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Expressiveness, decidability, and undecidability of Interval Temporal Logic

Candidate:
Dario Della Monica

Supervisor:
Angelo Montanari

Co-Supervisors:
Guido Sciavicco
Valentin Goranko

Author's e-mail: dario.dellamonica@uniud.it

Author's address:

Dipartimento di Matematica e Informatica
Università degli Studi di Udine
Via delle Scienze, 206
33100 Udine
Italia

To my familiy and
to the memory of my grandma Rachele

Abstract

Interval Temporal Logics are formalisms particularly suitable to express temporal properties. Unlike standard temporal logics, they use intervals, instead of points, as primitive ontological entities. The most studied propositional interval temporal logic is Halpern and Shoham's Modal Logic of Time Intervals (HS for short). It features a modal operator for each possible ordering relation between pairs of intervals (the so-called Allen's relations). HS is very expressive, but its satisfiability problem turns out to be highly undecidable (over most classes of linear orders). The three main contributions of this thesis are the following ones. First, we provide a complete classification of HS fragments with respect to their relative expressive power in the class of all linear orders. Second, we systematically investigate the decidable/undecidable status of the satisfiability problem for a number of previously unclassified HS fragments, showing that, once more, undecidability is the rule and decidability the exception. Pairing the results given here with existing ones, the long-standing goal of obtaining a complete classification of all HS fragments with respect to their satisfiability problem is now almost achieved. Third, we study metric, hybrid, and first-order extensions of Propositional Neighborhood Logic (over natural numbers), a meaningful and well-studied decidable fragment of HS.

Le logiche temporali ad intervalli sono formalismi particolarmente adatti ad esprimere proprietà temporali. Diversamente dalle logiche temporali standard, usano gli intervalli, anziché i punti, come entità ontologiche primitive. La logica proposizionale temporale ad intervalli più studiata è la logica modale degli intervalli temporali di Halpern e Shoham (HS in breve). Essa include un operatore modale per ogni possibile relazione d'ordine tra coppie di intervalli (relazioni di Allen). HS è molto espressiva, ma il suo problema della soddisfacibilità è fortemente indecidibile (sulla maggior parte degli ordini lineari). I tre contributi principali di questa tesi sono i seguenti. In primo luogo, forniamo una classificazione completa dei frammenti di HS rispetto al loro potere espressivo nella classe di tutti gli ordini lineari. In secondo luogo, investighiamo sistematicamente lo stato (decidibile/indecidibile) del problema della soddisfacibilità per vari frammenti di HS non ancora classificati, mostrando, ancora una volta, che l'indecidibilità è la regola e la decidibilità l'eccezione. Unendo i risultati dati qui con quelli già esistenti, l'obiettivo (di vecchia data) di ottenere una classificazione completa di tutti i frammenti di HS rispetto al

problema della soddisfacibilità è ormai quasi realizzato. In terzo luogo, studiamo estensioni metriche, ibride e al prim'ordine della Logica Proposizionale delle Vicinanze (sui numeri naturali), un frammento decidibile significativo e molto studiato di HS.

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Introduction

At the beginning, it was the darkness. Then, logicians made the light, they became curious, and moved toward the darkness... as close as they could.

In this dissertation, the light *is the decidability and the* darkness *the undecidability of the satisfiability problem. In the area of interval temporal logics, undecidability is the rule and decidability the exception. Most of this thesis moves along the boundary between decidability and undecidability with respect to a well-studied family of logics, namely, Halpern and Shoham's Modal Logic of Time Intervals (*HS*, for short) and its fragments. In particular, we contribute to shape such a boundary, often flowing into undecidability, and when we run into a decidable fragment, well... we try to extend it and move toward the boundary, as close as we can! We also address some expressiveness issues for this class of formalisms, by providing a complete classification of all* HS *fragments with respect to their relative expressive power in the class of all linear orders.*

Temporal reasoning plays a major role in computer science. According to the standard approach, time points are taken as the basic temporal entities and temporal domains are represented as ordered sequences of time points. This work moves from a different, more natural perspective on time, according to which the primitive ontological entity is the time interval instead of the time point. Such an alternative approach can be justified by means of philosophical as well as practical arguments.

