe that $\bigcup_\iota F_\iota$ is an ideal: $0 \in \bigcup_\iota F_\iota$ since 0 belongs to each F_ι; if $x \in \bigcup_\iota F_\iota$ and $y \leqslant x$, then there exists $F_{\bar\iota}$ such that $x \in F_{\bar\iota}$ and therefore $y \in F_{\bar\iota} \subseteq \bigcup_\iota F_\iota$. Lastly, if $x, y \in \bigcup_\iota F_\iota$ and $F_{\iota_1}, F_{\iota_2} \subseteq \bigcup_\iota F_\iota$ are such that $x \in F_{\iota_1}$ and $y \in F_{\iota_2}$, since $\{F_\iota\}_\iota$ is totally-ordered we have either $F_{\iota_1} \subseteq F_{\iota_2}$ or $F_{\iota_1} \supseteq F_{\iota_2}$. In the first case, $x \in F_{\iota_1} \subseteq F_{\iota_2} \ni y$ and then $x \oplus y \in F_{\iota_2} \subseteq \bigcup_\iota F_\iota$; the other case is proved in a similar fashion. This shows that $\bigcup_\iota F_\iota$ is an ideal. Furthermore, $\bigcup_\iota F_\iota \in \mathcal{F}$. Indeed, if $I \subseteq F_\iota$ for all ι, then $I \subseteq \bigcup_\iota F_\iota$, and if $a \notin F_\iota$ for all ι, then $a \notin \bigcup_\iota F_\iota$. Clearly, $\bigcup_\iota F_\iota$ is an upper bound for $\{F_\iota\}_\iota$. By Zorn's Lemma $\mathcal{F}$ has a maximal element, in other words there exists an ideal $\mathfrak{p}$ of A which is maximal with respect to the properties $I \subseteq \mathfrak{p}$ and $a \notin \mathfrak{p}$. We prove that $\mathfrak{p}$ is prime. Suppose by contradiction that there exist $x, y \in A$ such that $x \ominus y \notin \mathfrak{p}$ and $y \ominus x \notin \mathfrak{p}$. Since $J \subseteq \mathfrak{p} \subset \langle \mathfrak{p} \cup x \ominus y \rangle$, and $\mathfrak{p}$ is maximal between the ideals satisfying $J \subseteq \mathfrak{p}$ and $a \notin \mathfrak{p}$, we conclude that $a \in \langle \mathfrak{p} \cup x \ominus y \rangle$. That is, there exist $m \in \mathbb{N}$ and $p \in \mathfrak{p}$ such that $a \leqslant m(x \ominus y) \oplus p$. For the same reason, upon considering the ideal $\langle \mathfrak{p} \cup y \ominus x \rangle$, we can find $n \in \mathbb{N}$ and $q \in \mathfrak{p}$ satisfying $a \leqslant n(y \ominus x) \oplus q$. Set $t := \max(m, n)$ and $u := p \oplus q \in \mathfrak{p}$. Then $a \leqslant t(x \ominus y) \oplus u$ and $a \leqslant t(y \ominus x) \oplus u$, whence by Proposition 2.2.11

$$a \leqslant (t(x \ominus y) \oplus u) \wedge (t(y \ominus x) \oplus u) = (t(x \ominus y) \wedge t(y \ominus x)) \oplus u.$$

However $(x \ominus y) \wedge (y \ominus x) = 0$ by [21, Proposition 1.1.7], and Lemma 2.2.13 implies $(t(x \ominus y) \wedge t(y \ominus x)) = 0$. It follows that $a \leqslant 0 \oplus u = u$, but $u \in \mathfrak{p}$ and $a \leqslant u$, so that $a \in \mathfrak{p}$ that is a contradiction. □

Corollary 2.2.35. *Every proper ideal of an MV-algebra is the intersection of prime ideals.*

Proof. Let I be a proper ideal of an MV-algebra A. By Proposition 2.2.34, for each $a \in A \setminus J$ there exists a prime ideal $\mathfrak{p}_a$ such that $J \subseteq \mathfrak{p}_a$ and $a \notin \mathfrak{p}_a$. Consider the family $\{\mathfrak{p}_a\}_{a \in A \setminus J}$. We shall prove that $\bigcap_{a \in A \setminus J} \mathfrak{p}_a = J$. Clearly $\bigcap_{a \in A \setminus J} \mathfrak{p}_a \supseteq J$, because $\mathfrak{p}_a \supseteq J$ for every $a \in A \setminus J$. To prove the other inclusion, assume by contradiction that there is $x \in (\bigcap_{a \in A \setminus J} \mathfrak{p}_a) \setminus J$. In particular $x \in A \setminus J$, whence there exists a prime ideal $\mathfrak{p}$ such that $J \subseteq \mathfrak{p}$ and $x \notin \mathfrak{p}$. Then $\mathfrak{p} \in \{\mathfrak{p}_a\}_{a \in A \setminus J}$, but this is absurd since $x \in \bigcap_{a \in A \setminus J} \mathfrak{p}_a$ entails $x \in \mathfrak{p}$. □

Corollary 2.2.36. *Every non-trivial MV-algebra has a maximal ideal.*

Proof. In any non-trivial MV-algebra A, $\{0\}$ is a proper ideal; by Proposition 2.2.34, there is a prime ideal $\mathfrak{p}$ of A extending $\{0\}$. Lemma 2.2.33 implies that there exists a unique maximal ideal extending $\mathfrak{p}$, so that A has a maximal ideal. □

2.2.3 Subdirect representation

Let $\{A_i\}_{i \in I}$ be a family of MV-algebras, where I is an arbitrary set. The class of MV-algebras is equationally defined, thus Birkhoff's theorem [18, Theorem 11.9] entails that, in particular, an arbitrary product of MV-algebras is again an MV-algebra. Explicitly, the cartesian products of the sets A_i, denoted by $\prod_{i\in I} A_i$, can be endowed with (pointwise) operations in the following way. If we think of an element $f \in \prod_{i\in I} A_i$ as a function $f\colon I \to \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$ for every $i \in I$, then we can define

$$(\neg f)(i) := \neg f(i) \text{ and } (f \oplus g)(i) := f(i) \oplus g(i),$$

and $\bar{0}$ is the function $\bar{0}\colon i \mapsto 0_i$, for all $i \in I$. The algebra $(\prod_{i\in I} A_i, \oplus, \neg, \bar{0})$ is an MV-algebra, and it is called the *direct product* of the family $\{A_i\}_{i\in I}$. Further, the function $\pi_j\colon \prod_{i\in I} A_i \to A_j$ given by $f \mapsto f(j)$ is an MV-homomorphism, called jth *projection*.

Definition 2.2.37. We say that the MV-algebra A is a *subdirect product* of the family of MV-algebras $\{A_i\}_{i\in I}$ if, and only if, there exists an injective MV-homomorphism $h\colon A \to \prod_{i\in I} A_i$ such that, for each $j \in I$, the composition $\pi_j \circ h\colon A \to A_j$ is surjective.

Remark 2.2.38. If A is a subdirect product of $\{A_i\}_{i\in I}$, then A is isomorphic to the subalgebra $h(A)$ of the product $\prod_{i\in I} A_i$; furthermore, for every $j \in I$ the restriction $\pi_{j_{|h(A)}}\colon h(A) \to A_j$ is surjective. Saying that the MV-homomorphism $\pi_j \circ h$ is surjective, means that for every $a_i \in A_i$ there exists an element of the subalgebra $h(A)$ whose ith component is a_i; in other words, there is $f \in h(A)$ such that $f(i) = a_i$.

Theorem 2.2.39. *An MV-algebra A is a subdirect product of the family $\{A_i\}_{i\in I}$ if, and only if, there exists a family $\{J_i\}_{i\in I}$ of ideals of A such that $A_i \cong A/J_i$ for each $i \in I$, and $\bigcap_{i\in I} J_i = \{0\}$.*

Proof. Suppose that A is a subdirect product of the family $\{A_i\}_{i\in I}$, so that there exists an injective MV-homomorphism $h\colon A \to \prod_{i\in I} A_i$ such that, for every $j \in I$, the MV-homomorphism $\pi_j \circ h\colon A \to A_j$ is surjective. If $j \in I$, define $J_j := \ker \pi_j \circ h$. By the first isomorphism theorem [18, Theorem 6.12] we have $\frac{A}{\ker \pi_j \circ h} = \frac{A}{J_j} \cong A_j$. It suffices to prove that $\bigcap_{i\in I} J_i = \{0\}$. One of the two inclusions is trivial. For the other inclusion, assume that $x \in \bigcap_{i\in I} J_i$; this means that, for all $i \in I$, $x \in J_i = \ker \pi_j \circ h$ if, and only if, $\pi_j \circ h(x) = 0$. Then $h(x) = 0 \in \prod_{i\in I} A_i$, whence $x = 0$ because h is injective. We conclude that $\bigcap_{i\in I} J_i = \{0\}$. Conversely, let $\{J_i\}_{i\in I}$ be a family of ideals of A such that $A_i \cong A/J_i$ for each $i \in I$, and $\bigcap_{i\in I} J_i = \{0\}$. Fix an isomorphism $\varepsilon_i\colon A/J_i \cong A_i$ for every $i \in I$, and define an MV-homomorphism $h\colon A \to \prod_{i\in I} A_i$ as $(h(x))(i) := \varepsilon_i(\frac{x}{J_i})$. We have

$$x \in \ker h \Leftrightarrow (h(x))(i) = 0\ \forall i \in I \Leftrightarrow \varepsilon_i(\frac{x}{J_i}) = 0\ \forall i \in I \Leftrightarrow$$

$$\frac{x}{J_i} = 0\ \forall i \in I \Leftrightarrow x \in J_i\ \forall i \in I \Leftrightarrow x \in \bigcap_{i\in I} J_i = \{0\}$$

so that, by Lemma 2.2.24.(3), h is injective. To conclude, it is enough to show that, for every $j \in I$, the MV-homomorphism $\pi_j \circ h\colon A \to A_j$ is surjective. If $y \in A_j$, since the map $\varepsilon_j\colon A/J_j \to A_j$ is surjective, there exists $\frac{a}{J_j} \in A/J_j$ satisfying $\varepsilon_j(\frac{a}{J_j}) = y$. Therefore $\pi_j \circ h(a) = (h(a))(j) = \varepsilon_j(\frac{a}{J_j}) = y$. □

Remark 2.2.40. We have just seen that any family of ideals $\{J_i\}_{i\in I}$ of A, whose intersection is trivial, gives a representation of A as a subdirect product of the family $\{A/J_i\}_{i\in I}$. More generally ([15, Corollary 1, p. 140]) the (isomorphic) representations of a finitary algebra A as a subdirect product are in bijective correspondence with the families of congruences $\{\theta_i\}_{i\in I}$ on A such that $\bigcap_{i\in I}\theta_i = 0$, where $0 := \{(x,y)\in A\times A \mid x = y\}$ is the trivial congruence.

The following is known as *Chang's subdirect representation theorem*:

Theorem 2.2.41. *Every MV-algebra is a subdirect product of MV-chains.*

Proof. If the MV-algebra is trivial, it is a chain and there is nothing to prove. By Theorem 2.2.39, together with Remark 2.2.31, it suffices to show that the intersection of all prime ideals of a non-trivial MV-algebra A is $\{0\}$. But $\{0\}$ is itself an ideal, and by Corollary 2.2.35 the theorem is proved. □

2.2.4 Radical and infinitesimal elements

Definition 2.2.42. An MV-algebra A is *simple* if its only ideals are the trivial ideal $\{0\}$ and the improper ideal A.

Remark 2.2.43. In every simple non-trivial MV-algebra, the ideal $\{0\}$ is maximal. Therefore it is the unique maximal ideal, since any other ideal extends it.

Theorem 2.2.44. *Let A be an MV-algebra. The following are equivalent.*

1. *A is simple and non-trivial.*
2. *A is isomorphic to a subalgebra of the standard MV-algebra* $[0,1]$.

Proof. See [21, Theorem 3.5.1]. □

Remark 2.2.45. By Theorem 2.2.44, it follows that every simple MV-algebra is totally-ordered. Moreover, the cardinality of a simple MV-algebra is not greater than $\aleph_1$, since it can be embedded in the MV-algebra $[0,1]$.

Proposition 2.2.46. *If A is an MV-algebra and J is an ideal of A, there is a bijective order-preserving correspondence between ideals of A extending J, and ideals of A/J.*

Proof. Denote by q the map which sends an ideal $I \subseteq A$ such that $J \subseteq I$, to the set $q(I) := I/J \subseteq A/J$. We shall prove that q is a bijective order-preserving correspondence. First, we check that q is well-defined, i.e. that I/J is an ideal of A/J. The element 0 is in I/J since $\ker q = J \subseteq I$; if $\frac{x}{J}, \frac{y}{J} \in I/J$, then $x, y \in I$ implies $x \oplus y \in I$ because I is closed under finite $\oplus$-sums, whence $\frac{x}{J} \oplus \frac{y}{J} = \frac{x\oplus y}{J} \in I/J$. If $\frac{x}{J} \in I/J$, $\frac{y}{J} \in A/J$ satisfy $\frac{y}{J} \leqslant \frac{x}{J}$, it follows that $\frac{y}{J} \leqslant \frac{x}{J}$ if, and only if, $\frac{y}{J} \ominus \frac{x}{J} = 0$ if, and only if, $y \ominus x \in J \subseteq I$. Thus $x \oplus (y \ominus x) \in I$ if, and only if, $y \oplus (x \ominus y) \in I$ by Axiom **MV6**. This shows that $y \leqslant y \oplus (x \ominus y) \in I$, so that $y \in I$, i.e. $\frac{y}{J} \in I/J$. We conclude that I/J is an ideal. Regarding the bijectivity of q, for every ideal I extending J, we have

$$q^{-1}(q(I)) = \{x \in A \mid q(x) \in q_J(I)\} = \{x \in A \mid \tfrac{x}{J} \in I/J\} = \{x \in A \mid x \in I\} = I.$$

By Lemma 2.2.24.(1), if $K \subseteq A/J$ is an ideal, then $q^{-1}(K)$ is an ideal of A; consequently $J = \ker q = q^{-1}(0) \subseteq q^{-1}(K)$ because $0 \in K$, and lastly

$$q(q^{-1}(K)) = \{q(x) \mid x \in q^{-1}(K)\} = \{q(x) \mid q(x) \in K\} = K.$$

Finally, we show that q is order-preserving. If I_1, I_2 are ideals of A extending J, and $I_1 \subseteq I_2$, then $I_1/J = q(I_1) \subseteq q(I_2) = I_2/J$. On the other hand, if K_1, K_2 are ideals of A/J satisfying $K_1 \subseteq K_2$, it follows

$$q^{-1}(K_1) = \{x \in A \mid \tfrac{x}{J} \in K_1\} \subseteq \{x \in A \mid \tfrac{x}{J} \in K_2\} = q^{-1}(K_2).$$

□

Corollary 2.2.47. *An ideal $\mathfrak{m} \subseteq A$ is maximal if, and only if, the quotient MV-algebra $A/\mathfrak{m}$ is simple and non-trivial.*

Proof. This follows at once from Proposition 2.2.46. □

Definition 2.2.48. For an arbitrary MV-algebra A, the *radical* ideal of A, denoted by $\operatorname{Rad} A$, is the intersection of all the maximal ideals of A. An MV-algebra is *semisimple* if its radical is trivial.

In the following proof we use the fact that all subalgebras of $[0, 1]$ are simple.

Theorem 2.2.49. *Let A be an MV-algebra. The following are equivalent.*

1. *A is semisimple and non-trivial.*
2. *A is a subdirect product of subalgebras of $[0, 1]$.*

Proof. By Theorem 2.2.39, A is a subdirect product of the family $\{A_i\}_{i \in I}$ of subalgebras of $[0, 1]$ if, and only if, there exists a family $\{J_i\}_{i \in I}$ of ideals of A such that $A_i \cong A/J_i$ for every $i \in I$, and $\bigcap_{i \in I} J_i = \{0\}$. By Corollary 2.2.47, this happens precisely when the family $\{J_i\}_{i \in I}$ is a family of maximal ideals of A whose intersection is trivial; in turn, this is equivalent to asking that $\operatorname{Rad} A = \{0\}$, i.e. that A is semisimple. □

The following statement, dealing with semisimple quotients, should be compared with Corollary 2.2.47.

Lemma 2.2.50. *Let A be a non-trivial MV-algebra, and let J be an ideal of A. Then J is an intersection of maximal ideals if, and only if, A/J is semisimple.*

Proof. Let $q \colon A \to A/J$ be the quotient map, and suppose that A/J is a semisimple MV-algebra. Upon denoting with $\{\mathfrak{m}_i\}_{i \in I}$ the family of all the maximal ideals of A/J, we have $\bigcap_{i \in I} \mathfrak{m}_i = \{0\}$, and

$$J = \ker q = q^{-1}(0) = q^{-1}\Big(\bigcap_{i \in I} \mathfrak{m}_i\Big) = \bigcap_{i \in I} q^{-1}(\mathfrak{m}_i).$$

It suffices to show that each ideal $q^{-1}(\mathfrak{m}_i)$ is maximal. If $I \subseteq A$ is an ideal of A such that $q^{-1}(\mathfrak{m}_i) \subset I$, then Proposition 2.2.46 entails that $\mathfrak{m}_i = q(q^{-1}(\mathfrak{m}_i)) \subset q(I) = I/J$. Since $\mathfrak{m}_i$ is maximal, we deduce that $I/J = A/J$, hence $I = q^{-1}(q(I)) = q^{-1}(q(A)) = A$. We have proved that J is the intersection of maximal ideals of A. In the other direction, assume that J is the intersection of a family $\{\mathfrak{m}_i\}_{i\in I}$ of maximal ideals of A, and let $\{\mathfrak{m}_\iota^J\}_{\iota\in I'}$ be the family of all the maximal ideals of A containing J. Clearly $\{\mathfrak{m}_i\}_{i\in I} \subseteq \{\mathfrak{m}_\iota^J\}_{\iota\in I'}$, whence $\bigcap_{\iota\in I'} \mathfrak{m}_\iota^J \subseteq \bigcap_{i\in I} \mathfrak{m}_i = J$. On the other hand, $J \subseteq \bigcap_{\iota\in I'} \mathfrak{m}_\iota^J$, so that J coincides with the intersection of all maximal ideals extending J. Applying the same argument as above, along with Proposition 2.2.46, we have that $\{q(\mathfrak{m}_\iota^J)\}_{\iota\in I'}$ is the family of all maximal ideals of A/J, and

$$\operatorname{Rad} A/J = \bigcap_{\iota\in I'} q(\mathfrak{m}_\iota^J) = q\Big(\bigcap_{\iota\in I'} \mathfrak{m}_\iota^J\Big) = q(J) = \{0\}.$$

□

The elements of the radical ideal of an MV-algebra can be characterised as infinitely small elements, in the following sense.

Definition 2.2.51. Let A be an MV-algebra. An element $a \in A$ is *infinitesimal* if, and only if, $a \neq 0$ and $na \leqslant \neg a$ for every $n \in \mathbb{N}$. The set of infinitesimal elements of A is denoted by $\operatorname{infinit} A$.

Proposition 2.2.52. *In an arbitrary MV-algebra A,* $\operatorname{Rad} A = \operatorname{infinit} A \cup \{0\}$.

Proof. We begin showing that $\operatorname{infinit} A \cup \{0\} \subseteq \operatorname{Rad} A$. It is elementary that $0 \in \operatorname{Rad} A$, therefore we prove that $\operatorname{infinit} A \subseteq \operatorname{Rad} A$. Let $a \in \operatorname{infinit} A$ be an infinitesimal element and assume, by contradiction, that $a \notin \operatorname{Rad} A$; in other words, there exists a maximal ideal $\mathfrak{m}$ of A such that $a \notin \mathfrak{m}$. The ideal $\langle \mathfrak{m} \cup \{a\}\rangle$ strictly contains $\mathfrak{m}$, and $\mathfrak{m}$ is maximal, hence $\langle \mathfrak{m} \cup \{a\}\rangle$ is improper. In particular, $1 \in \langle \mathfrak{m} \cup \{a\}\rangle$, so that there exist $n \in \mathbb{N}$ and $z \in \mathfrak{m}$ satisfying $1 = na \oplus z$. The element a is infinitesimal, thus $na \leqslant \neg a$ and, by Lemma 2.2.9.(1), $a \leqslant \neg na$. Lemma 2.2.7.(1) implies that $1 = na \oplus z$ if, and only if, $\neg na \leqslant z$, whence $a \leqslant \neg na \leqslant z \in \mathfrak{m}$ and $a \in \mathfrak{m}$ because $\mathfrak{m}$ is downward closed. This is a contradiction. In order to prove the other inclusion, namely $\operatorname{Rad} A \subseteq \operatorname{infinit} A \cup \{0\}$, let us suppose by contradiction that $a \in \operatorname{Rad} A$ is a non-zero element which is not infinitesimal. This means that there exists $m \in \mathbb{N}$ such that $ma \nleqslant \neg a$, or equivalently, $ma \ominus \neg a \neq 0$. By Proposition 2.2.34 there is a prime ideal $\mathfrak{p}$ (extending the trivial ideal $\{0\}$, and) which does not contain the element $ma \ominus \neg a$. Since $\mathfrak{p}$ is a prime ideal, we have $\neg a \ominus ma \in \mathfrak{p}$. By Lemma 2.2.33 there exists a maximal ideal $\mathfrak{m} \subseteq A$ extending $\mathfrak{p}$, that is $\neg a \ominus ma \in \mathfrak{p} \subseteq \mathfrak{m}$. Observe that

$$\neg a \ominus ma = \neg a \odot \neg ma = \neg(\neg\neg a \oplus \neg\neg ma) = \neg(a \oplus ma) = \neg(m+1)a.$$

Consequently $(m+1)a \notin \mathfrak{m}$, for otherwise we would have, by Remark 2.2.5,

$$1 = (m+1)a \oplus \neg(m+1)a \in \mathfrak{m}$$

and therefore $\mathfrak{m} = A$ which is not possible. The ideal $\mathfrak{m}$ is closed under finite $\oplus$-sums and $(m+1)a \notin \mathfrak{m}$, hence $a \notin \mathfrak{m}$. Then $a \notin \operatorname{Rad} A$, a contradiction. □

In view of the characterisation of the radical ideal provided by Proposition 2.2.52, an MV-algebra is seen to be semisimple if, ad only if, it does not contain any infinitesimal element. The next result is stated for later use.

Lemma 2.2.53. *If A is an MV-algebra and $x, y \in \operatorname{Rad} A$, then $x \odot y = 0$.*

Proof. Let $x, y \in \operatorname{Rad} A$ be elements of the radical ideal. By Proposition 2.2.14 and Chang's subdirect representation Theorem 2.2.41, one can show that

$$x \oplus y \oplus (x \odot y) = x \oplus y.$$

Hence, by Lemma 2.2.6 it suffices to prove $(x \oplus y) \odot (x \odot y) = 0$. Notice that

$$\begin{aligned}(x \oplus y) \odot (x \odot y) &= (x \oplus y) \odot \neg(\neg x \oplus \neg y)\\ &= \neg(\neg(x \oplus y) \oplus (\neg x \oplus \neg y)).\end{aligned}$$

This means that

$$\begin{aligned}(x \oplus y) \odot (x \odot y) = 0 &\Leftrightarrow\\ \neg(\neg(x \oplus y) \oplus (\neg x \oplus \neg y)) = 0 &\Leftrightarrow\\ \neg(x \oplus y) \oplus (\neg x \oplus \neg y) = 1 &\Leftrightarrow\\ x \oplus y \leqslant \neg x \oplus \neg y. & \qquad\qquad \text{(Lemma 2.2.7)}\end{aligned}$$

Proposition 2.2.52 entails $x \leqslant \neg x$ and $y \leqslant \neg y$, in other words there are $z_1, z_2 \in A$ such that $x \oplus z_1 = \neg x$ and $y \oplus z_2 = \neg y$. Then

$$(x \oplus y) \oplus (z_1 \oplus z_2) = (x \oplus z_1) \oplus (y \oplus z_2) = \neg x \oplus \neg y,$$

that is equivalent, by Lemma 2.2.7, to $x \oplus y \leqslant \neg x \oplus \neg y$. □

2.2.5 Representing semisimple and free MV-algebras

Semisimple MV-algebras admit a sharp characterisation: they can be represented as algebras of continuous functions on some compact Hausdorff space, as stated below in Proposition 2.2.69. It turns out that this representation of semisimple MV-algebras can be deduced by the representation of a proper subfamily, that of free MV-algebras, since every MV-algebra is a quotient of a free one. For this reason, the fundamental result in this connection is McNaughton's Theorem 2.2.64 which identifies the free MV-algebra over κ generators with the algebra of all continuous piecewise linear functions, with integer coefficients, on the Tychonoff cube $[0, 1]^\kappa$ (which is a compact Hausdorff space, with respect to the product topology).

We briefly recall the general (category-theoretic) notion of free object. Let (C, U) be a *concrete category*, i.e. C is a category and $U\colon \mathsf{C} \to \mathsf{Set}$ is a faithful functor. Given an object X of Set and an object A of C, we say that A is a *free object* over X *with respect to* U, if there exists a function $i\colon X \to U(A)$ satisfying the following *universal property*: for all objects B of C and all functions $f\colon X \to U(B)$, there exists a unique morphism $g\colon A \to B$ in C such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & U(A) \arrow[d, "U(g)"] \\ & U(B) \end{tikzcd}$$

If the previous conditions are satisfied, X is called a *set of generators* for A, and the function i is thought of as the insertion of generators. Free objects, when they exist, are unique up to isomorphism and only depend on the cardinality of the generating set:

Proposition 2.2.54. *Let X, Y be sets, and assume that there exists a bijection between X and Y. If (C, U) is a concrete category and FX, FY are objects of C which are free on X and Y, respectively, then $FX \cong FY$.*

Proof. We first prove the case $X = Y$, i.e. we prove that FX, if it exists, is unique to within a unique isomorphism. By definition, there are two functions $i\colon X \to U(FX)$ and $\bar{i}\colon X \to U(FY)$ with the universal property; since FX is free on X, there exists a unique morphism $\phi\colon FX \to FY$ such that $U(\phi) \circ i = \bar{i}$. But FY is also free on X, hence there is a unique morphism $\psi\colon FY \to FX$ satisfying the condition $U(\psi) \circ \bar{i} = i$.

$$\begin{tikzcd} & U(FX) \arrow[d, dashed, shift right, "U(\phi)"'] \\ X \arrow[ur, "i"] \arrow[dr, "\bar{i}"'] & \\ & U(FY) \arrow[u, dashed, shift right, "U(\psi)"'] \end{tikzcd}$$

It is clear that the identity morphisms $1_{FX}, 1_{FY}$ are the unique morphisms satisfying $U(1_{FX}) \circ i = i$ and $U(1_{FY}) \circ \bar{i} = \bar{i}$, respectively. However $U(\psi \circ \phi) \circ i = i$ and $U(\phi \circ \psi) \circ \bar{i} = \bar{i}$; we conclude that $\psi \circ \phi = 1_{FX}$ and $\phi \circ \psi = 1_{FY}$, that is $FX \cong FY$. Assume now that $\sigma\colon X \to Y$ is a bijection. We shall prove that the object FY, along with the function $\bar{i} \circ \sigma\colon X \to U(FY)$, is free on X; the argument above will imply that FX and FY are isomorphic objects. Suppose we are given an object A of C, and a function $f\colon X \to U(A)$.

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[drr, "f"'] & Y \arrow[r, "\bar{i}"] \arrow[dr, "f \circ \sigma^{-1}"] & U(FY) \arrow[d, dashed, "U(g)"] \\ & & U(A) \end{tikzcd}$$

Since FY is free on Y, there exists a unique morphism $g\colon FY \to A$ such that $U(g) \circ \bar{i} = f \circ \sigma^{-1}$, which is equivalent to $U(g) \circ \bar{i} \circ \sigma = f$. In other words, FY is free on X. $\square$

In most cases, for example when the category C is a variety of algebras (with homomorphisms as morphisms), the faithful functor $U\colon \mathsf{C} \to \mathsf{Set}$ is defined to be the usual underlying-set functor. Free objects in finitary varieties of algebras always exist by a theorem of Birkhoff [18, Theorem 10.12]. Specifically, they are constructed as algebras of terms for the language (see Lemma 5.1.1). Another way of representing free algebras is through term functions, as we shall see in Proposition 2.2.60 below, in the case of MV-algebras.

Remark 2.2.55. More generally, it is known that (possibly non-finitary) varieties of algebras correspond exactly, up to equivalence, to categories which are monadic over Set (see [49], or [54, Theorem 5.40 p. 66, Theorem 5.45 p. 68]). In particular, in the latter categories the underlying-set functor U admits a left adjoint functor $F \dashv U\colon \mathsf{C} \to \mathsf{Set}$ mapping a set X to the free object $F(X)$ on X. Therefore, free objects exist also in infinitary varieties of algebras.

Given an arbitrary cardinal κ, the free MV-algebra over a set of κ generators will be denoted by Free_κ. It is clear, from the previous discussion, that the algebra Free_κ always exists and it is unique up to isomorphism.

Remark 2.2.56. Every MV-algebra is isomorphic to a quotient of some free MV-algebra. Assume that the MV-algebra A is generated by no more than κ elements, and consider a function $X \to U(A)$, where X is a set of cardinality κ. The universal property of the free MV-algebra Free_κ provides a (clearly surjective) MV-homomorphism $f\colon \mathrm{Free}_\kappa \to A$. By the first isomorphism theorem [18, Theorem 6.12] A is isomorphic to the quotient $\mathrm{Free}_\kappa / \ker f$ where $\ker f$ is, by Lemma 2.2.24.(1), an ideal of Free_κ. More generally, it is clear that the argument has nothing to do with MV-algebras specifically, and thus applies to all varieties of (possibly infinitary) algebras.

The first step toward a representation of free MV-algebras consists in showing that Free_κ can be identified with the algebra of *term functions* on the MV-algebra $[0,1]^\kappa$.

Definition 2.2.57. Fix a cardinal κ and consider a set $\{x_\alpha\}_{\alpha<\kappa}$ of propositional variables (i.e. there are distinct variables $x_1, x_2, \dots, x_\alpha, \dots$ for every ordinal number $\alpha < \kappa$). The set of MV-terms is defined inductively as follows.

1. The constant 0 and the variables x_α are MV-terms, for every ordinal $\alpha < \kappa$.
2. If τ is an MV-term, then $\neg\tau$ is an MV-term.
3. If σ, τ are MV-terms, then $(\sigma \oplus \tau)$ is an MV-term.

Given an MV-algebra A, every term τ for the language of MV-algebras gives rise to a function $\tau^A\colon A^\kappa \to A$ as follows.

Definition 2.2.58. Let τ be an MV-term in the variables $\{x_\alpha\}_{\alpha<\kappa}$. The *term function* $\tau^A\colon A^\kappa \to A$ is defined inductively on the number of connectives of τ as:

1. $x_\alpha^A := \pi_\alpha$ (i.e., variables are interpreted as projections).
2. $0^A :=$ constant function of value 0 on A^κ.
3. $(\neg\tau)^A := \neg(\tau^A)$.
4. $(\sigma \oplus \tau)^A := \sigma^A \oplus \tau^A$.

We write $\mathrm{Term}_\kappa^{\mathrm{A}}$ for the set of all term functions on A^κ.

Remark 2.2.59. Every term function is, by construction, a function on A^κ depending on a *finite* number of variables. Upon equipping $\mathrm{Term}_\kappa^{\mathrm{A}}$ with pointwise operations, it is easily seen that it is an MV-algebra. Specifically, it is a subalgebra of the MV-algebra A^{A^κ} of all the functions from A^κ to A.

Denote by $\mathrm{Proj}_\kappa^{\mathrm{A}}$ the set of projections $\{\pi_\alpha \colon A^\kappa \to A\}_{\alpha<\kappa}$. Then

Proposition 2.2.60. *For each cardinal κ, $\mathrm{Term}_\kappa^{[0,1]}$ is the free MV-algebra over the generating set $\mathrm{Proj}_\kappa^{[0,1]}$, in other words $\mathrm{Term}_\kappa^{[0,1]} \cong \mathrm{Free}_\kappa$. Moreover, there is a unique such isomorphism that extends the bijection $\mathrm{Proj}_\kappa^{[0,1]} \to \{x_\alpha\}_{\alpha<\kappa}$ given by $\pi_\alpha \mapsto x_\alpha$.*

Proof. See [21, Proposition 3.1.4]. □

Remark 2.2.61. Proposition 2.2.60 is a non-trivial result. In particular it amounts essentially to Chang's Completeness Theorem that $[0,1]$ generates the variety of MV-algebras, i.e. that $\mathbb{HSP}([0,1])$ is the entire variety of MV-algebras.

A concrete representation of the MV-algebra Free_κ can be obtained by characterising, intrinsically, those functions $[0,1]^\kappa \to [0,1]$ that arise as term functions.

Definition 2.2.62. Let $n \geqslant 1$ be an arbitrary fixed integer. A function $f \colon [0,1]^n \to [0,1]$ is a *McNaughton function* if the following conditions hold.

1. f is continuous.
2. f is piecewise linear with integer coefficients; in other words, there are finitely many linear polynomials with integer coefficients $p_1, \ldots, p_k$ (the *linear constituents* of f) such that, for all $y \in [0,1]^n$, there exists $j \in \{1, \ldots, k\}$ such that $f(y) = p_j(y)$.

This definition can be generalised by considering an arbitrary cardinal number κ. A function $g \colon [0,1]^\kappa \to [0,1]$ is a *McNaughton function* if there exist ordinal numbers $\alpha_1, \ldots, \alpha_n < \kappa$ and a McNaughton function $f \colon [0,1]^n \to [0,1]$ such that, for all $y = (y_1, y_2, \ldots) \in [0,1]^\kappa$, $g(y) = f(y_{\alpha_1}, \ldots, y_{\alpha_n})$. The set of all McNaughton functions on the Tychonoff cube $[0,1]^\kappa$, endowed with pointwise operations, is an MV-algebra. In fact, it is a subalgebra of the MV-algebra $[0,1]^{[0,1]^\kappa}$.

Proposition 2.2.63. *For every cardinal number κ, the term functions $f \in \mathrm{Term}_\kappa^{[0,1]}$ are McNaughton functions.*

Proof. Clearly projections are McNaughton functions, and so is the constant function of value 0 on $[0,1]^\kappa$. Let $(\neg\tau)^{[0,1]}$ be a term function. By the inductive hypothesis, $\tau^{[0,1]}$ has linear constituents $p_1,\ldots,p_k$; it is easy to show that $(\neg\tau)^{[0,1]}$ is a McNaughton function with linear constituents $1-p_1,\ldots,1-p_k$. Finally, if $\sigma^{[0,1]},\tau^{[0,1]}$ are term functions with linear constituents $p_1,\ldots,p_k$ and $q_1,\ldots,q_l$ respectively, then $(\sigma\oplus\tau)^{[0,1]}$ is a McNaughton function whose linear constituents are among the polynomials $\{1,p_i+q_j\}$, for $1\leqslant i\leqslant k$ and $1\leqslant j\leqslant l$. □

The converse, namely that for every continuous piecewise linear function with integer coefficients $f\colon[0,1]^n\to[0,1]$ there exists an MV-term τ over the variables $x_1,\ldots,x_n$ such that $\tau^{[0,1]}=f$, was proved in 1951 by McNaughton [55, Theorem 2]. This results gives an MV-isomorphism between $\mathrm{Term}_\kappa^{[0,1]}$ and the MV-algebra of McNaughton functions on $[0,1]^\kappa$. By applying Proposition 2.2.60, we conclude that

Theorem 2.2.64 (McNaughton)**.** *For every cardinal κ, the free MV-algebra* Free_κ *is isomorphic to the MV-algebra of McNaughton functions on* $[0,1]^\kappa$*. Moreover, there is a unique such isomorphism that sends x_α to π_α, for all $\alpha<\kappa$.*

Proof. See [21, Theorem 9.1.5]. □

Remark 2.2.65. Henceforth, we identify Free_κ with its image under the isomorphism of Theorem 2.2.64.

Recall that, if X is a topological space, then $\mathrm{C}(X,[0,1])$ is an MV-algebra (see Example 2.2.4). Moreover, it is easy to see that, as a subalgebra of $[0,1]^X$, it is a subdirect product of subalgebras of $[0,1]$. Thus, by Theorem 2.2.49, $\mathrm{C}(X,[0,1])$ is a semisimple MV-algebra.

Definition 2.2.66. Let X be a topological space. An MV-subalgebra A of $[0,1]^X$ is said to be *separating* if, for every pair of distinct points $y,z\in X$, there exists $f\in A$ such that $f(y)=0$ and $f(z)\neq 0$.

Separating algebras of functions enjoy the following property.

Lemma 2.2.67. *Let X be a compact Hausdorff space, let A be a separating subalgebra of the MV-algebra* $\mathrm{C}(X,[0,1])$*, and let J be an ideal of A. The ideal J is an intersection of maximal ideals if, and only if, the quotient A/J is isomorphic to the MV-algebra whose elements are restrictions of functions of A to the closed set $\bigcap_{f\in J}f^{-1}(0)$, and whose operations are defined pointwise from those of the standard MV-algebra* $[0,1]$*.*

Proof. See [21, Proposition 3.4.5]. □

Proposition 2.2.68. *Let κ be a cardinal number. The MV-algebra* Free_κ *is a separating subalgebra of* $\mathrm{C}([0,1]^\kappa,[0,1])$*.*

Proof. As stated in Remark 2.2.65, we regard Free_κ as the algebra of McNaughton functions on $[0,1]^\kappa$. In particular, it is a subalgebra of $\mathrm{C}([0,1]^\kappa,[0,1])$. We prove that it is separating. Let $y=(y_1,y_2,\ldots)$ and $z=(z_1,z_2,\ldots)$ be distinct points of $[0,1]^\kappa$ and let $\alpha<\kappa$ be an ordinal number such that $y_\alpha\neq z_\alpha$. Assuming without loss of generality that $y_\alpha<z_\alpha$, pick a rational number r such that $y_\alpha<r<z_\alpha$ and define the polynomial $p(x_\alpha):=ax_\alpha-ar$, where a is a positive integer such that ar is also an integer. It is easy to see that the truncation $(p\vee 0)\wedge 1$ is a McNaughton function, hence an element of Free_κ, and that it takes value 0 in y, but not in z. □

Proposition 2.2.69. *An MV-algebra A is semisimple if, and only if, it is isomorphic to a separating subalgebra of* $\mathrm{C}(X,[0,1])$ *for some compact Hausdorff space X.*

Proof. By Remark 2.2.56 there exist a cardinal number κ and an ideal J of Free_κ such that A is isomorphic to the quotient MV-algebra Free_κ/J. Proposition 2.2.68 states that Free_κ is a separating subalgebra of $\mathrm{C}([0,1]^\kappa,[0,1])$, where $[0,1]^\kappa$, equipped with the product topology, is a compact Hausdorff space. By Lemma 2.2.50 and Lemma 2.2.67, Free_κ/J is semisimple if, and only if, it is isomorphic to the MV-algebra B whose elements are McNaughton functions restricted to the closed set $\bigcap_{f\in J}f^{-1}(0)$. Define $X:=\bigcap_{f\in J}f^{-1}(0)$. This is a closed set in a compact space, hence it is compact. Moreover B is a separating subalgebra of $\mathrm{C}(X,[0,1])$, because Free_κ is separating by Proposition 2.2.68. □

We conclude this section by showing that, for a semisimple MV-algebra A, the compact Hausdorff space in the statement of Proposition 2.2.69 can be recovered (up to homeomorphism) from the MV-algebra A. In fact, the connection is deeper: for every MV-algebra there is a naturally associated compact Hausdorff space, unique to within a homeomorphism.

Consider an arbitrary MV-algebra A, and let $\mathrm{Max}\,A$ denote the set of all maximal ideals of A, i.e.

$$\mathrm{Max}\,A:=\{\mathfrak{m}\subseteq A\mid \mathfrak{m}\text{ is a maximal ideal of }A\}.$$

We define the *spectral topology* on $\mathrm{Max}\,A$ (sometimes called *hull-kernel*, or *Stone-Zariski* topology) by giving, as a subbasis of closed sets, the family of all subsets of the form

$$F_a:=\{\mathfrak{m}\in\mathrm{Max}\,A\mid a\in\mathfrak{m}\}\subseteq\mathrm{Max}\,A$$

for all $a\in A$. The topological space $\mathrm{Max}\,A$ is called the *maximal spectrum* of the MV-algebra A.

Remark 2.2.70. Since every maximal ideal is prime, the family $\{F_a\}_{a\in A}$ is closed under finite unions by Lemma 2.2.22. Indeed, $F_a\cup F_b=F_{a\wedge b}$ for all $a,b\in A$. Hence this subbasis is, in fact, a basis for the topology of $\mathrm{Max}\,A$. Moreover, the basis of closed sets is closed under finite intersections because, for all $a,b\in A$, $F_a\cap F_b=F_{a\oplus b}$. Finally, we remark that $F_0=\mathrm{Max}\,A$ and $F_1=\varnothing$.

Lemma 2.2.71. *For every MV-algebra A,* $\mathrm{Max}\,A$ *is a compact Hausdorff space.*

Proof. If A is the trivial MV-algebra, then $\operatorname{Max} A$ is the empty space, which is a compact Hausdorff space. On the other hand, if A is non-trivial, then $\operatorname{Max} A$ is non-empty by Corollary 2.2.36. Every two distinct maximal ideals $\mathfrak{m}, \mathfrak{n} \in \operatorname{Max} A$ are incomparable, meaning that there exist elements $a, b \in A$ such that $a \in \mathfrak{m} \setminus \mathfrak{n}$ and $b \in \mathfrak{n} \setminus \mathfrak{m}$. It is possible to show that, for all $x, y \in A$,

$$(x \ominus (x \wedge y)) \wedge (y \ominus (x \wedge y)) = 0. \tag{2.1}$$

Indeed, it is elementary that (2.1) holds for MV-chains, and the general case follows by Theorem 2.2.41. Then every maximal ideal of A contains the element $(a \ominus (a \wedge b)) \wedge (b \ominus (a \wedge b))$, and Lemma 2.2.22 entails

$$F_{a\ominus(a\wedge b)} \cup F_{b\ominus(a\wedge b)} = \operatorname{Max} A.$$

Since $a \ominus (a \wedge b) \leqslant a \in \mathfrak{m}$ and $b \ominus (a \wedge b) \leqslant b \in \mathfrak{n}$, it is clear that $\operatorname{Max} A \setminus F_{a\ominus(a\wedge b)}$ and $\operatorname{Max} A \setminus F_{b\ominus(a\wedge b)}$ are disjoint open sets separating the points $\mathfrak{m}, \mathfrak{n} \in \operatorname{Max} A$. To show that $\operatorname{Max} A$ is compact, we shall prove that every family of closed sets with the finite intersection property (i.e. every finite subfamily has non-empty intersection) has non-empty intersection. Let $\mathcal{F}$ be such a family of closed sets. By a standard application of Zorn's Lemma, upon considering the collection of all the families of closed sets extending $\mathcal{F}$, with the finite intersection property, we can find a maximal element $\mathcal{G}$ of this collection. Observe that $\bigcap \mathcal{G} \subseteq \bigcap \mathcal{F}$, thus it suffices to show that $\mathcal{G}$ has non-empty intersection. Since every closed set in $\operatorname{Max} A$ is the intersection of basic closed sets, we can assume without loss of generality that every element of $\mathcal{G}$ is a basic closed set. Since $\mathcal{G}$ cannot be empty, there is $a \in A$ such that $F_a \in \mathcal{G}$. Define the set

$$\mathfrak{n} := \{a \in A \mid F_a \in \mathcal{G}\}.$$

This is an ideal of A: the element 0 belongs to $\mathfrak{n}$ because $\mathcal{G}$ is maximal; if $a \in \mathfrak{n}$ and $b \in A$ is such that $b \leqslant a$, then

$$F_b = \{\mathfrak{m} \in \operatorname{Max} A \mid b \in \mathfrak{m}\} \supseteq \{\mathfrak{m} \in \operatorname{Max} A \mid a \in \mathfrak{m}\} = F_a \in \mathcal{G}$$

so that $F_b \in \mathcal{G}$ because the latter is maximal. Finally, if $a, b \in \mathfrak{n}$, then $F_{a\oplus b} = F_a \cap F_b \in \mathcal{G}$ (see Remark 2.2.70) again by the maximality of $\mathcal{G}$ and the fact that $\mathcal{G}$ has the finite intersection property. We shall prove that $\mathfrak{n}$ is a maximal ideal, in other words $\mathfrak{n} \in \operatorname{Max} A$. If $I \subseteq A$ is an ideal of A that strictly contains $\mathfrak{n}$, then there exists $k \in I \setminus \mathfrak{n}$. In particular $F_k \notin \mathcal{G}$; by the maximality of $\mathcal{G}$, we deduce that there exists a family $\{F_a\}_{a\in A'} \subseteq \mathcal{G}$ such that $A' \subseteq A$ is a finite subset, and $\left(\bigcap_{a\in A'} F_a\right) \cap F_k = \varnothing$. Upon denoting $s := \bigoplus_{a\in A'} a$, Remark 2.2.70 shows that $s \oplus k = 1$. However $F_s \in \mathcal{G}$ by the maximality of $\mathcal{G}$, hence $s \in \mathfrak{n} \subset I$; the latter is closed under finite $\oplus$-sums, so that $1 \in I$. In other words $I = A$, and $\mathfrak{n}$ is a maximal ideal. It is elementary to see that $\bigcap \mathcal{G} = \{\mathfrak{n}\}$. □

Let A be a non-trivial MV-algebra, and let $\mathfrak{m} \in \operatorname{Max} A$ be a maximal ideal of A. By Corollary 2.2.47 and Theorem 2.2.44, the quotient MV-algebra $A/\mathfrak{m}$ is isomorphic to a subalgebra of $[0, 1]$. This means that there exists an injective MV-homomorphism

$\iota_{\mathfrak{m}}\colon A/\mathfrak{m} \to [0,1]$. In fact, we shall see in Theorem 2.3.30 below, that such MV-embedding is unique. By composing with the quotient map $A \to A/\mathfrak{m}$, we obtain an MV-homomorphism such that, for every $a \in A$,

$$A \to A/\mathfrak{m} \to [0,1], \quad a \mapsto \tfrac{a}{\mathfrak{m}} \mapsto \iota_{\mathfrak{m}}\left(\tfrac{a}{\mathfrak{m}}\right).$$

Now, we change the point of view: we fix an element $a \in A$, and let the maximal ideal $\mathfrak{m}$ vary among the points of $\operatorname{Max} A$. In this way, we get a function

$$\widehat{a}\colon \operatorname{Max} A \to [0,1], \quad \widehat{a}(\mathfrak{m}) := \iota_{\mathfrak{m}}\left(\tfrac{a}{\mathfrak{m}}\right).$$

For each $a \in A$, it is possible to see that $\widehat{a}$ is a continuous function on $\operatorname{Max} A$, and the correspondence $a \mapsto \widehat{a}$ defines an MV-homomorphism between the MV-algebra A and the MV-algebra $\mathrm{C}(\operatorname{Max} A, [0,1])$ (see [57, Theorem 4.16]). This map, denoted by

$$\widehat{\cdot}\colon A \to \mathrm{C}(\operatorname{Max} A, [0,1]),$$

is *natural*, meaning that for every MV-homomorphisms $h\colon A \to B$ the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "\widehat{\cdot}"] \arrow[d, "h"'] & \mathrm{C}(\operatorname{Max} A, [0,1]) \arrow[d, "- \circ h^{-1}"] \\
B \arrow[r, "\widehat{\cdot}"'] & \mathrm{C}(\operatorname{Max} B, [0,1])
\end{tikzcd}$$

The image of the MV-homomorphism $\widehat{\cdot}\colon A \to \mathrm{C}(\operatorname{Max} A, [0,1])$ is a subalgebra of the MV-algebra $\mathrm{C}(\operatorname{Max} A, [0,1])$, denoted by $\widehat{A}$; its underlying set is

$$\{\widehat{a} \in \mathrm{C}(\operatorname{Max} A, [0,1]) \mid a \in A\}.$$

We remark that $\widehat{A}$ is a separating subalgebra of $\mathrm{C}(\operatorname{Max} A, [0,1])$. Indeed, if $\mathfrak{m}, \mathfrak{n} \in \operatorname{Max} A$ are distinct (hence incomparable) maximal ideals, there exists $a \in \mathfrak{m} \setminus \mathfrak{n}$, so that

$$\widehat{a}(\mathfrak{m}) = \iota_{\mathfrak{m}}\left(\tfrac{a}{\mathfrak{m}}\right) = \iota_{\mathfrak{m}}(0) = 0 \quad \text{and} \quad \widehat{a}(\mathfrak{n}) = \iota_{\mathfrak{n}}\left(\tfrac{a}{\mathfrak{n}}\right) \neq 0,$$

because $\iota_{\mathfrak{n}}$ (unlike $\widehat{\cdot}$) is injective. In other words, the element $\widehat{a} \in \widehat{A}$ separates the points $\mathfrak{m}, \mathfrak{n} \in \operatorname{Max} A$.

The next result subsumes Proposition 2.2.69.

Theorem 2.2.72. *An MV-algebra A is semisimple if, and only if, the natural MV-homomorphism*

$$\widehat{\cdot}\colon A \to \mathrm{C}(\operatorname{Max} A, [0,1])$$

is injective, in which case A is isomorphic to the separating subalgebra $\widehat{A}$ of $\mathrm{C}(\operatorname{Max} A, [0,1])$.

Proof. If A is a semisimple MV-algebra, i.e. the intersection of all maximal ideals of A is the trivial ideal $\{0\}$, and $a \in A$ is a non-zero element, there exists $\mathfrak{m} \in \operatorname{Max} A$ such that $a \notin \mathfrak{m}$. Then $\widehat{a}$ is not the constant function of value 0 on $\operatorname{Max} A$, since

$\widehat{a}(\mathfrak{m}) = \iota_{\mathfrak{m}}\left(\frac{a}{\mathfrak{m}}\right) \neq 0$ because $\iota_{\mathfrak{m}}$ is injective. By Lemma 2.2.24.(3), the natural MV-homomorphism $\widehat{\cdot}\colon A \to \mathrm{C}(\operatorname{Max} A, [0,1])$ is injective. On the other hand, upon assuming the injectivity of the latter, we find that A is isomorphic to the separating subalgebra $\widehat{A}$ of $\mathrm{C}(\operatorname{Max} A, [0,1])$. Then Proposition 2.2.69 entails that A is a semisimple MV-algebra. □

Proposition 2.2.73. *If X is a compact Hausdorff space and A is a separating MV-subalgebra of $\mathrm{C}(X,[0,1])$, then $\operatorname{Max} A$ is homeomorphic to X.*

Proof. See [57, Theorem 4.16]. □

We briefly describe the homeomorphism of Proposition 2.2.73, in the case $A = \mathrm{C}(X,[0,1])$. For every subset $S \subseteq X$, define

$$\mathbb{I}(S) := \{f \in \mathrm{C}(X,[0,1]) \mid f(x) = 0 \text{ for all } x \in S\}.$$

It is elementary that $\mathbb{I}(S)$ is an ideal of the MV-algebra $\mathrm{C}(X,[0,1])$. Moreover, it is possible to show that the ideal $\mathbb{I}(S)$ is maximal if, and only if, S is a singleton and, in fact, every maximal ideal of $\mathrm{C}(X,[0,1])$ is of the form $\mathbb{I}(\{p\})$ for some $p \in X$ (this is an MV-algebraic version of Stone-Kolmogorov-Gelfand lemma [31]). In other words, there is a bijection

$$\mu_X\colon X \to \operatorname{Max}\mathrm{C}(X,[0,1]), \quad \mu_X(p) := \mathbb{I}(\{p\}).$$

We claim that, for every compact Hausdorff space X, μ_X is a homeomorphism. To see that μ_X is continuous, consider a basic closed set

$$F_f := \{\mathfrak{m} \in \operatorname{Max}\mathrm{C}(X,[0,1]) \mid f \in \mathfrak{m}\},$$

for some $f \in \mathrm{C}(X,[0,1])$. We prove that its preimage under μ_X is a closed subset of X.

$$\mu_X^{-1}(F_f) = \{p \in X \mid \mathbb{I}(\{p\}) \in F_f\} = \{p \in X \mid f \in \mathbb{I}(\{p\})\} = \{p \in X \mid f(p) = 0\} = f^{-1}(0),$$

where the latter is closed, being the preimage of a point in a Hausdorff space under a continuous function. Since a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, the claim is proved.