On the one hand, philosophical roots of interval-based temporal reasoning can be dated back to Zeno and Aristotle. The nature of time has always been one of the favourite subjects among philosophers, in particular, the discussion whether time instants or time periods should be regarded as the primary objects of temporal ontology has a distinct philosophical flavour. A comprehensive study and logical analysis of point-based and interval-based ontologies, languages, and logical systems can be found in [99]. As a matter of fact, real-world events have an intrinsic *duration* and thus "durationless" points are not suitable to properly deal with them. In addition, in an interval-based setting, several philosophical and logical paradoxes disappear [99], like the Zeno's flying arrow paradox ("if at each *instant* the flying arrow stands still, how is movement possible?") and the dividing instant dilemma ("if the light is on and it is turned off, what is its state at the *instant* between the two events?").

On the other hand, from a technical point of view, interval temporal logics turn out to be (much) more expressive than point-based ones [101] and more appropriate for a number of applications. In particular, interval-based temporal reasoning naturally arises in artificial intelligence (temporal knowledge representation, systems for temporal planning and maintenance, qualitative reasoning, theories of events), temporal databases (event modeling, temporal aggregation), computational linguistics (analysis of progressive tenses, semantics and processing of natural languages), and formal specification and verification of complex systems (hardware verification, protocol analysis) [54]. One of the first examples of an interval-based formalism has been given by Moszkowski in [87], where Propositional Interval Temporal Logic (PITL) is proposed as a meaningful formalism for the specification and verification of hardware components. An extension of PITL, called Duration Calculus (DC), exploits the additional notion of duration of an event (state) during an interval of time, in order to reason about design and requirements for time-critical systems [37].

Despite the relevance of interval-based temporal reasoning, interval temporal logics are far less studied and popular than point-based ones because of their higher conceptual and computational complexity (relations between intervals are more complex than those between points). Even if, in principle, it is possible to consider interval relations of any ariety, as a matter of fact in the literature we only find interval temporal logics based on binary or ternary interval relations. In particular, to the best of our knowledge, the only logics based on ternary interval relations are Venema's CDT logic and its fragments [63, 102]. Much more work has been done on binary relations. A systematic analysis of the variety of such relations (over linear orderings) was first accomplished by Allen [3], who explored the use of interval reasoning in systems for time management and planning. Halpern and Shoham's Modal Logic of Time Intervals HS, introduced in [58], can be viewed as the logic of Allen's relations [3], since it features one modal operator for each one of the 12 basic temporal relations (excluding identity) that may hold between any pair of intervals (on a linear ordering).

Unfortunately, as already pointed out, in the area of interval logics undecidability is the rule and decidability the exception, and this does not come as a surprise, since formulae of these logics are evaluated over intervals, that is, pairs of points. As a consequence, formulae translate into binary relations over the underlying ordering, and the validity/satisfiability problem translates to the validity/satisfiability problem of the dyadic fragment of second-order logic.

The case of HS is paradigmatic. In [59], Halpern and Shoham show that such a logic is undecidable under very weak assumptions on the class of interval structures over which it is interpreted. They prove that validity in HS over the classes of all linear models, all discrete linear models, and all dense linear models is undecidable. They also prove that validity in HS over any of the orderings of the natural numbers, integers, or reals is not even recursively axiomatizable.

For a long time, results of this nature have discouraged attempts for practical applications and further research on interval temporal logics. The search for a way out was basically confined to the identification of severe syntactic and/or semantic restrictions to impose on the logic to obtain decidable fragments. As an example, in [87] Moszkowski shows that PITL decidability can be recovered by constraining atomic propositions to be point-wise and defining truth at an interval as truth at its initial point (locality). However, in these cases interval temporal logics are actually reducible to point-based ones, thus loosing most of their (interval) peculiarities.