2.3 The equivalence Γ

Denote by MV the category of MV-algebras and MV-homomorphisms, and recall that $\ell\mathsf{Grp_u}$ denotes the category of unital ℓ-groups and unital ℓ-homomorphisms. In this section we shall introduce Mundici's equivalence between the categories MV and $\ell\mathsf{Grp_u}$. From a logical point of view, this categorical equivalence allows us to translate results from the theory of unital ℓ-groups, which is not even first-order axiomatisable, into results about the variety of MV-algebras, and conversely. Yosida duality for (a certain class of) unital ℓ-groups, along with the possibility of reducing to an equational language via Mundici's equivalence, point the way for the duality obtained in Chapter 4.

Firstly we will introduce the functor $\Gamma\colon \ell\mathsf{Grp_u} \to \mathsf{MV}$ mapping a unital ℓ-group to its unit interval. The fact that this functor is an equivalence is proved by giving a quasi-inverse $\Xi\colon \mathsf{MV} \to \ell\mathsf{Grp_u}$. The construction of the latter requires the notion of a good sequence in an MV-algebra, which we introduce before defining the functor Ξ.

Given an ℓ-group G and an element $G \ni u \geqslant 0$, consider the set

$$[0,u] := \{x \in G \mid 0 \leqslant x \leqslant u\}.$$

This can be endowed with MV-algebraic operations in the following manner: for every $x, y \in [0,u]$, define

$$x \oplus y := u \wedge (x+y), \quad \text{and} \quad \neg x := u - x.$$

It can be checked that the axioms for an MV-algebra are satisfied, in other words:

Lemma 2.3.1. *If G is an ℓ-group and $G \ni u \geqslant 0$, then $\Gamma(G,u) := ([0,u], \oplus, \neg, 0)$ is an MV-algebra.*

Proof. See [21, Proposition 2.1.2]. □

So far, we have not assumed that the ℓ-group G is unital. Further, the element u of Lemma 2.3.1 is an arbitrary non-negative element of the ℓ-group G. When G is a unital ℓ-group and u is a strong order unit for G, then the MV-algebra $\Gamma(G,u)$ is called the *unit interval* of G.

Example 2.3.2. $\Gamma(\mathbb{R}, 1) \cong [0,1]$, the standard MV-algebra.

Example 2.3.3. For any compact Hausdorff space X, $\Gamma(\mathrm{C}(X,\mathbb{R})) \cong \mathrm{C}(X,[0,1])$ (cf. Examples 2.1.58 and 2.2.4).

Remark 2.3.4. Let G be an ℓ-group, and let $u \geqslant 0$ be a non-negative element of G. The set $S := \{x \in G \mid |x| \leqslant nu \text{ for some } n \in \mathbb{N}\}$ is contained in G, in fact S is an ideal of G, generated by u. Moreover, u is a strong order unit of S. Observe that

$$\begin{aligned}
\Gamma(S,u) &= \{x \in S \mid 0 \leqslant x \leqslant u\} \\
&= \{x \in G \mid |x| \leqslant nu \ \text{ for some } \ n \in \mathbb{N}, \ \text{ and } \ 0 \leqslant x \leqslant u\} \\
&= \{x \in G \mid 0 \leqslant x \leqslant u\} \\
&= \Gamma(G,u).
\end{aligned}$$

For this reason, when dealing with the MV-algebra $\Gamma(G,u)$, where G is a unital ℓ-group and u is a positive element of G, we shall always assume that u is a strong order unit for G, so that $S = G$.

The following facts are elementary.

Lemma 2.3.5. *If $f\colon (G,u) \to (H,v)$ is a unital ℓ-homomorphism, then*

$$\Gamma(f) := f_{|[0,u]}\colon \Gamma(G,u) \to \Gamma(H,v)$$

is an MV-homomorphism. □

Corollary 2.3.6. $\Gamma\colon \ell\mathsf{Grp_u} \to \mathsf{MV}$ *is a functor from the category of unital ℓ-groups to the category of MV-algebras.* □

We now turn to good sequences.

Definition 2.3.7. A *good sequence* in an MV-algebra A is a sequence $\boldsymbol{a} = (a_1, a_2, a_3, \ldots)$ where, for all i, $a_i \in A$, $a_i \oplus a_{i+1} = a_i$ and there exists $n_0 \in \mathbb{N}$ such that $a_n = 0$ for every $n > n_0$.

Notation 2.3.8. A sequence $(a_1, a_2, \ldots, a_n, 0, 0, \ldots)$ is denoted simply by $(a_1, a_2, \ldots, a_n)$. In other words, we do not distinguish between the sequence $(a_1, a_2, \ldots, a_n)$ and the sequence $(a_1, a_2, \ldots, a_n, 0^m)$ for any $m \in \mathbb{N}$, where 0^m represents a m-tuple of zeroes. If $a \in A$, then $(a, 0, \ldots, 0, \ldots)$ is a good sequence, and it is written as (a).

Example 2.3.9. If A is a Boolean algebra (see Example 2.2.2), then a good sequence in A is a non-increasing eventually zero sequence of elements. This is due to the fact that, in any idempotent MV-algebra, $x \oplus y = x \vee y$ [21, Theorem 1.5.3].

Good sequences in totally-ordered MV-algebras admit a sharp characterisation.

Proposition 2.3.10. *Every good sequence in an MV-chain A is of the form $(1^p, a)$, for some $p \in \mathbb{N}$ and $a \in A$.*

Proof. Let $(a_1, a_2, a_3, \ldots)$ be a good sequence in A. In particular, for all i, the condition $a_i \oplus a_{i+1} = a_i$ is satisfied. By Proposition 2.2.14.(2) we must have either $a_i = 1$ or $a_{i+1} = 0$. Since the identity $x \oplus 1 = 1$ holds in any MV-algebra, an arbitrary good sequence in A is of the form $(1^p, a, 0, 0, \ldots) = (1^p, a)$ for some $p \in \mathbb{N}$ and $a \in A$. □

Lemma 2.3.11. *Suppose that the MV-algebra A is a subdirect product of the family $\{A_i\}_{i \in I}$. The sequence $\boldsymbol{a} = (a_1, \ldots, a_n, \ldots)$ of elements of A is a good sequence if, and only if, for every $i \in I$*

$$\boldsymbol{a}_i = (\pi_i(a_1), \ldots, \pi_i(a_n), \ldots)$$

is a good sequence in A_i, and there exists $n_0 \in \mathbb{N}$ such that $\pi_i(a_n) = 0$ for all $n > n_0$ and for all $i \in I$.

Proof. It is sufficient to observe that $a_n \oplus a_{n+1} = a_n$ if, and only if, $\pi_i(a_n) \oplus \pi_i(a_{n+1}) = \pi_i(a_n \oplus a_{n+1}) = \pi_i(a_n)$ for every $i \in I$. Therefore $\boldsymbol{a}$ is a good sequence in A if, and only if, each $\boldsymbol{a}_i$ is a good sequence in A_i and there exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$ and for all $i \in I$, $\pi_i(a_n) = 0$. □

Definition 2.3.12. Given good sequences $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_m)$, define their *sum* $\boldsymbol{a} + \boldsymbol{b}$ as the sequence $\boldsymbol{c} = (c_1, c_2, c_3, \ldots)$, where

$$c_i := a_i \oplus (a_{i-1} \odot b_1) \oplus \cdots \oplus (a_1 \odot b_{i-1}) \oplus b_i.$$

Remark 2.3.13. Consider an MV-chain A, and good sequences $\boldsymbol{a}, \boldsymbol{b}$ in A. Proposition 2.3.10 states that there exist $p, q \in \mathbb{N}$ and $a, b \in A$ such that $\boldsymbol{a} = (1^p, a)$ and $\boldsymbol{b} = (1^q, b)$. Applying the definition of the sum of two good sequences, we find

$$(1^p, a) + (1^q, b) = (1^{p+q}, a \oplus b, a \odot b).$$

By Proposition 2.2.14.(1) $a \oplus b < 1$ if, and only if $a \odot b = 0$. Hence we must have either $(1^p, a) + (1^q, b) = (1^{p+q+1}, a \odot b)$ or $(1^p, a) + (1^q, b) = (1^{p+q}, a \oplus b)$. In both cases, the sequence $\boldsymbol{a} + \boldsymbol{b}$ is again a good sequence in the MV-chain A. In fact, this is true for an arbitrary MV-algebra:

Proposition 2.3.14. *If $\boldsymbol{a}$ and $\boldsymbol{b}$ are good sequences in the MV-algebra A, then $\boldsymbol{a} + \boldsymbol{b}$ is a good sequence in A.*

Proof. If A is trivial, then $(0) + (0) = (0)$ is a good sequence. Hence we assume that A is non-trivial. By Theorem 2.2.41 the MV-algebra A is a subdirect product of a family of MV-chains $\{A_i\}_{i \in I}$. Since $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_m)$ are good sequences in A, by Lemma 2.3.11 the sum $\boldsymbol{a} + \boldsymbol{b} = ((a+b)_1, \ldots, (a+b)_{m+n}, \ldots)$ is a good sequence in A if, and only if, for every $i \in I$, $\pi_i(\boldsymbol{a} + \boldsymbol{b}) = (\pi_i(a+b)_1, \ldots, \pi_i(a+b)_{m+n}, \ldots)$ is a good sequence in A_i. By hypothesis $\boldsymbol{a}$ and $\boldsymbol{b}$ are good sequences in A, hence

$$\pi_i(\boldsymbol{a}) = (\pi_i(a_1), \ldots, \pi_i(a_n)),$$

and

$$\pi_i(\boldsymbol{b}) = (\pi_i(b_1), \ldots, \pi_i(b_m))$$

are good sequences in A_i. Since A_i is totally-ordered, Remark 2.3.13 implies that $\pi_i(\boldsymbol{a}) + \pi_i(\boldsymbol{b})$ is a good sequence in A_i. We conclude that, for every $i \in I$, $\pi_i(\boldsymbol{a} + \boldsymbol{b}) = \pi_i(\boldsymbol{a}) + \pi_i(\boldsymbol{b})$ is a good sequence in A_i. $\square$

Proposition 2.3.15. *Let A be an MV-algebra, and let M_A denote the set of good sequences in A. Then $(M_A, +, (0))$ is a commutative monoid (where the operation $+$ is the sum of good sequences). Further, for good sequences $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in M_A$, the following hold.*

1. *If $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c}$, then $\boldsymbol{b} = \boldsymbol{c}$.*
2. *If $\boldsymbol{a} + \boldsymbol{b} = (0)$, then $\boldsymbol{a} = \boldsymbol{b} = (0)$.*

Proof. See [21, Proposition 2.3.1]. $\square$

Remark 2.3.16. If $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_m)$ are good sequences, we can assume without loss of generality that $m = n$. In fact, if e.g. $m < n$, it is enough to consider $\boldsymbol{b} = (b_1, \ldots, b_m, 0^{n-m})$ (cf. Notation 2.3.8).

Proposition 2.3.17. *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be good sequences in the MV-algebra A. The following are equivalent.*

1. *There exists $\boldsymbol{c} \in M_A$ such that $\boldsymbol{b} + \boldsymbol{c} = \boldsymbol{a}$.*

2. *$b_i \leqslant a_i$ for all $i = 1, \ldots, n$.*

Proof. See [21, Proposition 2.3.2]. □

Proposition 2.3.17 allows us to equip the commutative monoid M_A with a partial order: we agree to set $\boldsymbol{b} \leqslant \boldsymbol{a}$ if, and only if, the good sequences $\boldsymbol{a}, \boldsymbol{b}$ satisfy the equivalent conditions above. In fact, it is elementary that the relation $\leqslant$ is a reflexive, transitive, and antisymmetric relation on M_A.

Lemma 2.3.18. *Let $\boldsymbol{a}, \boldsymbol{b}$ be good sequences in the MV-algebra A.*

1. *If $\boldsymbol{b} \leqslant \boldsymbol{a}$, then there exists a unique $\boldsymbol{c} \in M_A$ satisfying $\boldsymbol{b} + \boldsymbol{c} = \boldsymbol{a}$.*
2. *The partial order $\leqslant$ on M_A is invariant under translations, i.e. if $\boldsymbol{b} \leqslant \boldsymbol{a}$, then $\boldsymbol{b} + \boldsymbol{d} \leqslant \boldsymbol{a} + \boldsymbol{d}$ for every $\boldsymbol{d} \in M_A$.*

Proof. Item 1 follows at once by Proposition 2.3.15.(1) for, if $\boldsymbol{c}, \boldsymbol{c}' \in M_A$ are such that $\boldsymbol{b} + \boldsymbol{c} = \boldsymbol{a} = \boldsymbol{b} + \boldsymbol{c}'$, then $\boldsymbol{c} = \boldsymbol{c}'$. Concerning item 2, if $\boldsymbol{b} \leqslant \boldsymbol{a}$, then there exists $\boldsymbol{c} \in M_A$ satisfying $\boldsymbol{b} + \boldsymbol{c} = \boldsymbol{a}$, so that the good sequences $\boldsymbol{b} + \boldsymbol{c}$ and $\boldsymbol{a}$ coincide termwise. If $\boldsymbol{d} \in M_A$ is an arbitrary good sequence, each term of the sequence $\boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}$ is equal to the corresponding term of the sequence $\boldsymbol{a} + \boldsymbol{d}$, in other words $\boldsymbol{b} + \boldsymbol{d} \leqslant \boldsymbol{a} + \boldsymbol{d}$. □

It is possible to show that the partially ordered set $(M_A, \leqslant)$ is a lattice [21, Proposition 2.3.5]. Indeed, if $\boldsymbol{a}, \boldsymbol{b} \in M_A$, the following are good sequences

$$\boldsymbol{a} \vee \boldsymbol{b} := (a_1 \vee b_1, \ldots, a_n \vee b_n, \ldots),$$

$$\boldsymbol{a} \wedge \boldsymbol{b} := (a_1 \wedge b_1, \ldots, a_n \wedge b_n, \ldots)$$

and they are the least upper bound and the greatest lower bound of the pair $\boldsymbol{a}, \boldsymbol{b}$, respectively.

In the following, starting with an MV-algebra A, we shall construct an ℓ-group G_A whose positive cone is isomorphic to the lattice-ordered commutative monoid M_A.

Remark 2.3.19. Given the lattice-ordered commutative monoid $\mathbb{N} \cup \{0\}$, we can construct an ℓ-group G such that $G^+ \cong \mathbb{N} \cup \{0\}$, as the quotient of $\mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}$ by the equivalence relation $(m, n) \sim (p, q)$ if, and only if, $m + q = p + n$. Clearly $G \cong \mathbb{Z}$, and we obtained the integers by considering differences of natural numbers.

The previous remark represents the motivation for the next

Definition 2.3.20. The equivalence relation $\sim$ on $M_A \times M_A$ is defined as follows: if $(\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{c}, \boldsymbol{d}) \in M_A \times M_A$, then $(\boldsymbol{a}, \boldsymbol{b}) \sim (\boldsymbol{c}, \boldsymbol{d})$ if, and only if, $\boldsymbol{a} + \boldsymbol{d} = \boldsymbol{c} + \boldsymbol{b}$. Denote by

$$[\boldsymbol{a}, \boldsymbol{b}] := \{(\boldsymbol{c}, \boldsymbol{d}) \in M_A \times M_A \mid (\boldsymbol{a}, \boldsymbol{b}) \sim (\boldsymbol{c}, \boldsymbol{d})\}$$

the equivalence class of the element $(\boldsymbol{a}, \boldsymbol{b})$, and by G_A the set $M_A \times M_A / \sim$ of equivalence classes.

Lemma 2.3.21. *The structure* $(G_A, [(0),(0)], +)$ *is an abelian group, where*

$$[\boldsymbol{a},\boldsymbol{b}] + [\boldsymbol{c},\boldsymbol{d}] := [\boldsymbol{a}+\boldsymbol{c}, \boldsymbol{b}+\boldsymbol{d}],$$

$$-[\boldsymbol{a},\boldsymbol{b}] := [\boldsymbol{b},\boldsymbol{a}].$$

Proof. It is easy to check that $(G_A, [(0),(0)], +)$ is an abelian monoid, since so is M_A. Therefore it suffices to observe that

$$[\boldsymbol{a},\boldsymbol{b}] + (-[\boldsymbol{a},\boldsymbol{b}]) = [\boldsymbol{a},\boldsymbol{b}] + [\boldsymbol{b},\boldsymbol{a}] = [\boldsymbol{a}+\boldsymbol{b}, \boldsymbol{a}+\boldsymbol{b}] = [(0),(0)]$$

because $\boldsymbol{a}+\boldsymbol{b}+(0) = (0)+\boldsymbol{a}+\boldsymbol{b}$. □

The next step is defining an order on the abelian group G_A which induces a lattice structure on G_A.

Definition 2.3.22. Let A be an MV-algebra, and let $[\boldsymbol{a},\boldsymbol{b}], [\boldsymbol{c},\boldsymbol{d}] \in G_A$. Set $[\boldsymbol{a},\boldsymbol{b}] \preceq [\boldsymbol{c},\boldsymbol{d}]$ if, and only if, there exists $\boldsymbol{e} \in M_A$ such that $[\boldsymbol{c},\boldsymbol{d}] - [\boldsymbol{a},\boldsymbol{b}] = [\boldsymbol{e},(0)]$.

Remark 2.3.23. Notice that, for all $[\boldsymbol{a},\boldsymbol{b}], [\boldsymbol{c},\boldsymbol{d}] \in G_A$,

$$\begin{aligned}
[\mathbf{a},\mathbf{b}] \preceq [\mathbf{c},\mathbf{d}] \;&\Leftrightarrow\; [\mathbf{c},\mathbf{d}] - [\mathbf{a},\mathbf{b}] = [\mathbf{e},(0)]\\
&\Leftrightarrow\; [\mathbf{c}+\mathbf{b}, \mathbf{a}+\mathbf{d}] = [\mathbf{e},(0)]\\
&\Leftrightarrow\; \mathbf{c}+\mathbf{b}+(0) = \mathbf{e}+\mathbf{a}+\mathbf{d}\\
&\Leftrightarrow\; \mathbf{a}+\mathbf{d}+\mathbf{e} = \mathbf{c}+\mathbf{b}\\
&\Leftrightarrow\; \mathbf{a}+\mathbf{d} \leqslant \mathbf{c}+\mathbf{b},
\end{aligned}$$

where $\leqslant$ is the partial order on the monoid M_A. Consequently, it is clear that the map $\psi\colon M_A \to G_A$ defined by $\psi(\boldsymbol{a}) := [\boldsymbol{a},(0)]$ induces an isomorphism between the monoid M_A and the positive cone of the partially ordered group G_A, namely

$$\begin{aligned}
G_A^+ &= \{[\boldsymbol{c},\boldsymbol{d}] \in G_A \mid 0 \preceq [\boldsymbol{c},\boldsymbol{d}]\}\\
&= \{[\boldsymbol{c},\boldsymbol{d}] \in G_A \mid \boldsymbol{c} \geqslant \boldsymbol{d}\}.
\end{aligned}$$

Indeed, it is obvious that ψ is injective and that $\psi(M_A) = \{[\boldsymbol{a},(0)] \mid \boldsymbol{a} \in M_A\} \subseteq G_A^+$; on the other hand $G_A^+ \subseteq \psi(M_A)$ by Proposition 2.3.18.(1).

Proposition 2.3.24. *The partial order* $\preceq$ *on* G_A *is invariant under translations, and* $(G_A, \preceq)$ *is a lattice. For every pair of elements* $[\boldsymbol{a},\boldsymbol{b}], [\boldsymbol{c},\boldsymbol{d}] \in G_A$*, the least upper bound and greatest lower bound are given, respectively, by*

$$[\boldsymbol{a},\boldsymbol{b}] \vee [\boldsymbol{c},\boldsymbol{d}] = [(\boldsymbol{a}+\boldsymbol{d}) \vee (\boldsymbol{c}+\boldsymbol{b}), \boldsymbol{b}+\boldsymbol{d}],$$

$$[\boldsymbol{a},\boldsymbol{b}] \wedge [\boldsymbol{c},\boldsymbol{d}] = [(\boldsymbol{a}+\boldsymbol{d}) \wedge (\boldsymbol{c}+\boldsymbol{b}), \boldsymbol{b}+\boldsymbol{d}].$$

Proof. See [21, Proposition 2.4.2]. □

The ℓ-group G_A is called the *Chang* ℓ*-group* associated to the MV-algebra A.

Proposition 2.3.25. *The element $u_A := [(1),(0)]$ is a strong order unit for G_A.*

Proof. Let $[\boldsymbol{a},\boldsymbol{b}] \in G_A$. We only prove the result for the case in which $[\boldsymbol{a},\boldsymbol{b}]$ belongs to the positive cone of G_A, and leave the rest to the reader. By Remark 2.3.23 there exists $\boldsymbol{e} = (e_1, e_2, \ldots) \in M_A$ such that $[\boldsymbol{a},\boldsymbol{b}] = [\boldsymbol{e},(0)]$. Let $m \in \mathbb{N}$ be such that $e_n = 0$ for every $n > m$. We conclude that $[\boldsymbol{e},(0)] \preceq mu_A = [1^m,(0)]$ because

$$[\boldsymbol{e},(0)] \preceq [1^m,(0)] \;\Leftrightarrow\; \boldsymbol{e} + (0) \leqslant 1^m + (0) \;\Leftrightarrow\; \boldsymbol{e} \leqslant 1^m \;\Leftrightarrow\; e_i \leqslant 1 \;\; \forall i \leqslant m.$$

□

Corollary 2.3.26. *If A is an MV-algebra, then $\Xi(A) := (G_A, u_A)$ is a unital ℓ-group.* □

Remark 2.3.27. The map $\phi\colon A \to \Gamma(G_A, u_A)$, given by $\phi(a) := [(a),(0)]$, is an MV-isomorphism between the MV-algebra A and the unit interval of the unital ℓ-group G_A. Indeed, by Remark 2.3.23, the underlying set of $\Gamma(G_A, u_A)$ is

$$\{[\boldsymbol{a},\boldsymbol{b}] \in G_A \mid [(0),(0)] \preceq [\boldsymbol{a},\boldsymbol{b}] \preceq [(1),(0)]\} = \{[(c),(0)] \in G_A \mid c \in A\},$$

which is in bijection with A. This bijection extends to an MV-isomorphism [21, Theorem 2.4.5]. Moreover, the MV-algebra A is totally-ordered if, and only if, the unital ℓ-group G_A is totally-ordered. On the one hand, if A is an MV-chain, then the monoid M_A is also totally-ordered, and so is G_A. On the other hand, if G_A is totally-ordered, then its unit interval $\Gamma(G_A, u_A)$ is totally-ordered. But the latter is isomorphic to A, hence A is an MV-chain.

With respect to morphisms, if $h\colon A \to B$ is an MV-homomorphism, one can show that the map $h^*\colon M_A \to M_B$, sending the good sequence $\boldsymbol{a} = (a_1, a_2, \ldots) \in M_A$ to $h^*(\boldsymbol{a}) := (h(a_1), h(a_2), \ldots)$, is both a monoid homomorphism and a lattice homomorphism. Further, the map $\Xi(h)\colon (G_A, u_A) \to (G_B, u_B)$, given by $\Xi(h)[\boldsymbol{a},\boldsymbol{b}] := [h^*(\boldsymbol{a}), h^*(\boldsymbol{b})]$, is a unital ℓ-homomorphism (see [21, pp. 139–140]). Composition of MV-homomorphisms and identities are easily seen to be preserved. Summing up:

Corollary 2.3.28. $\Xi\colon \mathsf{MV} \to \ell\mathsf{Grp_u}$ *is a functor from the category of MV-algebras to the category of unital ℓ-groups.* □

Remarkably, in 1986 Mundici [56, Theorem 3.9] proved:

Theorem 2.3.29. *The functor $\Gamma\colon \ell\mathsf{Grp_u} \to \mathsf{MV}$ is an equivalence between the category of unital ℓ-groups and the category of MV-algebras, whose quasi-inverse is $\Xi\colon \mathsf{MV} \to \ell\mathsf{Grp_u}$.*

Proof. See [21, Theorem 7.1.2, Theorem 7.1.7]. □

By Theorem 2.1.57, along with the functor Γ, it follows at once that

Theorem 2.3.30. *Every non-trivial semisimple MV-chain admits a unique MV-embedding in the standard MV-algebra* $[0,1]$*. In particular, if* A *is any non-trivial MV-algebra, and* $\mathfrak{m} \in \operatorname{Max} A$*, then there is a unique MV-embedding*

$$\mathfrak{h}_{\mathfrak{m}} \colon \tfrac{A}{\mathfrak{m}} \to [0,1].$$

□

Chapter 3

Yosida duality

What we refer to, in modern terms, as the investigation of the dual of the category KHaus, was traditionally tackled through the study of the continuous functions over a given space. In this framework, the main rôle is played by $\mathrm{C}(X)$, the collection of all the continuous ($\mathbb{R}$ or $\mathbb{C}$)-valued functions on the compact Hausdorff space X. Depending on the structure we endow the codomain with, $\mathrm{C}(X)$ can be regarded in rather different ways. The literature on $\mathrm{C}(X)$ as a ring is extensive, see for example [33], but many other structures have been taken into consideration (lattice, vector space, Banach algebra, etc.). Historically, the three main representation theorems concerning $\mathrm{C}(X)$ were proved at the beginning of the forties. In 1941, Kakutani [45] gave an order-theoretic characterisation of the unital real Banach lattices (=unital lattice-ordered real Banach spaces) of the form $\mathrm{C}(X, \mathbb{R})$. It is clear that, in his work, the non-algebraic concept of norm plays a crucial rôle. Although the approach adopted by Kakutani was more general, we remark that this representation result was announced simultaneously in [47] by M. and S. Krein. The related duality between KHaus and the category whose objects are some unital real Banach lattices (called *M-spaces*) and whose morphisms are unital vector lattice homomorphisms, was made explicit in [8]. In the same year, Yosida showed in the landmark paper [67] that a vector lattice (=lattice-ordered real vector space) with a strong order unit is isomorphic to $\mathrm{C}(X, \mathbb{R})$, for some compact Hausdorff space X if, and only if, it is archimedean and norm-complete. In sharp contrast with the Kakutani-Krein-Krein result, the norm on the vector lattice is not a primitive operator, but it is induced by the strong order unit. Yosida's representation theorem extends to a categorical duality with KHaus, and even if the dual class of vector lattices fails being equationally definable, it still involves algebraic language only. Finally, in 1943, on the way to a representation theorem for (possibly non-commutative) complex C*-algebras, Gelfand and Neumark [32] proved that a complex unital C*-algebra can be represented as the family of all continuous $\mathbb{C}$-valued functions on a compact Hausdorff space if, and only if, it is commutative. This result gives rise to the well-known Gelfand-Neumark duality (see Section 6.2). As in Kakutani's representation result, the norm is a primitive element in the structure of a C*-algebra.

What follows is an account of Yosida's representation theorem, in the language of ℓ-groups, and of the related categorical duality.

3.1 The Hölder-Yosida construction

Recall that, by a unital ℓ-group, we understand an abelian lattice-ordered group with a distinguished strong order unit (Definitions 2.1.1 and 2.1.52). In Example 2.1.59 we have seen that not every archimedean unital ℓ-group is isomorphic to an ℓ-group of the form $\mathrm{C}(X, \mathbb{R})$ for some compact Hausdorff space X. Indeed, a counterexample is provided by the unital archimedean ℓ-group $(\mathbb{Z}, 1)$. Nevertheless, $\mathbb{Z}$ can be embedded in the ℓ-group $\mathrm{C}(\{p\}, \mathbb{R}) \cong \mathbb{R}$, where $\{p\}$ is any one-point space. The Hölder-Yosida construction shows that every archimedean unital ℓ-group admits such an embedding.

Remark 3.1.1. The Hölder-Yosida construction is the analogue for unital ℓ-groups of Chang's representation result for semisimple MV-algebras, see Theorem 2.2.72. Indeed, it is possible to obtain one result from the other by applying the functor Γ of Theorem 2.3.29. In the special case of totally-ordered structures this is carried out in Theorem 2.3.30.

Let (G, u) be an archimedean unital ℓ-group. We shall find a compact Hausdorff space X, along with an embedding $(G, u) \to (\mathrm{C}(X, \mathbb{R}), 1_X)$. The idea is the same described in the case of MV-algebras at the end of Section 2.2, mutatis mutandis. Define the set

$$\operatorname{Max} G := \{\mathfrak{m} \subseteq G \mid \mathfrak{m} \text{ is a maximal ideal of } G\}.$$

By Lemma 2.1.44, for every maximal ideal $\mathfrak{m} \in \operatorname{Max} G$, there exists an injective ℓ-homomorphism $\frac{G}{\mathfrak{m}} \to \mathbb{R}$; further, Theorem 2.1.57 states that there exists a unique unital embedding

$$\mathfrak{h}_{\mathfrak{m}} \colon (\tfrac{G}{\mathfrak{m}}, \tfrac{u}{\mathfrak{m}}) \to (\mathbb{R}, 1).$$

The composition of maps

$$g \mapsto \tfrac{g}{\mathfrak{m}} \mapsto \mathfrak{h}_{\mathfrak{m}}\left(\tfrac{g}{\mathfrak{m}}\right)$$

shows that, once we fix a maximal ideal $\mathfrak{m}$, we can associate to each element $g \in G$ a real number $\mathfrak{h}_{\mathfrak{m}}\left(\frac{g}{\mathfrak{m}}\right) \in \mathbb{R}$. Reversing the point of view, given an element $g \in G$, we can define a function

$$\widehat{g} \colon \operatorname{Max} G \to \mathbb{R}, \quad \widehat{g}(\mathfrak{m}) := \mathfrak{h}_{\mathfrak{m}}\left(\tfrac{g}{\mathfrak{m}}\right).$$

Here, the element $g \in G$ is fixed, while $\mathfrak{m}$ varies among the elements of $\operatorname{Max} G$. In view of the previous construction, it makes sense to ask whether maximal ideals exist in an arbitrary unital ℓ-group.

Lemma 3.1.2. *If (G, u) is a non-trivial unital ℓ-group, then there exists a maximal ideal in G.*

Proof. The argument is a standard application of Zorn's Lemma, therefore we shall just give a sketch of the proof. Assume that the ℓ-group (G, u) is non-trivial, i.e. $G \neq \{0 = u\}$. Hence $\{0\}$ is a proper ideal of G. Let $\mathcal{D}$ denote the set of proper ideals of G extending the trivial ideal $\{0\}$. The set $\mathcal{D}$ is partially ordered by set-theoretic inclusion, and it is non-empty because $\{0\}$ is in $\mathcal{D}$. Let D be a totally-ordered subset of $\mathcal{D}$. One proves that $\bigcup_{I_j \in D} I_j$ is an upper bound for D, and $\bigcup_{I_j \in D} I_j \in \mathcal{D}$. Then Zorn's lemma implies that there exists a maximal element in $\mathcal{D}$, in other words there exists a maximal ideal in G. $\square$

Remark 3.1.3. In Lemma 3.1.2 the unital assumption is crucial. Indeed, there exist non-unital ℓ-groups which do not admit any maximal ideal (see [51, Example 27.8]).

Having defined the set $\operatorname{Max} G$, we shall equip it with a topology. The latter should be defined in such a way that the topological space $\operatorname{Max} G$ is a compact Hausdorff space, and each function $\widehat{g}\colon \operatorname{Max} G \to \mathbb{R}$ is continuous. Instead of giving this topology immediately, we show how it arises, naturally, from the comparison with the ℓ-group $\mathrm{C}(X, \mathbb{R})$.

Notation 3.1.4. In this chapter, X will always denote a compact Hausdorff space. Moreover, we write $\mathrm{C}(X)$ for the ℓ-group $\mathrm{C}(X, \mathbb{R})$. Finally, by an ideal we understand an ℓ-ideal.

In order to study the (maximal) ideals of $\mathrm{C}(X)$, consider a subset $K \subseteq X$ and define

$$\mathbb{I}(K) := \{f \in \mathrm{C}(X) \mid f(x) = 0 \text{ for all } x \in K\}.$$

Remark 3.1.5. Note that the operator $\mathbb{I}$ reverses inclusions. If $K_1 \subseteq K_2$ are subsets of X and $f \in \mathbb{I}(K_2)$, then $f(x) = 0$ for all $x \in K_1 \subseteq K_2$. This shows that $f \in \mathbb{I}(K_1)$, that is $\mathbb{I}(K_2) \subseteq \mathbb{I}(K_1)$.

Proposition 3.1.6. *If $K \subseteq X$ is an arbitrary subset, then $\mathbb{I}(K)$ is an ideal of $\mathrm{C}(X)$.*

Proof. We must prove that $\mathbb{I}(K)$ is a convex subgroup and sublattice of $\mathrm{C}(X)$. If $f, g \in \mathbb{I}(K)$, then $-f \in \mathbb{I}(K)$ and $f + g \in \mathbb{I}(K)$, therefore $\mathbb{I}(K)$ is a subgroup of $\mathrm{C}(X)$. It is also a sublattice, since it is clearly closed under the operations $\wedge, \vee$. Lastly, it is convex: if $f \geqslant g \geqslant h$, where $f, h \in \mathbb{I}(K)$, then for all $x \in K$, $0 = f(x) \geqslant g(x) \geqslant h(x) = 0$, whence $g \in \mathbb{I}(K)$. $\square$

We can always assume without loss of generality that the subset K in the statement of the preceding proposition is closed:

Lemma 3.1.7. *Let $K \subseteq X$ be an arbitrary subset. Then $\mathbb{I}(K) = \mathbb{I}(\overline{K})$, where $\overline{K}$ is the topological closure of K in X.*

Proof. One of the two inclusions follows at once by Remark 3.1.5, upon observing that $K \subseteq \overline{K}$. On the other hand, if $f_{|K} = 0$, the continuity of f implies $f_{|\overline{K}} = 0$. $\square$

The question arises, which are those closed subsets $K \subseteq X$ for which $\mathbb{I}(K)$ is a maximal ideal.

Lemma 3.1.8 (Gelfand-Kolmogorov)**.** *The ideal $\mathbb{I}(K)$ of $\mathrm{C}(X)$ is maximal if, and only if, K is a one-point set.*

Proof. This was first proved, regarding $\mathrm{C}(X)$ as a ring, in [31]. In the case of ℓ-groups the same arguments apply, mutatis mutandis. $\square$

We have shown how to construct an ideal of $\mathrm{C}(X)$ starting from a (closed) subset of X. The converse is also possible. If $I \subseteq \mathrm{C}(X)$ is an arbitrary subset, define

$$\mathbb{V}(I) := \{x \in X \mid f(x) = 0 \text{ for all } f \in I\}.$$

The set $\mathbb{V}(I)$ is called the *zero set*, or *vanishing locus*, of I.

Remark 3.1.9. The operator $\mathbb{V}$, as the operator $\mathbb{I}$, reverses inclusions. If $I_1 \subseteq I_2$ are subsets of $\mathrm{C}(X)$ and $x \in X$ is such that $f(x) = 0$ for all $f \in I_2$, then $f(x) = 0$ for all $f \in I_1 \subseteq I_2$. In other words, $\mathbb{V}(I_2) \subseteq \mathbb{V}(I_1)$.

Proposition 3.1.10. *If $I \subseteq \mathrm{C}(X)$ is an arbitrary subset, then $\mathbb{V}(I)$ is a closed subset of X.*

Proof. Observe that $\mathbb{V}(I) = \bigcap_{f \in I} f^{-1}(0)$, where each $f^{-1}(0)$ is closed since it is the continuous preimage of a closed set (every one-point space in a T_1-space is closed). An arbitrary intersection of closed sets is closed, therefore $\mathbb{V}(I)$ is closed. □

In fact we can always assume without loss of generality that the subset I in the statement of the foregoing proposition is an ideal:

Lemma 3.1.11. *Let $I \subseteq \mathrm{C}(X)$ be an arbitrary subset. Then $\mathbb{V}(I) = \mathbb{V}(\langle I \rangle)$, where $\langle I \rangle$ is the ideal generated by I.*

Proof. Since $I \subseteq \langle I \rangle$, one of the two inclusions follows at once from Remark 3.1.9. On the other hand, if $x \in \mathbb{V}(I)$ and

$$g \in \langle I \rangle = \{h \in \mathrm{C}(X) \mid \exists n \in \mathbb{N},\ \exists f \in I \text{ such that } -n|f| \leqslant h \leqslant n|f|\},$$

we have $0 = -n|f(x)| \leqslant g(x) \leqslant n|f(x)| = 0$, i.e. $x \in \mathbb{V}(\langle I \rangle)$. □

Remark 3.1.12. Every closed subset of X is of the form $\mathbb{V}(I)$ for some ideal $I \subseteq \mathrm{C}(X)$. Indeed, if $S \subseteq X$ is an arbitrary subset, it is easy to see that $\mathbb{V}(\mathbb{I}(S)) = \overline{S}$. In particular, if $K \subseteq X$ is a closed subset, then $K = \mathbb{V}(\mathbb{I}(K))$.

The next step consists in defining the concept of vanishing locus for an arbitrary ℓ-group G. Let us consider $g \in G$ and $\mathfrak{m} \in \operatorname{Max} G$. We agree to say that the element g *vanishes* in $\mathfrak{m}$, if $\widehat{g}(\mathfrak{m}) := \mathfrak{h}_{\mathfrak{m}}(\frac{g}{\mathfrak{m}}) = 0 \in \mathbb{R}$. Observe that, since the ℓ-homomorphism $\mathfrak{h}_{\mathfrak{m}}$ is injective, g vanishes in $\mathfrak{m}$ if, and only if, $g \in \mathfrak{m}$. Define

$$\mathbb{V}(g) := \{\mathfrak{m} \in \operatorname{Max} G \mid \widehat{g}(\mathfrak{m}) = 0\}.$$

The *vanishing locus* of the subset $S \subseteq G$ is

$$\mathbb{V}(S) := \bigcap_{g \in S} \mathbb{V}(g) = \{\mathfrak{m} \in \operatorname{Max} G \mid \widehat{g}(\mathfrak{m}) = 0 \text{ for all } g \in S\}.$$

Remark 3.1.13. We can write, equivalently,

$$\mathbb{V}(g) = \{\mathfrak{m} \in \operatorname{Max} G \mid g \in \mathfrak{m}\},$$

and

$$\mathbb{V}(S) = \{\mathfrak{m} \in \operatorname{Max} G \mid S \subseteq \mathfrak{m}\}.$$

Again, we can suppose without loss of generality that S is an ideal of G:

Lemma 3.1.14. *If $S \subseteq G$, then $\mathbb{V}(S) = \mathbb{V}(\langle S \rangle)$.*

Proof. If $\mathfrak{m} \in \mathbb{V}(\langle S \rangle)$, then $\widehat{g}(\mathfrak{m}) = 0$ for all $g \in S \subseteq \langle S \rangle$, so that $\mathfrak{m} \in \mathbb{V}(S)$. In the other direction, if $\mathfrak{m} \in \mathbb{V}(S)$ and $g' \in \langle S \rangle$, there exist $n \in \mathbb{N}$ and $g \in S$ such that $-n|g| \leqslant g' \leqslant n|g|$. Since $g \in S$, we have $g \in \mathfrak{m}$; consequently $-n|g|, n|g| \in \mathfrak{m}$ because the latter is an ℓ-ideal. However $\mathfrak{m}$ is convex, therefore $g' \in \mathfrak{m}$, that is $\widehat{g'}(\mathfrak{m}) = 0$ if, and only if, $\mathfrak{m} \in \mathbb{V}(\langle S \rangle)$. □

This motivates the following

Definition 3.1.15. Let G be an ℓ-group. A subset $T \subseteq \operatorname{Max} G$ is *closed* if, and only if, it is of the form $\mathbb{V}(I)$ for some ideal $I \subseteq G$.

Lemma 3.1.16. *Let $\mathcal{I}$ denote the collection of all the ideals of the ℓ-group G. The family of closed sets $\{\mathbb{V}(I)\}_{I \in \mathcal{I}}$ is a topology for* Max *G, called the* spectral topology. *A basis of closed sets for the latter is given by $\{\mathbb{V}(g)\}_{g \in G}$.*

Proof. See [13, Théorème 10.1.4, Proposition 10.1.7]. □

In particular, for all positive elements $g, h \in G$ we have $\mathbb{V}(g \wedge h) = \mathbb{V}(g) \cap \mathbb{V}(h)$ [13, Lemme 10.1.1]. The set $\operatorname{Max} G$, equipped with the spectral topology, is called the *maximal spectrum* of the ℓ-group G.

Proposition 3.1.17. *Let G be an ℓ-group, and let $\operatorname{Max} G$ be its maximal spectrum. The following hold.*

1. *$\operatorname{Max} G$ is Hausdorff.*
2. *$\operatorname{Max} G$ is compact if, and only if, G is unital.*
3. *For all $g \in G$, the function $\widehat{g}\colon \operatorname{Max} G \to \mathbb{R}$ is continuous.*

Proof. If G is the trivial group, there is nothing to prove. To check that the space $\operatorname{Max} G$ is Hausdorff, let $\mathfrak{m}, \mathfrak{n} \in \operatorname{Max} G$ be distinct points. Distinct maximal ideals are incomparable, hence there are positive elements $a, b \in G$ such that $a \in \mathfrak{m} \backslash \mathfrak{n}$ and $b \in \mathfrak{n} \backslash \mathfrak{m}$. By Lemma 2.1.4 we have

$$(a - (a \wedge b)) \wedge (b - (a \wedge b)) = 0.$$

Note that $a-(a\wedge b)\in\mathfrak{m}$ because $0\leqslant a-(a\wedge b)\leqslant a$, and similarly $b-(a\wedge b)\in\mathfrak{n}$. Further, $a\wedge b\in\mathfrak{m}$ since $0\leqslant a\wedge b\leqslant a$, so that $b-(a\wedge b)\notin\mathfrak{m}$, for otherwise $b=(a\wedge b)+(b-(a\wedge b))\in\mathfrak{m}$. In an analogous way, one can see that $a-(a\wedge b)\notin\mathfrak{n}$. Upon observing that

$$\mathbb{V}(a-(a\wedge b))\cap\mathbb{V}(b-(a\wedge b))=\mathbb{V}((a-(a\wedge b))\wedge(b-(a\wedge b)))=\mathbb{V}(0)=\operatorname{Max}G,$$

we see that $\operatorname{Max}G\setminus\mathbb{V}(a-(a\wedge b))$ and $\operatorname{Max}G\setminus\mathbb{V}(b-(a\wedge b))$ are disjoint open sets separating the elements $\mathfrak{m},\mathfrak{n}\in\operatorname{Max}G$. Item 2 is proved in [13, Proposition 10.1.6], while item 3 in [13, Corollaire 13.2.6]. $\square$

3.2 Yosida map

For an ℓ-group G, we shall give precise conditions under which the *Yosida map*

$$\mathrm{Y}\colon G\to\mathrm{C}(\operatorname{Max}G),\quad g\mapsto(\widehat{g}\colon\ \operatorname{Max}G\to\mathbb{R})$$

is an injective and surjective ℓ-homomorphism.

It is elementary that Y is an ℓ-homomorphism precisely because, for all $\mathfrak{m}\in\operatorname{Max}G$, the quotient map $q_{\mathfrak{m}}\colon G\to\frac{G}{\mathfrak{m}}$ is an ℓ-homomorphism. For example, given $f,g\in G$, $\widehat{f+g}=\widehat{f}+\widehat{g}$ if, and only if, for all $\mathfrak{m}\in\operatorname{Max}G$, $\frac{f+g}{\mathfrak{m}}=\frac{f}{\mathfrak{m}}+\frac{g}{\mathfrak{m}}$ if, and only if, $q_{\mathfrak{m}}(f+g)=q_{\mathfrak{m}}(f)+q_{\mathfrak{m}}(g)$. Further, if G is unital, with distinguished strong order unit $u\in G$, it is clear that $\mathrm{Y}\colon(G,u)\to(\mathrm{C}(\operatorname{Max}G),1_{\operatorname{Max}G})$ is a unital ℓ-homomorphism, since each embedding $\mathfrak{h}_{\mathfrak{m}}\colon(\frac{G}{\mathfrak{m}},\frac{u}{\mathfrak{m}})\to(\mathbb{R},1)$ is unital. The following commutative diagram shows that the Yosida map can be thought of as the *diffusion* of Hölder's embedding to the whole ℓ-group.

$$\begin{tikzcd}
G \arrow[r, "\mathrm{Y}"] \arrow[d, "q_{\mathfrak{m}}"'] & \mathrm{C}(\operatorname{Max}G) \arrow[d, "q_{\mathbb{I}(\mathfrak{m})}"] \\
\frac{G}{\mathfrak{m}} \arrow[r, "\mathfrak{h}_{\mathfrak{m}}"'] & \frac{\mathrm{C}(\operatorname{Max}G)}{\mathbb{I}(\mathfrak{m})}\cong\mathbb{R}
\end{tikzcd}$$

Note that, considering the quotient with respect to the ideal $\mathbb{I}(\mathfrak{m})$ means looking at the value of a function at the point $\mathfrak{m}\in\operatorname{Max}G$. Indeed $\frac{f}{\mathfrak{m}}=\frac{g}{\mathfrak{m}}\in\frac{\mathrm{C}(\operatorname{Max}G)}{\mathbb{I}(\mathfrak{m})}$ if, and only if, $f(\mathfrak{m})=g(\mathfrak{m})$. Hence the equivalence class $\frac{f}{\mathfrak{m}}$ is completely determined by the real value $f(\mathfrak{m})\in\mathbb{R}$. It suffices to consider the constant functions on $\operatorname{Max}G$, to conclude that the quotient of the ℓ-group $\mathrm{C}(\operatorname{Max}G)$ by the ideal $\mathbb{I}(\mathfrak{m})$ is isomorphic to $\mathbb{R}$.

The next result should be compared with Theorem 2.2.72.

Proposition 3.2.1. *A unital ℓ-group (G,u) is archimedean if, and only if, the Yosida map $\mathrm{Y}\colon(G,u)\to(\mathrm{C}(\operatorname{Max}G),1_{\operatorname{Max}G})$ is an injective ℓ-homomorphism.*

A few preliminary facts are needed, in order to prove the foregoing proposition.

Definition 3.2.2. The *radical* ideal of an ℓ-group G is defined as the intersection of all its maximal ideals. In symbols,

$$\operatorname{Rad} G := \bigcap \operatorname{Max} G.$$

Lemma 3.2.3. *Let (G, u) be a unital ℓ-group. The following are equivalent.*

1. *G is archimedean.*
2. *The radical of G is trivial, i.e.* $\operatorname{Rad} G = \{0\}$.

Proof. This was first proved in [68, Theorem 1], for unital vector lattices (=unital lattice-ordered real vector spaces). The proof for unital vector lattices can be adapted, by using the unital Hölder's Theorem 2.1.57 in place of [68, Lemma 1]. □

Remark 3.2.4. We remark that the analogue of Lemma 3.2.3 does not hold if G is not unital. A counterexample is provided in [68, §2].

Corollary 3.2.5. *Let (G, u) be an archimedean unital ℓ-group. Then, for all $g \in G$, $g \neq 0$ if, and only if, there exists $\mathfrak{m} \in \operatorname{Max} G$ such that $\widehat{g}(\mathfrak{m}) \neq 0$.*

Proof. If the unital ℓ-group is trivial, there is nothing to prove. If it is not trivial, Lemma 3.2.3 entails $\bigcap \operatorname{Max} G = \{0\}$. It follows at once that $g \neq 0$ if, and only if, there exists $\mathfrak{m} \in \operatorname{Max} G$ such that $g \notin \mathfrak{m}$ or, equivalently, such that $\widehat{g}(\mathfrak{m}) \neq 0$. □

Remark 3.2.6. In an archimedean unital ℓ-group (G, u), whenever we are given a non-zero element $g \in G$, we can find a maximal ideal $\mathfrak{m} \in \operatorname{Max} G$ such that $\widehat{g}(\mathfrak{m}) \neq 0$. If the unital ℓ-group (G, u) is non-trivial and non-archimedean, there are non-zero elements $h \in G$ such that $\widehat{h}$ is the constant function of value 0 on the maximal spectrum of G. These elements are precisely those $h \in G$ which satisfy $0 < n|h| \leqslant u$ for all $n \in \mathbb{N}$. They can be equivalently described as those elements $h \in G$ for which there is $g \in G$ such that $0 < n|h| \leqslant g$ for all $n \in \mathbb{N}$.

Proof of Proposition 3.2.1. If the ℓ-homomorphism Y is injective, then (G, U) is isomorphic to the ℓ-subgroup $(\mathrm{Y}(G), 1_{\operatorname{Max} G})$ of $\mathrm{C}(\operatorname{Max} G)$. The ℓ-group $\mathrm{C}(\operatorname{Max} G)$ is archimedean by Lemma 2.1.22, hence so is $G \cong \mathrm{Y}(G)$. On the other hand, assume that G is an archimedean unital ℓ-group, and consider distinct elements $f, g \in G$. Then $f - g \neq 0$, and by Corollary 3.2.5 there exists a maximal ideal $\mathfrak{m} \in \operatorname{Max} G$ such that $\widehat{f - g}(\mathfrak{m}) \neq 0$. Upon recalling that the Yosida map is an ℓ-homomorphism, we have $\widehat{f - g}(\mathfrak{m}) \neq 0$ if, and only if, $\widehat{f}(\mathfrak{m}) - \widehat{g}(\mathfrak{m}) \neq 0$ if, and only if, $\widehat{f}(\mathfrak{m}) \neq \widehat{g}(\mathfrak{m})$. This shows that $\widehat{f} \neq \widehat{g}$. □

Example 3.2.7. Let us consider the unital ℓ-group $(\mathbb{Z}, 1)$. It is clearly archimedean, hence the Yosida map provides an embedding $\mathrm{Y} \colon (\mathbb{Z}, 1) \to (\mathrm{C}(\operatorname{Max} \mathbb{Z}), 1_{\operatorname{Max} \mathbb{Z}})$. As observed in Example 2.1.48, the ℓ-group $\mathbb{Z}$ is simple, so that $\operatorname{Max} \mathbb{Z} = \{\{0\}\}$. In this situation, the Yosida map is essentially the inclusion $(\mathbb{Z}, 1) \hookrightarrow (\mathbb{R}, 1) \cong (\mathrm{C}(\operatorname{Max} \mathbb{Z}), 1_{\operatorname{Max} \mathbb{Z}})$. The latter is not surjective; in particular, $\mathrm{C}(\operatorname{Max} \mathbb{Z})$ is a completion of $\mathbb{Z}$, in a sense which we shall now make precise.

In the study of the family $\mathrm{C}(X)$ of continuous functions on the compact Hausdorff space X, the *uniform* norm is often taken into consideration. Given $f \in \mathrm{C}(X)$, the latter norm is defined by

$$\|f\|_\infty := \sup_{x\in X} |f(x)|.$$

Regarding $(\mathrm{C}(X), 1_X)$ as a unital ℓ-group, we can introduce another norm, in the following way:

$$\begin{aligned}\|f\|_{1_X} &:= \sup\{\tfrac{p}{q} \in \mathbb{Q} \mid p,q \in \mathbb{N},\ q \neq 0 \ \text{ and } \ |f| \geqslant \tfrac{p}{q}\cdot 1_X\}\\ &= \sup\{\tfrac{p}{q} \in \mathbb{Q} \mid p,q \in \mathbb{N},\ q \neq 0 \ \text{ and } \ q|f| \geqslant p1_X\}.\end{aligned}$$

It is elementary that, for all $f \in \mathrm{C}(X)$, $\|f\|_\infty = \|f\|_{1_X}$. This motivates the following

Definition 3.2.8. Let (G, u) be a unital ℓ-group. For all $g \in G$, the *seminorm induced by the unit* u on G, is given by

$$\begin{aligned}\|g\|_u &:= \inf\{\tfrac{p}{q} \in \mathbb{Q} \mid p,q \in \mathbb{N},\ q \neq 0 \ \text{ and } \ q|g| \leqslant pu\}\\ &= \sup\{\tfrac{p}{q} \in \mathbb{Q} \mid p,q \in \mathbb{N},\ q \neq 0 \ \text{ and } \ q|g| \geqslant pu\} \in \mathbb{R}.\end{aligned}$$

It is easy to check that $\|\cdot\|_u \colon G \to \mathbb{R}^+$ is, in fact, a seminorm, i.e. it satisfies all axioms for a norm except, possibly, the *faithfulness* condition: $\|g\|_u = 0$ implies $g = 0$. The next statement, which follows from the very definition of the seminorm induced by the unit, tells us when this seminorm is, in fact, a norm.

Lemma 3.2.9. *Let (G, u) be a unital ℓ-group. Then*

$$\{g \in G \mid \|g\|_u = 0\} = \operatorname{Rad} G.$$

In particular, $\|\cdot\|_u$ is a norm if, and only if, (G, u) is archimedean. □

Remark 3.2.10. Recall that $\mathrm{C}(X)$, with the metric induced by the uniform norm, or equivalently by the norm $\|\cdot\|_{1_X}$, is a Cauchy-complete metric space. Upon introducing multiplication by real numbers, $\mathrm{C}(X)$ can be regarded as a Cauchy-complete metric linear space.

Example 3.2.11. The norm induced by the unit on the unital ℓ-group $(\mathbb{Q}, 1)$ is the usual absolute value. However, $\mathbb{Q}$ is not Cauchy-complete, and the Yosida embedding $(\mathbb{Q}, 1) \to (\mathrm{C}(\mathbb{Q}), 1_{\operatorname{Max}\mathbb{Q}}) \cong (\mathbb{R}, 1)$ is essentially the completion of $\mathbb{Q}$ with respect to the norm.

Example 3.2.12. The norm induced by the unit on the unital ℓ-group $(\mathbb{Z}, 1)$ is again the absolute value. Now, $\mathbb{Z}$ is complete with respect to this norm, however it is not a divisible group. Recall that

Definition 3.2.13. An abelian group G is *divisible* if, for all $g \in G$ and for all $m \in \mathbb{N}$, there exists $h \in G$ such that $g = mh$.

We remark that every ℓ-group is torsion-free [34, Corollary 0.1.2]. It is standard that every torsion-free abelian group G admits a canonical embedding into a divisible abelian group, called the *divisible hull* of G.

Remark 3.2.14. Consider an archimedean unital ℓ-group (G, u), along with the norm $\|\cdot\|_u$ induced by the unit. One can check that the operations $+, \wedge, \vee$ are continuous with respect to the topology induced by (the metric induced by) the norm. Therefore, we can extend the operations above to the norm-completion of the ℓ-group (G, u).

Example 3.2.15. With reference to Examples 3.2.11 and 3.2.12, the divisible hull of $(\mathbb{Z}, 1)$ is $(\mathbb{Q}, 1)$, and the latter is not norm-complete. Its completion is $(\mathbb{R}, 1)$, which is isomorphic to $(\mathrm{C}(X), 1_X)$, where X is an arbitrary one-point space.

We are now ready to characterise those archimedean unital ℓ-groups for which the Yosida map is surjective, hence an ℓ-isomorphism.

Proposition 3.2.16. *Let (G, u) be a non-trivial archimedean unital ℓ-group. The following are equivalent.*

1. *The Yosida map $\mathrm{Y}\colon (G, u) \to (\mathrm{C}(\mathrm{Max}\, G), 1_{\mathrm{Max}\, G})$ is a surjective ℓ-homomorphism.*
2. *G is divisible, and complete in the norm $\|\cdot\|_u$ induced by the unit.*

The proof of this result relies, in an essential way, on the following lattice-theoretic version of the Stone-Weierstrass theorem. Recall that a set S of continuous real-valued functions on a topological space X *separates* points of X if, for each pair of distinct points $x, y \in X$, there is $f \in S$ such that $f(x) \neq f(y)$.

Theorem 3.2.17. *Let S denote a subset of the family $\mathrm{C}(X)$ of continuous real-valued functions on a compact Hausdorff space X. Suppose that S contains the function 1_X, is closed under the operations $+, \wedge, \vee$ and under multiplication by rational numbers, and separates points of X. Then S is dense in $\mathrm{C}(X)$ with respect to the uniform norm.*

Proof. See [6, Theorem 11.3]. $\square$

Proof of Proposition 3.2.16. The map Y is an injective ℓ-homomorphism by Proposition 3.2.1. Assuming that it is also surjective, we conclude that G is divisible and complete, since it is ℓ-isomorphic to the divisible complete ℓ-group $\mathrm{C}(\mathrm{Max}\, G)$. Conversely, suppose that G is divisible, and complete with respect to the norm $\|\cdot\|_u$. We must prove that $\mathrm{Y}(G) = \mathrm{C}(\mathrm{Max}\, G)$. The set $\mathrm{Y}(G)$ contains $1_{\mathrm{Max}\, G} = \widehat{u}$, is closed under the operations $+, \wedge, \vee$, and it is closed under multiplication by rational constants since G is divisible. Lastly, $\mathrm{Y}(G)$ separates points of $\mathrm{Max}\, G$: if $\mathfrak{m}, \mathfrak{n}$ are distinct points of $\mathrm{Max}\, G$, then there exists an element $f \in G$ such that $f \in \mathfrak{m} \setminus \mathfrak{n}$. The function $\widehat{f}$ separates the two points, because $\widehat{f}(\mathfrak{m}) = 0 \neq \widehat{f}(\mathfrak{n})$. Theorem 3.2.17 entails that $\mathrm{Y}(G)$ is dense in $\mathrm{C}(\mathrm{Max}\, G)$ with respect to the uniform norm $\|\cdot\|_\infty = \|\cdot\|_{1_{\mathrm{Max}\, G}}$. By hypothesis G is complete in the norm $\|\cdot\|_u$, hence the ℓ-group $\mathrm{Y}(G)$, which is isomorphic to G by Proposition 3.2.1, is complete in the norm $\|\cdot\|_{\widehat{u}} = \|\cdot\|_{1_{\mathrm{Max}\, G}}$. We conclude that $\mathrm{Y}(G) = \mathrm{C}(\mathrm{Max}\, G)$. $\square$

Proposition 3.2.16, along with Proposition 3.2.1, provide a sharp characterisation of the unital ℓ-groups of continuous functions on some compact Hausdorff space. We shall state it for future reference.

Theorem 3.2.18 (Yosida's representation theorem)**.** *A unital ℓ-group (G, u) is represented by a compact Hausdorff space X, i.e. $(G, u) \cong (\mathrm{C}(X), 1_X)$, if, and only if, the following hold.*

1. *G is archimedean.*
2. *G is divisible.*
3. *G is complete with respect to the norm $\|\cdot\|_u$ induced by the unit.* □

Observe that, if a unital ℓ-group (G, u) is represented by a compact Hausdorff space X, then it is represented by its maximal spectrum $\operatorname{Max} G$. Indeed, if (G, u) is represented by X, then it is a complete, divisible, and archimedean unital ℓ-group, since it is isomorphic to $(\mathrm{C}(X), 1_X)$. Then Propositions 3.2.1 and 3.2.16 imply $(G, u) \cong (\mathrm{C}(\operatorname{Max} G), 1_{\operatorname{Max} G})$.

3.3 The categorical duality

Recall that $\ell\mathsf{Grp}_\mathsf{u}$ denotes the category of unital ℓ-groups and unital ℓ-homomorphisms, while KHaus denotes the category of compact Hausdorff spaces and continuous maps. In this section we shall see that Yosida's representation theorem induces a dual equivalence between the categories KHaus and $\ell\mathsf{Grp}_\mathsf{u}$. We begin by making explicit the functorial correspondences introduced in the preceding section.

The following fact was observed in Example 2.1.58.

Lemma 3.3.1. *If X is a compact Hausdorff space, then*

$$\mathcal{C}(X) := (\mathrm{C}(X), 1_X)$$

is a unital ℓ-group. □

Lemma 3.3.2. *If $f\colon X \to Y$ is a continuous function between compact Hausdorff spaces, then*

$$\mathcal{C}(f) := - \circ f\colon (\mathrm{C}(Y), 1_Y) \to (\mathrm{C}(X), 1_X)$$

is a unital ℓ-homomorphism.

Proof. We check that $- \circ f\colon (\mathrm{C}(Y), 1_Y) \to (\mathrm{C}(X), 1_X)$ is a group homomorphism and a lattice homomorphism, and it preserves the strong order unit. If $g, g' \in \mathrm{C}(Y)$ and $x \in X$, then

$$((g + g') \circ f)(x) = (g + g')(f(x)) = g(f(x)) + g'(f(x)) = (g \circ f)(x) + (g' \circ f)(x).$$

This shows that $\mathcal{C}(f)$ is a group homomorphism. To see that it is a lattice homomorphism it suffices to observe that, for all $x \in X$,

$$((g \wedge g') \circ f)(x) = (g \wedge g')(f(x))$$

$$
\begin{aligned}
&= \min\left(g(f(x)), g'(f(x))\right) \\
&= (g(f(x))) \wedge (g'(f(x))) \\
&= ((g \circ f)(x)) \wedge ((g' \circ f)(x)).
\end{aligned}
$$

In other words, $\mathcal{C}(f)(g \wedge g') = \mathcal{C}(f)(g) \wedge \mathcal{C}(f)(g')$. A similar argument shows that $\mathcal{C}(f)(g \vee g') = \mathcal{C}(f)(g) \vee \mathcal{C}(f)(g')$. Moreover, the ℓ-homomorphism $\mathcal{C}(f)$ is unital since, for all $x \in X$, $(1_Y \circ f)(x) = 1_Y(f(x)) = 1$. That is, $\mathcal{C}(f)(1_Y) = 1_X$. □

It is immediate to verify that the correspondence $X \mapsto \mathcal{C}(X)$ defines a functor.

Corollary 3.3.3. $\mathcal{C}\colon \mathsf{KHaus} \to \ell\mathsf{Grp_u}$ *is a contravariant functor from the category of compact Hausdorff spaces to the category of unital ℓ-groups.* □

The next result was proved in Proposition 3.1.17.

Lemma 3.3.4. *If (G, u) is a unital ℓ-group, then*

$$
\mathcal{M}(G) := \operatorname{Max} G
$$

is a compact Hausdorff space, with respect to the spectral topology. □

Lemma 3.3.5. *If $h\colon (G, u) \to (H, v)$ is a unital ℓ-homomorphism, then*

$$
\mathcal{M}(h) := h^{-1}\colon \operatorname{Max} H \to \operatorname{Max} G
$$

is a continuous function.

Proof. Clearly, if $\mathfrak{m} \subseteq H$ is an ideal of H, i.e. a convex subgroup and sublattice, then $h^{-1}(\mathfrak{m})$ is a subgroup and sublattice of G. To see that $h^{-1}(\mathfrak{m})$ is convex, let $a, c \in h^{-1}(\mathfrak{m})$ and pick $b \in G$ such that $a \leqslant b \leqslant c$. The ℓ-homomorphism h is order-preserving by Lemma 2.1.11.(1), hence $h(a) \leqslant h(b) \leqslant h(c)$. The elements $h(a), h(c)$ belong to the convex set $\mathfrak{m}$, therefore $h(b) \in \mathfrak{m}$. We conclude that $h^{-1}(\mathfrak{m})$ is an ideal of G. Assume, further, that $\mathfrak{m} \in \operatorname{Max} H$ is a maximal ideal of H, and let $x \in G \setminus h^{-1}(\mathfrak{m})$. Since $h(x) \notin \mathfrak{m}$, by Lemma 2.1.55 there exists $n \in \mathbb{N}$ such that $h(u - nx) = v - nh(x) \in \mathfrak{m}$, whence $u - nx \in h^{-1}(\mathfrak{m})$. Again by Lemma 2.1.55, we conclude that $h^{-1}(\mathfrak{m})$ is a maximal ideal of G. We now prove that the function h^{-1} is continuous, i.e. that the preimage of a closed subset of $\operatorname{Max} G$, under the function h^{-1}, is a closed subset of $\operatorname{Max} H$. An arbitrary closed subset of $\operatorname{Max} G$ is of the form

$$
\mathbb{V}(I) = \{\mathfrak{m} \in \operatorname{Max} G \mid g \in \mathfrak{m} \text{ for all } g \in I\},
$$

where $I \subseteq G$ is an ideal of G. Its preimage is

$$
\begin{aligned}
(h^{-1})^{-1}(\mathbb{V}(I)) &= \{\mathfrak{n} \in \operatorname{Max} H \mid h^{-1}(\mathfrak{n}) \in \mathbb{V}(I)\} \\
&= \{\mathfrak{n} \in \operatorname{Max} H \mid g \in h^{-1}(\mathfrak{n}) \text{ for all } g \in I\} \\
&= \{\mathfrak{n} \in \operatorname{Max} H \mid h(g) \in \mathfrak{n} \text{ for all } g \in I\} \\
&= \mathbb{V}(h(I)).
\end{aligned}
$$

Lemma 3.1.14 entails $\mathbb{V}(h(I)) = \mathbb{V}(\langle h(I)\rangle)$, so that the preimage of $\mathbb{V}(I)$ under the function h^{-1} is the closed subset $\mathbb{V}(\langle h(I)\rangle) \subseteq \operatorname{Max} H$. □

Again, the correspondence $G \mapsto \mathcal{M}(G)$ defines a functor:

Corollary 3.3.6. $\mathcal{M}\colon \ell\mathsf{Grp_u} \to \mathsf{KHaus}$ *is a contravariant functor from the category of unital ℓ-groups to the category of compact Hausdorff spaces.* □

In view of Theorem 3.2.18, we are interested in the full subcategory of $\ell\mathsf{Grp_u}$ whose objects are the complete, divisible, and archimedean unital ℓ-groups. Denote this subcategory by YAlg; for the sake of brevity, we refer to its objects as *Yosida algebras*. Henceforth, we consider the restrictions of the above functors to the category YAlg, that is $\mathcal{M}\colon \mathsf{YAlg} \to \mathsf{KHaus}$, and $\mathcal{C}\colon \mathsf{KHaus} \to \mathsf{YAlg}$. The next two results state that they are quasi-inverse functors.

Proposition 3.3.7. *There exists a natural isomorphism*

$$\mu\colon \mathrm{Id}_{\mathsf{KHaus}} \to \mathcal{M} \circ \mathcal{C},$$

where $\mathrm{Id}_{\mathsf{KHaus}}$ *is the identity functor on the category* KHaus.

Proof. The first step consists in showing that a compact Hausdorff space X is homeomorphic to the maximal spectrum $\operatorname{Max} \mathrm{C}(X)$. Note that every maximal ideal $\mathfrak{m} \in \operatorname{Max} \mathrm{C}(X)$ is of the form $\mathbb{I}(S)$ for some closed subset $S \subseteq X$. Specifically, $S = \mathbb{V}(\mathfrak{m})$, indeed

$$\begin{aligned}\mathbb{I}(\mathbb{V}(\mathfrak{m})) &= \mathbb{I}(\{x \in X \mid f(x) = 0 \text{ for all } f \in \mathfrak{m}\}) \\ &= \{f \in \mathrm{C}(X) \mid f(x) = 0 \text{ for all } x \in X \text{ such that } f(x) = 0 \text{ for all } f \in \mathfrak{m}\} \supseteq \mathfrak{m}.\end{aligned}$$

However, $\mathbb{I}(\mathbb{V}(\mathfrak{m})) \neq \mathrm{C}(X)$, so that $\mathfrak{m} = \mathbb{I}(\mathbb{V}(\mathfrak{m}))$ by the maximality of $\mathfrak{m}$. Lemma 3.1.8 states that the ideal $\mathbb{I}(S)$ is maximal if, and only if, $S = \{p\}$ for some $p \in X$. Therefore there is a surjective map $\mu_X\colon X \to \operatorname{Max} \mathrm{C}(X)$ defined by

$$\mu_X\colon p \mapsto \mathbb{I}(\{p\}) = \{f \in \mathrm{C}(X) \mid f(p) = 0\}.$$

The map μ_X is injective by Urysohn's lemma [28, Theorem 1.5.11] (recall that every compact Hausdorff space is normal [28, Theorem 3.1.9], hence Urysohn's lemma applies). In fact, it turns out that the bijection μ_X is a homeomorphism. In order to prove the continuity, pick a closed subset $\mathbb{V}(I) \subseteq \operatorname{Max} \mathrm{C}(X)$, where $I \subseteq \mathrm{C}(X)$ is an ideal. We show that its preimage under μ_X is a closed subset of X.

$$\begin{aligned}\mu_X^{-1}(\mathbb{V}(I)) &= \mu_X^{-1}\left(\{\mathfrak{m} \in \operatorname{Max} \mathrm{C}(X) \mid f(\mathfrak{m}) = 0 \text{ for all } f \in I\}\right) \\ &= \{p \in X \mid f(\mathbb{I}(\{p\})) = 0 \text{ for all } f \in I\} \\ &= \{p \in X \mid f \in \mathbb{I}(\{p\}) \text{ for all } f \in I\} \\ &= \{p \in X \mid f(p) = 0 \text{ for all } f \in I\} \\ &= \bigcap_{f \in I} f^{-1}(0).\end{aligned}$$

Each zero set $f^{-1}(0)$ is closed, being the preimage of a point in a T_1-space under a continuous function. An arbitrary intersection of closed sets is closed, hence $\mu_X \colon X \to \operatorname{Max} \mathrm{C}(X)$ is continuous. To check that its inverse μ_X^{-1} is continuous, we will prove that, for any closed subset $K \subseteq X$, $\mu_X(K)$ is closed in $\operatorname{Max} \mathrm{C}(X)$. The set K is compact, because it is a closed subset of the compact space X. The continuous image of a compact set is compact, therefore $\mu_X(K)$ is a compact subset of the Hausdorff space $\operatorname{Max} \mathrm{C}(X)$, i.e. $\mu_X(K)$ is closed. Define a natural transformation $\mu \colon \mathrm{Id}_{\mathsf{KHaus}} \to \mathcal{M} \circ \mathcal{C}$ in the following way: for each compact Hausdorff space X, the component $(\mu)_X$ of μ at X is the morphism $(\mu)_X := \mu_X$. We have just proved that every such component is an isomorphism in KHaus, hence it suffices to show that μ is a natural transformation. Let $f \colon X \to Y$ be a continuous function between compact Hausdorff spaces: we must prove that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\mu_X"] \arrow[d, "f"'] & \operatorname{Max} \mathrm{C}(X) \arrow[d, "(\mathcal{M}\circ\mathcal{C})(f)"] \\
Y \arrow[r, "\mu_Y"'] & \operatorname{Max} \mathrm{C}(Y)
\end{tikzcd}$$

We remark that, given a maximal ideal $\mathfrak{m} \in \operatorname{Max} \mathrm{C}(X)$,

$$(\mathcal{M} \circ \mathcal{C})(f)(\mathfrak{m}) = (- \circ f)^{-1}(\mathfrak{m}) = \{h \in \mathrm{C}(Y) \mid h \circ f \in \mathfrak{m}\}.$$

We conclude that, for all $p \in X$,

$$\begin{aligned}
(\mathcal{M} \circ \mathcal{C})(f) \circ \mu_X(p) &= (\mathcal{M} \circ \mathcal{C})(f)(\mathbb{I}(\{p\})) \\
&= \{h \in \mathrm{C}(Y) \mid h \circ f \in \mathbb{I}(\{p\})\} \\
&= \{h \in \mathrm{C}(Y) \mid (h \circ f)(p) = 0\} \\
&= \{h \in \mathrm{C}(Y) \mid h(f(p)) = 0\} \\
&= \mathbb{I}(\{f(p)\}) \\
&= (\mu_Y \circ f)(p).
\end{aligned}$$

□

Proposition 3.3.8. *There exists a natural isomorphism*

$$\nu \colon \mathrm{Id}_{\mathsf{YAlg}} \to \mathcal{C} \circ \mathcal{M},$$

where $\mathrm{Id}_{\mathsf{YAlg}}$ *is the identity functor on the category* YAlg.

Proof. Let (G, u) be a Yosida algebra. Upon denoting by

$$\mathrm{Y}_{(G,u)} \colon (G, u) \to (\mathrm{C}(\operatorname{Max} G), 1_{\operatorname{Max} G})$$

the Yosida map, we know by Propositions 3.2.1 and 3.2.16 that $\mathrm{Y}_{(G,u)}$ is a bijective unital ℓ-homomorphism. Define a natural transformation $\nu \colon \mathrm{Id}_{\mathsf{YAlg}} \to \mathcal{C} \circ \mathcal{M}$ whose component at (G,u) is the isomorphism $(\nu)_{(G,u)} := \mathrm{Y}_{(G,u)}$ in the category YAlg. To prove the statement, it suffices to show that ν is a natural transformation, i.e. that given a unital ℓ-homomorphism $h\colon (G,u) \to (H,v)$, the following diagram is commutative.

$$\begin{tikzcd}
(G,u) \arrow[r, "\mathrm{Y}_{(G,u)}"] \arrow[d, "h"'] & (\mathrm{C}(\operatorname{Max} G), 1_{\operatorname{Max} G}) \arrow[d, "(\mathcal{C}\circ\mathcal{M})(h)"] \\
(H,v) \arrow[r, "\mathrm{Y}_{(H,v)}"'] & (\mathrm{C}(\operatorname{Max} H), 1_{\operatorname{Max} H})
\end{tikzcd}$$

For all $g \in G$, we have

$$(\mathrm{Y}_{(H,v)} \circ h)(g) = \mathrm{Y}_{(H,v)}(h(g)) = \mathrm{Y}_{(H,v)}(h(g)) = \widehat{h(g)},$$

and

$$(\mathcal{C}\circ\mathcal{M})(h) \circ \mathrm{Y}_{(G,u)}(g) = (\mathcal{C}\circ\mathcal{M})(h)(\mathrm{Y}_{(G,u)}(g)) = (\mathcal{C}\circ\mathcal{M})(h)(\widehat{g}) = \widehat{g} \circ h^{-1}.$$

However, $\widehat{h(g)} = \widehat{g} \circ h^{-1}$ if, and only if, for all $\mathfrak{m} \in \operatorname{Max} H$, the condition $\widehat{h(g)}(\mathfrak{m}) = \widehat{g} \circ h^{-1}(\mathfrak{m})$ is satisfied. In turn, this happens if, and only if,

$$\mathfrak{h}_{\mathfrak{m}}\left(\frac{h(g)}{\mathfrak{m}}\right) = \mathfrak{h}_{h^{-1}(\mathfrak{m})}\left(\frac{g}{h^{-1}(\mathfrak{m})}\right),$$

where $\mathfrak{h}_{\mathfrak{m}}, \mathfrak{h}_{h^{-1}(\mathfrak{m})}$ are the unique unital ℓ-embeddings

$$\mathfrak{h}_{\mathfrak{m}}\colon \frac{H}{\mathfrak{m}} \to \mathbb{R}, \quad \mathfrak{h}_{h^{-1}(\mathfrak{m})}\colon \frac{G}{h^{-1}(\mathfrak{m})} \to \mathbb{R}$$

provided by the unital Hölder's Theorem 2.1.57. On the other hand, the ℓ-homomorphism $h\colon (G,u) \to (H,v)$ induces an ℓ-homomorphism $\frac{G}{h^{-1}(\mathfrak{m})} \to \frac{H}{\mathfrak{m}}$, which we denote again by h. The latter homomorphism is injective: if $g \in \frac{G}{h^{-1}(\mathfrak{m})}$ is such that $h(g) = 0 \in \frac{H}{\mathfrak{m}}$, then $h(g) \in \mathfrak{m}$, i.e. $g \in h^{-1}(\mathfrak{m})$. Thus $g = 0 \in \frac{G}{h^{-1}(\mathfrak{m})}$. We have a diagram

$$\begin{tikzcd}
\frac{H}{\mathfrak{m}} \arrow[r, "\mathfrak{h}_{\mathfrak{m}}"] & \mathbb{R} \\
\frac{G}{h^{-1}(\mathfrak{m})} \arrow[u, "h"] \arrow[ur, "\mathfrak{h}_{h^{-1}(\mathfrak{m})}"'] &
\end{tikzcd}$$

It is elementary that the composition $\mathfrak{h}_{\mathfrak{m}} \circ h$ is an injective unital ℓ-homomorphism from $\frac{G}{h^{-1}(\mathfrak{m})}$ to $\mathbb{R}$. By Theorem 2.1.57 there is only one such embedding, whence $\mathfrak{h}_{h^{-1}(\mathfrak{m})} = \mathfrak{h}_{\mathfrak{m}} \circ h$. To complete the proof, observe that

$$\mathfrak{h}_{h^{-1}(\mathfrak{m})}\left(\frac{g}{h^{-1}(\mathfrak{m})}\right) = (\mathfrak{h}_{\mathfrak{m}} \circ h)\left(\frac{g}{h^{-1}(\mathfrak{m})}\right) = \mathfrak{h}_{\mathfrak{m}}\left(\frac{h(g)}{\mathfrak{m}}\right).$$

$\square$

We have just proved the existence of a duality between compact Hausdorff spaces and Yosida algebras, i.e. complete, divisible, and archimedean unital ℓ-groups.

Theorem 3.3.9 (Yosida duality)**.** *The category* KHaus *of compact Hausdorff spaces is dually equivalent to the category* YAlg *of Yosida algebras via the functors* $\mathcal{C}$ *and* $\mathcal{M}$*.* $\square$

Chapter 4

δ-algebras

4.1 Summary of MV-algebraic results

We recall here those results about MV-algebras that will be used in the present chapter.

Lemma 4.1.1. *Let A be an MV-algebra and let $x, y \in A$. The following conditions are equivalent.*

1. $\neg x \oplus y = 1$.
2. $x \odot \neg y = 0$, *i.e.* $x \ominus y = 0$.
3. $y = x \oplus (y \ominus x)$.
4. *There exists $z \in A$ such that $x \oplus z = y$.*

Upon defining, for all $x, y \in A$, $x \leqslant y$ if x, y satisfy the equivalent conditions above, the partially ordered set $(A, \leqslant)$ is a lattice.

Proof. See Lemma 2.2.7 and Remark 2.2.10. $\square$

Lemma 4.1.2. *If A is an MV-algebra and $x, y \in A$, then the following hold.*

1. $x \leqslant y$ *if, and only if,* $\neg y \leqslant \neg x$.
2. *If $x \leqslant y$ then, for all $z \in A$, $x \oplus z \leqslant y \oplus z$ and $x \odot z \leqslant y \odot z$.*
3. *For all $x, y \in A$ and for all $n \in \mathbb{N}$, if $x \leqslant y$ then $nx \leqslant ny$.*

Proof. For items 1 and 2 please see Lemma 2.2.9. Item 3 is an elementary consequence of item 2. $\square$

Recall that the *Chang distance* on an MV-algebra A is the function $d\colon A \times A \to A$ defined by

$$d(x, y) := (x \ominus y) \oplus (y \ominus x).$$

Lemma 4.1.3. *If A is an MV-algebra, and $x, y \in A$, the following hold.*

1. $d(x, y) = 0$ *if, and only if,* $x = y$.
2. *If* $x \leqslant y$, *then* $y = x \oplus d(x, y)$.
3. *For all* $x, y, z \in A$, *if* $x \leqslant y \leqslant z$ *then* $d(x, z) \geqslant d(y, z)$.
4. *For every MV-homomorphism* $h\colon A \to B$, $h(d(x, y)) = d(h(x), h(y))$.

Proof. For item 1 see Proposition 2.2.28. Item 2 is a direct consequence of Axiom **MV6**. Items 3 and 4 are straightforward verifications. □

Lemma 4.1.4. *The following identities hold in any MV-algebra A, for all $x, y, z \in A$.*

1. $x \oplus y \oplus (x \odot y) = x \oplus y$.
2. $(x \ominus y) \oplus ((x \oplus \neg y) \odot y) = x$.
3. $x \ominus (y \oplus z) = (x \ominus y) \ominus z$.

Proof. Items 1 and 2 follow by Proposition 2.2.14 and Chang's subdirect representation Theorem 2.2.41. Item 3 is an easy computation. □

Lemma 4.1.5. *For arbitrary elements x, y, z in an MV-algebra A,*

$$\text{if } x \oplus z = y \oplus z \text{ and } x \odot z = 0 = y \odot z, \text{ then } x = y.$$

In particular, if $y = x \oplus y$ and $x \odot y = 0$, then $x = 0$.

Proof. This is Lemma 2.2.6. □

Recall that a subset I of an MV-algebra A is an *ideal* of A, provided that it is non-empty, downward-closed, and closed under finite $\oplus$-sums. Those proper ideals of A which are are not strictly contained in any proper ideal are called *maximal*. The family of all the maximal ideals of A is denoted by $\operatorname{Max} A$. The *radical* ideal of A is the intersection of all the maximal ideals in A, in symbols $\operatorname{Rad} A := \bigcap \operatorname{Max} A$. A non-zero element $x \in A$ is *infinitesimal* if it satisfies $nx \leqslant \neg x$ for all $n \in \mathbb{N}$.

Proposition 4.1.6. *For every MV-algebra A,*

$$\operatorname{Rad} A = \{x \in A \mid nx \leqslant \neg x \text{ for all } n \in \mathbb{N}\}.$$

Proof. See Proposition 2.2.52. □

Lemma 4.1.7. *If A is an MV-algebra and $x, y \in \operatorname{Rad} A$, then $x \odot y = 0$.*

Proof. See Lemma 2.2.53. $\square$

Lemma 4.1.8. *Given an arbitrary MV-homomorphism $h\colon A \to B$, if $\mathfrak{m} \in \operatorname{Max} B$ then $h^{-1}(\mathfrak{m}) \in \operatorname{Max} A$.*

Proof. See Lemma 2.2.25. $\square$

The set $\operatorname{Max} A$ of all the maximal ideals of the MV-algebra A can be equipped with the Stone-Zariski topology, as described at the end of Section 2.2. A basis of closed sets for this topology is given by the sets of the form

$$F_a := \{\mathfrak{m} \in \operatorname{Max} A \mid a \in \mathfrak{m}\},$$

for $a \in A$. The topological space $\operatorname{Max} A$ is called the *maximal spectrum* of A.

Proposition 4.1.9. *If A is an MV-algebra, then $\operatorname{Max} A$ is a compact Hausdorff space.*

Proof. This is Lemma 2.2.71. $\square$

Recall that an MV-algebra A is *semisimple* if $\operatorname{Rad} A = \{0\}$, i.e. A has no infinitesimal elements. Semisimple MV-algebras can be characterised in the following way.

Proposition 4.1.10. *Let A be an MV-algebra. Then A is semisimple if, and only if, it is isomorphic to a separating subalgebra of $\mathrm{C}(X, [0,1])$ for some compact Hausdorff space X. In this case, the space X is homeomorphic to the maximal spectrum $\operatorname{Max} A$ of the MV-algebra A.*

Proof. See Propositions 2.2.69 and 2.2.73. $\square$

Finally, we recall Mundici's equivalence:

Theorem 4.1.11. *The functor $\Gamma\colon \ell\mathsf{Grp_u} \to \mathsf{MV}$ is an equivalence between the category of unital ℓ-groups and the category of MV-algebras.*

Proof. See Theorem 2.3.29. $\square$

4.2 Definition and basic results

The language of δ-algebras is obtained from the language of MV-algebras by adding an infinitary function symbol. Specifically, let $\mathcal{L}_\Delta := \{\delta, \oplus, \neg, 0\}$ be a language formed by a function symbol δ of arity $\aleph_0$, a binary function symbol $\oplus$, a unary function symbol $\neg$ and a constant 0.

Notation 4.2.1. The infinitary function symbol δ takes as argument a countable sequence of terms. We write $\vec{x}, \vec{y}, \vec{f}$ and $\vec{0}$ as a shorthand for $x_1, x_2, x_3, \ldots$, $y_1, y_2, y_3, \ldots$, $f_1, f_2, f_3, \ldots$ and $0, 0, 0, \ldots$, respectively.

For the sake of simplicity, define a unary operation $f_{\frac{1}{2}}$ by setting

$$f_{\frac{1}{2}}(x) := \delta(x, \vec{0}).$$

Definition 4.2.2. A *δ-algebra* is an algebra $(A, \delta, \oplus, \neg, 0)$ such that $(A, \oplus, \neg, 0)$ is an MV-algebra, and the following identities are satisfied.

A1 $d\left(\delta(\vec{x}), \delta(x_1, \vec{0})\right) = \delta(0, x_2, x_3, \ldots)$.

A2 $f_{\frac{1}{2}}(\delta(\vec{x})) = \delta(f_{\frac{1}{2}}(x_1), f_{\frac{1}{2}}(x_2), f_{\frac{1}{2}}(x_3), \ldots)$.

A3 $\delta\,(x, x, x, \ldots) = x$.

A4 $\delta(0, \vec{x}) = f_{\frac{1}{2}}(\delta(\vec{x}))$.

A5 $\delta(x_1 \oplus t_1, x_2 \oplus t_2, x_3 \oplus t_3, \ldots) \geqslant \delta(x_1, x_2, x_3, \ldots)$.

A6 $f_{\frac{1}{2}}(x \ominus y) = f_{\frac{1}{2}}(x) \ominus f_{\frac{1}{2}}(y)$.

Remark 4.2.3. Since every MV-algebra is equipped with a natural lattice order (see Lemma 4.1.1), Axiom **A5** can be clearly written in an equational form.

Starting from the operation $f_{\frac{1}{2}}$, we can define a unary operation $f_{\frac{1}{2^n}}$ for each positive integer $n \in \mathbb{N}$.

$$f_{\frac{1}{2^n}}(x) := \underbrace{f_{\frac{1}{2}}(\cdots(f_{\frac{1}{2}}}_{n\,\text{times}}(x))\cdots).$$

With respect to these derived operations, an identity analogous to Axiom **A2** holds.

Lemma 4.2.4. *Let A be δ-algebra, and let $\{x_i\}_{i\in\mathbb{N}} \subseteq A$. Then, for all $n \in \mathbb{N}$,*

$$f_{\frac{1}{2^n}}(\delta(\vec{x})) = \delta(f_{\frac{1}{2^n}}(x_1), f_{\frac{1}{2^n}}(x_2), f_{\frac{1}{2^n}}(x_3), \ldots).$$

Proof. By induction on $n \in \mathbb{N}$. If $n = 1$, the statement is precisely Axiom **A2**, hence we suppose $n > 1$.

$$\begin{aligned}
f_{\frac{1}{2^n}}(\delta(\vec{x})) &= f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(\delta(\vec{x}))) \\
&= f_{\frac{1}{2}}(\delta(f_{\frac{1}{2^{n-1}}}(x_1), f_{\frac{1}{2^{n-1}}}(x_2), f_{\frac{1}{2^{n-1}}}(x_3), \ldots)) && \text{(inductive hypothesis)} \\
&= \delta(f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(x_1)), f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(x_2)), f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(x_3)), \ldots) && (\mathbf{A2}) \\
&= \delta(f_{\frac{1}{2^n}}(x_1), f_{\frac{1}{2^n}}(x_2), f_{\frac{1}{2^n}}(x_3), \ldots).
\end{aligned}$$

□

Lemma 4.2.5. *If A is a δ-algebra and $x \in A$, then*

$$f_{\frac{1}{2^n}}(x) = \delta(\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, x, \vec{0}).$$

Proof. We argue by induction on $n \in \mathbb{N}$. If $n = 1$, the statement coincides with the definition of $f_{\frac{1}{2}}$. If $n > 1$, then

$$\begin{aligned} f_{\frac{1}{2^n}}(x) &= f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(x)) \\ &= f_{\frac{1}{2}}(\delta(\underbrace{0, \ldots, 0}_{n-2 \text{ times}}, x, \vec{0})) && \text{(inductive hypothesis)} \\ &= \delta(\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, x, \vec{0}). && (\mathbf{A4}) \end{aligned}$$

□

Lemma 4.2.6. *In a δ-algebra A, for every $\{x_i\}_{i \in \mathbb{N}} \subseteq A$,*

$$f_{\frac{1}{2^n}}(\delta(\vec{x})) = \delta(\underbrace{0, \ldots, 0}_{n \text{ times}}, \vec{x}).$$

Proof. For $n = 1$, the statement holds by Axiom **A4**. Hence, we suppose $n > 1$. Then we have

$$\begin{aligned} \delta(\underbrace{0, \ldots, 0}_{n \text{ times}}, \vec{x}) &= f_{\frac{1}{2}}(\delta(\underbrace{0, \ldots, 0}_{n-1 \text{ times}}, \vec{x})) && (\mathbf{A4}) \\ &= f_{\frac{1}{2}}\left(f_{\frac{1}{2^{n-1}}}(\delta(\vec{x}))\right) && \text{(inductive hypothesis)} \\ &= f_{\frac{1}{2^n}}(\delta(\vec{x})). \end{aligned}$$

□

Lemma 4.2.7. *Let A be a δ-algebra. For all $\{x_i\}_{i \in \mathbb{N}} \subseteq A$, and for all $n \in \mathbb{N}$,*

$$\delta(\vec{x}) \geqslant \delta(x_1, x_2, \ldots, x_n, \vec{0}).$$

Moreover, $\delta(\vec{x}) = \delta(x_1, \vec{0}) \oplus \delta(0, x_2, x_3, \ldots)$.

Proof. The first part of the statement follows at once by Axiom **A5**, upon defining $t_1 := 0, \ldots, t_n := 0$, and $t_i := x_i$ for all $i > n$. Indeed, for any $\{x_i\}_{i \in \mathbb{N}} \subseteq A$, we have

$$\begin{aligned} \delta(x_1, x_2, x_3, \ldots) &= \delta(x_1 \oplus 0, x_2 \oplus 0, \ldots, x_n \oplus 0, 0 \oplus x_{n+1}, 0 \oplus x_{n+2}, \ldots) \\ &\geqslant \delta(x_1, x_2, \ldots, x_n, \vec{0}). && (\mathbf{A5}) \end{aligned}$$

In order to prove the second part, observe that $\delta(\vec{x}) \geqslant \delta(x_1, \vec{0})$ by the argument above. Consequently,

$$\delta(\vec{x}) = \delta(x_1, \vec{0}) \oplus d(\delta(\vec{x}), \delta(x_1, \vec{0})) \qquad \text{(Lemma 4.1.3.(2))}$$

$$= \delta(x_1, \vec{0}) \oplus \delta(0, x_2, x_3, \ldots). \qquad \textbf{(A1)}$$

□

Proposition 4.2.8. *The following identities hold in an arbitrary δ-algebra.*

1. $\bigoplus_{i=1}^{n} \delta(0, \ldots, x_i, \vec{0}) = \delta(x_1, x_2, \ldots, x_n, \vec{0})$.
2. $\delta(\vec{x}) = \delta(x_1, x_2, \ldots, x_n, \vec{0}) \oplus \delta(0, 0, \ldots, x_{n+1}, x_{n+2}, \ldots)$.
3. $\delta(x, y, y, y, \ldots) = f_{\frac{1}{2}}(x) \oplus f_{\frac{1}{2}}(y)$.
4. $\delta(x_1, x_2, \ldots, x_n, \vec{0}) = \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i)$.
5. $f_{\frac{1}{2}}(x) \oplus f_{\frac{1}{2}}(x) = x$.
6. $d\Big(\delta(\vec{x}), \delta(x_1, \ldots, x_n, \vec{0})\Big) = \delta(0, \ldots, x_{n+1}, x_{n+2}, \ldots)$.
7. $\neg f_{\frac{1}{2}}(1) = f_{\frac{1}{2}}(1)$.
8. $f_{\frac{1}{2^n}}(x \ominus y) = f_{\frac{1}{2^n}}(x) \ominus f_{\frac{1}{2^n}}(y)$.

Proof.

1. The equation holds trivially for $n = 1$, and it is true if $n = 2$, by Lemma 4.2.7. Assuming that $n > 2$ we show that, for all $k = 1, \ldots, n-2$,

$$\bigoplus_{i=n-k}^{n} \delta(0, \ldots, x_i, \vec{0}) = \delta(0, \ldots, x_{n-k}, x_{n-k+1}, \ldots, x_n, \vec{0}). \qquad (4.1)$$

If $k = 1$, then

$$\begin{aligned}
\bigoplus_{i=n-1}^{n} \delta(0, \ldots, x_i, \vec{0}) &= \delta(0, \ldots, x_{n-1}, \vec{0}) \oplus \delta(0, \ldots, x_n, \vec{0}) \\
&= f_{\frac{1}{2^{n-2}}}(\delta(x_{n-1}, \vec{0})) \oplus f_{\frac{1}{2^{n-2}}}(\delta(0, x_n, \vec{0})) && \text{(Lemma 4.2.6)} \\
&= \delta(f_{\frac{1}{2^{n-2}}}(x_{n-1}), \vec{0}) \oplus \delta(0, f_{\frac{1}{2^{n-2}}}(x_n), \vec{0}) && \text{(Lemma 4.2.4)} \\
&= \delta(f_{\frac{1}{2^{n-2}}}(x_{n-1}), f_{\frac{1}{2^{n-2}}}(x_n), \vec{0}) && \text{(Lemma 4.2.7)} \\
&= f_{\frac{1}{2^{n-2}}}(\delta(x_{n-1}, x_n, \vec{0})) && \text{(Lemma 4.2.4)} \\
&= \delta(0, \ldots, x_{n-1}, x_n, \vec{0}). && \text{(Lemma 4.2.6)}
\end{aligned}$$

Now suppose (4.1) is true for $1 \leqslant k < n-1$. We prove that it is true for $k+1$.

$$\begin{aligned}
\bigoplus_{i=n-(k+1)}^{n} \delta(0, \ldots, x_i, \vec{0}) &= \delta(0, \ldots, x_{n-k-1}, \vec{0}) \oplus \bigoplus_{i=n-k}^{n} \delta(0, \ldots, x_i, \vec{0}) \\
&= \delta(0, \ldots, x_{n-k-1}, \vec{0}) \oplus \delta(0, \ldots, x_{n-k}, \ldots, x_n, \vec{0}) \\
& \text{(inductive hypothesis)}
\end{aligned}$$

$$= f_{\frac{1}{2^{n-k-2}}}(\delta(x_{n-k-1}, \vec{0})) \oplus f_{\frac{1}{2^{n-k-2}}}(\delta(0, x_{n-k}, \ldots, x_n, \vec{0}))$$
(Lemma 4.2.6)
$$= \delta(f_{\frac{1}{2^{n-k-2}}}(x_{n-k-1}), \vec{0}) \oplus \delta(0, f_{\frac{1}{2^{n-k-2}}}(x_{n-k}), \ldots, f_{\frac{1}{2^{n-k-2}}}(x_n), \vec{0})$$
(Lemma 4.2.4)
$$= \delta(f_{\frac{1}{2^{n-k-2}}}(x_{n-k-1}), f_{\frac{1}{2^{n-k-2}}}(x_{n-k}), \ldots, f_{\frac{1}{2^{n-k-2}}}(x_n), \vec{0})$$
(Lemma 4.2.7)
$$= f_{\frac{1}{2^{n-k-2}}}(\delta(x_{n-k-1}, x_{n-k}, \ldots, x_n, \vec{0})) \quad \text{(Lemma 4.2.4)}$$
$$= \delta(0, \ldots, x_{n-k-1}, x_{n-k}, \ldots, x_n, \vec{0}). \quad \text{(Lemma 4.2.6)}$$

In particular, for $k = n - 2$, we have

$$\bigoplus_{i=n-(n-2)}^{n} \delta(0, \ldots, x_i, \vec{0}) = \delta(0, x_2, x_3, \ldots, x_n, \vec{0}).$$

Therefore

$$\begin{aligned}\bigoplus_{i=1}^{n} \delta(0, \ldots, x_i, \vec{0}) &= \delta(x_1, \vec{0}) \oplus \bigoplus_{i=2}^{n} \delta(0, \ldots, x_i, \vec{0}) \\ &= \delta(x_1, \vec{0}) \oplus \delta(0, x_2, x_3, \ldots, x_n, \vec{0}) \\ &= \delta(x_1, x_2, \ldots, x_n, \vec{0}). \qquad \text{(Lemma 4.2.7)}\end{aligned}$$

2. We proceed by induction on $n \in \mathbb{N}$. For $n = 1$ the identity holds by Lemma 4.2.7, hence let $n > 1$.

$$\delta(x_1, \ldots, x_n, \vec{0}) \oplus \delta(0, \ldots, x_{n+1}, x_{n+2}, \ldots) =$$
$$= \bigoplus_{i=1}^{n} \delta(0, \ldots, x_i, \vec{0}) \oplus \delta(0, \ldots, x_{n+1}, x_{n+2}, \ldots) \quad \text{(Proposition 4.2.8.(1))}$$
$$= \bigoplus_{i=1}^{n-1} \delta(0, \ldots, x_i, \vec{0}) \oplus \delta(0, \ldots, x_n, \vec{0}) \oplus \delta(0, \ldots, x_{n+1}, x_{n+2}, \ldots)$$
$$= \delta(x_1, \ldots, x_{n-1}, \vec{0}) \oplus f_{\frac{1}{2^{n-1}}}(\delta(x_n, \vec{0})) \oplus f_{\frac{1}{2^{n-1}}}(\delta(0, x_{n+1}, x_{n+2}, \ldots))$$
(Proposition 4.2.8.(1), Lemma 4.2.6)
$$= \delta(x_1, \ldots, x_{n-1}, \vec{0}) \oplus \delta(f_{\frac{1}{2^{n-1}}}(x_n), \vec{0}) \oplus \delta(0, f_{\frac{1}{2^{n-1}}}(x_{n+1}), f_{\frac{1}{2^{n-1}}}(x_{n+2}), \ldots)$$
(Lemma 4.2.4)
$$= \delta(x_1, \ldots, x_{n-1}, \vec{0}) \oplus \delta(f_{\frac{1}{2^{n-1}}}(x_n), f_{\frac{1}{2^{n-1}}}(x_{n+1}), f_{\frac{1}{2^{n-1}}}(x_{n+2}), \ldots) \quad \text{(Lemma 4.2.7)}$$
$$= \delta(x_1, \ldots, x_{n-1}, \vec{0}) \oplus f_{\frac{1}{2^{n-1}}}(\delta(x_n, x_{n+1}, x_{n+2}, \ldots)) \quad \text{(Lemma 4.2.4)}$$
$$= \delta(x_1, \ldots, x_{n-1}, \vec{0}) \oplus \delta(0, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots) \quad \text{(Lemma 4.2.6)}$$
$$= \delta(\vec{x}). \quad \text{(inductive hypothesis)}$$

3. This is easily proved in the following way.

$$\delta(x, y, y, y, \ldots) = \delta(x, \vec{0}) \oplus \delta(0, y, y, y, \ldots) \quad \text{(Proposition 4.2.8.(2))}$$

$$= f_{\frac{1}{2}}(x) \oplus f_{\frac{1}{2}}(\delta(y, y, y, \ldots)) \qquad (\mathbf{A4})$$
$$= f_{\frac{1}{2}}(x) \oplus f_{\frac{1}{2}}(y). \qquad (\mathbf{A3})$$

4. The equation holds, since

$$\delta(x_1, x_2, \ldots, x_n, \vec{0}) = \bigoplus_{i=1}^{n} \delta(0, \ldots, x_i, \vec{0}) \qquad \text{(Proposition 4.2.8.(1))}$$
$$= \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i). \qquad \text{(Lemma 4.2.5)}$$

5. By an easy computation,

$$f_{\frac{1}{2}}(x) \oplus f_{\frac{1}{2}}(x) = \delta(x, x, x, \ldots) \qquad \text{(Proposition 4.2.8.(3))}$$
$$= x. \qquad (\mathbf{A3})$$

6. Arguing by induction on $n \in \mathbb{N}$, the statement is true for $n = 1$ by Axiom **A1**. Assuming $n > 1$,

$$d\left(\delta(\vec{x}), \delta(x_1, \ldots, x_n, \vec{0})\right) = \delta(\vec{x}) \ominus \delta(x_1, \ldots, x_n, \vec{0})$$
$$\text{(Lemma 4.2.7, Lemma 4.1.1)}$$
$$= \delta(\vec{x}) \ominus \left(\delta(x_1, \ldots, x_{n-1}, \vec{0}) \oplus \delta(0, \ldots, x_n, \vec{0})\right) \qquad \text{(Proposition 4.2.8.(2))}$$
$$= \left(\delta(\vec{x}) \ominus \delta(x_1, \ldots, x_{n-1}, \vec{0})\right) \ominus \delta(0, \ldots, x_n, \vec{0}) \qquad \text{(Lemma 4.1.4.(3))}$$
$$= \delta(0, \ldots, x_n, x_{n+1}, \ldots) \ominus \delta(0, \ldots, x_n, \vec{0}) \qquad \text{(inductive hypothesis)}$$
$$= f_{\frac{1}{2^{n-1}}}(\delta(x_n, x_{n+1}, \ldots)) \ominus f_{\frac{1}{2^{n-1}}}(\delta(x_n, \vec{0})) \qquad \text{(Lemma 4.2.6)}$$
$$= \delta(f_{\frac{1}{2^{n-1}}}(x_n), f_{\frac{1}{2^{n-1}}}(x_{n+1}), \ldots) \ominus \delta(f_{\frac{1}{2^{n-1}}}(x_n), \vec{0}) \qquad \text{(Lemma 4.2.4)}$$
$$= d\left(\delta(f_{\frac{1}{2^{n-1}}}(x_n), f_{\frac{1}{2^{n-1}}}(x_{n+1}), \ldots), \delta(f_{\frac{1}{2^{n-1}}}(x_n), \vec{0})\right) \qquad \text{(Lemma 4.2.7, Lemma 4.1.1)}$$
$$= \delta(0, f_{\frac{1}{2^{n-1}}}(x_{n+1}), f_{\frac{1}{2^{n-1}}}(x_{n+2}), \ldots) \qquad (\mathbf{A1})$$
$$= f_{\frac{1}{2^{n-1}}}(\delta(0, x_{n+1}, x_{n+2}, \ldots)) \qquad \text{(Lemma 4.2.4)}$$
$$= \delta(0, \ldots, x_{n+1}, x_{n+2}, \ldots). \qquad \text{(Lemma 4.2.6)}$$

7. Recall that $1 := \neg 0$. In any MV-algebra we have

$$1 \ominus x = \neg(\neg 1 \oplus x) = \neg x. \qquad (4.2)$$

It follows

$$\neg f_{\frac{1}{2}}(1) = 1 \ominus f_{\frac{1}{2}}(1) \qquad (4.2)$$
$$= \delta(1, 1, 1, \ldots) \ominus \delta(1, \vec{0}) \qquad (\mathbf{A3})$$
$$= d\left(\delta(1, 1, 1, \ldots), \delta(1, \vec{0})\right) \qquad \text{(Lemma 4.2.7, Lemma 4.1.1)}$$

$$
\begin{aligned}
&= \delta(0,1,1,1,\ldots) && (\mathbf{A1})\\
&= f_{\frac{1}{2}}(\delta(1,1,1,\ldots)) && (\mathbf{A4})\\
&= f_{\frac{1}{2}}(1). && (\mathbf{A3})
\end{aligned}
$$

8. For $n = 1$ this is Axiom **A6**. If $n > 1$,

$$
\begin{aligned}
f_{\frac{1}{2^n}}(x \ominus y) &= f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(x \ominus y)) &&\\
&= f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(x) \ominus f_{\frac{1}{2^{n-1}}}(y)) && \text{(inductive hypothesis)}\\
&= f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(x)) \ominus f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(y)) && (\mathbf{A6})\\
&= f_{\frac{1}{2^n}}(x) \ominus f_{\frac{1}{2^n}}(y). &&
\end{aligned}
$$

□

Lemma 4.2.9. *Let A be a δ-algebra, and let $x, y \in A$. Then $f_{\frac{1}{2}}(x) \leqslant x$, and*

$$x \leqslant y \ \textit{ if, and only if, } \ f_{\frac{1}{2}}(x) \leqslant f_{\frac{1}{2}}(y).$$

Finally, $x = 0$ if, and only if, $f_{\frac{1}{2}}(x) = 0$.