A renewed interest for interval temporal logics has been recently stimulated by the discovery of expressive decidable fragments of HS. Propositional interval logics of temporal neighborhood (PNL), as well as propositional interval logics of the subinterval relation, are meaningful fragments of HS, that allow one to express fairly natural relations between intervals, which turn out to be decidable when interpreted over various classes of interval temporal structures. As an effect, the identification of expressive enough decidable fragments of HS has been added to the current research agenda for (interval) temporal logic. While the algebra of Allen's relations, the so-called Allen's Interval Algebra, has been extensively studied and its fragments have been completely classified with respect to their relative expressive power as well as to computational complexity [40, 71] (tractability/intractability of the consistency problem of fragments of Interval Algebra), the logical counterpart of these problems for HS is considerably harder.

The aim of this dissertation is to address expressivity, decidability, and undecidability issues for HS fragments. As already pointed out, the language of HS features 12 modal operators, one for each Allen's relation (excluding identity). By restricting the set of modalities, 2^{12} syntactically different languages arise. The main goal of this thesis is to classify such a set of languages with respect to both expressive power and (un)decidability of the satisfiability problem. After a preliminary chapter, where we introduce basic notions (Chapter 1), in Chapter 2 we compares the expressiveness of the fragments of HS over the class of all linear orders. By massively exploiting the notion of bisimulation between interval models, we establish a complete set of inter-definability equations between modal operators of HS, thus obtaining a complete classification of the family of fragments of HS with respect to their expressiveness. Using that result we have found that there are exactly 1347 expressively different such fragments out of $2^{12} = 4096$ sets of modal operators in HS. This result is very interesting also from the perspective of the most challenging open problem in the area of interval temporal logics, that is, getting a complete classification of HS fragments with respect to decidability/undecidability of their satisfiability problem. Indeed, it allows one to properly identify the (small) set of HS fragments for which the decidability of the satisfiability problem is still an open problem. We address such a problem in Chapter 3, where a number of undecidability results, based on reductions from (suitable versions of) the tiling problem, is provided. Our results hold under a very weak assumption on the class of linear orders, namely the existence of an infinite (ascending or descending) sequence of points. In particular, they hold over the class of all (resp., all discrte, all dense) linear orders, as well as over linear orders based on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. Finally, we also show how to adapt the proof in order to deal with classes of finite linear orders. Even if there still are some open cases, the classification is now very close to be complete. Besides the search for the precise boundary between decidability and undecidability, another question that can be naturally raised is the following one: is it possible to extend (decidable) HS fragments, whose modalities are purely qualitative, with metric features? In Chapter 4, the decidable fragment PNL of HS is extended with metric constructs. More precisely, we introduce a family of metric extensions of PNL, we study their (relative) expressive power, and we prove the decidability of the most expressive logic of the family. Quite surprisingly, decidability is also preserved when the bidimensional (spatial) version of the logic is considered. In Chapter 5, we analyze further extensions of (metric versions of) PNL with classical machinery, namely, hybrid and first-order constructs. In these cases, even very weak extensions immediately yields undecidability. Finally, in the last chapter (Conclusions), we summarize some open problems and we outline possible future research directions.