Proof. Notice that

$$f_{\frac{1}{2}}(x) = \delta(x, \vec{0}) \leqslant \delta(x, x, x, \ldots) = x. \qquad (\mathbf{A3}, \mathbf{A5})$$

Next, we prove that $f_{\frac{1}{2}}(x) = 0$ if, and only if, $x = 0$. One of the two implications follows at once by **A3**. To prove the other one observe that, if $f_{\frac{1}{2}}(x) = 0$, then

$$x = f_{\frac{1}{2}}(x) \oplus f_{\frac{1}{2}}(x) = 0 \oplus 0 = 0$$

by Proposition 4.2.8.(5). Now suppose $x \leqslant y$. Axiom **A5** entails

$$f_{\frac{1}{2}}(x) = \delta(x, \vec{0}) \leqslant \delta(y, \vec{0}) = f_{\frac{1}{2}}(y).$$

Conversely, assuming $f_{\frac{1}{2}}(x) \leqslant f_{\frac{1}{2}}(y)$, by Proposition 4.2.8.(5) and Lemma 4.1.2.(3) we have

$$x = f_{\frac{1}{2}}(x) \oplus f_{\frac{1}{2}}(x) \leqslant f_{\frac{1}{2}}(y) \oplus f_{\frac{1}{2}}(y) = y.$$

□

Remark 4.2.10. It is easy to see that, if $x \leqslant y$, then $f_{\frac{1}{2^n}}(x) \leqslant f_{\frac{1}{2^n}}(y)$ for all $n \in \mathbb{N}$. This was proved in Lemma 4.2.9 for $n = 1$. If $n > 1$, by using the inductive hypothesis $f_{\frac{1}{2^{n-1}}}(x) \leqslant f_{\frac{1}{2^{n-1}}}(y)$, we find

$$
\begin{aligned}
f_{\frac{1}{2^n}}(x) &= f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(x)) &&\\
&\leqslant f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(y)) && \text{(Lemma 4.2.9)}\\
&= f_{\frac{1}{2^n}}(y). &&
\end{aligned}
$$

Reasoning by induction on $n \in \mathbb{N}$, it is also clear that $f_{\frac{1}{2^n}}(x) \leqslant x$ for all $n \in \mathbb{N}$. This is true for $n = 1$, by Lemma 4.2.9. If $n > 1$, by the induction hypothesis $f_{\frac{1}{2^{n-1}}}(x) \leqslant x$, along with Lemma 4.2.9,

$$f_{\frac{1}{2^n}}(x) = f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(x)) \leqslant f_{\frac{1}{2^{n-1}}}(x) \leqslant x.$$

Lemma 4.2.11. *Let A be a δ-algebra, let $x \in A$ and let $m, n \in \mathbb{N}$ be such that $m < n$. Then*

$$\underbrace{f_{\frac{1}{2^n}}(x) \oplus \cdots \oplus f_{\frac{1}{2^n}}(x)}_{2^m \textit{ times}} = f_{\frac{1}{2^{n-m}}}(x).$$

Proof. Let us fix an arbitrary positive integer $n \in \mathbb{N}$. We shall proceed by induction on $m = 1, \ldots, n-1$. If $m = 1$, then

$$\begin{aligned} f_{\frac{1}{2^n}}(x) \oplus f_{\frac{1}{2^n}}(x) &= f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(x)) \oplus f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(x)) \\ &= f_{\frac{1}{2^{n-1}}}(x). && \text{(Proposition 4.2.8.(5))} \end{aligned}$$

Now, if $1 < m < n$, we see that

$$\begin{aligned} \underbrace{f_{\frac{1}{2^n}}(x) \oplus \cdots \oplus f_{\frac{1}{2^n}}(x)}_{2^m \text{ times}} &= (\underbrace{f_{\frac{1}{2^n}}(x) \oplus \cdots \oplus f_{\frac{1}{2^n}}(x)}_{2^{m-1} \text{ times}}) \oplus (\underbrace{f_{\frac{1}{2^n}}(x) \oplus \cdots \oplus f_{\frac{1}{2^n}}(x)}_{2^{m-1} \text{ times}}) \\ &= f_{\frac{1}{2^{n-m+1}}}(x) \oplus f_{\frac{1}{2^{n-m+1}}}(x) && \text{(inductive hypothesis)} \\ &= f_{\frac{1}{2}}(f_{\frac{1}{2^{n-m}}}(x)) \oplus f_{\frac{1}{2}}(f_{\frac{1}{2^{n-m}}}(x)) \\ &= f_{\frac{1}{2^{n-m}}}(x). && \text{(Proposition 4.2.8.(5))} \end{aligned}$$

□

Corollary 4.2.12. *If A is a δ-algebra then, for all $x \in A$ and for all $n \in \mathbb{N}$,*

$$\underbrace{f_{\frac{1}{2^n}}(x) \oplus \cdots \oplus f_{\frac{1}{2^n}}(x)}_{2^n \textit{ times}} = x.$$

Proof. The case $n = 1$ is proved in Proposition 4.2.8.(5). Hence, assuming $n > 1$, we have

$$\begin{aligned} \underbrace{f_{\frac{1}{2^n}}(x) \oplus \cdots \oplus f_{\frac{1}{2^n}}(x)}_{2^n \text{ times}} &= (\underbrace{f_{\frac{1}{2^n}}(x) \oplus \cdots \oplus f_{\frac{1}{2^n}}(x)}_{2^{n-1} \text{ times}}) \oplus (\underbrace{f_{\frac{1}{2^n}}(x) \oplus \cdots \oplus f_{\frac{1}{2^n}}(x)}_{2^{n-1} \text{ times}}) \\ &= f_{\frac{1}{2^{n-(n-1)}}}(x) \oplus f_{\frac{1}{2^{n-(n-1)}}}(x) && \text{(Lemma 4.2.11)} \\ &= f_{\frac{1}{2}}(x) \oplus f_{\frac{1}{2}}(x) \\ &= x. && \text{(Proposition 4.2.8.(5))} \end{aligned}$$

□

4.3 Every δ-algebra is semisimple

Given a δ-algebra $(A, \delta, \oplus, \neg, 0)$, we can consider the maximal spectrum of its MV-algebraic reduct $(A, \oplus, \neg, 0)$. We continue to denote it by $\operatorname{Max} A$. By the *radical* of a δ-algebra we understand the radical $\operatorname{Rad} A$ of its underlying MV-algebra. A δ-algebra is *semisimple* if its radical is trivial, i.e. $\operatorname{Rad} A = \{0\}$. In this section we prove that every δ-algebra is semisimple, in other words every δ-algebra has a semisimple MV-algebraic reduct.

Lemma 4.3.1. *If A is a δ-algebra, and $x \in A$ satisfies $x \leqslant f_{\frac{1}{2^n}}(1)$ for all $n \in \mathbb{N}$, then $x \in \operatorname{Rad} A$.*

Proof. Notice that

$$x \leqslant f_{\frac{1}{2}}(1) \ \Leftrightarrow\ x \ominus f_{\frac{1}{2}}(1) = 0 \ \Leftrightarrow\ x \odot \neg f_{\frac{1}{2}}(1) = 0,$$

so that, by Proposition 4.2.8.(7),

$$x \odot f_{\frac{1}{2}}(1) = 0. \tag{4.3}$$

Let us fix an arbitrary $n \in \mathbb{N}$. Then $x \leqslant f_{\frac{1}{2^{n+1}}}(1)$ entails

$$\begin{aligned} nx &\leqslant \underbrace{f_{\frac{1}{2^{n+1}}}(1) \oplus \cdots \oplus f_{\frac{1}{2^{n+1}}}(1)}_{n \text{ times}} && \text{(Lemma 4.1.2.(3))} \\ &\leqslant \underbrace{f_{\frac{1}{2^{n+1}}}(1) \oplus \cdots \oplus f_{\frac{1}{2^{n+1}}}(1)}_{2^n \text{ times}} \\ &= f_{\frac{1}{2}}(1). && \text{(Lemma 4.2.11)} \end{aligned}$$

By (4.3) and Lemma 4.1.2.(2) we have

$$x \odot nx \leqslant x \odot f_{\frac{1}{2}}(u) = 0,$$

therefore

$$x \odot nx = 0 \ \Leftrightarrow\ nx \ominus \neg x = 0 \ \Leftrightarrow\ nx \leqslant \neg x.$$

Then Proposition 4.1.6 implies $x \in \operatorname{Rad} A$. □

Lemma 4.3.2. *If A is a δ-algebra and $x \in \operatorname{Rad} A$ then, for all $n \in \mathbb{N}$,*

$$f_{\frac{1}{2^n}}(2^n x) = x.$$

Proof. The proof goes by induction on $n \in \mathbb{N}$. In order to prove the case $n = 1$, note that

$$\begin{aligned} f_{\frac{1}{2}}(x \oplus x) &= f_{\frac{1}{2}}(\neg(\neg x \ominus x)) \\ &= f_{\frac{1}{2}}(1 \ominus (\neg x \ominus x)) \end{aligned}$$

$$
\begin{aligned}
&= f_{\frac{1}{2}}(1) \ominus f_{\frac{1}{2}}(\neg x \ominus x) && (\mathbf{A6})\\
&= f_{\frac{1}{2}}(1) \ominus (f_{\frac{1}{2}}(\neg x) \ominus f_{\frac{1}{2}}(x)) && (\mathbf{A6})\\
&= f_{\frac{1}{2}}(1) \ominus (f_{\frac{1}{2}}(1 \ominus x) \ominus f_{\frac{1}{2}}(x))\\
&= f_{\frac{1}{2}}(1) \ominus ((f_{\frac{1}{2}}(1) \ominus f_{\frac{1}{2}}(x)) \ominus f_{\frac{1}{2}}(x)) && (\mathbf{A6})\\
&= f_{\frac{1}{2}}(1) \ominus (\neg(\neg f_{\frac{1}{2}}(1) \oplus f_{\frac{1}{2}}(x)) \ominus f_{\frac{1}{2}}(x))\\
&= f_{\frac{1}{2}}(1) \ominus \neg((\neg f_{\frac{1}{2}}(1) \oplus f_{\frac{1}{2}}(x)) \oplus f_{\frac{1}{2}}(x))\\
&= f_{\frac{1}{2}}(1) \odot (\neg f_{\frac{1}{2}}(1) \oplus f_{\frac{1}{2}}(x) \oplus f_{\frac{1}{2}}(x))\\
&= (\neg f_{\frac{1}{2}}(1) \oplus f_{\frac{1}{2}}(x) \oplus f_{\frac{1}{2}}(x)) \ominus \neg f_{\frac{1}{2}}(1)\\
&= (f_{\frac{1}{2}}(1) \oplus f_{\frac{1}{2}}(x) \oplus f_{\frac{1}{2}}(x)) \ominus f_{\frac{1}{2}}(1) && (\text{Proposition 4.2.8.(7)})\\
&= (x \oplus f_{\frac{1}{2}}(1)) \ominus f_{\frac{1}{2}}(1). && (\text{Proposition 4.2.8.(5)})
\end{aligned}
$$

We have just showed that

$$f_{\frac{1}{2}}(x \oplus x) = (x \oplus f_{\frac{1}{2}}(1)) \ominus f_{\frac{1}{2}}(1). \tag{4.4}$$

However, by Lemma 4.1.4.(2), we know that

$$[(x \oplus f_{\frac{1}{2}}(1)) \ominus f_{\frac{1}{2}}(1)] \oplus [(x \oplus f_{\frac{1}{2}}(1) \oplus \neg f_{\frac{1}{2}}(1)) \odot f_{\frac{1}{2}}(1)] = x \oplus f_{\frac{1}{2}}(1)$$

if, and only if,

$$[(x \oplus f_{\frac{1}{2}}(1)) \ominus f_{\frac{1}{2}}(1)] \oplus [(x \oplus 1) \odot f_{\frac{1}{2}}(1)] = x \oplus f_{\frac{1}{2}}(1).$$

In turn, this is equivalent to

$$[(x \oplus f_{\frac{1}{2}}(1)) \ominus f_{\frac{1}{2}}(1)] \oplus f_{\frac{1}{2}}(1) = x \oplus f_{\frac{1}{2}}(1). \tag{4.5}$$

We shall see that the hypotheses of Lemma 4.1.5 are satisfied, that is

$$[(x \oplus f_{\frac{1}{2}}(1)) \ominus f_{\frac{1}{2}}(1)] \odot f_{\frac{1}{2}}(1) = 0, \tag{4.6}$$

and

$$x \odot f_{\frac{1}{2}}(1) = 0. \tag{4.7}$$

A straightforward calculation proves that (4.6) holds:

$$
\begin{aligned}
[(x \oplus f_{\frac{1}{2}}(1)) \ominus f_{\frac{1}{2}}(1)] \odot f_{\frac{1}{2}}(1) &= \neg[\neg(x \oplus f_{\frac{1}{2}}(1)) \oplus f_{\frac{1}{2}}(1)] \odot f_{\frac{1}{2}}(1)\\
&= \neg[\neg(x \oplus f_{\frac{1}{2}}(1)) \oplus f_{\frac{1}{2}}(1) \oplus \neg f_{\frac{1}{2}}(1)]\\
&= \neg[\neg(x \oplus f_{\frac{1}{2}}(1)) \oplus 1]\\
&= \neg 1\\
&= 0.
\end{aligned}
$$

Concerning equation (4.7) we observe that, by Lemma 4.1.4.(1),

$$x \oplus f_{\frac{1}{2}}(1) \oplus (x \odot f_{\frac{1}{2}}(1)) = x \oplus f_{\frac{1}{2}}(1).$$

Hence, by Lemma 4.1.5, it suffices to show that

$$(x \oplus f_{\frac{1}{2}}(1)) \odot (x \odot f_{\frac{1}{2}}(1)) = 0. \tag{4.8}$$

We have

$$(x \oplus f_{\frac{1}{2}}(1)) \odot (x \odot f_{\frac{1}{2}}(1)) = 0$$

if, and only if,

$$\neg[\neg(x \oplus f_{\frac{1}{2}}(1)) \oplus (\neg x \oplus \neg f_{\frac{1}{2}}(1))] = 0$$

if, and only if,

$$\neg(x \oplus f_{\frac{1}{2}}(1)) \oplus (\neg x \oplus \neg f_{\frac{1}{2}}(1)) = 1.$$

By Lemma 4.1.1 and Proposition 4.2.8.(7), this is equivalent to

$$x \oplus f_{\frac{1}{2}}(1) \leqslant \neg x \oplus f_{\frac{1}{2}}(1).$$

However, the latter inequality follows by Lemma 4.1.2.(2) upon observing that, if $x \in \operatorname{Rad} A$, then $x \leqslant \neg x$. Hence (4.8) is proved, and applying Lemma 4.1.5 to (4.5) we see that

$$(x \oplus f_{\frac{1}{2}}(1)) \ominus f_{\frac{1}{2}}(1) = x.$$

Finally, by (4.4) we have

$$f_{\frac{1}{2}}(x \oplus x) = x. \tag{4.9}$$

Now, for every positive integer n, we conclude that

$$\begin{aligned} f_{\frac{1}{2^n}}(2^n x) &= f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(2^{n-1}(2x))) \\ &= f_{\frac{1}{2}}(2x) && \text{(inductive hypothesis)} \\ &= x. && (4.9) \end{aligned}$$

□

Remark 4.3.3. It is easy to show that the converse of Lemma 4.3.1 holds. Indeed, suppose that A is a δ-algebra, and let $x \in \operatorname{Rad} A$. Then, for all $n \in \mathbb{N}$, we have

$$2^n x \leqslant \neg x \leqslant 1,$$

so that, by Remark 4.2.10 and Lemma 4.3.2,

$$x = f_{\frac{1}{2^n}}(2^n x) \leqslant f_{\frac{1}{2^n}}(1).$$

In other terms, the radical of a δ-algebra can be characterised as follows.

$$\operatorname{Rad} A = \{x \in A \mid x \leqslant f_{\frac{1}{2^n}}(1) \ \text{ for all } n \in \mathbb{N}\}.$$

The next result is fundamental in proving that the radical ideal of a δ-algebra is trivial, i.e. that the only element $x \in A$ satisfying $x \leqslant f_{\frac{1}{2^n}}(1)$, for all $n \in \mathbb{N}$, is $x = 0$.

Proposition 4.3.4. *In any δ-algebra A,*

$$\textit{if } \{x_i\}_{i\in\mathbb{N}} \subseteq \operatorname{Rad} A, \quad \textit{then } \delta(\vec{x}) \in \operatorname{Rad} A.$$

Proof. By Lemma 4.3.1, it is sufficient to prove that, for all $n \in \mathbb{N}$, $\delta(\vec{x}) \leqslant f_{\frac{1}{2^n}}(1)$. Fix an arbitrary positive integer $n \in \mathbb{N}$, and set $m := n + 1$. Then Lemma 4.2.9 entails

$$f_{\frac{1}{2^m}}(1) = f_{\frac{1}{2}}(f_{\frac{1}{2^n}}(1)) \leqslant f_{\frac{1}{2^n}}(1),$$

hence

$$f_{\frac{1}{2^m}}(1) \ominus f_{\frac{1}{2^n}}(1) = 0$$

by Lemma 4.1.1. Now,

$$\begin{aligned} d(f_{\frac{1}{2^m}}(1), f_{\frac{1}{2^n}}(1)) &= (f_{\frac{1}{2^m}}(1) \ominus f_{\frac{1}{2^n}}(1)) \oplus (f_{\frac{1}{2^n}}(1) \ominus f_{\frac{1}{2^m}}(1)) \\ &= f_{\frac{1}{2^n}}(1) \ominus f_{\frac{1}{2^m}}(1) \\ &= f_{\frac{1}{2^n}}(1 \ominus f_{\frac{1}{2}}(1)) && \text{(Proposition 4.2.8.(8))} \\ &= f_{\frac{1}{2^n}}(\neg f_{\frac{1}{2}}(1)) \\ &= f_{\frac{1}{2^n}}(f_{\frac{1}{2}}(1)) && \text{(Proposition 4.2.8.(7))} \\ &= f_{\frac{1}{2^m}}(1). \end{aligned}$$

Therefore, by Lemma 4.1.3.(2),

$$f_{\frac{1}{2^n}}(1) = f_{\frac{1}{2^m}}(1) \oplus f_{\frac{1}{2^m}}(1). \tag{4.10}$$

We claim that

$$\delta(x_1, x_2, \ldots, x_m, \vec{0}) \leqslant f_{\frac{1}{2^m}}(1). \tag{4.11}$$

Indeed, we have

$$\begin{aligned} \delta(x_1, x_2, \ldots, x_m, \vec{0}) &= \delta(f_{\frac{1}{2^m}}(2^m x_1), f_{\frac{1}{2^m}}(2^m x_2), \ldots, f_{\frac{1}{2^m}}(2^m x_m), \vec{0}) && \text{(Lemma 4.3.2)} \\ &= f_{\frac{1}{2^m}}(\delta(2^m x_1, 2^m x_2, \ldots, 2^m x_m, \vec{0})) && \text{(Lemma 4.2.4)} \\ &\leqslant f_{\frac{1}{2^m}}(1). && \text{(Remark 4.2.10)} \end{aligned}$$

The proposition is then proved by the following computation.

$$\begin{aligned} \delta(\vec{x}) &= \delta(x_1, x_2, \ldots, x_m, \vec{0}) \oplus \delta(0, \ldots, 0, x_{m+1}, x_{m+2}, \ldots) && \text{(Proposition 4.2.8.(2))} \\ &= \delta(x_1, x_2, \ldots, x_m, \vec{0}) \oplus f_{\frac{1}{2^m}}(\delta(x_{m+1}, x_{m+2}, \ldots)) && \text{(Lemma 4.2.6)} \\ &\leqslant f_{\frac{1}{2^m}}(1) \oplus f_{\frac{1}{2^m}}(\delta(1, 1, 1, \ldots)) && \text{((4.11), } \mathbf{A5}\text{, Lemma 4.1.2.(2))} \\ &= f_{\frac{1}{2^m}}(1) \oplus f_{\frac{1}{2^m}}(1) && (\mathbf{A3}) \\ &= f_{\frac{1}{2^n}}(1). && \text{(4.10)} \end{aligned}$$

□

We can finally prove the main result of this section.

Theorem 4.3.5. *Every δ-algebra is semisimple.*

Proof. Pick an arbitrary element $x \in \operatorname{Rad} A$. Recall that $\operatorname{Rad} A$ is an ideal of the underlying MV-algebra of A. In particular, it is closed with respect to $\oplus$-sums, so that $nx \in \operatorname{Rad} A$ for all $n \in \mathbb{N}$. Consider the countable sequence

$$\vec{t} := \{2^i x\}_{i \in \mathbb{N}} \subseteq \operatorname{Rad} A.$$

Then $\delta(\vec{t}) \in \operatorname{Rad} A$ by Proposition 4.3.4, and

$$\begin{aligned}
\delta(\vec{t}) &= \delta(2x, \vec{0}) \oplus \delta(0, 2^2x, 2^3x, 2^4x, \ldots) && \text{(Proposition 4.2.8.(2))}\\
&= f_{\frac{1}{2}}(2x) \oplus f_{\frac{1}{2}}(\delta(2^2x, 2^3x, 2^4x, \ldots)) && (\mathbf{A4})\\
&= x \oplus f_{\frac{1}{2}}(\delta(2(2x), 2(2^2x), 2(2^3x), \ldots)) && \text{(Lemma 4.3.2)}\\
&= x \oplus \delta(f_{\frac{1}{2}}(2(2x)), f_{\frac{1}{2}}(2(2^2x)), f_{\frac{1}{2}}(2(2^3x)), \ldots) && (\mathbf{A2})\\
&= x \oplus \delta(2x, 2^2x, 2^3x, \ldots) && \text{(Lemma 4.3.2)}\\
&= x \oplus \delta(\vec{t}).
\end{aligned}$$

We have just proved that

$$\delta(\vec{t}) = x \oplus \delta(\vec{t}). \tag{4.12}$$

Since $x, \delta(\vec{t}) \in \operatorname{Rad} A$, Lemma 4.1.7 implies $x \odot \delta(\vec{t}) = 0$. Applying Lemma 4.1.5 to equation (4.12), we conclude that $x = 0$, i.e. $\operatorname{Rad} A = \{0\}$. □

Corollary 4.3.6. *Every δ-algebra is isomorphic, as an MV-algebra, to a separating subalgebra of* $\mathrm{C}(X, [0,1])$ *for some compact Hausdorff space X.*

Proof. This follows at once from Theorem 4.3.5 and Proposition 4.1.10. □

4.4 The representation theorem

Proposition 4.4.1. *Let A be a δ-algebra, and let $\{x_i\}_{i \in \mathbb{N}} \subseteq A$. If $\bigvee_{n \in \mathbb{N}} \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i)$ exists in A, then*

$$\delta(\vec{x}) = \bigvee_{n \in \mathbb{N}} \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i).$$

Proof. By Proposition 4.2.8.(1),(4) we know that, for all $n \in \mathbb{N}$,

$$\delta(\vec{x}) \geqslant \delta(x_1, x_2, \ldots, x_n, \vec{0}) = \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i).$$

Hence, by definition of least upper bound,

$$\delta(\vec{x}) \geqslant \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i).$$

Upon defining the element

$$t := d\left(\bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i), \delta(\vec{x})\right),$$

Lemma 4.1.3.(2) implies

$$\delta(\vec{x}) = \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i) \oplus t. \tag{4.13}$$

We prove that t is an infinitesimal element. Clearly, we have

$$\delta(x_1, x_2, \ldots, x_n, \vec{0}) = \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i) \leqslant \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i) \leqslant \delta(\vec{x}).$$

Then Lemma 4.1.3.(3) entails, for all $n \in \mathbb{N}$,

$$d(\delta(x_1, x_2, \ldots, x_n, \vec{0}), \delta(\vec{x})) \geqslant d\left(\bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i), \delta(\vec{x})\right).$$

The latter holds if, and only if, by Proposition 4.2.8.(6),

$$\delta(0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots) \geqslant t.$$

Applying Lemma 4.2.6 and Remark 4.2.10 we see that, for all $n \in \mathbb{N}$,

$$t \leqslant f_{\frac{1}{2^n}}(\delta(x_{n+1}, x_{n+2}, \ldots)) \leqslant f_{\frac{1}{2^n}}(1),$$

hence $t \in \mathrm{Rad}\, A$ by Lemma 4.3.1. Since A is semisimple by Theorem 4.3.5, we must have $t = 0$. Therefore (4.13) states that

$$\delta(\vec{x}) = \bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i).$$

$\square$

Lemma 4.4.2. *Let A be a δ-algebra, identified with a suitable subalgebra of* $\mathrm{C}(X, [0,1])$*, for some compact Hausdorff space X (cf. Corollary 4.3.6). Then, for all $n \in \mathbb{N}$, the operation $f_{\frac{1}{2^n}}$ is the pointwise multiplication by the rational number $\frac{1}{2^n}$.*

Proof. By Proposition 4.2.8.(5) we know that, for all $g \in A$ and for all $x \in X$,

$$f_{\frac{1}{2}}(g)(x) \oplus f_{\frac{1}{2}}(g)(x) = g(x). \tag{4.14}$$

Further, Proposition 4.2.8.(7) implies

$$f_{\frac{1}{2}}(1_X)(x) = \neg f_{\frac{1}{2}}(1_X)(x) = 1 - f_{\frac{1}{2}}(1_X)(x),$$

where 1_X is the constant function of value 1 on X. Hence, $f_{\frac{1}{2}}(1_X)$ is the constant function of value $\frac{1}{2}$ on X. Then, by Lemma 4.2.9,

$$f_{\frac{1}{2}}(g)(x) \leqslant f_{\frac{1}{2}}(1_X)(x) = \tfrac{1}{2},$$

so that

$$f_{\frac{1}{2}}(g)(x) \oplus f_{\frac{1}{2}}(g)(x) = f_{\frac{1}{2}}(g)(x) + f_{\frac{1}{2}}(g)(x)$$

in the standard MV-algebra $[0,1]$, because

$$f_{\frac{1}{2}}(g)(x) \oplus f_{\frac{1}{2}}(g)(x) = \min\left(1, f_{\frac{1}{2}}(g)(x) + f_{\frac{1}{2}}(g)(x)\right) = f_{\frac{1}{2}}(g)(x) + f_{\frac{1}{2}}(g)(x).$$

Then (4.14) states that

$$f_{\frac{1}{2}}(g)(x) + f_{\frac{1}{2}}(g)(x) = g(x),$$

which is equivalent to

$$f_{\frac{1}{2}}(g)(x) = \tfrac{g(x)}{2}. \tag{4.15}$$

In other words, $f_{\frac{1}{2}}$ is the pointwise multiplication by $\frac{1}{2}$. If we assume $n > 1$,

$$\begin{aligned}
f_{\frac{1}{2^n}}(g)(x) &= f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(g))(x) \\
&= f_{\frac{1}{2}}\left(\tfrac{g}{2^{n-1}}\right)(x) && \text{(inductive hypothesis)} \\
&= \frac{\frac{g(x)}{2^{n-1}}}{2} && (4.15) \\
&= \tfrac{g(x)}{2^n}.
\end{aligned}$$

□

Notation 4.4.3. Henceforth, Lemma 4.4.2 allows us to write $\bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} \frac{g_i}{2^i}$ in place of $\bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(g_i)$.

The following result tells us that the intended models, i.e. the families of all continuous $[0,1]$-valued functions on a compact Hausdorff space, are, in fact, δ-algebras.

Proposition 4.4.4. *Given any compact Hausdorff space X, the MV-algebra* $\mathrm{C}(X,[0,1])$ *is a δ-algebra if, for all $\{f_i\}_{i\in\mathbb{N}} \subseteq \mathrm{C}(X,[0,1])$, the infinitary operation δ is defined as*

$$\delta(\vec{f}) := \sum_{i=1}^{\infty} \frac{f_i}{2^i}.$$

Proof. We prove that Axioms **A1**-**A6** are satisfied in the MV-algebra $\mathrm{C}(X,[0,1])$. Notice that the operation δ is well-defined because the series above is uniformly convergent.

A1 If $\{g_i\}_{i\in\mathbb{N}} \subseteq \mathrm{C}(X,[0,1])$ and $x \in X$, the sequence of the partial sums $\left\{\sum_{i=1}^{n} \frac{g_i(x)}{2^i}\right\}_{n\in\mathbb{N}}$ is monotonically increasing, therefore $\delta(\vec{g})(x) \geqslant \delta(g_1, \vec{0}_X)(x)$. For all $x \in X$ we have

$$
\begin{aligned}
d\left(\delta(\vec{g})(x), \delta(g_1, \vec{0}_X)(x)\right) &= \max\left(0, \left(\sum_{i=1}^{\infty} \frac{g_i(x)}{2^i}\right) - \frac{g_1(x)}{2}\right) \\
&= \sum_{i=2}^{\infty} \frac{g_i(x)}{2^i} \\
&= \delta(0_X, g_2, g_3, \ldots)(x).
\end{aligned}
$$

A2 Let $\{g_i\}_{i\in\mathbb{N}} \subseteq \mathrm{C}(X,[0,1])$. For all $x \in X$,

$$
\begin{aligned}
f_{\frac{1}{2}}(\delta(\vec{g}))(x) &= \frac{\sum_{i=1}^{\infty} \frac{g_i(x)}{2^i}}{2} \\
&= \sum_{i=1}^{\infty} \frac{g_i(x)}{2^{i+1}} \\
&= \sum_{i=1}^{\infty} \frac{f_{\frac{1}{2}}(g_i(x))}{2^i} \\
&= \delta\left(f_{\frac{1}{2}}(g_1), f_{\frac{1}{2}}(g_2), f_{\frac{1}{2}}(g_3), \ldots\right)(x).
\end{aligned}
$$

A3 If $g \in \mathrm{C}(X,[0,1])$ then, for all $x \in X$,

$$
\delta(g,g,g,\ldots)(x)\left(\sum_{i=1}^{\infty} \frac{g}{2^i}\right)(x) = \sum_{i=1}^{\infty} \frac{g(x)}{2^i} = g(x) \cdot \sum_{i=1}^{\infty} \frac{1}{2^i} = g(x) \cdot 1 = g(x).
$$

A4 Let $\{g_i\}_{i\in\mathbb{N}} \subseteq \mathrm{C}(X,[0,1])$. Then, for all $x \in X$,

$$
\delta(0_X, \vec{g})(x) = \frac{0_X(x)}{2} + \sum_{i=1}^{\infty} \frac{g_i(x)}{2^{i+1}} = 0 + \frac{1}{2} \cdot \sum_{i=1}^{\infty} \frac{g_i(x)}{2^i} = \frac{\sum_{i=1}^{\infty} \frac{g_i(x)}{2^i}}{2} = f_{\frac{1}{2}}(\delta(\vec{g})).
$$

A5 Let us consider $\{f_i\}_{i\in\mathbb{N}}, \{g_i\}_{i\in\mathbb{N}} \subseteq \mathrm{C}(X,[0,1])$, and set $\vec{h} := \{f_i \oplus g_i\}_{i\in\mathbb{N}}$. Then, for all $x \in X$ and for all $n \in \mathbb{N}$,

$$
\delta(h_1, \ldots, h_n, \vec{0}_X)(x) = \sum_{i=1}^{n} \frac{h_i(x_i)}{2^i} \geqslant \sum_{i=1}^{n} \frac{f_i(x_i)}{2^i} = \delta(f_1, \ldots, f_n, \vec{0}_X)(x).
$$

Consequently,

$$
\delta(\vec{h})(x) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{h_i(x_i)}{2^i} \geqslant \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f_i(x_i)}{2^i} = \delta(\vec{f})(x).
$$

A6 If $g, h \in \mathrm{C}(X,[0,1])$ then, for all $x \in X$,

$$
f_{\frac{1}{2}}(g \ominus h)(x) = \frac{g \ominus h}{2}(x) = \frac{(g \ominus h)(x)}{2} = \frac{g(x) \ominus h(x)}{2} = \frac{\max(0, g(x) - h(x))}{2}.
$$

However,

$$\frac{\max(0, g(x) - h(x))}{2} = \max\left(0, \frac{g(x) - h(x)}{2}\right) = \max\left(0, \frac{g(x)}{2} - \frac{h(x)}{2}\right).$$

In turn, we have

$$\max\left(0, \frac{g(x)}{2} - \frac{h(x)}{2}\right) = \frac{g(x)}{2} \ominus \frac{h(x)}{2} = f_{\frac{1}{2}}(g)(x) \ominus f_{\frac{1}{2}}(h)(x) = (f_{\frac{1}{2}}(g) \ominus f_{\frac{1}{2}}(h))(x).$$

□

Corollary 4.4.5. *Let X be a compact Hausdorff space, and consider the MV-algebra $A := \mathrm{C}(X, [0,1])$. There is a unique way to define an infinitary operation δ on A, such that $(A, \delta, \oplus, \neg, 0)$ is a δ-algebra.*

Proof. Proposition 4.4.4 assures that such an infinitary operation δ can be defined on A, in such a way that $(A, \delta, \oplus, \neg, 0)$ is a δ-algebra. By Proposition 4.4.1, to prove uniqueness it suffices to show that, for all $\{f_i\}_{i\in\mathbb{N}} \subseteq A$, the join $\bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} \frac{f_i}{2^i}$ exists in A. Upon recalling that the uniform limit of continuous functions is a continuous function, for all $\{f_i\}_{i\in\mathbb{N}} \subseteq A$ we have

$$\bigvee_{n\in\mathbb{N}} \bigoplus_{i=1}^{n} \frac{f_i}{2^i} = \bigvee_{n\in\mathbb{N}} \sum_{i=1}^{n} \frac{f_i}{2^i} = \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{f_i}{2^i} = \sum_{i=1}^{\infty} \frac{f_i}{2^i} \in A.$$

Here we used the fact that $\bigvee_{n\in\mathbb{N}} \sum_{i=1}^{n} \frac{f_i}{2^i} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{f_i}{2^i}$. To see that this is true, denote

$$g_n := \sum_{i=1}^{n} \frac{f_i}{2^i}, \quad f := \lim_{n\to\infty} g_n,$$

and suppose that $h \in A$ satisfies $h(x) \geqslant g_n(x)$ for all $n \in \mathbb{N}$ and for all $x \in X$. We prove that, for all $x \in X$, $h(x) \geqslant f(x)$. Fix an arbitrary point $x \in X$, and observe that the sequence $\{g_n(x)\}_{n\in\mathbb{N}} \subseteq [0,1]$ is monotonically increasing and bounded. Hence its supremum exists, and satisfies

$$\sup_{n\in\mathbb{N}} g_n(x) = \lim_{n\to\infty} g_n(x).$$

The sequence $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly, therefore pointwise, to the function f. In particular, $\lim_{n\to\infty} g_n(x) = f(x)$. Finally, by definition of supremum, since $h(x) \geqslant g_n(x)$ for all $n \in \mathbb{N}$, we must have

$$h(x) \geqslant \sup_{n\in\mathbb{N}} g_n(x) = f(x).$$

□

Remark 4.4.6. If we consider the singleton $X = \{p\}$ as a compact Hausdorff space, then $\mathrm{C}(\{p\}, [0,1]) \cong [0,1]$. Therefore, Corollary 4.4.5 ensures that the standard MV-algebra

$[0,1]$ admits a unique structure of δ-algebra. This structure is obtained by defining, for all $\{x_i\}_{i\in\mathbb{N}} \subseteq [0,1]$,

$$\delta(\vec{x}) := \sum_{i=1}^{\infty} \frac{x_i}{2^i}.$$

Definition 4.4.7. Let A, B be δ-algebras, and denote by δ, δ' the interpretations of the infinitary operation, respectively in A and B. An MV-homomorphism $h\colon A \to B$ is a *δ-homomorphism* if, for all $\{x_i\}_{i\in\mathbb{N}} \subseteq A$,

$$h(\delta(\vec{x})) = \delta'(h(x_1), h(x_2), h(x_3), \ldots).$$

Notation 4.4.8. The sequence $\{h(x_i)\}_{i\in\mathbb{N}}$ in the definition above will sometimes be denoted by $h(\vec{x})$. With this notation, the MV-homomorphism h is a δ-homomorphism if, for all $\{x_i\}_{i\in\mathbb{N}} \subseteq A$, $h(\delta(\vec{x})) = \delta'(h(\vec{x}))$.

Lemma 4.4.9. *Consider the δ-algebra $[0,1]$, and let A be an arbitrary δ-algebra. If $h\colon A \to [0,1]$ is an MV-homomorphism then, for all $x \in A$ and for all $n \in \mathbb{N}$,*

$$f_{\frac{1}{2^n}}(h(x)) = h(f_{\frac{1}{2^n}}(x)).$$

Proof. Let us fix an arbitrary element $x \in A$. If $n = 1$, Proposition 4.2.8.(5) entails

$$f_{\frac{1}{2}}(h(x)) \oplus f_{\frac{1}{2}}(h(x)) = h(x),$$

and

$$h(f_{\frac{1}{2}}(x)) \oplus h(f_{\frac{1}{2}}(x)) = h(f_{\frac{1}{2}}(x) \oplus f_{\frac{1}{2}}(x)) = h(x).$$

Hence, $f_{\frac{1}{2}}(h(x)) \oplus f_{\frac{1}{2}}(h(x)) = h(f_{\frac{1}{2}}(x)) \oplus h(f_{\frac{1}{2}}(x))$. On the other hand, since

$$f_{\frac{1}{2}}(h(x)) \oplus f_{\frac{1}{2}}(h(x)) = h(x) \leqslant h(1) = 1,$$

and

$$h(f_{\frac{1}{2}}(x)) \oplus h(f_{\frac{1}{2}}(x)) = h(x) \leqslant h(1) = 1,$$

it is clear that

$$f_{\frac{1}{2}}(h(x)) + f_{\frac{1}{2}}(h(x)) = h(f_{\frac{1}{2}}(x)) + h(f_{\frac{1}{2}}(x)).$$

We conclude that

$$f_{\frac{1}{2}}(h(x)) = h(f_{\frac{1}{2}}(x)). \tag{4.16}$$

If $n > 1$, then

$$\begin{aligned}
f_{\frac{1}{2^n}}(h(x)) &= f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(h(x))) \\
&= f_{\frac{1}{2}}(h(f_{\frac{1}{2^{n-1}}}(x))) && \text{(inductive hypothesis)} \\
&= h(f_{\frac{1}{2}}(f_{\frac{1}{2^{n-1}}}(x))) && (4.16) \\
&= h(f_{\frac{1}{2^n}}(x)).
\end{aligned}$$

□

Proposition 4.4.10. *If A is a δ-algebra, then any MV-homomorphism $h\colon A \to [0,1]$ is a δ-homomorphism.*

Proof. Denote by δ the interpretation of the infinitary operation in A, and by δ' the interpretation in $[0,1]$. By Remark 4.4.6 we know that, for every $\{y_i\}_{i\in\mathbb{N}} \subseteq [0,1]$,

$$\delta'(\vec{y}) = \sum_{i=1}^{\infty} \frac{y_i}{2^i}. \tag{4.17}$$

Now, for all $= \{x_i\}_{i\in\mathbb{N}} \subseteq A$ and for all $n \in \mathbb{N}$, the following holds.

$$\begin{aligned}
h(\delta(\vec{x})) &= h(\delta(x_1, x_2, \ldots, x_n, \vec{0}) \oplus \delta(0, 0, \ldots, x_{n+1}, x_{n+2}, \ldots)) && \text{(Proposition 4.2.8.(2))}\\
&= h(\delta(x_1, x_2, \ldots, x_n, \vec{0})) \oplus h(\delta(0, 0, \ldots, x_{n+1}, x_{n+2}, \ldots))\\
&= h\left(\bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i)\right) \oplus h(\delta(0, 0, \ldots, x_{n+1}, x_{n+2}, \ldots)) && \text{(Proposition 4.2.8.(4))}\\
&= h\left(\bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(x_i)\right) \oplus h(f_{\frac{1}{2^n}}(\delta(x_{n+1}, x_{n+2}, \ldots))) && \text{(Lemma 4.2.6)}\\
&= \bigoplus_{i=1}^{n} f_{\frac{1}{2^i}}(h(x_i)) \oplus f_{\frac{1}{2^n}}(h(\delta(x_{n+1}, x_{n+2}, \ldots))) && \text{(Lemma 4.4.9)}\\
&= \bigoplus_{i=1}^{n} \frac{h(x_i)}{2^i} \oplus \frac{h(\delta(x_{n+1}, x_{n+2}, \ldots))}{2^n}. && \text{(Lemma 4.4.2)}
\end{aligned}$$

We have proved that

$$h(\delta(\vec{x})) = \bigoplus_{i=1}^{n} \frac{h(x_i)}{2^i} \oplus \frac{h(\delta(x_{n+1}, x_{n+2}, \ldots))}{2^n}. \tag{4.18}$$

Notice that, in the δ-algebra $[0,1]$, the element $\bigoplus_{i=1}^{n} \frac{h(x_i)}{2^i}$ coincides with $\sum_{i=1}^{n} \frac{h(x_i)}{2^i}$, because

$$\sum_{i=1}^{n} \frac{h(x_i)}{2^i} \leqslant \sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n} < 1.$$

Moreover,

$$\left(\sum_{i=1}^{n} \frac{h(x_i)}{2^i}\right) \oplus \frac{h(\delta(x_{n+1}, x_{n+2}, \ldots))}{2^n} = \left(\sum_{i=1}^{n} \frac{h(x_i)}{2^i}\right) + \frac{h(\delta(x_{n+1}, x_{n+2}, \ldots))}{2^n} \tag{4.19}$$

since

$$\left(\sum_{i=1}^{n} \frac{h(x_i)}{2^i}\right) + \frac{h(\delta(x_{n+1}, x_{n+2}, \ldots))}{2^n} \leqslant \left(1 - \frac{1}{2^n}\right) + \frac{1}{2^n} = 1.$$

We remark that the limits $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{h(x_i)}{2^i}$ and $\lim_{n\to\infty} \frac{h(\delta(x_{n+1}, x_{n+2}, \ldots))}{2^n}$ exist in $[0,1]$, and that $h(\delta(\vec{x})) \in [0,1]$ does not depend on $n \in \mathbb{N}$. Hence,

$$h(\delta(\vec{x})) = \lim_{n\to\infty} h(\delta(\vec{x}))$$

$$= \lim_{n\to\infty} \left(\bigoplus_{i=1}^{n} \frac{h(x_i)}{2^i} \oplus \frac{h(\delta(x_{n+1}, x_{n+2}, \ldots))}{2^n} \right) \tag{4.18}$$

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{h(x_i)}{2^i} + \frac{h(\delta(x_{n+1}, x_{n+2}, \ldots))}{2^n} \right) \tag{4.19}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{h(x_i)}{2^i} + \lim_{n\to\infty} \frac{h(\delta(x_{n+1}, x_{n+2}, \ldots))}{2^n}$$

$$= \sum_{i=1}^{\infty} \frac{h(x_i)}{2^i}$$

$$= \delta'(h(\vec{x})). \tag{4.17}$$

□

If A is a δ-algebra, Corollary 4.3.6 states that A is isomorphic, as an MV-algebra, to a subalgebra of $\mathrm{C}(X, [0,1])$ for some compact Hausdorff space X. In fact, by Proposition 4.1.10 there is a canonical choice for the compact Hausdorff space: this is the maximal spectrum $\operatorname{Max} A$ of the MV-algebraic reduct of A.

Proposition 4.4.11. *Let $(A, \delta, \oplus, \neg, 0)$ be a δ-algebra, identified with an MV-subalgebra of $\mathrm{C}(\operatorname{Max} A, [0,1])$. Then, for all $\{f_i\}_{i\in\mathbb{N}} \subseteq A$ and for all $\mathfrak{m} \in \operatorname{Max} A$,*

$$\delta(\vec{f})(\mathfrak{m}) = \sum_{i=1}^{\infty} \frac{f_i}{2^i}(\mathfrak{m}).$$

Proof. We briefly recall how the embedding $\widehat{\cdot}\colon A \to \mathrm{C}(\operatorname{Max} A, [0,1])$ is defined (for more details, please see Section 2.2). If $f \in A$ and $\mathfrak{m} \in \operatorname{Max} A$, then the function $\widehat{f}\colon \operatorname{Max} A \to [0,1]$ is given by $\widehat{f}(\mathfrak{m}) := \iota_{\mathfrak{m}}\left(\frac{f}{\mathfrak{m}}\right)$, where $\iota_{\mathfrak{m}}$ is the unique embedding $\iota_{\mathfrak{m}}\colon \frac{A}{\mathfrak{m}} \to [0,1]$. Then, Theorem 2.2.72 allows us to identify A with the subalgebra $\widehat{A}$ of $\mathrm{C}(\operatorname{Max} A, [0,1])$, i.e. we identify $f \in A$ with $\widehat{f}\colon \operatorname{Max} A \to [0,1]$. Let us fix an arbitrary point $\mathfrak{m} \in \operatorname{Max} A$, and denote by $q_{\mathfrak{m}}\colon A \to \frac{A}{\mathfrak{m}}$ the quotient map. We can consider the MV-homomorphism $h_{\mathfrak{m}}\colon A \to [0,1]$ given by

$$h_{\mathfrak{m}} := \iota_{\mathfrak{m}} \circ q_{\mathfrak{m}}.$$

We remark that $h_{\mathfrak{m}}$ is precisely the evaluation at the point $\mathfrak{m} \in \operatorname{Max} A$. Upon denoting with δ' the infinitary operation in the δ-algebra $[0,1]$, for all $\{f_i\}_{i\in\mathbb{N}} \subseteq A$ and for all $\mathfrak{m} \in \operatorname{Max} A$, we have

$$\begin{aligned} \delta(\vec{f})(\mathfrak{m}) &= \iota_{\mathfrak{m}}\left(\frac{\delta(\vec{f})}{\mathfrak{m}}\right) \\ &= h_{\mathfrak{m}}(\delta(\vec{f})) \\ &= \delta'(h_{\mathfrak{m}}(\vec{f})) && \text{(Proposition 4.4.10)} \\ &= \delta'(\vec{f}(\mathfrak{m})) \end{aligned}$$

$$= \sum_{i=1}^{\infty} \frac{f_i(\mathfrak{m})}{2^i}, \qquad \text{(Remark 4.4.6)}$$

where $\vec{f}(\mathfrak{m}) := \{f_i(\mathfrak{m})\}_{i\in\mathbb{N}} \subseteq [0,1]$. This proves the proposition. □

Corollary 4.4.12. *If A, B are δ-algebras and $h\colon A \to B$ is an MV-homomorphism, then h is a δ-homomorphism.*

Proof. In the case $B = [0,1]$, the statement holds by Proposition 4.4.10. The proof of the latter relies on the fact that the interpretation of the infinitary operation in the δ-algebra $[0,1]$ is known: pointwise, it is the series $\sum_{i=1}^{\infty} \frac{f_i}{2^i}$. It follows by Proposition 4.4.11, that the same argument applies here, mutatis mutandis. □

Proposition 4.4.13. *Let A be a δ-algebra such that $A \subseteq \mathrm{C}(X,[0,1])$ for some compact Hausdorff space X, and let $f \in \mathrm{C}(X,[0,1])$ be an arbitrary continuous function. Suppose that there exists a monotonically increasing sequence $\{s_i\}_{i\in\mathbb{N}} \subseteq A$ which converges uniformly to f, and satisfies $\|s_1\| \leqslant 1/2$, $\|s_i - s_{i-1}\| \leqslant 1/2^i$ for all $i \geqslant 2$. Then*

$$f = \delta(2s_1, 2^2(s_2 \ominus s_1), 2^3(s_3 \ominus s_2), \ldots, 2^i(s_i \ominus s_{i-1}), \ldots).$$

In particular, $f \in A$.