Chapter 2 is based on the paper: "Expressiveness of the Interval Logics of Allen's Relations on the Class of all Linear Orders: Complete Classification" (with Valentin Goranko, Angelo Montanari, and Guido Sciavicco), accepted for publication in the Proceedings of the *22nd International Joint Conference on Artificial Intelligence (IJCAI 2011)*. Chapter 3 is based on the papers: "Decidable and Undecidable Fragments of Halpern and Shohams Interval Temporal Logic: Towards a Complete Classification" (with Davide Bresolin, Valentin Goranko, Angelo Montanari, and Guido Sciavicco), published in the Proceedings of the *15th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning (LPAR 2008)*, "Undecidability of Interval Temporal Logics with the Overlap Modality" (with Davide Bresolin, Valentin Goranko, Angelo Montanari, and Guido Sciavicco), published in the Proceedings of the *16th International Symposium on Temporal Representation and Reasoning (TIME 2009)*, and "Undecidability of the Logic of Overlap Relation over Discrete Linear Orderings" (with Davide Bresolin, Valentin Goranko, Angelo Montanari, and Guido Sciavicco), published in *Elsevier Electronic Notes in Theoretical Computer Science (ENTCS)*. Chapter 4 is based on the papers: "Metric Propositional Neighborhood Logics: Expressiveness, Decidability, and Undecidability" (with Davide Bresolin, Valentin Goranko, Angelo Montanari, and Guido Sciavicco), published in the Proceedings of the *19th European Conference on Artificial Intelligence (ECAI 2010)*, "A decidable spatial generalization of Metric Interval Temporal Logic" (with Davide Bresolin, Angelo Montanari, Pietro Sala, and Guido Sciavicco), published in the Proceedings of the *17th International Symposium on Temporal Representation and Reasoning (TIME 2010)*, and "Metric Propositional Neighborhood Logics on Natural Numbers" (with Davide Bresolin, Valentin Goranko, Angelo Montanari, and Guido Sciavicco), published in *Software and Systems Modeling (SoSyM)*. Chapter 5 is based on the papers: "Hybrid Metric Propositional Neighborhood Logics with Interval Length Binders" (with Valentin Goranko and Guido Sciavicco), published in the Proceedings of the *International Workshop on Hybrid Logic and Applications (HyLo 2010)* (an improved version of the paper will also be published in *Elsevier Electronic Notes in Theoretical Computer Science (ENTCS)*), "On First-Order Propositional Neighborhood Logics: a

First Attempt" (with Guido Sciavicco), published in the Proceedings of the *ECAI Workshop on Spatio-Temporal Dynamics (STeDY 2010)*.

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Preliminaries

Given a strict partial ordering $\mathbb{D} = \langle D, < \rangle$, an *interval* in $\mathbb{D}$ is an ordered pair $[d_0, d_1]$ such that $d_0, d_1 \in D$ and $d_0 \leq d_1$. If $d_0 < d_1$, $[d_0, d_1]$ is called *strict* interval, otherwise, it is called *point interval*. Often, we will refer to all intervals on $\mathbb{D}$ as *non-strict intervals*, to include both strict and point intervals. A point d *belongs to an interval* $[d_0, d_1]$ if $d_0 \leq d \leq d_1$. Notice that the endpoints of an interval are included in it. The set of all non-strict intervals on $\mathbb{D}$ will be denoted by $\mathbb{I}(\mathbb{D})^+$, while the set of all strict intervals will be denoted by $\mathbb{I}(\mathbb{D})^-$. By $\mathbb{I}(\mathbb{D})$ we will denote either of these. For the purpose of this thesis, we will call a pair $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ an *interval structure*. According to the original definition of interval given by Allen in [3], an interval is an ordered pair of points $[a, b]$, with $a < b$, thus excluding point intervals. In addition, Allen systematically analyzes the possible binary relations holding between pairs of intervals on a linear order. As depicted in Table 1.1 (first two columns), besides the identity relation *equal* (denoted by $=$), there are the six relations *before* ($<$), *meets* (m), *overlaps* (o), *finishes* (f), *during* (d), *starts* (s), plus their inverses *later* ($>$), *met-by* (mi), *overlapped-by* (oi), *finished-by* (fi), *contains* (di), *started-by* (si). Such 13 relations are *mutually exclusive* and *jointly exhaustive*, meaning that exactly one Allen's relation holds between any given pair of intervals. Given two intervals $[d_0, d_1]$ and $[d_2, d_3]$, we will use the notation $[d_0, d_1]\{<\}[d_2, d_3]$ to denote that the interval $[d_0, d_1]$ is related to $[d_2, d_3]$ by means of the relation *before*, and similarly for the other relations.

Each Allen's relation (excluding the identity relation) gives rise to a corresponding unary modal operator. The logic HS, introduced by Halpern and Shoham in [58], features 12 modal operators corresponding to the Allen's relation. In the definition of their own formalism, Halpern and Shoham have choosen a different notation from the one used by Allen. For the sake of clarity, in Table 1.1 we compare the two notations. More interesting, in the Halpern and Shoham's work, the semantics of the logic HS is defined including point intervals. As a consequence, the relations corresponding to the modal operators of HS are neither mutually exclusive nor jointly exahustive anymore. As an example, in the original semantics of HS, given two intervals $[a, b]$ and $[b, c]$, with $a < b < c$, both the relations *overlaps* and *meets* hold between the two intervals. As another example, the intervals $[a, b]$ and $[c, c]$, with $b < c$ are not related by any of the Allen's relation.