Proof. The following argument is due to Isbell [42]. For simplicity, define the element $s_0 := 0 \in A$, and set

$$\{2s_1, 2^2(s_2 \ominus s_1), 2^3(s_3 \ominus s_2), \ldots, 2^i(s_i \ominus s_{i-1}), \ldots\} = \{2^i(s_i \ominus s_{i-1})\}_{i\in\mathbb{N}} =: \vec{s}.$$

We point out that the hypothesis $\|s_i - s_{i-1}\| \leqslant 1/2^i$ for all $i \geqslant 1$, implies $\vec{s} \subseteq A$. Further, the assumption $s_i \geqslant s_{i-1}$ for all $i \geqslant 1$ entails that the function $s_i \ominus s_{i-1}$ coincides with $s_i - s_{i-1}$, for all $i \in \mathbb{N}$. Then, for every $x \in X$,

$$\begin{aligned}
\delta(\vec{s})(x) &= \sum_{i=1}^{\infty} \frac{2^i(s_i \ominus s_{i-1})}{2^i}(x) && \text{(Proposition 4.4.11)}\\
&= \sum_{i=1}^{\infty} s_i(x) - s_{i-1}(x)\\
&= \lim_{n\to\infty} \sum_{i=1}^{n} s_i(x) - s_{i-1}(x)\\
&= \lim_{n\to\infty} s_n(x)\\
&= f(x),
\end{aligned}$$

where the last equality holds since the uniform convergence implies the pointwise convergence. We conclude that $f = \delta(\vec{s})$. □

Lemma 4.4.14. *Let $A \subseteq \mathrm{C}(X,[0,1])$ be a separating δ-algebra. Then A is dense in $\mathrm{C}(X,[0,1])$ with respect to the uniform norm.*

Proof. The δ-algebra A is closed under finite $\oplus$-sums and finite joins. Moreover, it contains the constant functions $0_X, 1_X$, respectively of value 0 and 1 on X. Given a dyadic rational number $\phi := \frac{m}{2^n} \in [0,1]$ and a function $f \in A$, the function ϕf belongs to A since it can be obtained as

$$\phi f = \underbrace{f_{\frac{1}{2^n}}(f) \oplus \cdots \oplus f_{\frac{1}{2^n}}(f)}_{m \text{ times}}.$$

More generally, if $r \in [0,1]$ is any real number in the unit interval, it is easy to see that rf is in A. Indeed, let $\vec{r} := \{r_i\}_{i\in\mathbb{N}} \in \{0,1\}^\omega$ be a binary expansion of r. Assume r is not a dyadic rational number, so that this expansion is unique, and define $\vec{f} := \{f_i\}_{i\in\mathbb{N}} \subseteq A$ by $f_i := 0_X$ if $r_i = 0$, and $f_i := f$ if $r_i = 1$. Consequently, for all $x \in X$,

$$\delta(\vec{f})(x) = \sum_{i=1}^{\infty} \frac{f_i(x)}{2^i} = \sum_{i=1}^{\infty} \frac{r_i f(x)}{2^i} = f(x) \cdot \sum_{i=1}^{\infty} \frac{r_i}{2^i} = rf(x).$$

Choosing $f = 1_X$ in the previous construction, we see that every constant function of value $r \in [0,1]$ is in A. Furthermore, A separates points of X; hence, by the lattice-theoretic version of the Stone-Weierstrass Theorem 3.2.17, together with the functor Γ, we conclude that A is dense in $\mathrm{C}(X,[0,1])$. $\square$

Lemma 4.4.15. *If $A \subseteq \mathrm{C}(X,[0,1])$ is a separating δ-algebra, and $f \in \mathrm{C}(X,[0,1])$, there exists a monotonically increasing sequence $\{f_i\}_{i\in\mathbb{N}} \subseteq A$ which converges uniformly to f.*

Proof. Recall by Theorem 4.1.11 that the functor Γ, from the category of unital ℓ-groups to the category of MV-algebras, is an equivalence. Its quasi-inverse is denoted by Ξ (see Section 2.3 for details). The MV-algebra A is dense in $\mathrm{C}(X,[0,1])$ by Lemma 4.4.14, hence it is possible to show that its enveloping unital ℓ-group $\Xi(A)$ is dense in $\mathrm{C}(X,\mathbb{R})$. Consider the strictly decreasing sequence $\{a_i = 1/2^i\}_{i\in\mathbb{N}}$. In particular, a_i tends to zero, as $i \to \infty$. We define two more sequences $\{b_i\}_{i\in\mathbb{N}}$ and $\{c_i\}_{i\in\mathbb{N}} \subseteq A$, such that, for all $i \in \mathbb{N}$,

$$b_i := \frac{1}{2}(a_i + a_{i+1}) = \frac{3}{2^{i+2}}, \quad c_i := \frac{1}{2}(a_i - a_{i+1}) = \frac{1}{2^{i+2}}.$$

Observe that b_i, c_i tend to 0 as $i \to \infty$, and

$$b_i + c_i = \frac{1}{2^i} = a_i, \quad b_i - c_i = \frac{1}{2^{i+1}} = a_{i+1}.$$

For each $i \in \mathbb{N}$, $f - b_i \in \mathrm{C}(X,\mathbb{R})$, hence the density of $\Xi(A)$ entails the existence of $g_i \in \Xi(A)$ such that $\|g_i - (f - b_i)\| < c_i$. This is equivalent to saying that, for all $x \in X$,

$$f(x) - \frac{1}{2^{i-1}} < g_i(x) < f(x) - \frac{1}{2^i}.$$

We remark that

$$f(x) - \frac{1}{2^{i-1}} < g_i(x) < f(x) - \frac{1}{2^i} < g_{i+1}(x) < f(x) - \frac{1}{2^{i+1}},$$

i.e. $\{g_i\}_{i\in\mathbb{N}} \subseteq \Xi(A)$ is a strictly monotonically increasing sequence. Upon defining

$$f_i := g_i \vee 0_X,$$

it is clear that $\{f_i\}_{i\in\mathbb{N}} \subseteq A$ is a monotonically increasing sequence. To prove that the latter converges uniformly to f, it suffices to notice that, for all $i \in \mathbb{N}$,

$$\|f_i - f\| \leqslant \|g_i - f\| = \|(g_i - f + b_i) - b_i\| \leqslant \|g_i - (f_i - b_i)\| + \|-b_i\| < \frac{1}{2^{i+2}} + \frac{3}{2^{i+2}} = \frac{1}{2^i}.$$

□

Theorem 4.4.16. *For every δ-algebra A, there exists a compact Hausdorff space X such that $A \cong \mathrm{C}(X, [0,1])$.*

Proof. By Corollary 4.3.6 we know that A is MV-isomorphic (=δ-isomorphic, by Corollary 4.4.12) to a separating δ-algebra $B \subseteq \mathrm{C}(X,[0,1])$, where the infinitary operation on B is induced by that of A. We prove that $B = \mathrm{C}(X,[0,1])$. Let $g \in \mathrm{C}(X,[0,1])$ be an arbitrary continuous function. By Lemma 4.4.15 there exists a monotonically increasing sequence $\{g_i\}_{i\in\mathbb{N}} \subseteq B$ that converges uniformly to g. Consider the sequence $\{f_{\frac{1}{2}}(g_i)\}_{i\in\mathbb{N}} = \{\frac{g_i}{2}\}_{i\in\mathbb{N}} \subseteq B$, and note that $\|\frac{g_i}{2}\| \leqslant \frac{1}{2}$ for all $i \in \mathbb{N}$. The sequence $\{\frac{g_i}{2}\}_{i\in\mathbb{N}}$ is again uniformly convergent, but its limit is $\frac{g}{2}$. It is possible to find a subsequence $\{s_i\}_{i\in\mathbb{N}} \subseteq \{\frac{g_i}{2}\}_{i\in\mathbb{N}}$ satisfying $\|s_i - s_{i-1}\| \leqslant \frac{1}{2^i}$ for all $i \in \mathbb{N}$. This subsequence satisfies the hypotheses of Proposition 4.4.13, therefore (with the notation of the proof of the latter) we have $\delta(\vec{s}) = \frac{g}{2}$. We conclude, by Proposition 4.2.8.(5), that

$$\delta(\vec{s}) \oplus \delta(\vec{s}) = f_{\frac{1}{2}}(g) \oplus f_{\frac{1}{2}}(g) = g.$$

This shows that $g \in B$, so that $A \cong B = \mathrm{C}(X,[0,1])$. □

Let us denote by Δ the category with δ-algebras as objects, and δ-homomorphisms as morphisms. Corollary 4.4.12 then states that Δ is a full subcategory of the category MV of MV-algebras. The following fact was proved in Proposition 4.4.4.

Lemma 4.4.17. *If X is a compact Hausdorff space, then*

$$\mathcal{C}(X) := \mathrm{C}(X,[0,1])$$

is a δ-algebra. □

Lemma 4.4.18. *If $\varphi\colon Y \to X$ is a continuous function between compact Hausdorff spaces, then*

$$\mathcal{C}(\varphi) := - \circ \varphi\colon \mathcal{C}(X) \to \mathcal{C}(Y)$$

is a δ-homomorphism.

Proof. Since Δ is a full subcategory of MV, it suffices to prove that $\mathcal{C}(\varphi)$ is an MV-homomorphism. Given $f, g \in \mathcal{C}(X)$ and $q \in Y$, we have

$$(0_X \circ \varphi)(q) = 0_X(\varphi(q)) = 0 = 0_Y(q),$$

$$((f \oplus g) \circ \varphi)(q) = (f \oplus g)(\varphi(q)) = f(\varphi(q)) \oplus g(\varphi(q)) = (f \circ \varphi)(q) \oplus (g \circ \varphi)(q),$$
$$(\neg f \circ \varphi)(q) = \neg f(\varphi(q)) = 1 - f(\varphi(q)) = \neg(f(\varphi(q))) = \neg(f \circ \varphi)(q).$$

□

It is elementary that $\mathcal{C}$ preserves compositions, and maps the identity function on a compact Hausdorff space X to the identity homomorphism of the δ-algebra $\mathrm{C}(X, [0,1])$.

Corollary 4.4.19. *$\mathcal{C}\colon \mathsf{KHaus} \to \Delta$ is a contravariant functor from the category of compact Hausdorff spaces to the category of δ-algebras.* □

On the other hand, Proposition 4.1.9 tells us that

Lemma 4.4.20. *If A is a δ-algebra, then its maximal spectrum $\operatorname{Max} A$ is a compact Hausdorff space.* □

Regarding the morphisms,

Lemma 4.4.21. *If $h\colon B \to A$ is a δ-homomorphism, then*

$$\mathcal{M}(h) := h^{-1}\colon \mathcal{M}(A) \to \mathcal{M}(B)$$

is a continuous function.

Proof. We remark that the map $\mathcal{M}(h)$ is well-defined by Lemma 4.1.8. Recall that a basis of closed sets for $\operatorname{Max} B$ is given by the sets of the form

$$F_b := \{\mathfrak{m} \in \operatorname{Max} B \mid b \in \mathfrak{m}\}.$$

To prove that $\mathcal{M}(h)$ is a continuous function, it is sufficient to show that the preimage of each basic closed set under the map h^{-1} is closed in $\mathcal{M}(A)$. Indeed, this is true because

$$\begin{aligned}(h^{-1})^{-1}(F_b) &= \{\mathfrak{m} \in \operatorname{Max} A \mid h^{-1}(\mathfrak{m}) \in F_b\}\\ &= \{\mathfrak{m} \in \operatorname{Max} A \mid b \in h^{-1}(\mathfrak{m})\}\\ &= \{\mathfrak{m} \in \operatorname{Max} A \mid h(b) \in \mathfrak{m}\}\\ &= F_{h(b)}.\end{aligned}$$

□

Again, it is easy to see that $\mathcal{M}$ preserves compositions and the identity:

Corollary 4.4.22. *$\mathcal{M}\colon \Delta \to \mathsf{KHaus}$ is a contravariant functor from the category of δ-algebras to the category of compact Hausdorff spaces.* □

We prove that the functors $\mathcal{M}$ and $\mathcal{C}$ are quasi-inverse.

Proposition 4.4.23. *There exists a natural isomorphism*

$$\mu \colon \mathrm{Id}_{\mathsf{KHaus}} \to \mathcal{M} \circ \mathcal{C},$$

where $\mathrm{Id}_{\mathsf{KHaus}}$ *denotes the identity functor on the category* KHaus*.*

Proof. If X is a compact Hausdorff space, recall by Proposition 2.2.73 that the map

$$\mu_X \colon X \to \operatorname{Max} \mathrm{C}(X,[0,1]), \quad \mu_X(p) := \mathbb{I}(\{p\})$$

is a homeomorphism, where

$$\mathbb{I}(\{p\}) := \{f \in \mathrm{C}(X,[0,1]) \mid f(p) = 0\}.$$

Define the component of μ at X as $(\mu)_X := \mu_X$. To prove the statement, it suffices to show that μ is a natural transformation. In other words, that the following diagram commutes, whenever $f \colon X \to Y$ is a continuous function between compact Hausdorff spaces.

$$\begin{tikzcd}
X \arrow[r, "\mu_X"] \arrow[d, "f"'] & \operatorname{Max} \mathrm{C}(X,[0,1]) \arrow[d, "(\mathcal{M}\circ\mathcal{C})(f)"] \\
Y \arrow[r, "\mu_Y"'] & \operatorname{Max} \mathrm{C}(Y,[0,1])
\end{tikzcd}$$

For every $\mathfrak{m} \in \operatorname{Max} \mathrm{C}(X,[0,1])$ we have

$$(\mathcal{M} \circ \mathcal{C})(f)(\mathfrak{m}) = (- \circ f)^{-1}(\mathfrak{m}) = \{g \in \mathrm{C}(Y,[0,1]) \mid g \circ f \in \mathfrak{m}\}.$$

Hence, for all $p \in X$,

$$\begin{aligned}
(\mathcal{M} \circ \mathcal{C})(f) \circ \mu_X(p) &= (\mathcal{M} \circ \mathcal{C})(f)(\mathbb{I}(\{p\})) \\
&= \{g \in \mathrm{C}(Y,[0,1]) \mid g \circ f \in \mathbb{I}(\{p\})\} \\
&= \{g \in \mathrm{C}(Y,[0,1]) \mid (g \circ f)(p) = 0\} \\
&= \{g \in \mathrm{C}(Y,[0,1]) \mid g(f(p)) = 0\} \\
&= \mathbb{I}(\{f(p)\}) \\
&= (\mu_Y \circ f)(p).
\end{aligned}$$

□

Proposition 4.4.24. *There exists a natural isomorphism*

$$\nu \colon \mathrm{Id}_{\Delta} \to \mathcal{C} \circ \mathcal{M},$$

where Id_{Δ} *denotes the identity functor on the category* Δ*.*

Proof. Given a δ-algebra A, Theorem 4.4.16 shows that the map

$$\nu_A\colon A \to \mathrm{C}(\operatorname{Max} A, [0,1]), \quad \nu_A(a) := \widehat{a}$$

is an isomorphism in the category Δ. Define $\nu\colon \mathrm{Id}_\Delta \to \mathcal{C} \circ \mathcal{M}$ by its components: we set $(\nu)_A := \nu_A$. To prove that ν is a natural isomorphism, it is enough to show that it is a natural transformation. That is, for every δ-homomorphism $h\colon A \to B$, the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\nu_A"] \arrow[d, "h"'] & \mathrm{C}(\operatorname{Max} A, [0,1]) \arrow[d, "(\mathcal{C}\circ\mathcal{M})(h)"] \\ B \arrow[r, "\nu_B"'] & \mathrm{C}(\operatorname{Max} B, [0,1]) \end{tikzcd}$$

Note that, for all $a \in A$,

$$(\nu_B \circ h)(a) = \nu_B(h(a)) = \widehat{h(a)},$$

and

$$\left((\mathcal{C} \circ \mathcal{M})(h) \circ \nu_A\right)(a) = (\mathcal{C} \circ \mathcal{M})(h)(\widehat{a}) = \widehat{a} \circ h^{-1}.$$

In other words, we must prove that, for all $\mathfrak{n} \in \operatorname{Max} B$,

$$\widehat{h(a)}(\mathfrak{n}) = (\widehat{a} \circ h^{-1})(\mathfrak{n}). \tag{4.20}$$

In turn, upon denoting by $\mathfrak{h}_\mathfrak{n}, \mathfrak{h}_{h^{-1}(\mathfrak{n})}$ the unique MV-embeddings

$$\mathfrak{h}_\mathfrak{n}\colon \tfrac{B}{\mathfrak{n}} \to [0,1], \quad \mathfrak{h}_{h^{-1}(\mathfrak{n})}\colon \tfrac{A}{h^{-1}(\mathfrak{n})} \to [0,1]$$

provided by Theorem 2.3.30, (4.20) is equivalent to

$$\mathfrak{h}_\mathfrak{n}\left(\frac{h(a)}{\mathfrak{n}}\right) = \mathfrak{h}_{h^{-1}(\mathfrak{n})}\left(\frac{a}{h^{-1}(\mathfrak{n})}\right).$$

Fix an arbitrary maximal ideal $\mathfrak{n} \in \operatorname{Max} B$. It is clear that $h\colon A \to B$ induces an MV-homomorphism $\frac{A}{h^{-1}(\mathfrak{n})} \to \frac{B}{\mathfrak{n}}$, that we continue to denote by h. The latter homomorphism is injective. Indeed, if $h(a) = 0 \in \frac{B}{\mathfrak{n}}$, then $h(a) \in \mathfrak{n}$, that is $a \in h^{-1}(\mathfrak{n})$. Thus $a = 0 \in \frac{A}{h^{-1}(\mathfrak{n})}$. It follows that the composition

$$\mathfrak{h}_\mathfrak{n} \circ h\colon \frac{A}{h^{-1}(\mathfrak{n})} \to [0,1]$$

is an MV-embedding. By Theorem 2.3.30 we conclude that $\mathfrak{h}_{h^{-1}(\mathfrak{n})} = \mathfrak{h}_\mathfrak{n} \circ h$, whence

$$\mathfrak{h}_{h^{-1}(\mathfrak{n})}\left(\frac{a}{h^{-1}(\mathfrak{n})}\right) = \mathfrak{h}_\mathfrak{n}\left(h\left(\frac{a}{h^{-1}(\mathfrak{n})}\right)\right) = \mathfrak{h}_\mathfrak{n}\left(\frac{h(a)}{\mathfrak{n}}\right).$$

$\square$

We have just proved

Theorem 4.4.25. *The category* KHaus *of compact Hausdorff spaces is dually equivalent to the category* Δ *of* δ*-algebras via the functors* $\mathcal{C}$ *and* $\mathcal{M}$*.* $\square$

Chapter 5

The Lawvere-Linton theory of δ-algebras

5.1 Algebraic theories

From the point of view of classical model theory, and in particular of universal algebra, an algebraic theory is a set $\mathbb{T}$ of equational formulæ in a language that contains only finitary function symbols. This approach has two limitations: firstly, it only deals with finitary operations; secondly, it depends on a specific presentation in terms of operations and equations. In 1959 Słomiński [64] rectified the first limitation by introducing *infinitary universal algebra*, that is the study of equationally defined classes of algebras admitting infinitary function symbols. In the sixties Lawvere [48] introduced the categorical notion of *algebraic theory* which rectified the second defect without, however, rectifying the first. Indeed we will see that (Lawvere) algebraic theories can only capture finitary equational theories. We give a brief motivation before introducing the abstract notion of algebraic theory.

Suppose we are given a *presentation of a finitary algebraic theory*, i.e. an equational theory $\mathbb{T}$ (whose formulæ are called *axioms*) on a finitary signature Σ without relation symbols, together with a countable set of variables $\mathbf{Var} = \{x_1, x_2, \ldots\}$. For each positive integer n, we denote by $\mathcal{F}_n$ the set of function symbols of arity n. The set $\mathbf{Term}$ of terms for the language Σ is inductively defined in the following way: every variable is in $\mathbf{Term}$; if $\alpha \in \mathcal{F}_n$ and $t_1, \ldots, t_n \in \mathbf{Term}$, then $\alpha(t_1, \ldots, t_n) \in \mathbf{Term}$. The theory $\mathbb{T}$ is then a set of pairs of terms (s, t), $s, t \in \mathbf{Term}$, where each such pair can informally be thought of as the equation $s = t$. In the following, we use the latter notation whenever convenient.

A Σ-*structure* is a set U together with an operation $\widehat{\alpha}\colon U^n \to U$ for each function symbol $\alpha \in \mathcal{F}_n$. Observe that any function $\varphi\colon \mathbf{Var} \to U$ can be extended to a function $\overline{\varphi}\colon \mathbf{Term} \to U$. Indeed, suppose that the map $\overline{\varphi}$ is defined on the terms $t_1, \ldots, t_n \in \mathbf{Term}$, and let $\alpha \in \mathcal{F}_n$ be a function symbol of arity n. Then, we define

$$\overline{\varphi}(\alpha(t_1, \ldots, t_n)) := \widehat{\alpha}(\overline{\varphi}(t_1), \ldots, \overline{\varphi}(t_n)).$$

A *model* of $\mathbb{T}$ is a Σ-structure U such that, for every function $\varphi\colon \mathbf{Var} \to U$ and for every pair $(s,t) \in \mathbb{T}$, the condition $\overline{\varphi}(s) = \overline{\varphi}(t)$ is satisfied. The category whose objects are models for a theory $\mathbb{T}$ and whose morphisms are maps preserving the operations, is denoted by $\mathsf{Mod}\,\mathbb{T}$. It is possible to construct models for $\mathbb{T}$ of a purely syntactic nature. Define the set $\mathbf{Ded}$ of all the equalities between terms deducible from the axioms of $\mathbb{T}$ (this is usually called the *deductive closure* of the axioms) by the following deduction rules.

1. Every axiom (i.e. element of $\mathbb{T}$) is in $\mathbf{Ded}$.
2. For each term $t \in \mathbf{Term}$, $t = t \in \mathbf{Ded}$.
3. If $t = u$, $u = v \in \mathbf{Ded}$, then $u = t \in \mathbf{Ded}$ and $t = v \in \mathbf{Ded}$.
4. If $t = t' \in \mathbf{Ded}$ and t occurs as a subterm of $r \in \mathbf{Term}$, upon denoting with r' the term we obtain by substituting t in such an occurrence with t', then $r = r' \in \mathbf{Ded}$.
5. Suppose $t = t' \in \mathbf{Ded}$ where t and t' are terms in which the variables $x_1, \dots x_m$ occur. If $t_1, \dots, t_m$ are arbitrary terms and s, s' denote the terms obtained respectively from t and t' by simultaneously substituting each occurrence of the variable x_i with the term t_i for every $i = 1, \dots, m$, then $s = s' \in \mathbf{Ded}$.
6. Nothing else is in $\mathbf{Ded}$.

We can define an equivalence relation on the set $\mathbf{Term}$, given by $t \sim t'$ if, and only if, $t = t' \in \mathbf{Ded}$. One can prove that $\sim$ is a congruence, i.e. it is an equivalence relation compatible with all the operations in $\mathcal{F}_n$, for all $n \in \mathbb{N}$. If $T_n \subseteq \mathbf{Term}$ is the set of terms in which only variables among $x_1, \dots, x_n$ appear, the quotient $T_n/\sim$ is denoted by F_n.

Lemma 5.1.1. *If $\mathbb{T}$ is the presentation of a finitary algebraic theory, then F_n is a model for $\mathbb{T}$ and it is free in $\mathsf{Mod}\,\mathbb{T}$ (with respect to the underlying-set functor) over a set with n generators.*

Proof. See [18, Theorem 10.12]. □

Remark 5.1.2. Lemma 5.1.1 holds more generally for presentations of infinitary algebraic theories, if the countability restriction on the set of variables is dropped [64, 8.3].

It is possible to prove that a coproduct of free models is again a free model and that the nth copower of the free model F_1 over one generator is isomorphic to the free model F_n [17, Lemma 3.2.7]. The following proposition provides the connection between classical universal algebra and categorical universal algebra.

Proposition 5.1.3. *Let $\mathbb{T}$ be the presentation of a finitary algebraic theory and let $\mathcal{F}$ be the full subcategory of $\mathsf{Mod}\,\mathbb{T}$ whose objects are the free finitely generated models F_n, for all $n \in \mathbb{N}$. The category $\mathcal{F}^{op}$ has finite products, and $\mathsf{Mod}\,\mathbb{T}$ is equivalent to the category $\mathsf{Set}_\omega^{\mathcal{F}^{\mathsf{op}}}$ of functors $\mathcal{F}^{op} \to \mathsf{Set}$ that preserve finite products, where the morphisms in the latter functor category are the natural transformations.*

Proof. See [17, Proposition 3.2.9]. □

The following definition, motivated by Proposition 5.1.3, is due to Lawvere [48].

Definition 5.1.4. A *finitary algebraic theory* (or *Lawvere theory*) is a small category $\mathcal{T}$ whose objects are given by a countable set $\{T^0, T^1, \ldots, T^n, \ldots\}$ where T^i is the ith power of T^1, for each positive integer i. A *model* of $\mathcal{T}$ is a functor $F \colon \mathcal{T} \to \mathsf{Set}$ preserving finite products. A *homomorphism* between two models for $\mathcal{T}$ is a natural transformation between the two functors.

Proposition 5.1.3 states that for every presentation $\mathbb{T}$ of a finitary algebraic theory there is an associated Lawvere theory $\mathcal{F}^{op}$, where $\mathcal{F}$ is the category of finitely generated free models for $\mathbb{T}$. Further, we can recover the category of models for $\mathbb{T}$, up to equivalence, as $\mathsf{Set}_\omega^{\mathcal{F}^{\mathsf{op}}}$. Conversely, it is possible to show that each Lawvere theory as in Definition 5.1.4 arises from an appropriate presentation $\mathbb{T}$. Lawvere theories succeed in making finitary equational theories independent of a specific presentation, in fact every Lawvere theory is determined by a class of equational axiomatisations which can be regarded as different presentations (by operations and equations) of that same theory. The previous definition can be generalized in a straightforward way.

Definition 5.1.5. For any infinite regular cardinal λ, a *λ-ary algebraic theory* (or *Lawvere-Linton theory*) is a small category $\mathcal{T}^\lambda$ with all λ-products, such that every object is the product of μ copies of a *fundamental object* F_1, for some cardinal $\mu < \lambda$.

Again, a *model* of $\mathcal{T}^\lambda$ is a functor to Set preserving λ-products, and a morphism between two models is a natural transformation. Now we can consider a presentation $\mathbb{T}$ of an algebraic theory on a λ-signature (see the Prologue for additional details on λ-signatures). In this case, if $\mathcal{F}$ is the full subcategory of $\mathsf{Mod}\,\mathbb{T}$ whose objects are all the free objects generated by a set of cardinality strictly smaller than λ, then $\mathcal{F}^{op}$ is the λ-ary algebraic theory associated to $\mathbb{T}$. Observe that, for $\lambda = \aleph_0$, we recover the Lawvere theory of $\mathbb{T}$. Denote by $\mathsf{Set}_\lambda^{\mathcal{F}^{\mathsf{op}}}$ the category of models of the λ-ary algebraic theory of $\mathbb{T}$. In the following, when a presentation $\mathbb{T}$ of an algebraic theory is given, we write $\mathcal{T}^\lambda$ for the λ-ary algebraic theory associated to $\mathbb{T}$.

Proposition 5.1.3 can be extended to theories on λ-ary signatures.

Proposition 5.1.6. *Let $\mathbb{T}$ be the presentation of an algebraic theory on a λ-signature Σ, and denote by $\mathcal{F}$ the full subcategory of* $\mathsf{Mod}\,\mathbb{T}$ *whose objects are all the free objects on sets of cardinality smaller than λ. Then* $\mathsf{Mod}\,\mathbb{T} \simeq \mathsf{Set}_\lambda^{\mathcal{F}^{\mathsf{op}}}$.

Proof. See [1, Theorem 3.30]. □

5.2 The case of δ-algebras: Hilbert cubes

In this section, we compute the Lawvere-Linton theory of δ-algebras which reduces, by Proposition 5.1.6, to the study of the $\aleph_1$-ary algebraic theory associated to δ-algebras.

The first step consists in finding an explicit description of the free δ-algebras. We will see that the (unique up to isomorphism) free δ-algebra on a set X of generators is

$$\mathrm{C}([0,1]^X, [0,1]).$$

This can be proved directly, applying McNaughton's Theorem 2.2.64 and Proposition 4.4.13. However, we shall prove a stronger result, namely that the category Δ of δ-algebras is a reflective subcategory of the category MV of MV-algebras, i.e. the inclusion functor $\Delta \to \mathsf{MV}$ has a left adjoint. A description of the free objects in Δ is then obtained as a consequence.

We observe that Yosida's representation Theorem 3.2.18 can be translated into the language of MV-algebras. For this purpose, we agree to say that an MV-algebra A is *divisible* if its enveloping unital ℓ-group $\Xi(A)$ is a divisible group. Moreover, recall that every unital ℓ-group can be equipped with a seminorm induced by the strong order unit (see Section 3.2). Therefore, we can consider the seminorm induced on its unit interval, so that any MV-algebra can be naturally equipped with a seminorm. The latter is a norm if, and only if, the MV-algebra is semisimple (cf. Lemma 3.2.9). An MV-algebra is *complete* if it is complete in the norm induced by the norm of the enveloping unital ℓ-group $\Xi(A)$.

Remark 5.2.1. The definitions above make reference to the enveloping unital ℓ-group of the MV-algebra A. It is possible to give equivalent MV-algebraic definitions in the following way. Let $n \in \mathbb{N}$ be an arbitrary positive integer. There exists a term τ_n in the language of MV-algebras such that, for all $y \in A$, $\tau_n(y) = 0$ if, and only if,

$$\underbrace{y \oplus \cdots \oplus y}_{n \text{ times}} = \underbrace{y + \cdots + y}_{n \text{ times}}$$

in the enveloping ℓ-group (see [3] for details). It is easy to check that an MV-algebra A is divisible if, and only if, for all $x \in A$ and for all $n \in \mathbb{N}$, there exists a unique $y \in A$ such that

$$x = ny, \quad \tau_n(y) = 0.$$

If A is semisimple, we know that it is isomorphic to the separating subalgebra $\widehat{A}$ of $\mathrm{C}(\operatorname{Max} A, [0,1])$ by Theorem 2.2.72. One can show that, in this situation, A is divisible if, and only if, for all $\widehat{a} \in \widehat{A}$ and for all $k \in \mathbb{Q} \cap [0,1]$, the function $k\widehat{a}$ belongs to $\widehat{A}$. It is also possible to define a seminorm on a divisible MV-algebra without mentioning the unital ℓ-group A. For every $x \in A$, set

$$\|x\|_1 := \inf \{\tfrac{p}{q} \in \mathbb{Q} \cap [0,1] \mid x \leqslant p\tfrac{1}{q}\}.$$

Here $p\frac{1}{q}$ denotes the iterated $\oplus$-sum p times of the unique element $y \in A$ such that

$$qy = 1 \ \text{ and } \ \tau_q(y) = 0.$$

It is clear that an MV-algebra A is complete if, and only if, it is complete in the norm $\|\cdot\|_1$; in fact, the norm $\|\cdot\|_1$ coincides with the norm inherited by the norm of $\Xi(A)$.

Theorem 5.2.2. *An MV-algebra A is represented by a compact Hausdorff space X, i.e. $A \cong \mathrm{C}(X, [0,1])$, if, and only if, the following hold.*

1. *A is semisimple.*
2. *A is divisible.*
3. *A is complete.*

Proof. This follows at once from Theorem 3.2.18, along with the functor Γ. □

In other words, we can identify Δ with the full subcategory of MV whose objects are semisimple, divisible, and complete MV-algebras. In order to see that the latter category is a reflective subcategory of MV, we proceed by steps. Firstly, we shall prove that the category of semisimple MV-algebras and MV-homomorphisms is a reflective subcategory of MV. The reflector, i.e. the left adjoint to the inclusion functor, is the functor that maps an MV-algebra to the semisimple (see Lemma 2.2.50) MV-algebra

$$\frac{A}{\operatorname{Rad} A}.$$

Furthermore, the image of an MV-homomorphism $f\colon A \to B$ via the reflector is the induced map between the quotients, namely $\frac{a}{\operatorname{Rad} A} \mapsto \frac{f(a)}{\operatorname{Rad} B}$. The latter map is well-defined:

Remark 5.2.3. If $f\colon A \to B$ is an MV-homomorphism, then $f(x) \in \operatorname{Rad} B$ whenever $x \in \operatorname{Rad} A$. Indeed, by Proposition 2.2.52, if $n \in \mathbb{N}$ and $x \in \operatorname{Rad} A$, then $nx \leqslant \neg x$. By Remark 2.2.17, for all $n \in \mathbb{N}$,

$$nf(x) = f(nx) \leqslant f(\neg x) = \neg f(x).$$

We conclude, by Proposition 2.2.52, that $f(x) \in \operatorname{Rad} B$.

Lemma 5.2.4. *The category of semisimple MV-algebras is a reflective subcategory of the category* MV *of MV-algebras.*

Proof. Let A be an MV-algebra, and let B be a semisimple MV-algebra. Denote with $q\colon A \to \frac{A}{\operatorname{Rad} A}$ the quotient map. We must prove that, for every MV-homomorphism $f\colon A \to B$, there exists a unique MV-homomorphism $\tilde{f}\colon \frac{A}{\operatorname{Rad} A} \to B$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[dr, "f"'] & \frac{A}{\operatorname{Rad} A} \arrow[d, dashed, "\tilde{f}"] \\ & B \end{tikzcd}$$

For every equivalence class $[a] \in \frac{A}{\operatorname{Rad} A}$, define $\tilde{f}([a]) := f(a)$. We prove that this map is well-defined. Suppose that $b \in A$ satisfies $[a] = [b]$. This means that $d(a, b) \in \operatorname{Rad} A$, so that $f(d(a,b)) \in \mathrm{Rad}\, B = \{0\}$ by Remark 5.2.3. However, it is elementary that $f(d(a,b)) = d(f(a), f(b))$, hence $f(a) = f(b)$ by Proposition 2.2.28.(1). It is clear that $\tilde{f}$ is an MV-homomorphism, since so is f, and that it is the unique map such that $f = \tilde{f} \circ q$. □

If A is an arbitrary MV-algebra, it is not always the case that the abelian group $\Xi(A)$ is divisible. However, the latter group can be embedded in a divisible abelian group. Since every ℓ-group is torsion-free [34, Corollary 0.1.2] there is a canonical such group, the divisible hull of $\Xi(A)$, that we denote by $\Xi(A)^d$. In other words, for all $a \in \Xi(A)^d$ and for all $n \in \mathbb{N}$, there exists $b \in \Xi(A)^d$ such that $a = nb$. The (unique) element b is sometimes denoted by $\frac{a}{n}$. It is elementary that $\Xi(A)^d \cong \mathbb{Q} \otimes \Xi(A)$. It is easy to show that $\Xi(A)^d$ is an ℓ-group where, for all $\frac{a}{m}, \frac{b}{n} \in \Xi(A)^d$,

$$\frac{a}{m} \wedge \frac{b}{n} = \frac{na \wedge mb}{mn}, \tag{5.1}$$

$$\frac{a}{m} \vee \frac{b}{n} = \frac{na \vee mb}{mn}.$$

We prove only (5.1). Assume that $x \in \Xi(A)^d$ satisfies $x \leqslant \frac{a}{m}$ and $x \leqslant \frac{b}{n}$. Then $mx \leqslant a$ and $nx \leqslant b$, so that $mnx \leqslant na$ and $mnx \leqslant mb$. It follows that $mnx \leqslant na \wedge mb$. If y is the element in $\Xi(A)^d$ such that $mny = na \wedge mb$, we have $mnx \leqslant mny$, whence

$$x \leqslant y = \frac{na \wedge mb}{mn}.$$

Define the *divisible hull* of the MV-algebra A as the MV-algebra

$$A^d := \Gamma(\Xi(A)^d).$$

We remark that $\Xi(A^d) \cong \Xi(A)^d$.

Lemma 5.2.5. *The category of semisimple and divisible MV-algebras is a reflective subcategory of the category of semisimple MV-algebras.*

Proof. Let A be a semisimple MV-algebra, and consider the functor that sends A to the semisimple and divisible MV-algebra A^d (the behaviour of the latter functor on morphisms is clear). Denote by $i\colon A \to A^d$ the inclusion map. We prove that this functor is a reflector, i.e. for every semisimple and divisible MV-algebra B, and for every MV-homomorphism $f\colon A \to B$, there exists a unique MV-homomorphism $f^d\colon A^d \to B$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & A^d \arrow[d, dashed, "f^d"] \\ & B \end{tikzcd}$$

However, since the functor $\Gamma\colon \ell\mathsf{Grp_u} \to \mathsf{MV}$ is an equivalence (Theorem 2.3.29), it suffices to observe that there exists a unique group homomorphism $g\colon \Xi(A)^d \to \Xi(B)$ such that the next diagram is commutative.

$$\begin{tikzcd}
\Xi(A) \arrow[r, "\Xi(i)"] \arrow[dr, "\Xi(f)"'] & \Xi(A)^d \arrow[d, "g"] \\
& \Xi(B)
\end{tikzcd}$$

It is easy to prove that g is an ℓ-homomorphism since it extends the ℓ-homomorphism f. Therefore, the MV-homomorphism $f^d := \Gamma(g)$ has the required property. $\square$

Let A be a semisimple MV-algebra, equipped with the norm induced by the norm of the ℓ-group $\Xi(A)$. It is possible to show that the MV-algebraic operations of A are continuous with respect to the norm. Hence, there is an induced structure of MV-algebra on the norm-completion of A. We denote this completion by A^c. It is elementary that $A^c \cong \Gamma(\Xi(A)^c)$, where $\Xi(A)^c$ is the completion of the enveloping ℓ-group with respect to the norm induced by the strong order unit.

Lemma 5.2.6. *The category of semisimple, divisible and complete MV-algebras is a reflective subcategory of the category of semisimple and divisible MV-algebras.*

Proof. If A is a semisimple and divisible MV-algebra, denote by $j\colon A \to A^c$ the inclusion map. We must prove that, for every semisimple, divisible, and complete MV-algebra B and for every MV-homomorphism $f\colon A \to B$, there exists a unique MV-homomorphism $f^c\colon A^c \to B$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[dr, "f"'] & A^c \arrow[d, dashed, "f^c"] \\
& B
\end{tikzcd}$$

If b is an arbitrary element of A^c and $\{a_i\}_{i\in\mathbb{N}} \subseteq A$ is a sequence converging to b in the norm of A, define $f^c(b) := \lim_{i\in\mathbb{N}} f(a_i)$. Since the operations of A are continuous with respect to the norm, it is clear that f^c is an MV-homomorphism from A^c to B. Now, suppose that $g^c\colon A^c \to B$ is another MV-homomorphism satisfying $f = g^c \circ j$. Then we have two unital ℓ-homomorphisms $\Xi(f^c), \Xi(g^c)\colon \Xi(A^c) \to \Xi(B)$ satisfying

$$\Xi(f^c) \circ \Xi(j) = \Xi(f) = \Xi(g^c) \circ \Xi(j). \tag{5.2}$$

Since $\Xi(A^c), \Xi(B)$ are complete, divisible, and archimedean ℓ-groups, by Propositions 3.2.16 and 3.2.1 we know that $\Xi(A^c) \cong \mathrm{C}(\operatorname{Max}\Xi(A), \mathbb{R})$ (because $\Xi(A^c) \cong \Xi(A)^c$) and $\Xi(B) \cong \mathrm{C}(\operatorname{Max}\Xi(B), \mathbb{R})$. However, by Yosida duality every unital ℓ-homomorphism from $\mathrm{C}(\operatorname{Max}\Xi(A), \mathbb{R})$ to $\mathrm{C}(\operatorname{Max}\Xi(B), \mathbb{R})$ is of the form $- \circ h$ for some continuous map $h\colon \operatorname{Max}\Xi(B) \to \operatorname{Max}\Xi(A)$. Let h, h' be continuous maps such that $\Xi(f^c) = - \circ h$

and $\Xi(g^c) = -\circ h'$. Since A is a semisimple MV-algebra, it follows that $\Xi(A)$ is an archimedean ℓ-group, so that $\Xi(A)$ is ℓ-isomorphic to the separating ℓ-subgroup

$$\mathrm{Y}(\Xi(A)) = \{\widehat{g} \in \mathrm{C}(\mathrm{Max}\,\Xi(A), \mathbb{R}) \mid g \in \Xi(A)\}$$

of $\mathrm{C}(\mathrm{Max}\,\Xi(A), \mathbb{R})$. Then, (5.2) entails the following:

$$\text{for all } \widehat{g} \in \mathrm{Y}(G), \;\; \widehat{g} \circ h = \widehat{g} \circ h'. \tag{5.3}$$

Assume by contradiction that $h \neq h'$, i.e. there exists $\mathfrak{m} \in \mathrm{Max}\,\Xi(B)$ such that $h(\mathfrak{m}) \neq h'(\mathfrak{m})$. The ℓ-group $\mathrm{Y}(G)$ separates points, hence there exists $\widehat{g} \in \mathrm{Y}(G)$ such that $\widehat{g}(h(\mathfrak{m})) \neq \widehat{g}(h'(\mathfrak{m}))$, which contradicts (5.3). We conclude that $h = h'$, that is $\Xi(f^c) = \Xi(g^c)$. However Ξ is an equivalence (in particular it is faithful), thus $f^c = g^c$. ☐

Theorem 5.2.7. *The category Δ of δ-algebras is a full reflective subcategory of the category* MV *of MV-algebras. The reflector is provided by the functor*

$$\mathrm{R}\colon \mathsf{MV} \to \Delta, \;\; \mathrm{R}(A) := \left(\left(\tfrac{A}{\mathrm{Rad}\,A}\right)^d\right)^c.$$

Proof. The category Δ is a full subcategory of MV by Corollary 4.4.12. For each MV-algebra A, $\mathrm{R}(A)$ is the completion of the divisible hull of the MV-algebra $\frac{A}{\mathrm{Rad}\,A}$. Denote by $l\colon A \to \mathrm{R}(A)$ the composition of the maps

$$A \xrightarrow{\;q\;} \tfrac{A}{\mathrm{Rad}\,A} \xrightarrow{\;i\;} \left(\tfrac{A}{\mathrm{Rad}\,A}\right)^d \xrightarrow{\;j\;} \left(\left(\tfrac{A}{\mathrm{Rad}\,A}\right)^d\right)^c,$$

where q is the quotient map and i, j are the inclusion maps. Now, let A be an arbitrary MV-algebra. To prove the theorem, it suffices to show that for every complete, divisible, and semisimple MV-algebra B and for every MV-homomorphism $f\colon A \to B$, there exists a unique map $g\colon \mathrm{R}(A) \to B$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "l"] \arrow[dr, "f"'] & \mathrm{R}(A) \arrow[d, dashed, "g"] \\
& B
\end{tikzcd}$$

However, by Lemmas 5.2.4, 5.2.5 and 5.2.6, the following diagram is commutative.

$$\begin{tikzcd}
A \arrow[r, "q"] \arrow[rrrd, "f"'] & \frac{A}{\mathrm{Rad}\,A} \arrow[r, "i"] \arrow[rrd, "\tilde{f}"] & \left(\frac{A}{\mathrm{Rad}\,A}\right)^d \arrow[r, "j"] \arrow[rd, "\tilde{f}^d"] & \left(\left(\frac{A}{\mathrm{Rad}\,A}\right)^d\right)^c \arrow[d, "(\tilde{f}^d)^c"] \\
& & & B
\end{tikzcd}$$

Hence, to conclude it is enough to set $g := (\tilde{f}^d)^c$. In fact, in order to prove the theorem it suffices to recall that a composition of adjoint functors is an adjoint functor [52, Theorem 1 p. 101], and apply Lemmas 5.2.4, 5.2.5 and 5.2.6. ☐

Proposition 5.2.8. *The free δ-algebra on a set X of cardinality κ is the δ-algebra*

$$\mathrm{C}([0,1]^\kappa,[0,1]).$$

Proof. Let X be a set of cardinality κ, and let Free_κ denote the free MV-algebra on X. In view of Theorem 2.2.64, we can identify Free_κ with the MV-algebra of McNaughton functions on $[0,1]^\kappa$, i.e. the MV-algebra of all the continuous piecewise linear functions with integer coefficients from $[0,1]^\kappa$ to $[0,1]$. Then, Theorem 5.2.7 implies that the free δ-algebra on the set X is

$$\left(\left(\frac{\mathrm{Free}_\kappa}{\mathrm{Rad}\,\mathrm{Free}_\kappa}\right)^d\right)^c.$$

However, it is elementary that Free_κ is semisimple, so that $\frac{\mathrm{Free}_\kappa}{\mathrm{Rad}\,\mathrm{Free}_\kappa} \cong \mathrm{Free}_\kappa$. We remark that $(\mathrm{Free}_\kappa)^d$ is the MV-algebra of all the continuous $[0,1]$-valued piecewise linear functions with rational coefficients on $[0,1]^\kappa$. By Proposition 2.2.68 the MV-algebra Free_κ, a fortiori its divisible hull, separates points. The lattice version of Stone-Weierstrass Theorem 3.2.17, along with the functor Γ, entail that $(\mathrm{Free}_\kappa)^d$ is dense in the MV-algebra $\mathrm{C}([0,1]^\kappa,[0,1])$. Therefore,

$$\left((\mathrm{Free}_\kappa)^d\right)^c \cong \mathrm{C}([0,1]^\kappa,[0,1]).$$

□

Corollary 5.2.9. *The Lawvere-Linton theory $\mathcal{T}^{\aleph_1}$ of δ-algebras is equivalent to the category whose objects are the cubes $[0,1]^\lambda$, for every countable cardinal λ, and whose maps are all the continuous maps between cubes.*

Proof. Let $\mathcal{F}$ denote the full subcategory of Δ whose objects are the free δ-algebras on a countable set of generators. Since the free δ-algebra on a set of cardinality κ is isomorphic to $\mathrm{C}([0,1]^\kappa,[0,1])$ by Proposition 5.2.8, an arbitrary object of $\mathcal{F}$ is of the form $\mathrm{C}([0,1]^\lambda,[0,1])$ where λ is a countable cardinal. However, by the duality between δ-algebras and compact Hausdorff spaces (Theorem 4.4.25), it is clear that the dual of the δ-algebra $\mathrm{C}([0,1]^\lambda,[0,1])$ is the space $[0,1]^\lambda$. We conclude that $\mathcal{F}^{op}$ is the full subcategory of KHaus whose objects are all the cubes $[0,1]^\lambda$, with λ a countable cardinal number. Furthermore, by Proposition 5.1.6, the category Δ of δ-algebras is equivalent to the category of functors from $\mathcal{F}^{op}$ to Set which preserve countable products. □

Remark 5.2.10. Every category that is monadic over Set is complete and cocomplete [11, Corollary 2 p. 118, Proposition 4 p. 320]. Hence, in particular, the category Δ is complete and cocomplete. Recall that a subcategory D of a category C is said to be *closed* in C *under limits* (respectively *under colimits*) if the following property is satisfied: the limit (respectively colimit) of every diagram in D, computed in C is, in fact, in D. Since right adjoint functors preserve limits, every reflective subcategory is clearly closed under limits in the ambient category. Therefore, the category Δ is closed under limits in MV, by Theorem 5.2.7. However, it is easy to see that Δ is not closed in MV under colimits. General category-theoretic results show that the underlying-set functor of a variety commutes with directed colimits if, and only if, the variety is finitary [1, p.

149]. This means that there is a directed diagram in Δ whose colimit is computed in different ways in the categories Δ and MV. We explicitly describe this colimit. Consider the family of all the finitely generated free δ-algebras, partially ordered by inclusion. In fact, the latter set is a directed partially oredered set. One can prove that the colimit of this diagram in the category MV is the free MV-algebra on $\aleph_0$ generators, i.e. the algebra of the McNaughton functions on the cube $[0,1]^{\aleph_0}$. However, the colimit of the diagram in the category Δ is the free δ-algebra on $\aleph_0$ generators $\mathrm{C}([0,1]^{\aleph_0},[0,1])$. In particular, the latter is the completion of the divisible hull of the former. Another example is the following. Consider the directed set in Δ whose objects are all the δ-algebras of the form $\mathrm{C}([0,\frac{1}{n}],[0,1])$, for $n \in \mathbb{N}$. It is possible to show that the colimit of the latter diagram in Δ is the δ-algebra $[0,1]$, while the colimit in MV is not semisimple.

We close this chapter by providing some universal-algebraic information on the category Δ, regarded as a variety of algebras.

Lemma 5.2.11. *The δ-algebra $[0,1]$ generates Δ as a variety, and as a quasivariety.*

Proof. It is enough to observe that every δ-algebra is isomorphic to an algebra of the form $\mathrm{C}(X,[0,1])$ for some compact Hausdorff space, by Theorem 4.4.16. This means that every δ-algebra is a subalgebra of the δ-algebra $[0,1]^X$ (where operations in the latter algebra are defined pointwise). In other words,

$$\mathbb{SP}([0,1]) = \Delta,$$

and the lemma is proved. $\square$

Lemma 5.2.12. *The variety of δ-algebras does not admit any proper non-trivial subvariety.*

Proof. Let V be a non-trivial subvariety of Δ. By Lemma 5.2.11 it suffices to prove that the δ-algebra $[0,1]$ belongs to V. Since V is non-trivial, there is a non-trivial algebra $\mathrm{C}(X,[0,1])$ in V, i.e. $X \neq \varnothing$. Then the subalgebra of $\mathrm{C}(X,[0,1])$ generated by $\neg 0 = 1$ is isomorphic to $[0,1]$. $\square$

Lemma 5.2.13. *The unique non-trivial simple δ-algebra is $[0,1]$.*

Proof. Suppose A is a non-trivial simple δ-algebra, i.e. the only congruences on A are the trivial one and the improper one. Then the MV-reduct of A is a simple MV-algebra, so that A is MV-isomorphic to a subalgebra of $[0,1]$ by Theorem 2.2.44. It is elementary that $A = [0,1]$. $\square$

Remark 5.2.14. Recall that an algebra A is a *subdirect product* of the family of algebras $\{A_i\}_{i\in I}$ if A is isomorphic to a subalgebra of the product $\prod_{i\in I} A_i$, i.e. there is an injective homomorphism $\alpha\colon A \to \prod_{i\in I} A_i$, and $\pi_i \circ \alpha(A) = A_i$ for all $i \in I$ (where $\pi_i\colon \prod_{i\in I} A_i \to A_i$ is the ith projection). An algebra A is *subdirectly irreducible* if, whenever it is represented as the subirect product of a family $\{A_i\}_{i\in I}$, there is $i \in I$ such that $A \cong A_i$. G. Birkhoff proved in [14] that every *finitary* algebra is a subdirect product of subdirectly irreducible algebras. Moreover, he remarked that this result cannot be extended, in general, to infinitary algebras. However, since every δ-algebra is a subdirect product of copies of $[0, 1]$ by Theorem 4.4.16, it is clear that the unique subdirectly irreducible δ-algebra is the δ-algebra $[0, 1]$. This shows that every δ-algebra is a subdirect product of subdirectly irreducible δ-algebras.