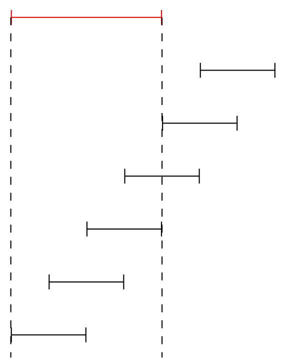
Interval's relations	**Allen's notation**	HS **notation**
	*equals* $\{=\}$	
	before $\{<\}$ / *after* $\{>\}$	$\langle L \rangle$ / $\langle \overline{L} \rangle$ (*Later*)
	meets $\{m\}$ / *met-by* $\{mi\}$	$\langle A \rangle$ / $\langle \overline{A} \rangle$ (*After*)
	overlaps $\{o\}$ / *overlapped-by* $\{oi\}$	$\langle O \rangle$ / $\langle \overline{O} \rangle$ (*Overlaps*)
	finished-by $\{fi\}$ / *finishes* $\{f\}$	$\langle E \rangle$ / $\langle \overline{E} \rangle$ (*Ends*)
	contains $\{di\}$ / *during* $\{d\}$	$\langle D \rangle$ / $\langle \overline{D} \rangle$ (*During*)
	started-by $\{si\}$ / *starts* $\{s\}$	$\langle B \rangle$ / $\langle \overline{B} \rangle$ (*Begins*)

Table 1.1: The thirteen possible relations between pairs of of intervals, with, respectively, the notation used by Allen to denote them (and their inverses), and the one used by Halpern and Shoham to denote the corresponding modal operators

In all systems considered here the intervals will be assumed *linear*, although this restriction can often be relaxed without essential complications. Thus, we will concentrate on partial orderings with the *linear interval property:*

$$\forall x \forall y (x < y \rightarrow \forall z_1 \forall z_2 (x < z_1 < y \wedge x < z_2 < y \rightarrow z_1 < z_2 \vee z_1 = z_2 \vee z_2 < z_1)),$$

that is, orderings in which every interval is linear. Clearly every linear ordering falls here. Fig. 1.1 shows a non-linear ordering with this property (on the left) and a non-linear ordering violating it (on the right).
An interval structure is said:

- **linear**, if every two points are comparable;
- **discrete**, if every point with a successor/predecessor has an immediate successor/predecessor along every path starting from/ending in it, that is,

  $$\forall x \forall y (x < y \rightarrow \exists z (x < z \wedge z \leq y \wedge \forall w (x < w \wedge w \leq y \rightarrow z \leq w))),$$

  and

  $$\forall x \forall y (x < y \rightarrow \exists z (x \leq z \wedge z < y \wedge \forall w (x \leq w \wedge w < y \rightarrow w \leq z)));$$

Figure 1.1: Linear interval property: on the left, an interval structure with the property; on the right, an interval structure violating it

- **dense**, if for every pair of different comparable points there exists another point in between:

$$\forall x \forall y(x < y \rightarrow \exists z(x < z \land z < y));$$

- **unbounded above** or **to right** (resp. **below** or **to left**), if every point has a successor (resp. predecessor);
- **finite**, if it has finitely many points;
- **Dedekind complete**, if every non-empty and bounded above set of points has a least upper bound.

Besides interval logics over the classes of linear, discrete, dense, (un)bounded, finite, and Dedekind complete interval structures, we will be discussing those interpreted on the single structures $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ with their usual orderings.

Before giving syntax and semantics of HS and its fragments, it is useful to mention different notions of *sub-interval* (resp., *super-interval*), that will be used in the next chapters. Given a partial ordering $\mathbb{D}$ and two intervals $[d_0, d_1]$ and $[s_0, s_1]$ in it:

- $[s_0, s_1]$ is a *strict sub-interval* (resp., *strict super-interval*) of $[d_0, d_1]$ if $d_0 < s_0$ (resp., $s_0 < d_0$) and $s_1 < d_1$ (resp., $d_1 < s_1$);
- $[s_0, s_1]$ is a *proper sub-interval* (resp., *proper super-interval*) of $[d_0, d_1]$ if $d_0 \leq s_0$ (resp., $s_0 \leq d_0$), $s_1 \leq d_1$ (resp., $d_1 \leq s_1$), and $[s_0, s_1] \neq [d_0, d_1]$.