Chapter 6

C*-algebras

6.1 Banach algebras

The theory of Banach algebras can be treated in four different ways, by studying either real commutative or non-commutative algebras, or complex commutative or non-commutative algebras. When considering C*-algebras (=Banach algebras with an involution satisfying the C*-identity), these four paths lead to four different representation theorems. These approaches are clearly exposed in the monograph [36]. Traditionally, the highest attention was paid to the complex case, in connection with the investigation of operator algebras. In this context, the two main representation results are due to Gelfand and Neumark. The first one states that every complex commutative unital C*-algebra is the algebra of all continuous $\mathbb{C}$-valued functions on a compact Hausdorff space [32, Lemma 1]. The second one allows to represent every complex (possibly non-commutative) unital C*-algebra as a (norm and adjoint)-closed algebra of *some* bounded operators on a complex Hilbert space [32, Theorem 1]. For a thorough treatment of the Gelfand-Neumark representation theorems, along with a historical introduction to the subject, the interested reader is referred to [26].

We will focus on the complex commutative case only. Since many central constructions in the theory of C*-algebras (e.g. the Gelfand transform) can be carried out more generally for Banach algebras, we shall begin studying the latter algebras. We will not restrict ourselves to C*-algebras, until their extra structure is needed (e.g. to show that the Gelfand transform of a C*-algebra is a *-isomorphism).

6.1.1 Introduction

Definition 6.1.1. A *complex commutative Banach algebra* is a complex Banach space A with a product (denoted by a dot $x \cdot y$ or by juxtaposition xy), satisfying the following conditions.

1. $xy = yx$.

2. $x(yz) = (xy)z$.
3. $x(y+z) = xy + xz$.
4. If $x_n \to x$, then $x_n y \to xy$ (and consequently, by commutativity, if $y_n \to y$, then $xy_n \to xy$), i.e. the product operation is continuous.

Remark 6.1.2. In item 4 of the previous definition, the notation $x_n \to x$ means that the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq A$ converges *in the metric induced by the norm* to the element $x \in A$. In general, when dealing with metric properties of a Banach algebra, we will always implicitly refer to the metric induced by the norm.

Notation 6.1.3. Henceforth, by a Banach algebra, we understand a complex commutative Banach algebra.

Definition 6.1.4. A *unit* of a Banach algebra A is an element $\mathsf{e} \in A$ such that $\mathsf{e}\, x = x$ for every $x \in A$ (the equality $x\,\mathsf{e} = x$ follows by commutativity). If A has a unit, then it is said to be a *unital* Banach algebra.

It is not always the case that a Banach algebra has a unit, however it is always possible to *add* it, in the following way. Consider the set $A_{\mathsf{e}} := A \times \mathbb{C}$ of pairs (a, λ), where a is an element of the algebra and λ is a complex number. We can equip this set with operations

$$(x_1, \lambda_1) + (x_2, \lambda_2) := (x_1 + x_2, \lambda_1 + \lambda_2),$$

$$\text{for all } \mu \in \mathbb{C}, \quad \mu(x, \lambda) := (\mu x, \mu\lambda),$$

$$(x_1, \lambda_1) \cdot (x_2, \lambda_2) := (x_1 x_2 + \lambda_1 x_2 + \lambda_2 x_1, \lambda_1 \lambda_2).$$

Furthermore, we can define a norm on A_{e}, by setting

$$\|(x, \lambda)\| := \|x\| + |\lambda|.$$

It is easy to verify that

Lemma 6.1.5. *A_{e} is a Banach algebra. Moreover, its subalgebra with underlying set $\{(x, 0) \in A_{\mathsf{e}} \mid x \in A\}$ is isomorphic and isometric to A.* □

Observe that the algebra A_{e} has a unit, namely the element $(0, 1)$. Indeed, for all $(x, \lambda) \in A_{\mathsf{e}}$,

$$(0, 1) \cdot (x, \lambda) = (0 \cdot x + 0 \cdot \lambda + 1 \cdot x, 1 \cdot \lambda) = (x, \lambda).$$

In other words, every Banach algebra A can be embedded in a unital Banach algebra A_{e}. Hence:

Notation 6.1.6. Throughout this chapter, unless stated otherwise we assume that every Banach algebra has a unit e. Therefore, by a Banach algebra we understand a unital Banach algebra. We do not require that the condition $\mathsf{e} \neq 0$ is satisfied, so that $\{0 = \mathsf{e}\}$ is a Banach algebra.

Some authors (see for example [16]) assume, as part of the definition of a Banach algebra, that the norm is submultiplicative, i.e. $\|xy\| \leqslant \|x\| \cdot \|y\|$. Another property of unital

Banach algebras, that is usually required, is that $\|\mathsf{e}\| = 1$. In fact, these conditions are not necessary, in the sense that given any Banach algebra A, there is an equivalent norm on A satisfying the conditions above. We first recall what it means for two norms to be equivalent. Given a normed space $(X, \|\cdot\|)$, we say that a norm $|||\cdot|||$ on X is *equivalent* to $\|\cdot\|$ if there exist real numbers $a, b > 0$ such that $a\|x\| \leqslant |||x||| \leqslant b\|x\|$ for every $x \in X$. Equivalent norms preserve convergence and limits of sequences, hence they induce the same topology.

Lemma 6.1.7. *Given a Banach algebra A, there exists an equivalent norm $\|\cdot\|$ on A such that $\|xy\| \leqslant \|x\| \cdot \|y\|$ for all $x, y \in A$, and $\|\mathsf{e}\| = 1$.*

Proof. The proof goes as follows: we shall construct a Banach algebra A' whose elements are some bounded linear operators from A to itself, and show that there is an isomorphism of linear spaces $T\colon A \to A'$ that is continuous with continuous inverse. The norm on A' will satisfy the required conditions, and so will the norm induced on A, that is $\|x\| := \|Tx\|$. For each element $x \in A$, define the operator $V_x\colon A \to A$ as $V_x y := x \cdot y$ for all $y \in A$. Clearly V_x is a linear operator, that is $V_x(y_1 + y_2) = V_x y_1 + V_x y_2$ and $V_x(\lambda y) = \lambda V_x y$, and it satisfies $V_{x_1 x_2} y = V_{x_1} y \cdot V_{x_2} y$. It is also a bounded operator: for any sequence $y_n \to y$, we have $V_x y_n = x y_n \to xy = V_x y$, hence V_x is bounded, because a linear operator between normed spaces is continuous if, and only if, it is bounded [50, Theorem 7A]. In this way, we get a collection of continuous operators $\{V_x\}_{x\in A}$ contained in the set $\mathsf{L}(A)$ of all bounded linear operators from A to itself. $\mathsf{L}(A)$ is a Banach space since A is a Banach space [50, 7B], and $\{V_x\}_{x\in A}$ is a linear submanifold of $\mathsf{L}(A)$. Denote $A' := \{V_x\}_{x\in A}$ and notice that, for all $V_x \in A'$, the identity $V_x(yz) = V_x y \cdot z$ holds since $V_x(yz) = x(yz) = (xy)z = (V_x y) \cdot z$. We will show that this property completely characterises the set A'. Assume that an operator $V\colon A \to A$ satisfies $V(xy) = V(y) \cdot z$ for all $y, z \in A$. We shall find an element $x \in A$ such that $V = V_x$. Consider $x := V\mathsf{e}$, where $\mathsf{e} \in A$ is the unit of A; then, for all $y \in A$, we have $Vy = V(\mathsf{e}\, y) = V\mathsf{e} \cdot y = x \cdot y$. In other words, we have shown that $V = V_x$, where $x = V\mathsf{e}$. Next, we prove that A' is norm-closed in $\mathsf{L}(A)$, so that it is a Banach space. Given $\{V_n\} \subseteq A'$ and $y \in A$, if $V_n y \to Vy$, then $V \in A'$; indeed, by the characteristic property, $V_n(y_1 y_2) = (V_n y_1) \cdot y_2$ for all $n \in \mathbb{N}$. Further $V_n \in A'$, hence there exists x_n such that $V_n y = x_n y$. The condition $V_n y \to Vy$ is equivalent to $x_n y \to xy$. For $y = \mathsf{e}$ we have $V_n \mathsf{e} = x_n \mathsf{e} = x_n$, and $x_n = V_n \mathsf{e} \to V\mathsf{e}$. Upon writing x for $V\mathsf{e}$, the sequence $x_n y = V_n y$ converges both to Vy and to xy. We conclude that $Vy = xy$, that is $V = V_x$, whence A' is closed under pointwise convergence. On the set A' there is a multiplication given by $V_x V_y = V_{xy}$, since $V_x V_y z = V_x(yz) = x(yz) = (xy)z = V_{xy} z$. The operator $\widehat{T}\colon A' \to A$ defined by $\widehat{T}\colon V_x \mapsto x$ is a bijective linear operator. It is also bounded, indeed

$$\|V_x\| = \sup_{\|y\|\leqslant 1} \|V_x y\| = \sup_{\|y\|\leqslant 1} \|xy\| \geqslant \left\| x \cdot \frac{\mathsf{e}}{\|\mathsf{e}\|} \right\| = \frac{1}{\|\mathsf{e}\|} \cdot \|x\|,$$

so that $\|x\| \leqslant \|\mathsf{e}\| \cdot \|V_x\|$, which says that $\widehat{T}$ is bounded by the constant $\|\mathsf{e}\|$. Since $\widehat{T}$ is a bijective bounded linear operator, by the Inverse Mapping theorem [22, Theorem 12.5 p. 91], the operator $T := \widehat{T}^{-1}\colon A \to A'$ is bounded. We conclude that T is a continuous isomorphism of linear spaces with continuous inverse. Finally, it is easy to see that the norm on A', namely the operator norm

$$\|V_x\| := \inf\{M \in \mathbb{R} \mid \|V_x y\| \leqslant M\|y\| \text{ for all } y \in A\},$$

satisfies $\|V_x V_y\| \leqslant \|V_x\| \cdot \|V_y\|$ for every $V_x, V_y \in A'$, and $\|T\mathsf{e}\| = \|V_\mathsf{e}\| = 1$. The equivalent norm on A, defined by $\|x\| := \|Tx\|$ satisfies the two conditions of the statement. □

Corollary 6.1.8. *Let A be a Banach algebra, let $\{x_n\}, \{y_n\} \subseteq A$, and let $x, y \in A$. If $x_n \to x$ and $y_n \to y$, then $x_n y_n \to xy$.*

Proof. Consider an equivalent norm on A satisfying $\|xy\| \leqslant \|x\| \cdot \|y\|$. Such a norm exists by Lemma 6.1.7. Then

$$\begin{aligned}\|x_n y_n - xy\| &= \|(x_n y_n - x_n y) + (x_n y - xy)\| \\ &\leqslant \|x_n y_n - x_n y\| + \|x_n y - xy\| \\ &= \|x_n(y_n - y)\| + \|y(x_n - x)\| \\ &\leqslant \|x_n\| \cdot \|y_n - y\| + \|y\| \cdot \|x_n - x\|.\end{aligned}$$

By hypothesis, $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$, as $n \to \infty$. Moreover $\|x\|, \|y\|$ are constants, hence $\|x_n y_n - xy\| \to 0$ as $n \to \infty$, i.e. $x_n y_n \to xy$. □

Henceforth, in view of Lemma 6.1.7, we assume that the norm on any Banach algebra satisfies the conditions $\|xy\| \leqslant \|x\| \cdot \|y\|$ and $\|\,\mathsf{e}\,\| = 1$.

Recall that a bounded operator $T\colon X \to Y$ between normed spaces is said to be *non-extensive* if $\|T\| \leqslant 1$, i.e. $\|Tx\| \leqslant \|x\|$ for all $x \in X$. A non-extensive operator is sometimes called a *weak contraction*. Further, recall that a map between unital complex algebras is a *homomorphism (of unital complex algebras)* if it preserves addition, multiplication, scalar multiplication, and the unit.

Definition 6.1.9. A map $f\colon A \to B$ between Banach algebras is a *Banach homomorphism*, provided that it is a non-extensive homomorphism of complex algebras.

In particular, every Banach homomorphism is continuous.

Remark 6.1.10. Observe that every bijective Banach homomorphism is an isometry. In fact, let T be a bijective bounded operator between Banach spaces. Then the inverse operator T^{-1} is a bounded by the Inverse Mapping theorem [22, Theorem 12.5 p. 91]. Moreover,

$$1 = \|TT^{-1}\| \leqslant \|T\| \cdot \|T^{-1}\|,$$

whence $\|T^{-1}\| \geqslant \|T\|^{-1} \geqslant 1$. If T is a bijective Banach homomorphism, then $1 \geqslant \|T\| \geqslant \|T^{-1}\|^{-1} \geqslant 1$, meaning that T is an isometry. In other words, a bijective homomorphism of complex algebras is a Banach isomorphism if, and only if, it is isometric.

We now give some examples of Banach algebras.

Example 6.1.11. Let $\mathrm{C}([0,1],\mathbb{C})$ be the set of all the $\mathbb{C}$-valued continuous functions on the real interval $[0,1]$. This is a (commutative) unital Banach algebra, whose norm is given by the uniform norm

$$\|f\|_\infty := \sup_{0\leqslant t\leqslant 1} |f(t)| = \max_{0\leqslant t\leqslant 1} |f(t)|,$$

for every $f \in \mathrm{C}([0,1],\mathbb{C})$. The multiplication of the algebra is the pointwise multiplication, and the unit is the constant function of value 1, i.e. $\mathsf{e} = 1_{[0,1]}$. The inequality $\|fg\| \leqslant \|f\|\cdot\|g\|$ is easily seen to be satisfied, and the unit e has obviously norm 1. More generally, these observations hold for an arbitrary compact Hausdorff space X, not only for the space $X = [0,1]$. In other terms, $\mathrm{C}(X,\mathbb{C})$ is a (commutative) unital Banach algebra.

Notation 6.1.12. In this chapter, the symbol $\mathrm{C}(X)$ will always denote the Banach algebra $\mathrm{C}(X,\mathbb{C})$ of all the $\mathbb{C}$-valued continuous functions on the compact Hausdorff space X.

Example 6.1.13. If X is a Banach space, and $\mathsf{L}(X)$ is the set of all bounded linear operators from X to itself, then $\mathsf{L}(X)$ is a non-commutative Banach algebra. In fact it is a Banach space by [50, 7B] if the norm of $T \in \mathsf{L}(X)$ is defined, as usual, as the *operator norm*

$$\|T\| := \inf\{M \in \mathbb{R} \mid \|Tx\| \leqslant M\|x\| \text{ for all } x \in X\}.$$

Furthermore, the multiplication on $\mathsf{L}(X)$ is taken to be the composition of operators, and the unit element is the identity operator on X. It is easy to see that, equipped with this structure, $\mathsf{L}(X)$ is a Banach algebra which is, in general, non-commutative.

Example 6.1.14. Let D_n denote the subset of $\mathrm{C}([0,1])$ whose elements admit continuous nth derivative. We remark that, if $f \in \mathsf{D}_n$, then f belongs to D_m for every $0 \leqslant m \leqslant n$. Define the following norm on the set D_n. For every element $f \in \mathsf{D}_n$,

$$\|f\| := \sum_{k=0}^{n} \max_{0\leqslant t\leqslant 1} |f^{(k)}(t)|.$$

The latter norm is not submultiplicative. Indeed, consider the function $f(t) := t$, and take $g = f$. Then $fg\colon t \mapsto t^2$, and

$$\|fg\| = \max_{0\leqslant t\leqslant 1} |t^2| + \max_{0\leqslant t\leqslant 1} |2t| + \max_{0\leqslant t\leqslant 1} |2| = 5,$$

while

$$\|f\|\cdot\|g\| = (\|f\|)^2 = \left(\max_{0\leqslant t\leqslant 1} |t| + \max_{0\leqslant t\leqslant 1} |1|\right)^2 = 4.$$

In this case we can consider an equivalent submultiplicative norm on D_n, namely

$$|\!|\!|f|\!|\!| := \sum_{k=0}^{n} \frac{\max_{0\leqslant t\leqslant 1} |f^{(k)}(t)|}{k!}.$$

It is possible to show that D_n is a Banach space, and a Banach algebra if multiplication is defined pointwise.

6.1.2 Spectrum and Gelfand-Mazur theorem

Proposition 6.1.15. *Let A be a Banach algebra, and let*

$$G := \{x \in A \mid \textit{there exists } x^{-1} \in A\}$$

be the set of invertible elements of A. The following hold.

1. *G is an open subset of A.*
2. *The map $x \mapsto x^{-1}$, from G to itself, is continuous.*

Proof. If A is the trivial Banach algebra $\{0 = \mathsf{e}\}$, there is nothing to prove. Otherwise, the set G is non-empty because $\mathsf{e} \in G$. We shall first prove that, for all $x \in A$ such that $\|x\| < 1$, there exists $(\mathsf{e} - x)^{-1} \in A$. In other words, a neighborhood of e consists of invertible elements. Consider the series

$$\mathsf{e} + x + x^2 + x^3 + \cdots$$

in the Banach algebra A: it is convergent, since it satisfies Cauchy's convergence test. Indeed, from the inequality $\|xy\| \leqslant \|x\| \cdot \|y\|$, it easily follows by induction that $\|x^i\| \leqslant \|x\|^i$ for every $i \in \mathbb{N}$. Upon denoting $q := \|x\|$, we have $\|x^i\| \leqslant q^i$. The condition $q < 1$ entails

$$\left\| \sum_{i=n+1}^{m} x^i \right\| \leqslant \sum_{i=n+1}^{m} \|x^i\| \leqslant \sum_{i=n+1}^{m} q^i \to 0, \quad \text{as } n \to \infty.$$

A Banach algebra is complete, so that $\mathsf{e} + \sum_{i=1}^{\infty} x^i =: y \in A$. We claim that $y = (\mathsf{e} - x)^{-1}$; this will show that $(\mathsf{e} - x)^{-1} \in A$. In fact, the equality $(\mathsf{e} - x)y = \mathsf{e}$ holds because

$$\begin{aligned}
(\mathsf{e} - x)y &= (\mathsf{e} - x) \lim_{n\to\infty} (\mathsf{e} + x + x^2 + \cdots + x^{n-1}) \\
&= \lim_{n\to\infty} (\mathsf{e} - x)(\mathsf{e} + x + x^2 + \cdots + x^{n-1}) \\
&= \lim_{n\to\infty} (\mathsf{e} + x + x^2 + \cdots + x^{n-1} - x - x^2 - x^3 - \cdots - x^n) \\
&= \lim_{n\to\infty} (\mathsf{e} - x^n) \\
&= \mathsf{e}\,.
\end{aligned}$$

Here x^n goes to 0 as $n \to \infty$, because the condition $q < 1$ implies $\|x^n\| \leqslant \|x\|^n \leqslant q^n$, i.e. $\lim_{n\to\infty} \|x^n\| = 0$. Now, to prove item 1, pick an element $x \in G$. We will show that the open neighborhood

$$\{x + h \mid h \in A,\ \|h\| < \tfrac{1}{2\|x^{-1}\|}\}$$

of x is contained in G, that is for every such h there exists $(x + h)^{-1} \in A$. We remark that $(x + h) = x(\mathsf{e} + x^{-1}h)$, and

$$\|x^{-1}h\| \leqslant \|x^{-1}\| \cdot \|h\| < \tfrac{\|x^{-1}\|}{2\|x^{-1}\|} = \tfrac{1}{2} < 1.$$

We proved above that, if $\|z\| < 1$, then $\mathsf{e} - z$ is invertible. Therefore the element $\mathsf{e} + x^{-1}h$ is invertible, and so is $x + h$ since

$$(x+h)x^{-1}(\mathsf{e} + x^{-1}h)^{-1} = (x+h)(x+h)^{-1} = \mathsf{e}\,.$$

For item 2, let h be an element of A satisfying the condition $\|h\| \leqslant \frac{1}{2\|x^{-1}\|}$. Then

$$\begin{aligned}\|(x+h)^{-1} - x^{-1}\| &= \|x^{-1}(\mathsf{e} + x^{-1}h)^{-1} - x^{-1}\| \\ &= \|x^{-1}((\mathsf{e} + x^{-1}h)^{-1} - \mathsf{e})\| \\ &\leqslant \|x^{-1}\| \cdot \|(\mathsf{e} + x^{-1}h)^{-1} - \mathsf{e}\,\|.\end{aligned}$$

Setting $y := x^{-1}h$, we find that the norm of the element y does not exceed 1, indeed we previously observed that $\|y\| \leqslant \frac{1}{2}$. Reasoning as above we have $(\mathsf{e} - y)^{-1} = \mathsf{e} + y + y^2 + y^3 + \cdots$, whence

$$(\mathsf{e} + y)^{-1} = (\mathsf{e} - (-y))^{-1} = \mathsf{e} - y + y^2 - y^3 + \cdots\,.$$

Consequently,

$$\begin{aligned}\|x^{-1}\| \cdot \|(\mathsf{e} + y)^{-1} - \mathsf{e}\,\| &= \|x^{-1}\| \cdot \| - y + y^2 - y^3 + y^4 - \cdots \| \\ &= \|x^{-1}\| \cdot \|y(-e + y - y^2 + y^3 - \cdots)\| \\ &\leqslant \|x^{-1}\| \cdot \|y\| \cdot \| - e + y - y^2 + y^3 - \cdots \| \\ &\leqslant \|x^{-1}\| \cdot \|x^{-1}\| \cdot \|h\| \cdot \| - e + y - y^2 + y^3 - \cdots \| \\ &\leqslant \|x^{-1}\|^2 \cdot \|h\| \cdot (\| - \mathsf{e}\,\| + \|y\| + \| - y^2\| + \|y^3\| + \cdots) \\ &\leqslant \|x^{-1}\|^2 \cdot \|h\| \cdot (1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots) \\ &= 2\|x^{-1}\|^2 \cdot \|h\|.\end{aligned}$$

To sum up, we showed that $\|(x+h)^{-1} - x^{-1}\| \leqslant 2\|x^{-1}\|^2 \cdot \|h\|$. It is clear that the real number $\|(x+h)^{-1} - x^{-1}\|$ goes to 0 as $h \to 0$. We conclude that the function $x \mapsto x^{-1}$ is continuous (in fact, it is Lipschitz continuous). □

We state, for future reference, a useful fact that we showed in the proof of the foregoing proposition.

Lemma 6.1.16. *Let A be a non-trivial Banach algebra, and let $x \in A$. If $\|x\| < 1$, then the element $\mathsf{e} - x$ is invertible.* □

Definition 6.1.17. Let A be a non-trivial Banach algebra, and let $x \in A$. The *spectrum* of x is the set of complex numbers $\lambda \in \mathbb{C}$ for which $x - \lambda\,\mathsf{e}$ is not invertible. In symbols,

$$\sigma_x := \{\lambda \in \mathbb{C} \mid (x - \lambda\,\mathsf{e})^{-1} \text{ does not exist}\} \subseteq \mathbb{C}.$$

The set $\Omega_x := \mathbb{C} \setminus \sigma_x$ is called the *resolvent set* of x. If $\lambda \in \Omega_x$, the element $R_\lambda := (x - \lambda\,\mathsf{e})^{-1} \in A$ is a *resolvent* for x.

Remark 6.1.18. If A is the trivial Banach algebra $\{0 = \mathsf{e}\}$, we set $\sigma_0 := \varnothing$.

Example 6.1.19. Consider the Banach algebra $\mathrm{C}(X)$, for some compact Hausdorff space X. If $f \in \mathrm{C}(X)$, then $\sigma_f = f(X)$. On the one hand, if $\lambda \in f(X)$, then the function $f - \lambda 1_X$ has a zero, hence it is not invertible. This shows that $f(X) \subseteq \sigma_f$. On the other hand, if $\lambda \notin f(X)$, then the function $f - \lambda 1_X$ never vanishes on the space X. Thus its inverse is a well-defined continuous function on X, so that $\lambda \notin \sigma_f$.

Lemma 6.1.20. *Let $x \in A$ be an element of a non-trivial Banach algebra, and let $\lambda \in \mathbb{C}$. If $|\lambda| > \|x\|$, then $\lambda \in \Omega_x$, i.e. $x - \lambda\,\mathsf{e}$ is invertible.*

Proof. Write $x - \lambda\,\mathsf{e} = \lambda\left(\frac{x}{\lambda} - \mathsf{e}\right)$. Then $\|\frac{x}{\lambda}\| = \frac{\|x\|}{|\lambda|} < 1$, because $|\lambda| > \|x\|$. By Lemma 6.1.16 the element $\frac{x}{\lambda} - \mathsf{e}$ is invertible. We conclude that $(x - \lambda\,\mathsf{e})^{-1} \in A$, since

$$(x - \lambda\,\mathsf{e})^{-1} = \left(\lambda\left(\tfrac{x}{\lambda} - \mathsf{e}\right)\right)^{-1} = \lambda^{-1}\left(\tfrac{x}{\lambda} - \mathsf{e}\right)^{-1}.$$

□

Corollary 6.1.21. *If A is a Banach algebra and $x \in A$, then σ_x is a bounded subset of $\mathbb{C}$.*

Proof. If A is the trivial Banach algebra $\{0 = \mathsf{e}\}$, then $\sigma_0 = \varnothing \subseteq \mathbb{C}$ is clearly bounded. If A is non-trivial and $\lambda \in \sigma_x$, then $|\lambda| \leqslant \|x\|$ by Lemma 6.1.20. □

Lemma 6.1.22. *If A is a Banach algebra and $x \in A$, then Ω_x is an open subset of $\mathbb{C}$.*

Proof. The statement is trivial if A is the trivial Banach algebra $\{0 = \mathsf{e}\}$. Let $\varphi\colon \mathbb{C} \to A$ be the map given by $\varphi(\lambda) := x - \lambda\,\mathsf{e}$. It is easy to see that φ is a continuous function. Further, it is elementary that $\Omega_x = \varphi^{-1}(G)$, where $G \subseteq A$ is the set of invertible elements of A. The set G is open by Proposition 6.1.15.(1), hence its preimage Ω_x is an open subset of $\mathbb{C}$. □

Corollary 6.1.23. *If A is a Banach algebra and $x \in A$, then σ_x is a closed subset of $\mathbb{C}$.* □

Corollary 6.1.24. *If A is a Banach algebra and $x \in A$, then σ_x is a compact subset of $\mathbb{C}$.*

Proof. This follows at once from the Heine-Borel theorem [5, Theorem 3.30], since σ_x is a closed and bounded subset of $\mathbb{C}$ by Corollaries 6.1.21 and 6.1.23. □

Lemma 6.1.25 (Hilbert Identity)**.** *Let A be a Banach algebra, and let $x \in A$. Then, for all $\lambda, \mu \in \Omega_x$,*

$$R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu.$$

Proof. Writing $(\lambda - \mu)\,\mathsf{e} = (x - \mu\,\mathsf{e}) - (x - \lambda\,\mathsf{e})$, and recalling that $R_\lambda = (x - \lambda\,\mathsf{e})^{-1}$ and $R_\mu = (x - \mu\,\mathsf{e})^{-1}$, we see that

$$R_\mu(\lambda - \mu)\,\mathsf{e}\,R_\lambda = R_\mu(x - \mu\,\mathsf{e})R_\lambda - R_\mu(x - \lambda\,\mathsf{e})R_\lambda$$

if, and only if,

$$(\lambda-\mu)R_\mu R_\lambda\,\mathsf{e} = \mathsf{e}\,R_\lambda - R_\mu\,\mathsf{e}\,.$$

Therefore $(\lambda-\mu)R_\mu R_\lambda = R_\lambda - R_\mu$. □

Corollary 6.1.26. *Let A be a Banach algebra, and let $x \in A$. The function $R\colon \Omega_x \to A$, defined by $R(\lambda) := R_\lambda$, is analytic on Ω_x.*

Proof. By Lemma 6.1.25, we know that

$$\tfrac{R_\lambda - R_\mu}{\lambda-\mu} = R_\mu R_\lambda.$$

For $\mu \to \lambda$, if the limit of the ratio exists, it coincides with the derivative of R evaluated in $\lambda \in \Omega_x$. Firstly, observe that, if $\mu \to \lambda$, then $R_\mu \to R_\lambda$. Indeed, assuming that $\mu \to \lambda$, we see that $(x-\mu\,\mathsf{e}) \to (x-\lambda\,\mathsf{e})$. Hence

$$R_\mu = (x-\mu\,\mathsf{e})^{-1} \to (x-\lambda\,\mathsf{e})^{-1} = R_\lambda,$$

by the continuity of the function $x \mapsto x^{-1}$ (Proposition 6.1.15.(2)). We conclude that $R_\mu R_\lambda \to (R_\lambda)^2$, i.e. R is differentiable in every point $\lambda \in \Omega_x$, with derivative $R'(\lambda) = (R_\lambda)^2$. □

Theorem 6.1.27. *Let $x \in A$ be an element of a non-trivial Banach algebra. Then the spectrum of x is non-empty, i.e. $\sigma_x \neq \varnothing$.*

Proof. Suppose by contradiction that there is $x \in A$ such that $\sigma_x = \varnothing$ or, equivalently, $\Omega_x = \mathbb{C}$. Then the function $R\colon \lambda \mapsto (x-\lambda\,\mathsf{e})^{-1}$ is analytic on the whole complex plane. Fix a real number K strictly greater than $\|x\|$, and assume that $|\lambda| > K$. We have

$$(x-\lambda\,\mathsf{e})^{-1} = \lambda^{-1}\left(\tfrac{x}{\lambda} - \mathsf{e}\right)^{-1},$$

where $\|\frac{x}{\lambda}\| = \frac{\|x\|}{|\lambda|} < 1$. As $|\lambda| \to \infty$, we have $\frac{x}{\lambda} - \mathsf{e} \to -\,\mathsf{e}$, and consequently $\left(\frac{x}{\lambda} - \mathsf{e}\right)^{-1} \to (-\,\mathsf{e})^{-1} = -\,\mathsf{e}$. Since $\left(\frac{x}{\lambda} - \mathsf{e}\right)^{-1}$ converges, $\|\left(\frac{x}{\lambda} - \mathsf{e}\right)^{-1}\|$ is bounded. It follows that $\|R(\lambda)\|$ is also bounded, because

$$\|R_\lambda\| = \tfrac{1}{|\lambda|}\|\left(\tfrac{x}{\lambda} - \mathsf{e}\right)^{-1}\|.$$

Now, we deal with the case $|\lambda| \leqslant K$. The function R is continuous, even differentiable, and the norm function $\|\cdot\|$ is continuous. Since the domain $\{\lambda \in \mathbb{C} \mid |\lambda| \leqslant K\}$ is compact, the function $\lambda \mapsto \|R(\lambda)\|$ is bounded. We have proved that, for all $\lambda \in \mathbb{C}$, the function $\|R(\lambda)\|$ is bounded. Liouville's theorem [4, p. 122] states that an analytic bounded function on $\mathbb{C}$ is constant. In our case this constant must be 0, because $R_\lambda = (x-\lambda\,\mathsf{e})^{-1} = \lambda^{-1}\left(\frac{x}{\lambda} - \mathsf{e}\right)^{-1} \to 0$, as $|\lambda|$ goes to infinity. In other words, $R\colon \Omega_x \to A$ is the constant function of value $0 \in A$. But this leads to a contradiction, namely

$$\mathsf{e} = (x-\lambda\,\mathsf{e})(x-\lambda\,\mathsf{e})^{-1} = (x-\lambda\,\mathsf{e})R_\lambda = 0.$$

□

Thus, for every element $x \in A$ of a non-trivial Banach algebra, there exists a complex number λ such that the element $x - \lambda\,\mathsf{e}$ is not invertible. As a consequence, we obtain the following

Theorem 6.1.28 (Gelfand-Mazur)**.** *Let A be a non-trivial Banach algebra in which every non-zero element is invertible. Then A is isomorphic and isometric to $\mathbb{C}$.*

Proof. Consider an arbitrary element $x \in A$. By Theorem 6.1.27 there exists $\lambda \in \sigma_x$. The element $x - \lambda\,\mathsf{e}$ is not invertible, hence $x - \lambda\,\mathsf{e} = 0$, i.e. $x = \lambda\,\mathsf{e}$. This means that, for every non-zero $x \in A$ there is a unique $\lambda_x \in \mathbb{C}$ such that $x = \lambda_x\,\mathsf{e}$. This allows to define a bijection $A \to \mathbb{C}$ by $x \mapsto \lambda_x$. It is easy to check that the latter is an isomorphism, and it is isometric since $\|x\| = \|\lambda_x\,\mathsf{e}\,\| = |\lambda_x| \cdot \|\,\mathsf{e}\,\| = |\lambda_x| \cdot 1 = |\lambda_x|$. □

6.1.3 Maximal ideals and multiplicative functionals

By a *subspace* of a Banach space X we understand a subset of X that is closed under sum and scalar multiplication, in other words a linear subspace of X. We do not require the subset be closed, i.e. that itself be a Banach space.

Definition 6.1.29. Given a Banach algebra A, an *ideal* of A is a subset $I \subseteq A$ such that I is a subspace of A and, for all $x \in A$, $xI \subseteq I$.

Any Banach algebra A contains two (possibly non-distinct) ideals, namely the *trivial* ideal $I = \{0\}$ and the *improper* ideal $I = A$.

Example 6.1.30. It is clear that the subset

$$I := \{f \in \mathrm{C}([0,1]) \mid f_{|[0,\frac{1}{2}]} = 0\}$$

is an ideal of the Banach algebra $\mathrm{C}([0,1])$.

The following result states that an arbitrary proper ideal consists of non-invertible elements.

Lemma 6.1.31. *If $I \subseteq A$ is a proper ideal then, for all $x \in I$, x^{-1} does not exist in A.*

Proof. Suppose, by contradiction, that there exists an invertible element $x \in I$, that is $x^{-1} \in A$. If $z \in A$ is an arbitrary element of the algebra, then $zx^{-1} \in A$ entails $(zx^{-1})I \subseteq I$. In particular, since $x \in I$, we get $z = (zx^{-1})x \in I$. Therefore $I = A$, but this contradicts the assumption that I is a proper ideal. □

On the other hand, any non-invertible element of a Banach algebra is contained in some proper ideal.

Lemma 6.1.32. *Let A be a Banach algebra, and let $x \in A$. If x is not invertible, then there is a proper ideal $I \subseteq A$ such that $x \in I$.*

Proof. Pick an element $x \in A$ which is not invertible, and consider the set

$$xA := \{xy \mid y \in A\}.$$

It is elementary that xA is an ideal of A. Moreover it is proper because $\mathsf{e} \notin xA$, since x is not invertible. Obviously we have $x = x\,\mathsf{e} \in xA$. □

Corollary 6.1.33. *A Banach algebra A does not admit any proper non-trivial ideal if, and only if, every non-zero element of A is invertible.* □

Remark 6.1.34. By Theorem 6.1.28 and Corollary 6.1.33 it follows at once that a Banach algebra does not admit any proper non-trivial ideal if, and only if, it is isomorphic and isometric to $\mathbb{C}$. This should be compared with Hölder's Theorem 2.1.23.

Definition 6.1.35. An ideal $\mathfrak{m} \subseteq A$ of a Banach algebra A is said to be *maximal* if it is proper and it is not strictly contained in any proper ideal of A. In other words, if $\mathfrak{m} \subseteq A$ is a proper ideal and $I \subseteq A$ is a proper ideal extending $\mathfrak{m}$, then $I = \mathfrak{m}$.

Maximal ideals play a key rôle in Gelfand representation theory of (commutative) Banach algebras; thus we shall now give a very important example of maximal ideal.

Example 6.1.36. Consider the Banach algebra $\mathrm{C}([0,1])$, and pick a point $t_0 \in [0,1]$. We will see that

$$\mathfrak{m}_{t_0} := \{f \in \mathrm{C}([0,1]) \mid f(t_0) = 0\}$$

is a maximal ideal of $\mathrm{C}([0,1])$, and every maximal ideal of the algebra is of this kind, for a unique t_0. One can easily check that $\mathfrak{m}_{t_0}$ is an ideal. Choosing $g \in \mathrm{C}([0,1])$ such that $g(t_0) \neq 0$, and $h \in \mathrm{C}([0,1])$, we can write

$$h(t) = \tfrac{h(t_0)}{g(t_0)} g(t) + \underbrace{\Big(h(t) - \tfrac{h(t_0)}{g(t_0)} g(t)\Big)}_{\in \mathfrak{m}_{t_0}},$$

so that an arbitrary continuous function h is combination of g with an element of $\mathfrak{m}_{t_0}$. In other terms, the codimension of $\mathfrak{m}_{t_0}$ is 1. If we add any element to $\mathfrak{m}_{t_0}$, we get something of codimension 0, which is the whole $\mathrm{C}([0,1])$. Hence $\mathfrak{m}_{t_0}$ is a maximal ideal. Viceversa, for every maximal ideal $\mathfrak{m} \subseteq A$, there exists $t_0 \in [0,1]$ such that $\mathfrak{m} = \mathfrak{m}_{t_0}$. Indeed, assume by contradiction that, for all $\tau \in [0,1]$, there exists a function $f_\tau \in \mathfrak{m}$ such that $f_\tau(\tau) \neq 0$. Then $|f_\tau(\tau)| > \delta_\tau > 0$ for some δ_τ. Consequently, for all $\tau \in [0,1]$, there is a continuous function $f_\tau \in \mathrm{C}([0,1])$, and a neighborhood U_τ of τ, such that $|f_\tau(t)| > \delta_\tau > 0$ for every $t \in U_\tau$. The family $\{U_\tau\}_{\tau\in[0,1]}$ is an open covering of the compact set $[0,1]$, hence there is a finite subcover $\{U_{\tau_i}\}_{i=1,\dots,n}$ of $[0,1]$. Notice that $f_{\tau_i} \in \mathfrak{m} \subset \mathrm{C}([0,1])$, whence $\overline{f_{\tau_i}} \in \mathrm{C}([0,1])$, and $f_{\tau_i} \cdot \overline{f_{\tau_i}} \in \mathfrak{m}$. It follows that

$$\sum_{i=1}^{n} f_{\tau_i}(t) \cdot \overline{f_{\tau_i}(t)} = \sum_{i=1}^{n} |f_{\tau_i}(t)|^2 \geqslant \min_{1\leqslant i\leqslant n} \delta_{\tau_i}^2 := \delta > 0.$$

Define $f(t) := \sum_{i=1}^{n} f_{\tau_i}(t) \cdot \overline{f_{\tau_i}(t)} \in \mathfrak{m}$. The inequality $|f(t)| \geqslant \delta$ entails that $f(t)$ is invertible, however a proper ideal cannot contain invertible elements by Lemma 6.1.31.

This shows that there is $t_0 \in [0,1]$ such that $\mathfrak{m} = \mathfrak{m}_{t_0}$. The real number t_0 is unique: indeed, assume that $\mathfrak{m}_{t_0} = \mathfrak{m} = \mathfrak{m}_{t_1}$ for some distinct $t_0, t_1 \in [0,1]$. This means that

$$\mathfrak{m} = \{f(t) \in \mathrm{C}([0,1]) \mid f(t_0) = 0 = f(t_1)\}.$$

By Urysohn's lemma [28, Theorem 1.5.11], the ideal $\mathfrak{m}_{t_1}$ strictly contains $\mathfrak{m}$, which contradicts the maximality of $\mathfrak{m}$. The same arguments apply to the study of the maximal ideals of the Banach algebra $\mathrm{C}(X)$, for an arbitrary compact Hausdorff space X.

Remark 6.1.37. Notice that, if I is a proper ideal of the Banach algebra A, then its closure $\overline{I}$ is again a proper ideal of A. The set G of invertible elements of A is open by Proposition 6.1.15, whence $A \setminus G$ is closed. Since $I \neq A$ and every element of an ideal is not invertible by Lemma 6.1.31, we see that $I \subset A \setminus G$. Then $\overline{I} \subseteq A \setminus G$, so that $\overline{I} \neq A$.

Corollary 6.1.38. *Every maximal ideal in a Banach algebra is closed.*

Proof. Let $\mathfrak{m}$ be a proper ideal of a Banach algebra A. Then $\overline{\mathfrak{m}}$ is a proper ideal containing $\mathfrak{m}$. By the maximality of $\mathfrak{m}$, we conclude that $\mathfrak{m} = \overline{\mathfrak{m}}$, i.e. $\mathfrak{m}$ is closed. $\square$

Theorem 6.1.39. *Every proper ideal of a Banach algebra is contained in some maximal ideal.*

Proof. Let I be a proper ideal of a Banach algebra A. If I is maximal, there is nothing to prove. Assuming that I is not maximal, we shall consider the set $\mathcal{M}$ of those proper ideals of A extending I, partially ordered by set-theoretic inclusion. The family $\mathcal{M}$ is non-empty because $I \in \mathcal{M}$. If $\{I_\alpha\}_\alpha$ is a totally-ordered subset of $\mathcal{M}$, define $I_0 := \bigcup_\alpha I_\alpha$. It is possible to see that $I_0 \in \mathcal{M}$ and that it is an upper bound for the family $\{I_\alpha\}_\alpha$. By Zorn's lemma there exists a maximal element $\mathfrak{m} \in \mathcal{M}$, that is a maximal ideal containing I. $\square$

Corollary 6.1.40. *An element of a Banach algebra A is invertible if, and only if, it is not contained in any maximal ideal of A.*

Proof. The statement is clearly true for the element $0 \in A$. Further, Lemma 6.1.31 states that proper ideals do not contain invertible elements. In particular, maximal ideals do not contain invertible elements. On the other hand, suppose that $0 \neq x \in A$ is not invertible. Then it is contained in some proper ideal I, by Lemma 6.1.32. But Theorem 6.1.39 entails that there is a maximal ideal $\mathfrak{m}$ such that $x \in I \subseteq \mathfrak{m}$, which is a contradiction. $\square$

Definition 6.1.41. A *multiplicative functional* on a Banach algebra A is a non-zero function $f\colon A \to \mathbb{C}$ satisfying $f(xy) = f(x)f(y)$ for all $x, y \in A$.

Example 6.1.42. Consider the Banach algebra $\mathrm{C}([0,1])$. Fix $t_0 \in [0,1]$ and define the map $f_{t_0}\colon \mathrm{C}([0,1]) \to \mathbb{C}$ by setting, for all $x \in \mathrm{C}([0,1])$, $f_{t_0}(x) := x(t_0)$. In other words, f_{t_0} is the evaluation at the point t_0. This is a multiplicative functional, because

$$f_{t_0}(xy) = xy(t_0) = x(t_0)y(t_0) = f_{t_0}(x)f_{t_0}(y).$$

Proposition 6.1.43. *If A is a Banach algebra and $f\colon A \to \mathbb{C}$ is a multiplicative functional, the following hold.*

1. $f(\mathsf{e}) = 1$.
2. *For every invertible element $x \in A$, $f(x^{-1}) = \frac{1}{f(x)}$.*

Proof. For item 1, we have $f(\mathsf{e}) = f(\mathsf{e}\cdot\mathsf{e}) = f(\mathsf{e})f(\mathsf{e})$, so that $f(\mathsf{e})(f(\mathsf{e}) - 1) = 0$. It follows that either $f(\mathsf{e}) = 1$ or $f(\mathsf{e}) = 0$. If $f(\mathsf{e}) = 0$ then, for all $x \in A$, we have

$$f(x) = f(\mathsf{e}\cdot x) = f(\mathsf{e}) \cdot f(x) = 0 \cdot f(x) = 0.$$

However we assumed that f is a non-zero function. Therefore $f(\mathsf{e}) = 1$. Using the latter identity in order to prove item 2, we see that

$$1 = f(\mathsf{e}) = f(xx^{-1}) = f(x)f(x^{-1}).$$

In other words, $f(x^{-1}) = \frac{1}{f(x)}$. □

Recall that a functional $f\colon A \to \mathbb{C}$ on a Banach algebra (more generally, on a $\mathbb{C}$-vector space) A is said to be *linear* if it satisfies

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y),$$

for all $x, y \in A$ and for all $\lambda, \mu \in \mathbb{C}$. We remark that there exist no linear multiplicative functionals on the trivial Banach algebra $\{0 = \mathsf{e}\}$. Indeed, one such functional f would satisfy $0 = f(0) = f(\mathsf{e}) = 1$.

Proposition 6.1.44. *If f is a multiplicative linear functional on a Banach algebra A, and $x \in A$ is such that $\|x\| < 1$, then $|f(x)| < 1$.*

Proof. Suppose, by contradiction, that $f(x) = \lambda$, where $|\lambda| \geqslant 1$. Observe that $\|\frac{x}{\lambda}\| < 1$, hence $\frac{x}{\lambda} - \mathsf{e}$ is invertible by Lemma 6.1.16. Consequently, the element $x - \lambda\,\mathsf{e}$ is invertible because $x - \lambda\,\mathsf{e} = \lambda\left(\frac{x}{\lambda} - \mathsf{e}\right)$. By Proposition 6.1.43.(2) we know that $f(x - \lambda\,\mathsf{e}) \neq 0$, whence $f(x) - f(\lambda\,\mathsf{e}) \neq 0$ if, and only if, $f(x) \neq \lambda f(\mathsf{e})$. However, Proposition 6.1.43.(1) ensures that $f(\mathsf{e}) = 1$. Thus $f(x) \neq \lambda$, a contradiction. □

Corollary 6.1.45. *Every multiplicative linear functional on a Banach algebra is bounded, with norm 1. In particular, it is continuous.*

Proof. Let A be a Banach algebra. Denote by $B := \{x \in A \mid \|x\| \leqslant 1\}$ the unit ball of A, and by B° the interior of B. Then

$$\|f\| = \sup_{x\in B} |f(x)| = \sup_{x \in B^\circ} |f(x)| \leqslant 1,$$

by Proposition 6.1.44. Finally, we notice that $\mathsf{e} \in B$ and $f(\mathsf{e}) = 1$ by Proposition 6.1.43.(1), whence $\|f\| = 1$. Since a linear functional is continuous if, and only if, it is bounded [46, Theorem 2 p. 77], f is continuous on A. □

Henceforth, we assume that every multiplicative functional on a Banach algebra is linear. For every such multiplicative functional $f\colon A \to \mathbb{C}$, we can define its *kernel* as

$$\ker f := \{x \in A \mid f(x) = 0\}.$$

The structure of the kernel of a multiplicative functional is made explicit in the following

Proposition 6.1.46. *If f is a multiplicative functional on a Banach algebra A, then $\ker f$ is a maximal ideal of A.*

Proof. It is elementary that $\ker f$ is an ideal of A. Moreover, $\ker f$ is proper since f is a non-zero function. Hence there exists $x \in A \setminus \ker f$. Furthermore, the codimension of $\ker f$ is 1, because every element $y \in A$ can be written as

$$y = \tfrac{f(y)}{f(x)}x + \left(y - \tfrac{f(y)}{f(x)}x\right),$$

where $y - \frac{f(y)}{f(x)}x \in \ker f$. This shows that $\ker f$ is maximal. $\square$

For any closed ideal I of a Banach algebra A, define the set $A/I := \{x+I \mid x \in A\}$. The latter is a Banach space by [50, Theorem 6B]. The family of cosets A/I can be equipped with the norm

$$\|x + I\| := \inf_{U \in x+I} \|U\|,$$

for every $x+I \in A/I$. If $x+I, y+I \in A/I$, we can even define a product in the following way: $(x + I)(y + I) := xy + I$. Then

$$\begin{aligned}
\|(x + I)(y + I)\| &= \inf_{U \in x+I,\ V \in y+I} \|UV\| \\
&\leqslant \inf_{U \in x+I,\ V \in y+I} \|U\| \cdot \|V\| \\
&= \inf_{U \in x+I} \|U\| \cdot \inf_{V \in y+I} \|V\| \\
&= \|x + I\| \cdot \|y + I\|.
\end{aligned}$$

Furthermore, $\mathsf{E} := \mathsf{e} + I$ is a unit for A/I. Indeed, for all $x + I \in A/I$,

$$\mathsf{E}(x + I) = (\mathsf{e} + I)(x + I) = (\mathsf{e}\, x) + I = x + I.$$

The norm of E does not exceed 1:

$$\|\,\mathsf{E}\,\| = \|\,\mathsf{e} + I\| = \inf_{U \in \mathsf{e} + I} \|U\| \leqslant \|\,\mathsf{e}\,\| = 1.$$

In fact, we shall prove that $\|\,\mathsf{E}\,\| = 1$. Suppose, by contradiction, that $\|\,\mathsf{E}\,\| < 1$, i.e. there exists $x \in I$ such that $\|\,\mathsf{e} + x\| < 1$. This means, by Lemma 6.1.16, that x is invertible; but the ideal I cannot contain invertible elements, by Lemma 6.1.31. Hence $\|\,\mathsf{E}\,\| = 1$. The Banach algebra A/I is well-defined, and it is called the *quotient* of the Banach algebra A with respect to its closed ideal I.

Remark 6.1.47. If $I \subseteq A$ is a closed ideal, then the quotient map $q\colon A \to A/I$, mapping $x \in A$ to $x + I \in A/I$, is continuous because it is clearly norm-decreasing. Further, q is multiplicative. Indeed, for all $x, y \in A$,

$$q(xy) = (xy) + I = (x + I)(y + I) = q(x)q(y).$$

Lemma 6.1.48. *Let I be a closed ideal of a Banach algebra A, let J be a closed ideal of the quotient A/I, and let $q\colon A \to A/I$ be the quotient map. Then $q^{-1}(J)$ is a closed ideal of A extending I.*

Proof. The fact that $q^{-1}(J)$ is an ideal of A is a straightforward computation. Further, $q^{-1}(J)$ is closed in A, being the preimage of a closed subset under a continuous function (see Remark 6.1.47). □

Proposition 6.1.46 states that the kernel of a multiplicative functional is a maximal ideal, so that to every multiplicative functional f we can associate the maximal ideal $\ker f$. The next result shows that the converse holds: to every maximal ideal $\mathfrak{m}$ we can associate a multiplicative functional, namely $q\colon A \to A/\mathfrak{m} \cong \mathbb{C}$.

Proposition 6.1.49. *If $\mathfrak{m}$ is a maximal ideal of a Banach algebra A, then $A/\mathfrak{m} \cong \mathbb{C}$.*

Proof. Let $q\colon A \to A/\mathfrak{m}$ denote the quotient map. By Corollary 6.1.33 and Theorem 6.1.28, it suffices to prove that A does not admit non-trivial proper ideals. Suppose, by contradiction, that there exists a proper non-trivial ideal $J \subseteq A/\mathfrak{m}$. By Remark 6.1.37 we can assume, without loss of generality, that J is a closed ideal. Then $q^{-1}(J)$ is a closed ideal of A by Lemma 6.1.48. Since J is non-trivial, the ideal $q^{-1}(J)$ strictly contains the maximal ideal $\mathfrak{m}$. Therefore $q^{-1}(J) = A$, i.e. J is improper, a contradiction. □

6.1.4 Gelfand transform

For a Banach algebra A, we agree to denote by $f_\mathfrak{m}$ the functional associated to the maximal ideal $\mathfrak{m}$ of A, as provided by Proposition 6.1.49. In more detail, $f_\mathfrak{m}$ is obtained as the composition of the quotient map $A \to A/\mathfrak{m}$ with the isometric isomorphism $A/\mathfrak{m} \cong \mathbb{C}$. The functional $f_\mathfrak{m}$ is obviously linear, and it is multiplicative by Remark 6.1.47. Now, observe that the correspondence between multiplicative functionals and maximal ideals, given by $f \to \ker f$ is surjective: every maximal ideal arises as the kernel of a multiplicative functional. Indeed, it is immediate to verify that $\mathfrak{m} = \ker f_\mathfrak{m}$. On the other hand, this correspondence is injective. If $f, g\colon A \to \mathbb{C}$ are multiplicative functionals satisfying $\ker f = \ker g$, then $g = \lambda f$ for some $\lambda \in \mathbb{C}$, by [22, Proposition A.1.4]. Then

$$1 = g(\mathsf{e}) = \lambda f(\mathsf{e}) = \lambda \cdot 1 = \lambda,$$

so that $f = g$. The existence of a bijection between the set of maximal ideals of a Banach algebra, and the set of multiplicative functionals on the algebra, allows us to define the

Gelfand transform in two, completely equivalent, ways. We shall work with the family of multiplicative functionals, denoted by

$$\Sigma := \{f\colon A \to \mathbb{C} \mid f \text{ is a multiplicative functional}\}.$$

In view of the foregoing discussion, Σ is called the *maximal spectrum* (or the *maximal ideal space*) of the Banach algebra A. The set Σ is contained in the collection W of all bounded linear functionals on the Banach algebra A. Equipped with pointwise operations, the space W is a $\mathbb{C}$-vector space, and it is usually called the *conjugate* (or *dual*) *space* of A. The space W can be turned into a topological space in two different ways. On the one hand, we have the topology induced by (the metric induced by) the usual operator norm. On the other hand, we can define the so-called *weak-star* topology on the conjugate space W, induced by the product topology on the space $\mathbb{C}^A$. Henceforth, we shall regard Σ as a topological space with respect to the weak-star topology. We remark that the maximal spectrum of the trivial Banach algebra $\{0 = \mathsf{e}\}$ is empty. The following easy result describes the convergence of sequences in the space W:

Lemma 6.1.50. *Let* W *be the space of bounded linear functionals on a Banach algebra* A*, equipped with the weak-star topology. A sequence* $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathsf{W}$ *converges to the element* $f \in \mathsf{W}$ *if, and only if,* $f_n(x) \to f(x)$ *for all* $x \in A$. □

Remarkably enough, we find

Proposition 6.1.51. *The maximal spectrum* Σ *of a Banach algebra is compact, with respect to the weak-star topology.*

Proof. If the Banach algebra is trivial, then its maximal spectrum is empty and there is nothing to prove. Hence, we shall assume that A is a non-trivial Banach algebra. Alaoglu's theorem [50, Theorem 9B] states that, if X is a normed space, then the closed unit ball in the conjugate space W (with respect to the operator norm) is compact in the weak-star topology. The maximal spectrum Σ of A is contained in the unit ball of W by Corollary 6.1.45, hence it suffices to prove that Σ is closed in W, because a closed subset of a compact space is compact. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq \Sigma$ be a sequence of multiplicative functionals converging to the functional $f_0 \in \mathsf{W}$. By Lemma 6.1.50, this means that $f_n(x) \to f_0(x)$ for every $x \in A$. We will prove that f_0 is a multiplicative functional, so that $f_0 \in \Sigma$. For all $x, y \in A$ we have $f_n(xy) \to f_0(xy)$. Moreover, by Corollary 6.1.8,

$$f_n(xy) = f_n(x) \cdot f_n(y) \to f_0(x) \cdot f_0(y).$$

This shows that, for all $x, y \in A$, $f_0(xy) = f_0(x) \cdot f_0(y)$. Thus Σ is compact. □

Remark 6.1.52. Proposition 6.1.51, as stated above, holds for *unital* (commutative) Banach algebras. In general, if a Banach algebra does not admit unit, its maximal spectrum is a locally compact space, i.e. a space in which every point has a closed compact neighborhood (see [50, Theorem 19B]).

Given a Banach algebra A and an element $x \in A$, define the function $\widehat{x}\colon \Sigma \to \mathbb{C}$ by setting, for all $f \in \Sigma$,

$$\widehat{x}(f) := f(x).$$

The function $\widehat{x}$ is continuous, and the map

$$\Lambda\colon A \to \mathrm{C}(\Sigma), \quad \Lambda(x) := \widehat{x}$$

is a multiplicative linear operator [22, Theorem 8.9 p. 220]. The operator Λ is called the *Gelfand transform*.

Proposition 6.1.53. *If A is a Banach algebra, then the Gelfand transform*

$$\Lambda\colon A \to \mathrm{C}(\Sigma)$$

has norm 1*. In particular, Λ is a continuous operator.*

Proof. By Corollary 6.1.45, for all $x \in A$, we have

$$\|\Lambda x\| = \|\widehat{x}\| = \sup_{f\in\Sigma} |\widehat{x}(f)| = \sup_{f\in\Sigma} |f(x)| \leqslant 1 \cdot \|x\| = \|x\|.$$

In other terms, $\|\Lambda\| \leqslant 1$. In fact $\|\Lambda\| = 1$, because $\|\Lambda\, \mathsf{e}\,\| = \|1_\Sigma\| = 1$. □

Theorem 6.1.54. *If A is a Banach algebra and $x \in A$, then $\sigma_x = \{f(x) \mid f \in \Sigma\}$. In other words, $\sigma_x = \widehat{x}(\Sigma)$.*

Proof. If A is the trivial Banach algebra, there is nothing to prove. Recall that, by definition of the spectrum of an element, $\lambda \in \sigma_x$ if the element $x - \lambda\,\mathsf{e}$ is not invertible. By Corollary 6.1.40, $\lambda \in \sigma_x$ if, and only if, there exists a maximal ideal $\mathfrak{m} \subseteq A$ such that $x - \lambda\,\mathsf{e} \in \mathfrak{m}$. Since every maximal ideal is the kernel of a multiplicative functional, this is equivalent to the existence of a multiplicative functional $f \in \Delta$ such that $x - \lambda\,\mathsf{e} \in \ker f$. In turn, $f(x - \lambda\,\mathsf{e}) = 0$, if, and only if,

$$f(x) = f(\lambda\,\mathsf{e}) = \lambda \cdot f(\mathsf{e}) = \lambda \cdot 1 = \lambda.$$

That is, $\lambda \in \{f(x) \mid f \in \Sigma\}$. □

Example 6.1.55. Consider the Banach algebra $\mathrm{C}([0,1])$. For every multiplicative functional h on A there exists a unique $t_0 \in [0,1]$ such that, for all $f \in \mathrm{C}([0,1])$, $h(f) = f(t_0)$ (cf. Example 6.1.36). Then Theorem 6.1.54 states nothing but

$$\sigma_f = \{f(t) \mid t \in [0,1]\} = f([0,1]).$$

Definition 6.1.56. The *radical* of a Banach algebra A is the set

$$\operatorname{Rad} A := \{x \in A \mid f(x) = 0 \ \ \forall f \in \Sigma\}.$$

The Banach algebra A is said to be *semisimple* if it has trivial radical, i.e. $\operatorname{Rad} A = \{0\}$.

Remark 6.1.57. By definition of radical, an element $x \in A$ belongs to $\operatorname{Rad} A$ if, and only if, $x \in \ker f$ for all $f \in \Sigma$. Given the bijection between maximal ideals of A and multiplicative functionals on A, this is equivalent to x belonging to every maximal ideal of A. Therefore, the radical of A is the intersection of all the maximal ideals of A:

$$\operatorname{Rad} A = \bigcap \{\mathfrak{m} \subseteq A \mid \mathfrak{m} \text{ is a maximal ideal}\}.$$

In fact, it is easy to see that $\operatorname{Rad} A$ is an ideal of A. Another way of describing the radical ideal is the following. Observe that, if $x \in A$ and $f \in \Sigma$, then $f(x) = 0$ if, and only if, $\widehat{x}(f) = 0$. This means that

$$\operatorname{Rad} A = \ker \Lambda.$$

The latter fact has an immediate consequence.

Lemma 6.1.58. *For every Banach algebra A, the Gelfand transform $\Lambda \colon A \to \mathrm{C}(\Sigma)$ is injective if, and only if, A is semisimple.* □

Therefore, if A is a semisimple Banach algebra, the Gelfand transform provides a Banach isomorphism between A and

$$\widehat{A} := \Lambda(A) = \{\widehat{x} \in \mathrm{C}(\Sigma) \mid x \in A\}.$$

The next formula allows us to compute the norm of the Gelfand transform of an element x, in terms of the norm of x.

Proposition 6.1.59. *For every element $x \in A$ of a Banach algebra,*

$$\|\widehat{x}\| = \lim_{n\to\infty} \sqrt[n]{\|x^n\|}.$$

Proof. See [50, Theorem 24A]. □

We conclude with one more characterisation of a semisimple Banach algebra A. We agree to say that a set T of bounded linear functionals on A is *total* if the only element $x \in A$ satisfying $f(x) = 0$, for all $f \in T$, is $x = 0$. From the very definition of radical, it follows

Lemma 6.1.60. *A Banach algebra is semisimple if, and only if, its maximal spectrum Σ is a total set.* □

6.1.5 Involution

Our main example of a Banach algebra is $\mathrm{C}([0,1])$ or, more generally, $\mathrm{C}(X)$ for some compact Hausdorff space X. If $f \colon X \to \mathbb{C}$ is a continuous function, it is promptly recognised that its complex conjugate $\overline{f} \colon X \to \mathbb{C}$ is again a continuous function. We recall that the function $\overline{f}$ is defined, for all $x \in X$, by $\overline{f}(x) := \overline{f(x)}$. The complex conjugation has an analogue in many other Banach algebras, leading to the notion of involution.

Definition 6.1.61. Let A be a Banach algebra. An operation ${}^*\colon A \to A$ is called an *involution* if it satisfies the following properties, for all $\lambda \in \mathbb{C}$ and for all $x, y \in A$.

1. $(x+y)^* = x^* + y^*$.
2. $(\lambda x)^* = \overline{\lambda} x^*$.
3. $(xy)^* = y^* x^*$.
4. $(x^*)^* = x$.

Example 6.1.62. Let us consider the algebra $\mathcal{B}(H)$ of all the bounded operators from a Hilbert space H to itself. This is a non-commutative Banach algebra, but nevertheless it admits an involution mapping an operator $D \in \mathcal{B}(H)$ to the adjoint operator D^* defined, for every $x, y \in H$, by

$$\langle Dx, y\rangle = \langle x, D^* y\rangle.$$

Definition 6.1.63. Let A, B be Banach algebras with an involution. A Banach homomorphism $f\colon A \to B$ is a *Banach *-homomorphism* if, for all $x \in A$, $f(x^*) = f(x)^*$.

Henceforth, we assume that any Banach algebra is endowed with an involution.

Definition 6.1.64. An element $x \in A$ of a Banach algebra is *self-adjoint*, provided that it is fixed by the involution, i.e. $x^* = x$.

Proposition 6.1.65. *If A is a Banach algebra and $x \in A$, then the following hold.*

1. *$x + x^*$, $i(x - x^*)$, and xx^* are self-adjoint elements.*
2. *There exist unique self-adjoint elements $u, v \in A$ such that $x = u + iv$.*

Proof. Item 1 is proved by means of straightforward computations.

$$(x + x^*)^* = x^* + x^{**} = x^* + x = x + x^*,$$

$$(i(x - x^*))^* = \overline{i}(x - x^*)^* = -i(x^* - x^{**}) = -i(x^* - x) = i(x - x^*),$$

$$(xx^*)^* = x^{**}x^* = xx^*.$$

In order to prove item 2, define

$$u := \tfrac{1}{2}(x + x^*), \text{ and } v := -\tfrac{i}{2}(x - x^*).$$

The elements u, v are self-adjoint, indeed

$$\left(\tfrac{1}{2}(x + x^*)\right)^* = \overline{\tfrac{1}{2}}(x + x^*)^* = \tfrac{1}{2}(x + x^*),$$

$$\left(-\tfrac{i}{2}(x - x^*)\right)^* = \overline{-\tfrac{1}{2}}(i(x - x^*))^* = -\tfrac{1}{2}(i(x - x^*)) = -\tfrac{i}{2}(x - x^*).$$

Moreover,

$$u + iv = \tfrac{1}{2}(x + x^*) + \tfrac{1}{2}(x - x^*) = \tfrac{1}{2}x + \tfrac{1}{2}x = x.$$

To prove the uniqueness of u and v, suppose that u', v' are self-adjoint elements satisfying $u + iv = x = u' + iv'$. Then

$$u - iv = (u + iv)^* = x^* = (u' + iv')^* = u' - iv',$$

and consequently $2u = x + x^* = 2u'$, so that $u = u'$. We conclude that $v = v'$ also holds. Self-adjoint elements in a Banach algebra play the same rôle of real numbers in $\mathbb{C}$. □

In the following, we state some more properties of the involution operation.

Proposition 6.1.66. *If A is a Banach algebra, the following hold.*

1. *The unit $\mathsf{e} \in A$ is a self-adjoint element.*
2. *For all $x \in A$, x is invertible if, ad only if, x^* is invertible. In this case, we have $(x^*)^{-1} = (x^{-1})^*$.*
3. *For all $x \in A$, $\sigma_{x^*} = \overline{\sigma_x} := \{\overline{\lambda} \in \mathbb{C} \mid \lambda \in \sigma_x\}$.*

Proof. Item 1 holds, since

$$\mathsf{e} = (\mathsf{e}^*)^* = (\mathsf{e}\,\mathsf{e}^*)^* = \mathsf{e}^*\,\mathsf{e}^{**} = \mathsf{e}^*\,\mathsf{e} = \mathsf{e}^*\,.$$

Regarding item 2, assume that there exists $x^{-1} \in A$. Then

$$\mathsf{e} = \mathsf{e}^* = (xx^{-1})^* = (x^{-1})^* x^*,$$

hence x^* is invertible and $(x^*)^{-1} = (x^{-1})^*$. On the other hand, suppose that there exists $(x^*)^{-1} \in A$. It follows

$$\mathsf{e} = \mathsf{e}^* = (x^*(x^*)^{-1})^* = ((x^*)^{-1})^* x^{**} = ((x^*)^{-1})^* x,$$

so that x is invertible. Lastly, in order to prove item 3, we shall equivalently show that $\lambda \notin \sigma_x$ if, and only if, $\lambda \notin \overline{\sigma_{x^*}}$, for all $\lambda \in \mathbb{C}$. Recall that $\lambda \notin \sigma_x$ if, and only if, $x - \lambda\,\mathsf{e}$ is invertible. By item 2, this happens precisely when $(x - \lambda\,\mathsf{e})^*$ is invertible, in which case $((x - \lambda\,\mathsf{e})^*)^{-1} = ((x - \lambda\,\mathsf{e})^{-1})^*$. Now,

$$(x - \lambda\,\mathsf{e})^* = x^* - \overline{\lambda}\,\mathsf{e}^* = x^* - \overline{\lambda}\,\mathsf{e}\,.$$

Hence there exists $((x - \lambda\,\mathsf{e})^*)^{-1}$ if, and only if, $\overline{\lambda} \notin \sigma_{x^*}$ if, and only if, $\lambda \notin \overline{\sigma_{x^*}}$. □

Proposition 6.1.67. *If A is a semisimple Banach algebra, then the involution operation on A is continuous, i.e. $x_n \to x$ entails $x_n^* \to x^*$.*

The proof of the proposition above requires the following

Lemma 6.1.68. *Let A be a Banach algebra, and let $h \in \Sigma$ be a multiplicative functional on A. Then the functional*

$$\phi\colon A \to \mathbb{C}, \quad \phi(x) := \overline{h(x^*)}$$

is multiplicative.

Proof. The operator ϕ is linear, because

$$\phi(x+y) = \overline{h((x+y)^*)} = \overline{h(x^*+y^*)} = \overline{h(x^*)+h(y^*)} = \overline{h(x^*)} + \overline{h(y^*)} = \phi(x)+\phi(y),$$

and, for all $\lambda \in \mathbb{C}$,

$$\phi(\lambda x) = \overline{h((\lambda x)^*)} = \overline{h(\overline{\lambda} x^*)} = \overline{\overline{\lambda} h(x^*)} = \lambda \overline{h(x^*)} = \lambda \phi(x).$$

It is bounded since h is a bounded operator, and $\|\phi\| = \|\overline{h}\| = \|h\|$. Finally, ϕ is multiplicative:

$$\phi(xy) = \overline{h((xy)^*)} = \overline{h(y^* x^*)} = \overline{h(y^*)h(x^*)} = \overline{h(y^*)} \cdot \overline{h(x^*)} = \overline{h(x^*)} \cdot \overline{h(y^*)} = \phi(x)\phi(y).$$

□

Proof of Proposition 6.1.67. By the Closed Graph theorem [22, Theorem 12.6 p. 91], every closed linear operator defined on a Banach space is bounded. Hence, to prove the boundedness of the involution operation $^* : A \to A$, which is equivalent to its continuity (see [46, Theorem 1 p. 96]), it suffices to show that it is closed. In other words, we shall prove that, if $x_n \to x$ and $x_n^* \to y$, then $y = x^*$. Pick a multiplicative functional $h \in \Sigma$, and consider the functional $\phi(x) := \overline{h(x^*)}$. By Lemma 6.1.68 ϕ is bounded, hence continuous. This means that, if $x_n \to x$, then $\lim_{n\to\infty} \phi(x_n) = \phi(x)$, i.e. $\lim_{n\to\infty} \overline{h(x_n^*)} = \overline{h(x^*)}$. However, since h is continuous by Corollary 6.1.45, and $x_n^* \to y$, we know that $\lim_{n\to\infty} \overline{h(x_n^*)} = \overline{h(y)}$. Therefore $\overline{h(x^*)} = \overline{h(y)}$. It follows that, for all $h \in \Sigma$, $h(x^*) = h(y)$, i.e. $h(x^* - y) = 0$. Since Σ is total by Lemma 6.1.60, we conclude that $x^* - y = 0$, that is $x^* = y$. □

6.2 Gelfand-Neumark duality

Definition 6.2.1. A *commutative unital* C**-algebra* is a commutative unital Banach algebra A with an involution satisfying, for all $x \in A$, the C**-identity*

$$\|xx^*\| = \|x\|^2.$$

Unless otherwise stated, by a C*-algebra we understand a commutative unital C*-algebra.

Example 6.2.2. The fundamental example of C*-algebra is the Banach algebra $\mathrm{C}(X)$, where X is a compact Hausdorff space and the involution is given by complex conjugation. The C*-identity is satisfied for all $f \in \mathrm{C}(X)$:

$$\|ff^*\| = \|f\overline{f}\| = \||f|^2\| = \sup_{x\in X} |f(x)|^2 = \sup_{x\in X} |f(x)| \cdot \sup_{x\in X} |f(x)| = \|f\| \cdot \|f\| = \|f\|^2.$$

In fact, the Gelfand-Neumark representation theorem states that every (commutative, unital) C*-algebra is of the kind $\mathrm{C}(X)$ for some compact Hausdorff space X. Further, there is a canonical choice for the space X: this is the maximal spectrum Σ of the algebra.

Lemma 6.2.3. *Let A be a C^*-algebra, and let $u \in A$ be a self-adjoint element. Then, for all $f \in \Sigma$, $f(u) \in \mathbb{R}$. In other words, $\sigma_u \subseteq \mathbb{R}$.*

Proof. Let $\alpha, \beta \in \mathbb{R}$ be such that $f(u) = \alpha + \beta i$. We shall prove that $\beta = 0$. Upon considering the element $z = u + it\,\mathsf{e} \in A$, where $t \in \mathbb{R}$, we have

$$z^* = u^* + \overline{it}\,\mathsf{e}^* = u - ite,$$

and

$$zz^* = (u + it\,\mathsf{e})(u - it\,\mathsf{e}) = u^2 + t^2\,\mathsf{e}^2 = u^2 + t^2\,\mathsf{e}\,.$$

Now, consider the real number $|f(z)|^2$. On the one hand,

$$\begin{aligned}|f(z)|^2 &= |f(u + it\,\mathsf{e})|^2 = |f(u) + h(it\,\mathsf{e})|^2 = |f(u) + ith(\mathsf{e})|^2\\ &= |f(u) + it \cdot 1|^2 = |f(u) + it|^2 = |\alpha + \beta i + it|^2 = |\alpha + (\beta + t)i|^2\\ &= \alpha^2 + (\beta + t)^2 = \alpha^2 + \beta^2 + 2\beta t + t^2.\end{aligned}$$

On the other hand, $|f(z)|^2 \leqslant \|z\|^2$ by Corollary 6.1.45, and

$$\|z\|^2 = \|zz^*\| = \|u^2 + t^2\,\mathsf{e}\,\| \leqslant \|u^2\| + \|t^2\,\mathsf{e}\,\| = \|u^2\| + t^2.$$

Therefore $\alpha^2 + \beta^2 + 2\beta t + t^2 \leqslant \|u^2\| + t^2$, i.e. $\alpha^2 + \beta^2 + 2\beta t \leqslant \|u^2\|$. Since u is fixed, the right-hand side of the inequality is constant, while the left-hand side depends on $t \in \mathbb{R}$. If we suppose by contradiction that $\beta \neq 0$, then the inequality does not hold for $t \to +\infty$ (if $\beta > 0$) or for $t \to -\infty$ (if $\beta < 0$). Hence $\beta = 0$, that is $f(u) \in \mathbb{R}$. □

Lemma 6.2.4. *If A is a C^*-algebra, then the Gelfand transform*

$$\Lambda\colon A \to \mathrm{C}(\Sigma)$$

is an injective isometric *-homomorphism.

Proof. We must prove that Λ is an injective isometric homomorphism of complex algebras, preserving the involution. It is elementary that it is a homomorphism of complex algebras; in particular, it preserves the unit by Proposition 6.1.43.(1). By the very definition of the Gelfand transform, Λ preserves the involution if, and only if, for all $x \in A$, $\widehat{x^*} = \overline{\widehat{x}}$. By Proposition 6.1.65 we can find self-adjoint elements $u, v \in A$ such that

$x = u + iv$. Then $x^* = u^* + \bar{i}v^* = u - iv$ and, since Λ is a linear operator, $\widehat{x^*} = \widehat{u} - i\widehat{v}$. We remark that Lemma 6.2.3 entails that, if $u \in A$ is a self-adjoint element, then

$$\widehat{u}(f) = f(u) = \overline{f(u)} = \overline{\widehat{u}}(f),$$

for all multiplicative functionals $f \in \Sigma$. In other terms, $\widehat{u} = \overline{\widehat{u}}$ for any self-adjoint element $u \in A$, whence

$$\widehat{x^*} = \widehat{u} - i\widehat{v} = \overline{\widehat{u}} + \overline{i\widehat{v}} = \overline{\widehat{x}}.$$

Stating that the Gelfand transform $\Lambda\colon A \to \mathrm{C}(\Sigma)$ is an isometry, means that, for all $x \in A$, $\|\Lambda x\| = \|x\|$. Assume that $y \in A$ is self-adjoint; we claim that $\|y^{2^n}\| = \|y\|^{2^n}$, for all $n \in \mathbb{N}$. If $n = 1$, then $\|y^2\| = \|yy^*\| = \|y\|^2$. For an arbitrary positive integer n, we have $\|y^{2^n}\| = \|(y^{2^{n-1}})^2\|$. Observe that the element $y^{2^{n-1}}$ is self-adjoint, since

$$\left(y^{2^{n-1}}\right)^* = (y^*)^{2^{n-1}} = y^{2^{n-1}}.$$

Thus $\|(y^{2^{n-1}})^2\| = \|y^{2^{n-1}}\|^2$ and, by the inductive hypothesis,

$$\|y^{2^{n-1}}\|^2 = \left(\|y\|^{2^{n-1}}\right)^2 = \|y\|^{2^n}.$$

Now, Proposition 6.1.59 shows that

$$\|\Lambda y\| = \|\widehat{y}\| = \lim_{n\to\infty} \sqrt[n]{\|y^n\|} = \lim_{n\to\infty} \sqrt[2^n]{\|y^{2^n}\|} = \lim_{n\to\infty} \sqrt[2^n]{\|y\|^{2^n}} = \|y\|.$$

If $x \in A$ is an arbitrary element, we obtain a self-adjoint element by considering $y := xx^*$. In this case, $\|y\| = \|\widehat{y}\|$ if, and only if,

$$\|xx^*\| = \|\widehat{xx^*}\| = \|\widehat{x}\widehat{x^*}\| = \|\widehat{x}\overline{\widehat{x}}\| = \||\widehat{x}|^2\| = \|\widehat{x}\|^2,$$

where we used the fact that Λ is a multiplicative operator. It follows that $\|x\|^2 = \|xx^*\| = \|\widehat{x}\|^2$, whence

$$\|x\| = \|\widehat{x}\| = \|\Lambda x\|.$$

To conclude, it suffices to show that the Λ is injective. Let $x \in A$ satisfy $\Lambda x = 0$ or, equivalently, $\|\Lambda x\| = 0$. Since Λ is an isometry, $\|x\| = \|\Lambda x\| = 0$, i.e. $x = 0$. $\square$

Then, Lemma 6.1.58 entails

Corollary 6.2.5. *Every* C^**-algebra is semisimple.* $\square$

The following representation result, due to Gelfand and Neumark [32, Lemma 1], is central in the theory of C^*-algebras.

Theorem 6.2.6 (Gelfand-Neumark)**.** *If A is a* C^**-algebra, then the Gelfand transform*

$$\Lambda\colon A \to \mathrm{C}(\Sigma)$$

is an isometric **-isomorphism.*

Proof. Lemma 6.2.4 states that the operator Λ is an injective isometric *-homomorphism. Now, we shall prove that it is surjective. In fact, it suffices to show that the range $\widehat{A} := \Lambda(A)$ is dense in $\mathrm{C}(\Sigma)$, since an isometric linear operator between Banach spaces has closed range. To prove the latter claim, assume that $\Lambda x_n \to y$ in $\widehat{A}$. Then,

$$\|\Lambda x_n - \Lambda x_m\| = \|\Lambda(x_n - x_m)\| = \|x_n - x_m\|.$$

The sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent in A, hence there exists $x \in A$ such that $x_n \to x$. By the continuity of the operator Λ, we have $\Lambda x_n \to \Lambda x$. Therefore $y = \Lambda x$, and the range $\widehat{A}$ is closed. In order to prove that $\widehat{A}$ is dense in $\mathrm{C}(\Sigma)$, we check that the hypotheses of the Stone-Weierstrass theorem [38, Theorem 7.34] are satisfied. Firstly, if $f \in \widehat{A}$ then $\overline{f} \in \widehat{A}$, indeed $\overline{\widehat{f}} = \widehat{f^*}$ (since Λ preserves the involution operation). The constant function 1_Σ belongs to $\widehat{A}$ because $1_\Sigma = \Lambda\,\mathsf{e} = \widehat{\mathsf{e}}$ (since Λ preserves the unit). To see that $\widehat{A}$ separates points of Σ, suppose that $f_1, f_2 \in \Sigma$ are distinct multiplicative functionals. This means, in particular, that there exists $x \in A$ such that $\widehat{x}(f_1) = f_1(x) \neq f_2(x) = \widehat{x}(f_2)$, i.e. the continuous function $\widehat{x}$ separates the points f_1 and f_2. Lastly, $\widehat{A}$ is clearly closed under addition and scalar multiplication, and it is closed under multiplication because Λ is a multiplicative operator (see [22, Theorem 8.9 p. 220]). □

Before proceeding to the next step, that is giving a functorial formulation of Gelfand-Neumark representation for C*-algebras, we provide a brief account of the *automatic continuity* of some maps between classes of Banach algebras. Recall that a Banach *-homomorphism is a function between Banach algebras (with involution) that is a non-extensive complex algebra homomorphism, preserving the involution. If the condition that the function be non-extensive is dropped, we speak of a **-homomorphism*. Finally, by a *homomorphism* between Banach algebras (possibly with involution), we understand a complex algebra homomorphism. The first result is

Lemma 6.2.7. *Let A be a (possibly non-commutative) Banach algebra, and let B be a semisimple Banach algebra. Then every homomorphism $f\colon A \to B$ is automatically continuous.*

Proof. See [23, Proposition 5.1.1]. □

Combining with Corollary 6.2.5, we see that every homomorphism between C*-algebras is continuous. In particular, every *-homomorphism between C*-algebras is continuous. In fact, the following stronger result holds.

Proposition 6.2.8. *Every *-homomorphism between* C**-algebras is non-extensive, i.e. is a Banach *-homomorphism.*

Proof. See [22, Proposition 1.11 p. 234]. □

Let us denote by C^* the category that has (complex, commutative, and unital) C*-algebras as objects, and *-homomorphisms (=Banach *-homomorphisms, by Proposition 6.2.8) as morphisms. As usual, KHaus denotes the category of compact Hausdorff spaces and continuous maps. The following easy fact was observed in Example 6.2.2. Here, recall that $\mathrm{C}(X)$ stands for $\mathrm{C}(X, \mathbb{C})$, the family of all continuous $\mathbb{C}$-valued functions on X.

Lemma 6.2.9. *If X is a compact Hausdorff space, then $\mathcal{C}(X) := \mathrm{C}(X)$ is a C^*-algebra.* ☐

The next result is elementary.

Lemma 6.2.10. *If $\varphi\colon X \to Y$ is a continuous function between compact Hausdorff spaces, then*

$$\mathcal{C}(\varphi) := - \circ \varphi\colon\ \mathrm{C}(Y) \to \mathrm{C}(X)$$

is a *-homomorphism. ☐

It can be easily seen that

Corollary 6.2.11. *$\mathcal{C}\colon \mathsf{KHaus} \to \mathsf{C}^*$ is a contravariant functor from the category of compact Hausdorff spaces to the category of C^*-algebras.* ☐

In the converse direction,

Lemma 6.2.12. *If A is a C^*-algebra, then $\mathcal{S}(A) := \Sigma_A$ is a compact Hausdorff space, where Σ_A denotes the maximal spectrum of A.*

Proof. The space Σ_A, equipped with the weak-star topology, is compact by Proposition 6.1.51. The complex field $\mathbb{C}$ is Hausdorff, so is the product $\mathbb{C}^A$, with respect to the product topology. Every subspace of a Hausdorff space is itself Hausdorff, whence Σ_A is a Hausdorff space. ☐

Lemma 6.2.13. *If $f\colon A \to B$ is a *-homomorphism between C^*-algebras, then*

$$\mathcal{S}(f) := - \circ f\colon \Sigma_B \to \Sigma_A$$

is a continuous map.

Proof. Assume that the sequence $\{h_n\}_{n\in\mathbb{N}} \subseteq \Sigma_B$ converges to the multiplicative functional $h_0 \in \Sigma_B$. We must prove that the sequence $\{\mathcal{S}(f)(h_n)\}_{n\in\mathbb{N}} \subseteq \Sigma_A$ converges to the element $\mathcal{S}(f)(h_0) \in \Sigma_A$. By Lemma 6.1.50 this happens if, and only if

$$\mathcal{S}(f)(h_n)(x) \to \mathcal{S}(f)(h_0)(x)$$

for all $x \in A$, that is

$$(h_n \circ f)(x) \to (h_0 \circ f)(x).$$

However, by hypothesis we know that $h_n(y) \to h_0(y)$ for all $y \in B$, whence, for all $x \in A$,

$$h_n(f(x)) \to h_0(f(x)).$$

☐

It is easy to verify that

Corollary 6.2.14. $\mathcal{S}\colon \mathsf{C}^* \to \mathsf{KHaus}$ *is a contravariant functor from the category of* C^**-algebras to the category of compact Hausdorff spaces.* $\square$

Proposition 6.2.15. *There exists a natural isomorphism*

$$\mu\colon \mathrm{Id}_{\mathsf{KHaus}} \to \mathcal{S} \circ \mathcal{C},$$

where $\mathrm{Id}_{\mathsf{KHaus}}$ *is the identity functor on the category* KHaus.

Proof. Let X be a compact Hausdorff space, and let $x_0 \in X$. It is clear that the map $h_{x_0}\colon \mathrm{C}(X) \to \mathbb{C}$ defined by $h_{x_0}(f) := f(x_0)$, for all $f \in \mathrm{C}(X)$, is a multiplicative functional. In other words, $h_{x_0} \in \Sigma_{\mathrm{C}(X)}$. Define the map

$$\mu_X\colon X \to \Sigma_{\mathrm{C}(X)}, \quad \mu_X(x_0) := h_{x_0}.$$

In view of the bijection between maximal ideals and multiplicative functionals, Example 6.1.36 entails that μ_X is bijective. Since the latter is a continuous function from a compact space to a Hausdorff space, it is closed. To prove that μ_X is a homeomorphism, it is enough to show that it is continuous. Assume that $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a sequence converging to $x_0 \in X$. Then the sequence $\{\mu_X(x_n)\}_{n\in\mathbb{N}} \subseteq \Sigma_{\mathrm{C}(X)}$ converges to $\mu_X(x_0)$ if, and only if, $h_{x_n} \to h_{x_0}$. By Lemma 6.1.50, this happens if, and only if, $h_{x_n}(f) \to h_{x_0}(f)$ for all $f \in \mathrm{C}(X)$, i.e. $f(x_n) \to f(x_0)$. However, this is true because f is continuous. We conclude that μ_X is a continuous map. For each compact Hausdorff space X, let $(\mu)_X := \mu_X$ be the component of μ at X. We have proved that every such component is an isomorphism in the category KHaus, therefore what is left to prove is that μ is a natural transformation. That is, for every continuous function $\varphi\colon X \to Y$ between compact Hausdorff spaces, the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\mu_X"] \arrow[d, "\varphi"'] & \Sigma_{\mathrm{C}(X)} \arrow[d, "(\mathcal{S}\circ\mathcal{C})(\varphi)"] \\
Y \arrow[r, "\mu_Y"'] & \Sigma_{\mathrm{C}(Y)}
\end{tikzcd}$$

Upon observing that, for every multiplicative functional $f \in \Sigma_{\mathrm{C}(X)}$,

$$(\mathcal{S} \circ \mathcal{C})(\varphi)(f) = f \circ (- \circ \varphi),$$

we see that, for all $x_0 \in X$,

$$\begin{aligned}
((\mathcal{S} \circ \mathcal{C})(\varphi) \circ \mu_X)(x_0) &= (\mathcal{S} \circ \mathcal{C})(\varphi)(h_{x_0}) \\
&= h_{x_0} \circ (- \circ \varphi).
\end{aligned}$$

Now, for every multiplicative functional $g \in \Sigma_{\mathrm{C}(Y)}$,

$$h_{x_0} \circ (- \circ \varphi)(g) = h_{x_0}(g \circ \varphi)$$

$$
\begin{aligned}
&= (g \circ \varphi)(x_0) \\
&= g(\varphi(x_0)) \\
&= h_{\varphi(x_0)}(g) \\
&= (\mu_Y \circ \varphi)(x_0)(g).
\end{aligned}
$$

In other words, $(\mathcal{S} \circ \mathcal{C})(\varphi) \circ \mu_X = \mu_Y \circ \varphi$. $\square$

Proposition 6.2.16. *There exists a natural isomorphism*

$$\nu\colon \mathrm{Id}_{\mathsf{C}^*} \to \mathcal{C} \circ \mathcal{S},$$

where $\mathrm{Id}_{\mathsf{C}^*}$ *is the identity functor on the category* C^**.*

Proof. If A is a C*-algebra, recall by Theorem 6.2.6 that the Gelfand transform

$$\Lambda_A\colon A \to \mathrm{C}(\Sigma_A)$$

is a *-isomorphism. Define the component of ν at A as the isomorphism $(\nu)_A := \Lambda_A$ in the category C^*. To prove the statement it suffices to show that ν is a natural transformation, i.e. for every *-homomorphism of C*-algebras $f\colon A \to B$, the next diagram is commutative.

$$
\begin{tikzcd}
A \arrow[r, "\Lambda_A"] \arrow[d, "f"'] & \mathrm{C}(\Sigma_A) \arrow[d, "(\mathcal{C}\circ\mathcal{S})(f)"] \\
B \arrow[r, "\Lambda_B"'] & \mathrm{C}(\Sigma_B)
\end{tikzcd}
$$

We remark that, for every continuous function $g \in \mathrm{C}(\Sigma_A)$,

$$(\mathcal{C} \circ \mathcal{S})(f)(g) = g \circ (- \circ f).$$

Then, for all $x \in A$,

$$
\begin{aligned}
((\mathcal{C} \circ \mathcal{S})(f) \circ \Lambda_A)(x) &= (\mathcal{C} \circ \mathcal{S})(f)(\widehat{x}) \\
&= \widehat{x} \circ (- \circ f).
\end{aligned}
$$

For all $\psi \in \mathrm{C}(\Sigma_B)$, we have

$$
\begin{aligned}
\widehat{x} \circ (- \circ f)(\psi) &= \widehat{x}(\psi \circ f) \\
&= (\psi \circ f)(x) \\
&= \psi(f(x)) \\
&= \widehat{f(x)}(\psi) \\
&= (\Lambda_B \circ f)(x)(\psi).
\end{aligned}
$$

We conclude that $(\mathcal{C} \circ \mathcal{S})(f) \circ \Lambda_A = \Lambda_B \circ f$. $\square$

We have proved that $\mathcal{C}$ and $\mathcal{S}$ are quasi-inverse functors:

Theorem 6.2.17 (Gelfand-Neumark duality)**.** *The category* KHaus *of compact Hausdorff spaces is dually equivalent to the category* C^* *of (complex, commutative, and unital)* C^**-algebras via the functors* $\mathcal{C}$ *and* $\mathcal{S}$*.* □

Remark 6.2.18. Consider a complex Banach algebra with involution. Upon substituting the complex field $\mathbb{C}$ with the real field $\mathbb{R}$, we obtain what is called a *real* Banach algebra with involution. Now, define a *real* C^**-algebra* to be a real Banach algebra A with involution such that, for all $x \in A$,

$$\|xx^*\| = \|x\|^2, \quad \text{and} \quad \mathsf{e} + xx^* \text{ is invertible.}$$

Unlike the complex case, the invertibility of the elements $1+xx^*$ does not follow from the other axioms. If the involution operation is allowed to be non-trivial, then every complex C^*-algebra is a real C^*-algebra. For instance, if we forget scalar multiplication by non-real numbers, the field $\mathbb{C}$ is a real C^*-algebra with complex conjugation as involution. However, if we require that the involution operation is the identity map, then $\mathbb{C}$ is not a real C^*-algebra anymore, because the element

$$1 + ii^* = 1 + i^2 = 0$$

is not invertible. An example of real C^*-algebra, with trivial involution, is provided by the family $\mathrm{C}(X, \mathbb{R})$ of all continuous $\mathbb{R}$-valued functions on some compact Hausdorff space X. Henceforth, by a real C^*-algebra, we understand a real C^*-algebra with trivial involution. For this class of algebras, an analogue of Gelfand-Neumark representation theorem holds. In fact, every real C^*-algebra A is isomorphic and isometric to a real C^*-algebra $\mathrm{C}(X, \mathbb{R})$, for some compact Hausdorff space X [36, Theorem 11.5]. As in the complex case, X can be taken to be the maximal spectrum of A. This representation theorem gives rise to a duality between KHaus and the category of real C^*-algebras and Banach homomorphisms. Composing with Gelfand-Neumark duality between KHaus and C^*, the category of real C^*-algebras is seen to be equivalent to the category of complex C^*-algebras. The functor from the former category to the latter, mapping a real C^*-algebra to a complex C^*-algebra, provides a *complexification* of an arbitrary real C^*-algebra (see [36, p. 71]). Going back to the complex case, we remark that Gelfand-Neumark duality can be seen as a specific duality between algebra and geometry: it relates a certain class of *commutative* algebras with a class of spaces. From this point of view, we are lead to think of compact Hausdorff spaces as *commutative spaces*. Consequently, the dual of the category of (possibly non-commutative) C^*-algebras identifies a class of mathematical objects that can be regarded as *non-commutative spaces*. For this reason, the study of the dual of the category of (possibly non-commutative) C^*-algebras goes under the name of *non-commutative geometry*. The latter is a topic of great interest in mathematics: sophisticated techniques have been developed, and deep results have been proved in this regard. However, a concrete realisation of the dual category at hand, is still lacking.

6.3 Monadicity of C^*

In 1971 Negrepontis showed [58] that the category C^* of (complex, commutative, and unital) C*-algebras is monadic over Set with respect to the *unit ball functor*, mapping a C*-algebra A to the set

$$\{x \in A \mid \|x\| \leqslant 1\}.$$

Remarkably, Van Osdol proved in [66] that the obvious extension of the latter functor to the category of (possibly non-commutative and non-unital) C*-algebras is monadic. In the same work, van Osdol showed that the unit ball functor from the category of (possibly non-commutative, unital) C*-algebras is monadic. Concerning the latter category, Pelletier and Rosický proved in [59] that it is monadic over Set with respect to two more functors: the hermitian unit ball functor, and the positive unit ball functor. If A is a (possibly non-commutative) C*-algebra, the *hermitian unit ball functor* associates to A the set

$$\{x \in A \mid \|x\| \leqslant 1, \;\; x = x^*\}.$$

The name is due to the fact that self-adjoint elements are also called *hermitian* elements. To define the positive unit ball functor, we need to introduce the notion of positive element in a C*-algebra. In fact, this notion will lead to the definition of a partial order on an arbitrary C*-algebra.

Notation 6.3.1. An order in a C*-algebra can be defined in a natural way, for both commutative and non-commutative C*-algebras. In this section we shall restrict to commutative C*-algebras only when necessary, hence by a C*-algebra we mean a possibly non-commutative unital C*-algebra. When the commutativity property is assumed, we state it explicitly.

A generalisation of Lemma 6.2.3 to the non-commutative case is needed. Here the spectrum of an element is defined as in the commutative case.

Lemma 6.3.2. *Let A be a* C*-*algebra, and let $x \in A$ be a self-adjoint element. Then $\sigma_x \subseteq \mathbb{R}$.*

Proof. See [36, Proposition 4.3]. □

Notation 6.3.3. The symbol $\mathbb{R}_{\geqslant 0}$ will denote the set of non-negative real numbers.

Definition 6.3.4. Let A be a C*-algebra. A self-adjoint element $x \in A$ is called *positive*, written as $x \geqslant 0$, if $\sigma_x \subseteq \mathbb{R}_{\geqslant 0}$.

Example 6.3.5. Consider the commutative C*-algebra $\mathrm{C}(X)$, for some compact Hausdorff space X. An easy generalisation of Example 6.1.55 shows that, for all $f \in \mathrm{C}(X)$, $\sigma_f = f(X)$. Therefore f is a positive element of $\mathrm{C}(X)$ if, and only if, $f(X) \subseteq \mathbb{R}_{\geqslant 0}$ if, and only if, f is a positive continuous $\mathbb{R}$-valued function on X.

Positive elements in a C*-algebra can be characterised in different ways.

Proposition 6.3.6. *If A is a* C*-*algebra and $x \in A$ is a self-adjoint element, the following are equivalent.*

1. $x \geqslant 0$.
2. *There exists a self-adjoint element $y \in A$ such that $x = y^2$.*
3. *There exists an element $z \in A$ such that $x = zz^*$.*

Proof. See [22, Theorem 3.6 p. 241]. □

Definition 6.3.7. Let A be a C*-algebra, and let $x, y \in A$. We set $x \leqslant y$ if x, y are self-adjoint elements and $y - x \geqslant 0$.

If A is a C*-algebra, then the *positive unit ball functor* maps A to the set

$$\{x \in A \mid \|x\| \leqslant 1, \;\; x \geqslant 0\}.$$

Example 6.3.8. Let X be a compact Hausdorff space, and consider the commutative C*-algebra $\mathrm{C}(X)$. It is clear that the unit ball of $\mathrm{C}(X)$ is the set of all functions that take values in the complex unit disc $\{\lambda \in \mathbb{C} \mid |\lambda| \leqslant 1\}$, while its hermitian unit ball is the set of functions whose range is contained in the real interval $[-1, 1]$. Finally, the positive unit ball of $\mathrm{C}(X)$ is the family of functions taking values in the unit interval $[0, 1]$.

It is easy to see that every *-homomorphism between C*-algebras is order-preserving (but, in general, not order-reflecting):

Lemma 6.3.9. *Let $f : A \to B$ be a *-homomorphism between* C**-algebras, and let $x, y \in A$ be self-adjoint elements. If $x \leqslant y$, then $f(x) \leqslant f(y)$.*

Proof. If $x \leqslant y$, then Proposition 6.3.6 entails that there exists a self-adjoint element $z \in A$ such that $z^2 = y - x$. Now,

$$f(y) - f(x) = f(y - x) = f(z^2) = f(z)^2,$$

where $f(z) \in B$ is self-adjoint since $f(z)^* = f(z^*) = f(z)$. In other words, $f(x) \leqslant f(y)$. □

Gelfand-Neumark duality (see Theorem 6.2.17) states that the category C^* is dually equivalent to KHaus via the functor $\mathcal{S}$. On the other hand, by Theorem 4.4.25, the category KHaus is dually equivalent to the category Δ of δ-algebras, which form a variety of infinitary algebras, via the functor $\mathcal{C}$. Given the correspondence between categories that are monadic over Set and varieties of possibly infinitary algebras (see [49], or [54, Theorem 5.40 p. 66, Theorem 5.45 p. 68]), it follows at once that C^* is monadic over Set with respect to the composition of the underlying-set functor $U \colon \Delta \to \mathsf{Set}$ with the equivalence $\mathcal{C} \circ \mathcal{S} \colon \mathsf{C}^* \to \Delta$. The aim of this section is to give a direct proof of the monadicity of the category C^* of commutative C*-algebras, with no reference to the dual category of KHaus. The key fact is that the monadic functor $U \circ \mathcal{C} \circ \mathcal{S} \colon \mathsf{C}^* \to \mathsf{Set}$ is naturally isomorphic to the positive unit ball functor.

Recall that, if x is an element of a commutative C*-algebra, then $\widehat{x}\colon \Sigma_A \to \mathbb{C}$ represents the Gelfand transform of x, defined on the maximal spectrum Σ_A of A.

Lemma 6.3.10. *Let A be a commutative* C**-algebra, and let $x \in A$. Then x is self-adjoint if, and only if, $\widehat{x}$ is $\mathbb{R}$-valued.*

Proof. If the element $x \in A$ is self-adjoint, then the continuous function $\widehat{x}\colon \Sigma_A \to \mathbb{C}$ is clearly $\mathbb{R}$-valued by Theorem 6.1.54 and Lemma 6.2.3. In the opposite direction, if $\widehat{x}$ is $\mathbb{R}$-valued, then

$$\widehat{x^*} = \overline{\widehat{x}} = \widehat{x}$$

because the Gelfand transform is a *-homomorphism. The latter is also injective, hence we conclude that $x^* = x$. □

The next result states that the Gelfand transform of a commutative C*-algebra, restricted to the family of self-adjoint elements, is order-preserving and order-reflecting.

Lemma 6.3.11. *Let A be a commutative* C**-algebra. If $x, y \in A$ are self-adjoint elements, then*

$$x \leqslant y \text{ if, and only if, } \widehat{x} \leqslant \widehat{y}.$$

Proof. One of the two directions follows at once from Lemma 6.3.9, because the Gelfand transform is a *-homomorphism. Conversely, assume that $x, y \in A$ are self-adjoint elements satisfying $\Lambda_A(x) \leqslant \Lambda_A(y)$. Then there exists a self-adjoint element $f \in \mathrm{C}(\Sigma_A)$, i.e. a continuous $\mathbb{R}$-valued function, such that

$$\Lambda_A(y - x) = \Lambda_A(y) - \Lambda_A(x) = f^2.$$

Recall that the map Λ_A is surjective by Theorem 6.2.6. If $z \in A$ satisfies $\Lambda_A(z) = f$, then the injectivity of Λ_A entails that $y - x = z^2$. The element z is self-adjoint by Lemma 6.3.10, therefore $x \leqslant y$ by Proposition 6.3.6. □

Given a C*-algebra A, denote the set of self-adjoint elements of A by

$$\mathrm{H}_A := \{x \in A \mid x^* = x\}.$$

Lemma 6.3.12. *For every* C**-algebra A, H_A is a partially ordered abelian group. Moreover, H_A is norm-closed in A.*

Proof. It is clear that, with respect to the sum of A, H_A is a partially ordered abelian group. We claim that the map $^*\colon A \to A$ is an isometry. Indeed, for all $x \in A$, $\|x\|^2 = \|xx^*\| \leqslant \|x\| \cdot \|x^*\|$, that is $\|x\| \leqslant \|x^*\|$. The latter inequality holds for the element $x := x^*$ as well, therefore $\|x^*\| \leqslant \|x^{**}\| = \|x\|$. Then the involution function is continuous, and so is the map $f\colon A \to A$ defined by $f(x) := x - x^*$. Therefore, $\mathrm{H}_A = f^{-1}(0)$ is closed in the topology induced by the norm. □

Proposition 6.3.13. *If A is a commutative* C**-algebra, then H_A is a unital ℓ-group.*

Proof. The set H_A is a partially ordered abelian group by Lemma 6.3.12. Now we prove that, for every pair of elements $x, y \in \mathrm{H}_A$, there exist a greatest lower bound $x \wedge y \in \mathrm{H}_A$ and a least upper bound $x \vee y \in \mathrm{H}_A$. Consider the element $w := \widehat{x} \wedge \widehat{y}$. It is elementary that $w \in \mathrm{C}(\Sigma_A)$ and it is $\mathbb{R}$-valued. By Theorem 6.2.6, the Gelfand transform $\Lambda_A : x \mapsto \widehat{x}$ is surjective, so that there exists $z \in A$ such that $\widehat{z} = w$. The element z belongs to H_A by Lemma 6.3.10, and it is clear from Lemma 6.3.11 that z is the greatest lower bound for the pair x, y. In a similar fashion, it is possible to define the least upper bound of x and y. The translation invariance property (see item 3 in Definition 2.1.1) is easily seen to hold, thus H_A is an ℓ-group. Notice that, if e is the unit of A, then e is a strong order unit for H_A. Indeed, we know that $\widehat{\mathsf{e}}$ is the constant function 1_{Σ_A} of value 1 on Σ_A, and the latter is a strong order unit for the ℓ-group $\mathrm{C}(\Sigma_A, \mathbb{R})$. Pick $x \in \mathrm{H}_A$, and consider its Gelfand transform $\widehat{x} \in \mathrm{C}(\Sigma_A, \mathbb{R})$. Then there exists $n \in \mathbb{N}$ such that $\widehat{x} \leqslant n\widehat{\mathsf{e}}$: this is equivalent, by Lemma 6.3.11, to $x \leqslant n\,\mathsf{e}$ (here $n\widehat{\mathsf{e}} = \widehat{n\,\mathsf{e}}$ by the linearity of the Gelfand transform). This proves that H_A is a unital ℓ-group. $\square$

Remark 6.3.14. In fact, H_A is a vector lattice (=lattice-ordered real vector space), since it is closed under multiplication by real numbers. Indeed, if $\lambda \in \mathbb{R}$ and $x \in \mathrm{H}_A$, then $(\lambda x)^* = \overline{\lambda} x^* = \lambda x$.