Since the Allen's relations *during* and *contains* refer to the former notion, for the purposes of this thesis we will use the expressions *contains* and *is contained* when we refer to the former definition, and the terms *sub-interval* and *super-interval* when we refer to the latter one.

1.1 Syntax and semantics of HS and its fragments

The language of (fragments of) HS includes the set of propositional letters $\mathcal{AP}$, the classical propositional connectives $\neg$ and $\land$ (all others, including the propositional constants $\top$ and $\bot$, are definable as usual), and a set of *interval temporal operators (modalities)* corresponding to the Allen's relations. Formulae are defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \lor \varphi \mid \langle X_1 \rangle \varphi \mid \ldots \mid \langle X_k \rangle \varphi,$$

As already pointed out, there are two different natural semantics for interval logics, namely, a *strict* one, which excludes point-intervals, and a *non-strict* one, which includes them. A *non-strict interval model* is a pair $M^+ = \langle \mathbb{I}(\mathbb{D})^+, V \rangle$, where

$\mathbb{I}(\mathbb{D})^+$ is the set of all non-strict intervals over $\mathbb{D}$ and $V : \mathbb{I}(\mathbb{D})^+ \to 2^{\mathcal{AP}}$ is a *valuation* assigning to each interval a set of atomic propositions considered true at it. Often, we will use the alternative (equivalent) notation for the valuation function $V : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{D})^+}$, assigning to each atomic poposition the set of intervals in which it is true. Respectively, a *strict interval model* is a structure $M^- = \langle \mathbb{I}(\mathbb{D})^-, V\rangle$ defined likewise. We will simply write $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$ when we do not wish to specify the strictness (assuming either version) or when it is sufficiently clear from the context. Formally, both the strict and non-strict semantics of HS modalities can be defined by the following rules:

- $M, [d_0, d_1] \Vdash \langle A\rangle\varphi$ iff $M, [d_1, d_2] \Vdash \varphi$ for some d_2;
- $M, [d_0, d_1] \Vdash \langle L\rangle\varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_1 < d_2$;
- $M, [d_0, d_1] \Vdash \langle B\rangle\varphi$ iff $M, [d_0, d_2] \Vdash \varphi$ for some d_2 such that $d_2 < d_1$;
- $M, [d_0, d_1] \Vdash \langle E\rangle\varphi$ iff $M, [d_2, d_1] \Vdash \varphi$ for some d_2 such that $d_0 < d_2$;
- $M, [d_0, d_1] \Vdash \langle D\rangle\varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_0 < d_2$ and $d_3 < d_1$;
- $M, [d_0, d_1] \Vdash \langle O\rangle\varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_0 < d_2 < d_1 < d_3$;
- $M, [d_0, d_1] \Vdash \langle \overline{A}\rangle\varphi$ iff $M, [d_2, d_0] \Vdash \varphi$ for some d_2;
- $M, [d_0, d_1] \Vdash \langle \overline{L}\rangle\varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_3 < d_0$;
- $M, [d_0, d_1] \Vdash \langle \overline{B}\rangle\varphi$ iff $M, [d_0, d_2] \Vdash \varphi$ for some d_2 such that $d_2 > d_1$;
- $M, [d_0, d_1] \Vdash \langle \overline{E}\rangle\varphi$ iff $M, [d_2, d_1] \Vdash \varphi$ for some d_2 such that $d_2 < d_0$;
- $M, [d_0, d_1] \Vdash \langle \overline{D}\rangle\varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_2 < d_0$ and $d_1 < d_3$;
- $M, [d_0, d_1] \Vdash \langle \overline{O}\rangle\varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_2 < d_0 < d_3 < d_1$.

For each one of the above-defined existential modalities, the corresponding universal modality is defined as usual, e.g., $[A]\varphi \equiv \neg\langle A\rangle\neg\varphi$. Moreover, in the non-strict semantics, it makes sense to consider the additional *modal constant for point intervals*, denoted π, interpreted as follows:

- $M^+, [d_0, d_1] \Vdash \pi$ iff $d_0 = d_1$.

Notice that, when the non-strict semantics is assumed (point intervals are allowed), the above semantic rules for the operators $\langle A\rangle$, $\langle\overline{A}\rangle$, $\langle L\rangle$, $\langle\overline{L}\rangle$, $\langle O\rangle$, and $\langle\overline{O}\rangle$ do not exactly match the original definition of Halpern and Shoham [59]. Indeed, in the Halpern and Shoham's definition, the operators $\langle A\rangle$ and $\langle\overline{A}\rangle$ are able to capture only strict intervals, that is, $\langle A\rangle\varphi$ is true over $[a, b]$ iff there exists a point c (strictly) greater than b such that φ is true over $[b, c]$, and symmetrically for the operator $\langle\overline{A}\rangle$. Consequently, also the operator $\langle L\rangle$ (resp., $\langle\overline{L}\rangle$), defined by Halpern and Shoham in terms of $\langle A\rangle$ (resp., $\langle\overline{A}\rangle$) as $\langle L\rangle\varphi \equiv \langle A\rangle\langle A\rangle\varphi$ (resp., $\langle\overline{L}\rangle\varphi \equiv \langle\overline{A}\rangle\langle\overline{A}\rangle\varphi$) is able to capture only strict interval in the future (resp., past) of the current one. Finally, the operator $\langle O\rangle$ (resp., $\langle\overline{O}\rangle$), defined in [59] in terms of the operators $\langle E\rangle$ and $\langle\overline{B}\rangle$ (resp., $\langle\overline{E}\rangle$ and $\langle B\rangle$) as $\langle O\rangle\varphi \equiv \langle E\rangle\langle\overline{B}\rangle\varphi$ (resp., $\langle\overline{O}\rangle\varphi \equiv \langle B\rangle\langle\overline{E}\rangle\varphi$) is able to capture also the so-called *right neighbors* (resp., *left neighbors*) of the current interval, that is, intervals starting (resp., ending) where the current interval ends (resp., starts), that is somehow unnatural. We have choosen to slightly modify such rules in order to get rid of some "bad behavior" of these operators, such as the ones described at the beginning of the chapter. Moreover, we believe that our definition represents a more natural extension of the Allen's relations to deal with point intervals. Nevertheless, it is still possible to define the operator $\langle O\rangle$ (resp., $\langle\overline{O}\rangle$) in terms of $\langle E\rangle$ and $\langle\overline{B}\rangle$ (resp., $\langle\overline{E}\rangle$ and $\langle B\rangle$) and, using the modal constant π, the operator $\langle L\rangle$ (resp., $\langle\overline{L}\rangle$) in terms of $\langle A\rangle$ (resp., $\langle\overline{A}\rangle$). However, it should not be difficult to adapt the results given here to deal with the original semantics definition provided by Halpern and Shoham.

The *truth of a formula over a given interval* $[a, b]$ *in a model* M is defined by structural induction on formulae:

- $M, [a, b] \Vdash \pi$ iff $a = b$;
- $M, [a, b] \Vdash p$ iff $p \in V([a, b])$, for all $p \in \mathcal{AP}$;
- $M, [a, b] \Vdash \neg\psi$ iff it is not the case that $M, [a, b] \Vdash \psi$;
- $M, [a, b] \Vdash \varphi \vee \psi$ iff $M, [a, b] \Vdash \varphi$ or $M, [a, b] \Vdash \psi$;
- $M, [a, b] \Vdash \langle X_i\rangle\psi$ iff there exists an interval $[c, d]$ such that $[a, b]\ R_{X_i}\ [c, d]$, and $M, [c, d] \Vdash \psi$,

where R_{X_i} is the binary interval relation corresponding to the modal operator $\langle X_i\rangle$ (Table 1.1). *Validity* and *satisfiability* are defined as usual, that is, given a formula φ of HS, we say that φ is *satisfiable* if there exists a model M and an interval $[a, b]$ such that $M, [a, b] \Vdash \varphi$, and that φ is *valid*, denoted $\models \varphi$, if it is true on every interval in every interval model. Two formulae φ and ψ are *equivalent*, denoted $\varphi \equiv \psi$, if $\models \varphi \leftrightarrow \psi$.