Proposition 6.3.13 states, in particular, that the set of self-adjoint elements of a commutative unital C^*-algebra is a lattice. Remarkably, in 1951 Sherman proved

Theorem 6.3.15. *If A is a (possibly non-unital)* C^**-algebra, then A is commutative if, and only if,* H_A *is lattice-ordered.*

Proof. See [63, Theorem 2]. $\square$

In the unital case, a generalisation of the previous theorem can be proved by means of the Riesz decomposition property. Recall that a partially ordered vector space V satisfies the *Riesz decomposition property* if, for all positive elements $f, g_1, g_2 \in V$ satisfying $f \leqslant g_1 + g_2$, there exist positive elements $f_1, f_2 \in V$ such that $f_1 \leqslant g_1$, $f_2 \leqslant g_2$, and $f = f_1 + f_2$. It is easy to see that

Lemma 6.3.16. *If A is a (unital) commutative* C^**-algebra, then* H_A *satisfies the Riesz decomposition property.*

Proof. Proposition 6.3.13 and Remark 6.3.14 show that H_A is a vector lattice. If $x, y_1, y_2 \in A$ are positive elements satisfying $x \leqslant y_1 + y_2$, then $\widehat{x}, \widehat{y_1}, \widehat{y_2} \in \mathrm{C}(\Sigma_A)$ are positive $\mathbb{R}$-valued functions satisfying $\widehat{x} \leqslant \widehat{y_1} + \widehat{y_2}$ by Lemma 6.3.11. Define

$$f_1 := \widehat{x} \wedge \widehat{y_1}, \text{ and } f_2 := \widehat{x} - f_1.$$

The elements $f_1, f_2 \in \mathrm{C}(\Sigma_A)$ are positive, and

$$f_1 + f_2 = (\widehat{x} \wedge \widehat{y_1}) + \widehat{x} - (\widehat{x} \wedge \widehat{y_1}) = \widehat{x}.$$

We shall prove that $f_1 \leqslant \widehat{y_1}$ and $f_2 \leqslant \widehat{y_2}$. The first inequality is obvious, for the second one assume, by contradiction, that there exists $p \in \Sigma_A$ such that $\widehat{y_2}(p) < f_2(p)$. Then

$$\widehat{y_2}(p) < f_2(p) = \widehat{x}(p) - f_1(p) = \widehat{x}(p) - \min\left(\widehat{x}(p), \widehat{y_1}(p)\right).$$

However, $\min\left(\widehat{x}(p), \widehat{y_1}(p)\right) = \widehat{y_1}(p)$, for otherwise $\widehat{y_2}(p) < 0$ that cannot be. Hence

$$\widehat{y_1}(p) + \widehat{y_2}(p) < \widehat{y_1}(p) + \widehat{x}(p) - \min\left(\widehat{x}(p), \widehat{y_1}(p)\right) = \widehat{y_1}(p) + \widehat{x}(p) - \widehat{y_1}(p) = \widehat{x}(p),$$

that is a contradiction because $\widehat{x} \leqslant \widehat{y_1} + \widehat{y_2}$. The Gelfand transform is surjective by Theorem 6.2.6, thus there exist $z_1, z_2 \in A$ such that $\widehat{z_1} = f_1$ and $\widehat{z_2} = f_2$. Lemma 6.3.11, along with the injectivity of the Gelfand transform, entail that the elements z_1, z_2 satisfy the Riesz decomposition property. $\square$

More generally, it can be proved that the condition above is sufficient.

Theorem 6.3.17. *A (unital)* C^**-algebra A is commutative if, and only if,* H_A *satisfies the Riesz decomposition property.*

Proof. See [29, Theorem 1]. $\square$

Lemma 6.3.18. *If $f\colon A \to B$ is a *-homomorphism between commutative* C^**-algebras, then*

$$\mathrm{H}(f) := f_{|\mathrm{H}_A} : \mathrm{H}_A \to \mathrm{H}_B$$

is a unital ℓ-homomorphism.

Proof. It is elementary that the map $\mathrm{H}(f)$ is well-defined, since f preserves the involution operation, and that it is a group homomorphism. We check that $\mathrm{H}(f)$ is also a lattice homomorphism. Since the Gelfand transform $\Lambda_B\colon B \to \mathrm{C}(\Sigma_B)$ is injective, for all $x, y \in \mathrm{H}_A$, the condition $f(x) \wedge f(y) = f(x \wedge y)$ is equivalent to

$$\widehat{f(x) \wedge f(y)} = \widehat{f(x \wedge y)}. \tag{6.1}$$

Since $f(x) \wedge f(y)$ is defined as the unique element of H_B such that $\widehat{f(x) \wedge f(y)} = \widehat{f(x)} \wedge \widehat{f(y)}$, (6.1) is equivalent to

$$(\widehat{f(x)} \wedge \widehat{f(y)})(h) = \widehat{f(x \wedge y)}(h) \text{ for all } h \in \Sigma_B.$$

In turn, this happens if, and only if,

$$h(f(x)) \wedge h(f(y)) = h(f(x \wedge y)) \text{ for all } h \in \Sigma_B. \tag{6.2}$$

However, it is clear that $h \circ f \in \Sigma_A$, so that

$$h(f(x)) \wedge h(f(y)) = \widehat{x}(h \circ f) \wedge \widehat{y}(h \circ f) = \widehat{x \wedge y}(h \circ f) = h(f(x \wedge y)),$$

since $x \wedge y$ is the unique element of H_A satisfying $\widehat{x \wedge y} = \widehat{x} \wedge \widehat{y}$. Then (6.2) is proved. The equality $f(x \vee y) = f(x) \vee f(y)$ can be proved in a similar way. Finally, it is elementary that the ℓ-homomorphism $\mathrm{H}(f)$ is unital. $\square$

It is easy to see that compositions and identity are preserved, hence:

Corollary 6.3.19. $\mathrm{H}\colon \mathsf{C}^* \to \ell\mathsf{Grp_u}$, *mapping a commutative* C^**-algebra* A *to the unital* ℓ*-group* H_A *of its self-adjoint elements, is a functor.* $\square$

By composing with the functor $\Gamma\colon \ell\mathsf{Grp_u} \to \mathsf{MV}$ (see Section 2.3), we obtain a functor

$$\mathrm{B}_+ := \Gamma \circ \mathrm{H}\colon \mathsf{C}^* \to \mathsf{MV}$$

that sends a commutative C^*-algebra A, with unit e, to the MV-algebra with underlying set

$$\{x \in A \mid x^* = x, \;\; 0 \leqslant x \leqslant \mathsf{e}\}.$$

Observe that every positive element is self-adjoint by definition. Furthermore, a positive element $x \in A$ satisfies $x \leqslant \mathsf{e}$ if, and only if, $\|x\| \leqslant 1$. Indeed, by Lemma 6.3.11, $x \leqslant \mathsf{e}$ if, and only if $\widehat{x} \leqslant 1_{\Sigma_A}$ if, and only if,

$$\sup_{f \in \Sigma_A} \widehat{x}(f) \leqslant \sup_{f \in \Sigma_A} 1_{\Sigma_A}(f).$$

Upon recalling that the Gelfand transform is an isometry by Theorem 6.2.6 and Remark 6.1.10, the latter inequality is equivalent to $\|x\| = \|\widehat{x}\| \leqslant 1$. In other words,

$$\{x \in A \mid x^* = x, \;\; 0 \leqslant x \leqslant \mathsf{e}\} = \{x \in A \mid \|x\| \leqslant 1, \;\; x \geqslant 0\}.$$

This shows that the positive unit ball of a commutative C^*-algebra admits a structure of MV-algebra. We will now prove that, in fact, it admits a structure of δ-algebra.

Remark 6.3.20. Notice that the set of positive elements in the unit ball of a C^*-algebra A is closed under multiplication by real numbers in $[0, 1]$. Indeed, if $\lambda \in [0, 1]$ and $x \in A$ belongs to the positive unit ball, then

$$\|\lambda x\| = |\lambda| \cdot \|x\| \leqslant \|x\| \leqslant 1.$$

Further, by Proposition 6.3.6 there exists a self-adjoint element $y \in A$ such that $x = y^2$. Then, it is elementary that the element $\sqrt{\lambda}y$ is self-adjoint and satisfies $(\sqrt{\lambda}y)^2 = \lambda x$. Again by Proposition 6.3.6, we conclude that λx is a positive element in the unit ball.

Let A be a commutative C^*-algebra, and consider the MV-algebra $\mathrm{B}_+(A)$. Define an infinitary operation $\delta\colon \mathrm{B}_+(A) \to \mathrm{B}_+(A)$, for all the countable sequences $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathrm{B}_+(A)$, as

$$\delta(\vec{x}) := \sum_{i=1}^{\infty} \frac{x_i}{2^i}.$$

In order to prove that δ is a well-defined operation, observe that $\frac{x_i}{2^i}$ belongs to $\mathrm{B}_+(A)$, for all $i \in \mathbb{N}$, by Remark 6.3.20. Moreover,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{x_i}{2^i} = \sum_{i=1}^{\infty} \tfrac{x_i}{2^i} \geqslant 0$$

because the set of positive elements of a C*-algebra is a closed cone [22, Proposition 3.7 p. 241]. To conclude, it suffices to show that the norm of $\|\delta(\vec{x})\|$ does not exceed 1:

$$\left\|\sum_{i=1}^{\infty} \tfrac{x_i}{2^i}\right\| = \lim_{n\to\infty} \left\|\sum_{i=1}^{n} \tfrac{x_i}{2^i}\right\| \leqslant \lim_{n\to\infty} \sum_{i=1}^{n} \left\|\tfrac{x_i}{2^i}\right\| \leqslant \lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{1}{2^i} = 1.$$

Proposition 6.3.21. *If A is a commutative* C*-*algebra, then* $\mathrm{B}_+(A)$ *is a δ-algebra.*

Proof. Upon defining the infinitary operation δ on $\mathrm{B}_+(A)$ as shown above, the proposition follows at once by Theorem 6.2.17 and Proposition 4.4.4. □

In view of the foregoing result, we adopt the notation $\mathrm{B}_+ \colon \mathsf{C}^* \to \Delta$.

With reference to Theorems 4.4.25 and 6.2.17,

Corollary 6.3.22. *The functor* $\mathrm{B}_+ \colon \mathsf{C}^* \to \Delta$ *is naturally isomorphic to the equivalence*

$$\mathcal{C} \circ \mathcal{S} \colon \mathsf{C}^* \to \mathsf{KHaus} \to \Delta.$$

Proof. Let A be a commutative C*-algebra. By Theorem 6.2.6, we know that there exists an isomorphisms $A \cong \mathrm{C}(\Sigma_A, \mathbb{C})$ in the category C^*. Therefore,

$$\begin{aligned} \mathrm{B}_+(A) &\cong \mathrm{B}_+(\mathrm{C}(\Sigma_A, \mathbb{C})) \\ &= \{f \in \mathrm{C}(\Sigma_A, \mathbb{C}) \mid \|f\| \leqslant 1,\ \ f \geqslant 0\} \\ &= \{f \in \mathrm{C}(\Sigma_A, \mathbb{C}) \mid f(\Sigma_A) \subseteq [0,1]\} \\ &= \mathrm{C}(\Sigma_A, [0,1]). \end{aligned}$$

This shows that there exists an isomorphism $\mu_A \colon \mathrm{B}_+(A) \to \mathcal{C} \circ \mathcal{S}(A)$ in the category Δ. Define $\mu \colon \mathrm{B}_+ \to \mathcal{C} \circ \mathcal{S}$ by setting, for every commutative C*-algebra A, $(\mu)_A := \mu_A$. Every such component is an isomorphism in Δ. The proof of the fact that μ is a natural transformation is left to the reader. It is clear that μ_A is nothing but the restriction of the Gelfand transform, sending $a \in \mathrm{B}_+(A)$ to $\widehat{a} \in \mathcal{C} \circ \mathcal{S}(A) = \mathrm{C}(\Sigma_A, [0,1])$. □

Theorem 6.3.23. *The category* C^* *of commutative (unital)* C*-*algebras is monadic over* Set *with respect to the positive unit ball functor.*

Proof. Denote by $U_+ \colon \mathsf{C}^* \to \mathsf{Set}$ the positive unit ball functor, mapping a commutative C*-algebra A to the set

$$\{x \in A \mid \|x\| \leqslant 1,\ \ x \geqslant 0\}.$$

The category Δ is a variety of infinitary algebras (of bounded arity), hence the underlying-set functor $U\colon \Delta \to \mathsf{Set}$ is monadic [54, Theorem 5.45 p. 68] (in fact, it is strictly monadic). This means the following. First, there is an adjoint pair $F \dashv U$, where $F\colon \mathsf{Set} \to \Delta$ maps a set of generators X to the free δ-algebra over X. In fact, we know that $FX \cong \mathrm{C}([0,1]^X, [0,1])$ by Proposition 5.2.8. Secondly, the category Δ is equivalent to the category $\mathsf{Set}^{U\circ F}$ of Eilenberg-Moore algebras for the monad $U \circ F\colon \mathsf{Set} \to \mathsf{Set}$. Denote by E the equivalence $\mathcal{C} \circ \mathcal{M}\colon \Delta \to \mathsf{C}^*$ (see Theorems 4.4.25 and 6.2.17). By Corollary 6.3.22, E and $\mathrm{B}_+\colon \mathsf{C}^* \to \Delta$ are quasi-inverse functors. Since adjoint functors are stable under composition [52, Theorem 1 p. 101], the functor $U \circ \mathrm{B}_+$ is right adjoint to $E \circ F$.

$$\begin{tikzcd} \mathsf{C}^* \arrow[r, "\mathrm{B}_+"] \arrow[rr, bend left, "U_+"] & \Delta \arrow[l, bend left, "E"] \arrow[r, "U"] & \mathsf{Set} \arrow[l, bend left, "F"] \end{tikzcd}$$

Let Set^T be the Eilenberg-Moore category for the monad $T := U \circ \mathrm{B}_+ \circ E \circ F$ on Set. The functor $\mathrm{B}_+ \circ E$ is naturally isomorphic to the identity functor 1_Δ, so that $U \circ \mathrm{B}_+ \circ E \circ F$ is naturally isomorphic to $U \circ F$. This natural isomorphism extends to an isomorphism of monads which induces a (concrete) isomorphism of the categories of algebras Set^T and $\mathsf{Set}^{U\circ F}$ [2, A.26]. Therefore,

$$\mathsf{Set}^T \cong \mathsf{Set}^{U\circ F} \simeq \Delta \simeq \mathsf{C}^*.$$

In other words, the functor $U \circ \mathrm{B}_+\colon \mathsf{C}^* \to \mathsf{Set}$ is monadic. However, it is elementary that $U \circ \mathrm{B}_+ = U_+$, hence we conclude that the positive unit ball functor $U_+\colon \mathsf{C}^* \to \mathsf{Set}$ is monadic. $\square$

Chapter 7

Epilogue

In Chapter 1 we presented some results concerning the axiomatisability of the dual category $\mathsf{KHaus}^{\mathsf{op}}$. On the one hand, there are the negative results due to Rosický and Banaschewski. They proved Bankston's conjecture, i.e. that KHaus is not dually equivalent to any elementary P-class of finitary algebras. Remarkably, Banaschewski proved that there is no full subcategory of KHaus extending the category St of Stone spaces that is dually equivalent to an elementary P-class of finitary algebras (see Theorem 1.2.11). In this respect, in Section 7.1 we prove an analogue of Banaschewski's theorem, i.e. that no full subcategory of KHaus extending the category St is dually equivalent to the category of models of a geometric theory of presheaf type. On the other hand, there are the positive results stating that the category $\mathsf{KHaus}^{\mathsf{op}}$ can be axiomatised in a certain extension of first-order logic (see Theorem 1.2.10). In this direction, we give an explicit axiomatisation of the category $\mathsf{KHaus}^{\mathsf{op}}$ in the infinitary language $\mathrm{L}_{\omega_1,\omega_1}$ over an algebraic signature, i.e. a signature with no relation symbols.

7.1 Axiomatisability of $\mathsf{KHaus}^{\mathsf{op}}$: one negative result

We prove that $\mathsf{KHaus}^{\mathsf{op}}$ is not axiomatisable by a geometric theory of presheaf type. A key step of the proof consists in showing that every finitely copresentable object in a full subcategory $\mathcal{F}$ of KHaus that extends St, is a finite discrete space. In the particular case in which $\mathcal{F} = \mathsf{KHaus}$, Gabriel and Ulmer proved that finitely copresentable objects coincide precisely with finite sets [30, p. 66]. Here we give a different proof which relies on the classical construction of the Gleason cover of a compact Hausdorff space.

Recall that a *Stone space* (or *Boolean space*) is a compact Hausdorff space whose topology admits a basis of clopen sets, i.e. sets that are both open and closed. For an arbitrary topological space, it is elementary that the collection of its clopen sets, endowed with set-theoretical operations, is a Boolean algebra. In 1936, in his seminal work [65], Stone proved that every Boolean algebra arises as the algebra of clopen sets of a Stone space, namely the space of its ultrafilters. This representation theorem, which extends to a categorical equivalence known as *Stone duality*, shows that every Boolean algebra is associated to a unique (up to homeomorphism) Stone space — the *dual space* of the Boolean algebra — and the converse holds as well.

A topological space is *extremally disconnected* if the closure of every open set is again open. It is clear that every compact extremally disconnected space is a Stone space. The following is a consequence of Stone's representation theorem.

Proposition 7.1.1. *A Boolean algebra is complete if, and only if, its dual space is extremally disconnected.*

Proof. See for example [35, p. 485]. □

Recall that a closed subset of a topological space is *regular closed* if it coincides with the closure of its interior. For example, any clopen subset is regular closed. It is well known that, given a topological space X, the collection of all regular closed subsets of X is a complete Boolean algebra with respect to inclusion (a proof can be found in [35, Lemma 3.1]). Denote by G_X the dual space of this Boolean algebra. By Proposition 7.1.1 not only is this a Stone space, but also an extremally disconnected space. Amongst other things, Gleason proved in [35, Theorem 3.2]

Theorem 7.1.2. *A compact Hausdorff space X is a continuous image of the extremally disconnected space G_X.* □

The space G_X is called the *Gleason cover* (or *absolute*) of the compact Hausdorff space X, and the continuous surjection $G_X \twoheadrightarrow X$ may be characterised by appropriate properties.

Lemma 7.1.3. *Let X be a topological space. Then X is a Stone space if, and only if, it is a cofiltered limit of finite discrete spaces.*

Proof. See [44, p. 236]. □

Recall that an object A of a category C is λ-*copresentable* if the contravariant functor $\mathsf{C}(-, A)\colon \mathsf{C} \to \mathsf{Set}$ preserves λ-cofiltered limits. If $\lambda = \aleph_0$, then we speak of *copresentable* object. Unfolding this definition, we find:

Lemma 7.1.4. *An object A is λ-copresentable in C precisely when, for every λ-cofiltered limit $G = \lim_{i \in I} G_i$, the following conditions hold.*

1. *For every morphism $h\colon G \to A$ there exists a canonical morphism $\alpha_j\colon G \to G_j$ of the limit such that h factors through α_j.*
2. *If $\phi_1, \phi_2\colon G_j \to A$ are such that $\phi_1 \circ \alpha_j = \phi_2 \circ \alpha_j$ for some canonical morphism $\alpha_j\colon G \to G_j$ of the limit, then there exists a morphism $g_{ij}\colon G_i \to G_j$ in the diagram such that $\phi_1 \circ g_{ij} = \phi_2 \circ g_{ij}$.*

Proof. Direct inspection. □

Theorem 7.1.5. *Assume that* F *is a full subcategory of* KHaus *extending* St. *If* F *is dually equivalent to a finitely accessible category, then* $\mathsf{F} = \mathsf{St}$.

Proof. Let C be a finitely accessible category which is dually equivalent to F. Every object in C is the filtered colimit of finitely presentable objects, so that every object in F is the cofiltered limit of finitely copresentable objects. We show that every finitely copresentable object in F is a finite set. Let F be an object of F, and denote by G its Gleason cover (see Theorem 7.1.2). The latter is, in particular, a Stone space, hence it is the cofiltered limit in KHaus of finite sets $\{G_i\}_{i \in I}$ by Lemma 7.1.3. Note that, since G is also an object of F and the full embedding $\mathsf{F} \hookrightarrow \mathsf{KHaus}$ reflects limits, G is the cofiltered limit in F of the finite sets $\{G_i\}_{i \in I}$.

$$\begin{tikzcd}
F & G \arrow[l, two heads, "\gamma"'] \arrow[d, bend left, "\alpha_i"'] \arrow[dd, bend left=60, "\alpha_j"] \\
 & G_i \arrow[d, "g_{ij}"] \\
 & G_j \arrow[uul, "\phi"]
\end{tikzcd}$$

If F is finitely copresentable, by Lemma 7.1.4 there exists a map $\phi \colon G_j \to F$ for some j, such that $\gamma = \phi \circ \alpha_j$. But γ is surjective, hence so is ϕ. This shows that F is finite, being the epimorphic image of a finite set. Thus, every object in F is the cofiltered limit of finite discrete sets. However, Stone spaces coincide with cofiltered limits in KHaus of finite discrete sets, and we have seen that cofiltered limits of finite sets in F are computed exactly as in KHaus. Hence an arbitrary object of F is a Stone space. Since St is a full subcategory of KHaus, we conclude that $\mathsf{F} = \mathsf{St}$. $\square$

It turns out that finitely accessible categories coincide, up to equivalence, with the categories of models of certain theories. Let Σ be a signature with no infinitary function symbols and no infinitary relation symbols. Denote by $\mathrm{L}^{\mathrm{g}}_{\infty,\omega}$ the fragment of the infinitary language $\mathrm{L}_{\infty,\omega}$ over the signature Σ in which only finitary conjunctions are allowed. A *geometric formula* is a formula in the language $\mathrm{L}^{\mathrm{g}}_{\infty,\omega}$ with only finitely many free variables, which is constructed by using only finitary conjunctions, possibly infinitary disjunctions, and existential quantifications. A *geometric theory* over the signature Σ is a set $\mathbb{T}$ of pairs (φ, ψ), called *axioms*, where φ, ψ are geometric formulæ. The pair (φ, ψ) can be thought of as the sentence

$$\forall x_1 x_2 \cdots x_n (\varphi \Rightarrow \psi),$$

where the free variables of φ and ψ are amongst $x_1, x_2, \ldots, x_n$. We shall now assume some knowledge of topos theory. It is possible to show that the geometric language is weak enough to be interpreted in any Grothendieck topos. Thus, we can consider the category of models of a geometric theory in an arbitrary Grothendieck topos. Given a geometric theory $\mathbb{T}$, we say that a Grothendieck topos $\mathcal{E}_{\mathbb{T}}$ is a *classifying topos* for $\mathbb{T}$ if, for any Grothendieck topos $\mathcal{E}$, the category of models of $\mathbb{T}$ in $\mathcal{E}$ is equivalent to the category of geometric morphisms from $\mathcal{E}$ to $\mathcal{E}_{\mathbb{T}}$. The following result is due to Joyal, Reyes and Makkai.

Theorem 7.1.6. *Every geometric theory $\mathbb{T}$ has a (unique up to equivalence) classifying topos $\mathcal{E}_{\mathbb{T}}$.*

Proof. See [43, p. 247]. □

In fact, every Grothendieck topos is the classifying topos of some geometric theory $\mathbb{T}$ (see for example [20, Theorem 7.11]). Recall that an example of Grothendieck topos is provided by the presheaf topos on a category C, i.e. the category of functors $\mathsf{C}^{\mathsf{op}} \to \mathsf{Set}$.

Definition 7.1.7. A geometric theory $\mathbb{T}$ is *of presheaf type* if its classifying topos is equivalent to a presheaf topos.

Notation 7.1.8. Let $\mathbb{T}$ be a geometric theory. We denote by $\mathsf{Mod}\,\mathbb{T}$ the category whose objects are models of $\mathbb{T}$ in the topos Set, and whose morphisms are homomorphisms preserving operations and relations.

We can now characterise finitely accessible categories as categories of models.

Theorem 7.1.9. *A category C is finitely accessible if, and only if, it is equivalent to $\mathsf{Mod}\,\mathbb{T}$ for some geometric theory of presheaf type $\mathbb{T}$.*

Proof. See for example [12, Proposition 0.1]. □

Specializing Theorem 7.1.5 for $\mathsf{F} = \mathsf{KHaus}$,

Corollary 7.1.10. *The dual category $\mathsf{KHaus}^{\mathsf{op}}$ is not axiomatisable by any geometric theory of presheaf type.* □

7.2 Axiomatisability of $\mathsf{KHaus}^{\mathsf{op}}$: one positive result

In Chapter 1 we saw that, as a consequence of Theorem 1.2.10, the dual category $\mathsf{KHaus}^{\mathsf{op}}$ can be axiomatised in the infinitary language $\mathrm{L}_{\omega_1,\omega_1}$. Specifically, $\mathsf{KHaus}^{\mathsf{op}}$ is equivalent to $\mathsf{Mod}\,\mathbb{T}$ for some limit theory $\mathbb{T}$ in $\mathrm{L}_{\omega_1,\omega_1}$ whose language possibly admits both function symbols and relation symbols. In this section we prove that $\mathbb{T}$ can be taken as a theory in $\mathrm{L}_{\omega_1,\omega_1}$ whose language does not contain relation symbols. However, we remark that the latter theory is not a limit theory.

Let X be an arbitrary topological space, and denote by $\Omega(X)$ the family of open sets in X. It is elementary that $\Omega(X)$ is a bounded distributive lattice with respect to inclusion. Recall that a subset I of a lattice A is an *ideal* if it satisfies the following conditions: I is non-empty, if $a, b \in I$ then $a \vee b \in I$, and $b \in I$ whenever $a \in I$ and $b \leqslant a$. A proper ideal $\mathfrak{m}$ of A is *maximal* if there is no proper ideal in A which strictly contains $\mathfrak{m}$.

Remark 7.2.1. A standard application of Zorn's Lemma shows that every non-trivial bounded distributive lattice has a maximal ideal [24, p. 237]. In fact, every lattice we will be dealing with in this section is bounded and distributive. For this reason, henceforth by a *lattice* we understand a bounded distributive lattice.

The set $\operatorname{Max} A$ of all the maximal ideals of a lattice A can be equipped with the *Stone-Zariski topology*. A subbasis of closed sets for the latter is given by the sets of the form

$$F_a := \{\mathfrak{m} \in \operatorname{Max} A \mid a \in \mathfrak{m}\},$$

for all $a \in A$ (see [37, pp. 99–102] for more details).

Theorem 7.2.2. *If A is a lattice, then $\operatorname{Max} A$ is a compact Hausdorff space with respect to the Stone-Zariski topology.*

Proof. See [44, p. 66]. □

If X is any T_1-space, we shall see that it is possible to define a compactification of X by means of a sublattice of $\Omega(X)$. If X satisfies additional topological properties, then the latter compactification coincides with the usual Stone-Čech compactification (see Theorem 7.2.11 below).

Definition 7.2.3. Let X be a topological space. A *Wallman basis* for X is a sublattice A of $\Omega(X)$ which is a basis for the topology of X and such that, whenever $U \in A$ and $x \in U$, there exists $V \in A$ such that $U \cup V = X$ and $x \notin V$.

Example 7.2.4. If X is a T_1-space, then every singleton in X is a closed subset. Thus $\Omega(X)$ is a Wallman basis for X, since the open set V in Definition 7.2.3 can be taken as $V := X \setminus \{x\}$. This shows that every T_1-space has a Wallman basis.

Lemma 7.2.5. *If X is a T_0-space and A is a Wallman basis for X, then the map*

$$\eta_A \colon X \to \operatorname{Max} A, \quad \eta_A(x) := \{U \in A \mid x \notin U\}$$

is an embedding with dense image.

Proof. See [44, p. 136]. □

Every possible choice of a Wallman basis A for a T_0-space X provides an embedding of X in the compact Hausdorff space $\operatorname{Max} A$ (cf. Theorem 7.2.2), called the *Wallman compactification* of X relative to A. We remark that there are topological spaces that do not admit any Wallman basis. However, in view of Example 7.2.4,

Corollary 7.2.6. *Every T_1-space has a Wallman compactification. In particular, every T_1-space can be embedded in a compact Hausdorff space.* □

We now turn to the investigation of a specific class of open subsets of a topological space X. Let us consider the set $\mathrm{C}(X, \mathbb{R})$ of all the continuous $\mathbb{R}$-valued functions on X. For every element $f \in \mathrm{C}(X, \mathbb{R})$, the *cozero-set* of f is defined as

$$\operatorname{coz} f := \{x \in X \mid f(x) \neq 0\} = f^{-1}(\mathbb{R} \setminus \{0\}).$$

The family of all the cozero-sets of the space X is denoted by

$$\operatorname{Coz}(X) := \{\operatorname{coz} f \subseteq X \mid f \in \mathrm{C}(X, \mathbb{R})\}.$$

We remark that, for any topological space X, we have $\varnothing, X \in \operatorname{Coz}(X)$ since $\varnothing = \operatorname{coz} 0_X$ and $X = \operatorname{coz} 1_X$.

Lemma 7.2.7. *If X is a topological space, then the following hold.*

1. *For every $\operatorname{coz} f \in \operatorname{Coz}(X)$ there exists a continuous function $g \colon X \to [0,1]$ such that $\operatorname{coz} f = \operatorname{coz} g$.*
2. $\operatorname{Coz}(X) \subseteq \Omega(X)$.
3. *The family $\operatorname{Coz}(X)$ is closed under finite intersections and countable unions. In particular, $\operatorname{Coz}(X)$ is a sublattice of $\Omega(X)$.*

Proof. In order to prove item 1, let $f \colon X \to \mathbb{R}$ be a continuous function on the space X. We define the function $g \colon X \to [0,1]$ as

$$g := \min(1, |f|).$$

It is elementary that g is a continuous functions with values in $[0,1]$ such that $\operatorname{coz} f = \operatorname{coz} g$. For item 2 it is sufficient to observe that every cozero-set is a continuous preimage of the open subset $\mathbb{R} \setminus \{0\}$ of $\mathbb{R}$. Lastly, suppose $\operatorname{coz} f, \operatorname{coz} g \in \operatorname{Coz}(X)$. Then it is easy to see that $\operatorname{coz} fg = \operatorname{coz} f \cap \operatorname{coz} g$. Moreover, if $\{f_i\}_{i \in \mathbb{N}} \subseteq \mathrm{C}(X, \mathbb{R})$ is a countable sequence of continuous functions (in view of the preceding item, we can assume $f_i(X) \subseteq [0,1]$ for every $i \in \mathbb{N}$), set $f := \sum_{i=1}^{\infty} \frac{f_i}{2^i}$. The function f is continuous since the latter series is uniformly convergent. It is clear that, for all $x \in X$, $f(x) = 0$ if, and only if, $f_i(x) = 0$ for each $i \in \mathbb{N}$. In other words,

$$\bigcup_{i=1}^{\infty} \operatorname{coz} f_i = \operatorname{coz} f = \operatorname{coz} \sum_{i=1}^{\infty} \frac{f_i}{2^i}.$$

□

As shown in Lemma 7.2.7.(2), a cozero-set in a topological space X is always an open subset. The converse is not true: in general, there exist open subsets of X that are not of the form $\operatorname{coz} f$ for any $f \in \mathrm{C}(X, \mathbb{R})$. We shall restrict our attention to a special class of topological spaces whose structure is reflected in the lattice $\operatorname{Coz}(X)$. Recall that a topological space is *completely regular* provided that it is a Hausdorff space such that, for all $x \in X$ and for all closed subsets $K \subseteq X$, there exists a continuous function $f \colon X \to \mathbb{R}$ such that $f(x) = 1$ and $f(y) = 0$ for every $y \in K$.

Remark 7.2.8. We remark that every compact Hausdorff space is completely regular. Indeed, let X be a compact Hausdorff space, let $x \in X$ and let $K \subseteq X$ be a closed subset. Every compact Hausdorff space is normal [28, Theorem 3.1.9], hence Urysohn's lemma [28, Theorem 1.5.11] applies: there exists a continuous function $f\colon X \to \mathbb{R}$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in K$.

Proposition 7.2.9. *Let X be a Hausdorff space. Then X is completely regular if, and only if,* $\mathrm{Coz}(X)$ *is a basis of open sets for the topology of X.*

Proof. See [33, p. 38]. □

Recall that a bounded distributive lattice $(A, \wedge, \vee, 0, 1)$ is *normal* if, for every pair of elements $b_1, b_2 \in A$ such that $b_1 \vee b_2 = 1$, there exist elements $c_1, c_2 \in A$ satisfying $c_1 \wedge c_2 = 0$, $c_1 \vee b_2 = 1$, $c_2 \vee b_1 = 1$. Given a Wallman basis A for a space X, we say that A is a *normal Wallman basis* for X if A is a normal lattice.

Lemma 7.2.10. *If X is a completely regular space, then* $\mathrm{Coz}(X)$ *is a normal Wallman basis for X.*

Proof. The set $\mathrm{Coz}(X)$ is a sublattice of $\Omega(X)$ by Lemma 7.2.7, and it is a basis of open sets for X by Proposition 7.2.9. In order to show that $\mathrm{Coz}(X)$ is a Wallman basis for X, it suffices to prove that for all $\mathrm{coz}\, f \in \mathrm{Coz}(X)$ and for all $x \in \mathrm{coz}\, f$ there exists $g\colon X \to \mathbb{R}$ such that $\mathrm{coz}\, f \cup \mathrm{coz}\, g = X$ and $x \notin \mathrm{coz}\, g$. Consider the closed subset $X \setminus \mathrm{coz}\, f$ of X. Since X is completely regular, there exists a continuous function $h\colon X \to \mathbb{R}$ such that $h(x) = 1$ and $h(y) = 0$ for all $y \in X \setminus \mathrm{coz}\, f$. Then the function $g := h - 1$ satisfies the required properties. For the normality of the lattice $\mathrm{Coz}(X)$, see [44, p. 137]. □

Theorem 7.2.11. *Let X be a completely regular space, and let A be a Wallman basis for X containing* $\mathrm{Coz}(X)$*. Then the Wallman compactification* $\mathrm{Max}\, A$ *of X is homeomorphic to the Stone-Čech compactification of X.*

Proof. See [44, p. 138]. □

The universal property of the Stone-Čech compactification [28, Theorem 3.6.1] entails that the Stone-Čech compactification of a compact Hausdorff space X is homeomorphic to X. Upon recalling that every compact Hausdorff space is completely regular by Remark 7.2.8, it follows at once

Corollary 7.2.12. *If X is a compact Hausdorff space, then* $\mathrm{Max}\,\mathrm{Coz}(X)$ *is homeomorphic to X.* □

Definition 7.2.13. An *Alexandroff algebra* is a bounded distributive lattice $(A, \wedge, \vee, 0, 1)$ satisfying the following conditions.

1. A is normal.
2. Countable joins exist in A.

3. Countable joins distribute over finite meets.

4. For every $a \in A$ there exist countable sequences $\{b_n\}_{n\in\mathbb{N}}, \{c_n\}_{n\in\mathbb{N}}$ in A such that $\bigvee_{n\in\mathbb{N}} c_n = a$, $b_n \wedge c_n = 0$ and $b_n \vee a = 1$ for every $n \in \mathbb{N}$.

An Alexandroff algebra A is *countably compact* if the following holds.

5. Whenever $\{c_n\}_{n\in\mathbb{N}}$ is a countable family in A satisfying $\bigvee_{n\in\mathbb{N}} c_n = 1$, there exists a finite subset $H \subseteq \mathbb{N}$ such that $\bigvee_{n\in H} c_n = 1$.

Lemma 7.2.14. *If X is a completely regular space, then* $\mathrm{Coz}(X)$ *is an Alexandroff algebra. If X is a compact Hausdorff space, then* $\mathrm{Coz}(X)$ *is a countably compact Alexandroff algebra.*

Proof. The first part of the statement is proved in [44, p. 140]. Now, assume that X is a compact Hausdorff space, and let $\{\mathrm{coz}\, f_i\}_{i\in\mathbb{N}} \subseteq \mathrm{Coz}(X)$ be a countable family satisfying $\bigcup_{i=1}^{\infty} \mathrm{coz}\, f_i = X$. Since $\{\mathrm{coz}\, f_i\}_{i\in\mathbb{N}}$ is a open covering of X, there exists a finite subset $H \subseteq \mathbb{N}$ such that $\bigcup_{i\in H} \mathrm{coz}\, f_i = X$. □

Let $\mathsf{Alex_c}$ denote the category with countably compact Alexandroff algebras as objects, and lattice homomorphisms preserving countable joins as morphisms. Then, for every compact Hausdorff space X, one can see that the correspondence $X \mapsto \mathrm{Coz}(X)$ defines a functor

$$\mathcal{C}\colon \mathsf{KHaus} \to \mathsf{Alex_c}.$$

The functor $\mathcal{C}$ sends a continuous map between compact Hausdorff spaces $f\colon X \to Y$ to the lattice homomorphism preserving countable joins

$$\mathcal{C}(f) := f^{-1}\colon\ \mathrm{Coz}(Y) \to \mathrm{Coz}(X).$$

On the other hand, it is possible to prove that the map sending a countably compact Alexandroff algebra A to the compact Hausdorff space $\mathrm{Max}\, A$ induces a functor

$$\mathcal{M}\colon \mathsf{Alex_c} \to \mathsf{KHaus}.$$

A lattice homomorphism preserving countable joins $h\colon A \to B$ is sent to the continuous map

$$\mathcal{M}(h) := h^{-1}\colon\ \mathrm{Max}\, B \to \mathrm{Max}\, A.$$

The interested reader is referred to [19, Section 3.2] for details.

Theorem 7.2.15 (Alexandroff duality)**.** *The category* KHaus *of compact Hausdorff spaces is dually equivalent to the category* $\mathsf{Alex_c}$ *of countably compact Alexandroff algebras via the functors $\mathcal{C}$ and $\mathcal{M}$.*

Proof. This theorem is proved in [19, Theorem 3.5] by means of topos-theoretic techniques. We remark that, for every compact Hausdorff space X, the component at X of the natural isomorphism $\mu\colon \mathrm{Id}_{\mathsf{KHaus}} \to \mathcal{M} \circ \mathcal{C}$ is provided by Corollary 7.2.12. □

We now give an axiomatisation of countably compact Alexandroff algebras in the language $\mathrm{L}_{\omega_1,\omega_1}$ over the (algebraic) lattice-theoretic signature given by the constants $0, 1$ and by the binary function symbols $\underline{\wedge}, \underline{\vee}$ (we underline the function symbols in order to distinguish them from the logical conjunction $\wedge$ and disjunction $\vee$). We agree to denote by $j(y, \{x_n\}_{n\in\mathbb{N}})$ the formula

$$\Big(\bigwedge_{n\in\mathbb{N}} y\underline{\wedge}x_n = x_n\Big) \wedge \Big(\forall z\Big(\Big(\bigwedge_{n\in\mathbb{N}} z\underline{\wedge}x_n = x_n\Big) \Rightarrow \Big(z\underline{\wedge}y = y\Big)\Big)\Big),$$

stating (semantically) that the element y is the join of the countable family $\{x_n\}_{n\in\mathbb{N}}$. Let $\mathbb{T}_{\mathsf{Alex}_\mathsf{c}}$ be the theory formed by the (equational) axioms for a bounded distributive lattice, along with the following five axioms.

$$(i)\quad \forall x_1\forall x_2\Big(\Big(x_1\underline{\vee}x_2 = 1\Big) \Rightarrow \exists y_1\exists y_2\Big(\Big(y_1\underline{\wedge}y_2 = 0\Big)\wedge\Big(y_1\underline{\vee}x_2 = 1\Big)\wedge\Big(y_2\underline{\vee}x_1 = 1\Big)\Big)\Big).$$

$$(ii)\quad \forall\{x_n\}_{n\in\mathbb{N}}\exists y\Big(j(y, \{x_n\}_{n\in\mathbb{N}})\Big).$$

$$(iii)\quad \forall\{x_n\}_{n\in\mathbb{N}}\forall w\forall y\forall z\Big(\Big(j(y, \{x_n\}_{n\in\mathbb{N}}) \wedge j(w, \{x_n\underline{\wedge}z\}_{n\in\mathbb{N}}\Big) \Rightarrow y\underline{\wedge}z = w\Big).$$

$$(iv)\quad \forall x\exists\{y_n\}_{n\in\mathbb{N}}\exists\{z_n\}_{n\in\mathbb{N}}\Big(j(x, \{z_n\}_{n\in\mathbb{N}}) \wedge \Big(\bigwedge_{n\in\mathbb{N}} y_n\underline{\wedge}z_n = 0\Big) \wedge \Big(\bigwedge_{n\in\mathbb{N}} y_n\underline{\wedge}x = 1\Big)\Big).$$

$$(v)\quad \bigvee_{n\in\mathbb{N}}\Big(\forall\{x_n\}_{n\in\mathbb{N}}\Big(j(1, \{x_n\}_{n\in\mathbb{N}}) \Rightarrow \exists y_1\cdots\exists y_n\Big(j(1, \{\tilde{y}_n\}_{n\in\mathbb{N}})\wedge\Big(\bigwedge_{k=1}^{n}\bigvee_{i\in\mathbb{N}} y_k = x_i\Big)\Big)\Big)\Big).$$

The countable sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}}$ in axiom (v) is defined by $\tilde{y}_i := y_i$ if $0 < i \leqslant n$ and $\tilde{y}_i := 0$ if $i > n$. A direct inspection shows that axioms (i)–(v) correspond precisely to items 1–5 of Definition 7.2.13. In other words,

Corollary 7.2.16. *The dual category* $\mathsf{KHaus}^{\mathsf{op}}$ *is equivalent to the category of models* $\mathsf{Mod}\,\mathbb{T}_{\mathsf{Alex}_\mathsf{c}}$ *of the theory* $\mathbb{T}_{\mathsf{Alex}_\mathsf{c}}$ *in the infinitary language* $\mathrm{L}_{\omega_1,\omega_1}$. □

Bibliography

[1] J. Adámek and J. Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[2] J. Adámek, J. Rosický, and E. M. Vitale. *Algebraic theories*, volume 184 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2011. A categorical introduction to general algebra, With a foreword by F. W. Lawvere.

[3] S. Aguzzoli. The complexity of McNaughton functions of one variable. *Adv. in Appl. Math.*, 21(1):58–77, 1998.

[4] L. V. Ahlfors. *Complex analysis*. McGraw-Hill Book Co., New York, third edition, 1978. An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.

[5] C. D. Aliprantis and K. C. Border. *Infinite dimensional analysis*. Springer, Berlin, third edition, 2006. A hitchhiker's guide.

[6] C. D. Aliprantis and O. Burkinshaw. *Principles of real analysis*. Academic Press, Inc., San Diego, CA, third edition, 1998.

[7] M. Artin, A. Grothendieck, and J.-L. Verdier. *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin-New York, 1972.

[8] B. Banaschewski. The duality between M-spaces and compact Hausdorff spaces. *Math. Nachr.*, 75:41–45, 1976.

[9] B. Banaschewski. More on compact Hausdorff spaces and finitary duality. *Canad. J. Math.*, 36(6):1113–1118, 1984.

[10] P. Bankston. Some obstacles to duality in topological algebra. *Canad. J. Math.*, 34(1):80–90, 1982.

[11] M. Barr and C. Wells. *Toposes, triples and theories*, volume 278 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[12] T. Beke. Theories of presheaf type. *J. Symbolic Logic*, 69(3):923–934, 2004.

[13] A. Bigard, K. Keimel, and S. Wolfenstein. *Groupes et Anneaux Reticules*, volume 608 of *Lecture notes in mathematics*. Springer, 1977.

[14] G. Birkhoff. Subdirect unions in universal algebra. *Bulletin (New Series) of the American Mathematical Society*, 50(10):764–768, 1944.

[15] G. Birkhoff. *Lattice theory*, volume 25 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, R.I., third edition, 1979.

[16] B. Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[17] F. Borceux. *Handbook of Categorical Algebra: Volume 2, Categories and Structures*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994.

[18] S. Burris and H. P. Sankappanavar. *A course in universal algebra*. Millennium Edition, 2012.

[19] O. Caramello. Gelfand spectra and Wallman compactifications. *arXiv:1204.3244 [math.CT]*, 2012.

[20] O. Caramello. Topos-theoretic background. Online notes, 2014.

[21] R. L. O. Cignoli, I. M. L. D'Ottaviano, and D. Mundici. *Algebraic foundations of many-valued reasoning*, volume 7 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2000.

[22] J. B. Conway. *A Course in Functional Analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer, 1990.

[23] H. G. Dales, P. Aiena, J. Eschmeier, K. Laursen, and G. A. Willis. *Introduction to Banach algebras, operators, and harmonic analysis*, volume 57 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2003.

[24] B. A. Davey and H. A. Priestley. *Introduction to lattices and order*. Cambridge University Press, New York, second edition, 2002.

[25] M. A. Dickmann. *Large infinitary languages*. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975. Model theory, Studies in Logic and the Foundations of Mathematics, Vol. 83.

[26] R. Doran and V. Belfi. *Characterizations of C^*-Algebras: the Gelfand-Naimark Theorems*, volume 101 of *Pure and Applied Mathematics*. Marcel Dekker, Inc., 1986.

[27] J. Duskin. Variations on Beck's tripleability criterion. In *Reports of the Midwest Category Seminar, III*, pages 74–129. Springer, Berlin, 1969.

[28] R. Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

[29] M. Fukamiya, Y. Misonou, and Z. Takeda. On order and commutativity of b^*-algebras. *Tohoku Mathematical Journal*, 6(1):89–93, 1954.

[30] P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Lecture Notes in Mathematics, Vol. 221. Springer-Verlag, Berlin-New York, 1971.

[31] I. M. Gelfand and A. N. Kolmogoroff. On rings of continuous functions on topological spaces. *Dokl. Akad. Nauk SSSR*, 22(1):11–15, 1939.

[32] I. M. Gelfand and M. Neumark. On the imbedding of normed rings into the ring of operators in hilbert space. *Matematiceskij Sbornik*, 54(2):197–217, 1943.

[33] L. Gillman and M. Jerison. *Rings of continuous functions*. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960.

[34] A. M. W. Glass and W. C. Holland. *Lattice-Ordered Groups: Advances and Techniques*. Mathematics and Its Applications. Springer, 1989.

[35] A. M. Gleason. Projective topological spaces. *Illinois J. Math.*, 2:482–489, 1958.

[36] K. R. Goodearl. *Notes on real and complex C^*-algebras*, volume 5 of *Shiva Mathematics Series*. Shiva Publishing Ltd., Nantwich, 1982.

[37] G. Grätzer. *General lattice theory*, volume 75 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[38] E. Hewitt and K. Stromberg. *Real and Abstract Analysis*. Springer-Verlag, 1965.

[39] O. Hölder. Die axiome der quantität und die lehre vom mass. *Berichten der mathematisch-physischen Classe der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig*, 53(4):1–64, 1901.

[40] O. Hölder. The axioms of quantity and the theory of measurement. *J. Math. Psych.*, 40(3):235–252, 1996. Translated from the 1901 German original and with notes by Joel Michell and Catherine Ernst, With an introduction by Michell.

[41] O. Hölder. The axioms of quantity and the theory of measurement. *J. Math. Psych.*, 41(4):345–356, 1997. Translated from the 1901 German original and with notes by Joel Michell and Catherine Ernst, With an introduction by Michell.

[42] J. Isbell. Generating the algebraic theory of $C(X)$. *Algebra Universalis*, 15(2):153–155, 1982.

[43] P. T. Johnstone. *Topos theory*. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1977. London Mathematical Society Monographs, Vol. 10.

[44] P. T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Reprint of the 1982 edition.

[45] S. Kakutani. Concrete representation of abstract (M)-spaces. (A characterization of the space of continuous functions.). *Ann. of Math. (2)*, 42:994–1024, 1941.

[46] A. N. Kolmogorov and S. V. Fomin. *Elements of the theory of functions and functional analysis*. Courier Dover Publications, 1999.

[47] M. Krein and S. Krein. On an inner characteristic of the set of all continuous functions defined on a bicompact Hausdorff space. *C. R. (Doklady) Acad. Sci. URSS (N.S.)*, 27:427–430, 1940.

[48] F. W. Lawvere. Functorial semantics of algebraic theories. *Proc. Nat. Acad. Sci. U.S.A.*, 50:869–872, 1963.

[49] F. E. J. Linton. Some aspects of equational categories. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 84–94. Springer, New York, 1966.

[50] L. H. Loomis. *An introduction to abstract harmonic analysis*. D. Van Nostrand Company, Inc., Toronto-New York-London, 1953.

[51] W. A. J. Luxemburg and A. C. Zaanen. *Riesz spaces. Vol. I*. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1971. North-Holland Mathematical Library.

[52] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer, 1978.

[53] M. Makkai and R. Paré. *Accessible categories: the foundations of categorical model theory*, volume 104 of *Contemporary Mathematics*. American Mathematical Society, 1989.

[54] E. G. Manes. *Algebraic theories*. Springer-Verlag, New York-Heidelberg, 1976. Graduate Texts in Mathematics, No. 26.

[55] R. McNaughton. A theorem about infinite-valued sentential logic. *J. Symbolic Logic*, 16:1–13, 1951.

[56] D. Mundici. Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus. *J. Funct. Anal.*, 65(1):15–63, 1986.

[57] D. Mundici. *Advanced Łukasiewicz Calculus and MV-Algebras*, volume 35 of *Trends in Logic*. Springer, 2011.

[58] J. W. Negrepontis. Duality in analysis from the point of view of triples. *J. Algebra*, 19:228–253, 1971.

[59] J. W. Pelletier and J. Rosický. Generating the equational theory of C^*-algebras and related categories. In *Categorical topology and its relation to analysis, algebra and combinatorics (Prague, 1988)*, pages 163–180. World Sci. Publ., Teaneck, NJ, 1989.

[60] J. W. Pelletier and J. Rosický. On the equational theory of C^*-algebras. *Algebra Universalis*, 30(2):275–284, 1993.

[61] J. Rosický. Elementary categories. *Arch. Math. (Basel)*, 52(3):284–288, 1989.

[62] Z. Semadeni. *Banach spaces of continuous functions.* Monografie matematyczne. PWN Polish Scientific Publishers, 1971.

[63] S. Sherman. Order in operator algebras. *Amer. J. Math.*, 73(1):227–232, 1951.

[64] J. Słomiński. The theory of abstract algebras with infinitary operations. *Rozprawy Mat.*, 18:67 pp., 1959.

[65] M. H. Stone. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40(1):37–111, 1936.

[66] D. H. Van Osdol. C^*-algebras and cohomology. In *Categorical topology (Toledo, Ohio, 1983)*, volume 5 of *Sigma Ser. Pure Math.*, pages 582–587. Heldermann, Berlin, 1984.

[67] K. Yosida. On vector lattice with a unit. *Proc. Imp. Acad. Tokyo*, 17(5):121–124, 1941.

[68] K. Yosida and M. Fukamiya. On vector lattice with a unit. II. *Proc. Imp. Acad. Tokyo*, 17:479–482, 1941.

Index