Some of the HS modalities are definable in terms of other ones. Depending on the considered (strict or non-strict) semantics, it is possible to identify two different minimal sets of modal operators that are expressive enough (jointly) to define all other operators. In the strict semantics, the six modalities $\langle A\rangle$, $\langle B\rangle$, $\langle E\rangle$, $\langle \overline{A}\rangle$, $\langle \overline{B}\rangle$, $\langle \overline{E}\rangle$ suffice to express all the other ones, as shown by the following equalities:

$$\begin{array}{ll} \langle L\rangle\varphi \equiv \langle A\rangle\langle A\rangle\varphi, & \langle \overline{L}\rangle\varphi \equiv \langle \overline{A}\rangle\langle \overline{A}\rangle\varphi, \\ \langle D\rangle\varphi \equiv \langle B\rangle\langle E\rangle\varphi, & \langle \overline{D}\rangle\varphi \equiv \langle \overline{B}\rangle\langle \overline{E}\rangle\varphi, \\ \langle O\rangle\varphi \equiv \langle E\rangle\langle \overline{B}\rangle\varphi, & \langle \overline{O}\rangle\varphi \equiv \langle B\rangle\langle \overline{E}\rangle\varphi. \end{array}$$

In the non-strict semantics, the four modalities $\langle B\rangle$, $\langle E\rangle$, $\langle \overline{B}\rangle$, $\langle \overline{E}\rangle$ are enough to express all the other operators, as shown by the following equalities:

$$\begin{array}{l} \langle A\rangle\varphi \equiv ([E]\bot \wedge (\varphi \vee \langle \overline{B}\rangle\varphi)) \vee \langle E\rangle([E]\bot \wedge (\varphi \vee \langle \overline{B}\rangle\varphi)), \\ \langle \overline{A}\rangle\varphi \equiv ([B]\bot \wedge (\varphi \vee \langle \overline{E}\rangle\varphi)) \vee \langle B\rangle([B]\bot \wedge (\varphi \vee \langle \overline{E}\rangle\varphi)), \\ \langle L\rangle\varphi \equiv \langle A\rangle(\langle E\rangle\top \wedge \langle A\rangle\varphi), \\ \langle \overline{L}\rangle\varphi \equiv \langle \overline{A}\rangle(\langle B\rangle\top \wedge \langle \overline{A}\rangle\varphi), \\ \langle D\rangle\varphi \equiv \langle B\rangle\langle E\rangle\varphi, \\ \langle \overline{D}\rangle\varphi \equiv \langle \overline{B}\rangle\langle \overline{E}\rangle\varphi, \\ \langle O\rangle\varphi \equiv \langle E\rangle(\langle E\rangle\top \wedge \langle \overline{B}\rangle\varphi), \\ \langle \overline{O}\rangle\varphi \equiv \langle B\rangle(\langle B\rangle\top \wedge \langle \overline{E}\rangle\varphi). \end{array}$$

Also the modal constant π is definable in terms of $\langle B\rangle$ and $\langle E\rangle$ as, respectively, $[B]\bot$ and $[E]\bot$.

The presence of π in the language allows one to interpret the strict semantics into the non-strict one by means of the translation:

- $\tau(p) = p$, for each $p \in \mathcal{AP}$;
- $\tau(\neg\phi) = \neg\tau(\phi)$;
- $\tau(\phi \wedge \psi) = \tau(\phi) \wedge \tau(\psi)$;
- $\tau(\langle X\rangle\,\phi) = \langle X\rangle\,(\neg\pi \wedge \tau(\phi))$ for any modality $\langle X_1\rangle, \ldots, \langle X_k\rangle$ of the language.

While HS features the whole set of modalities listed in Table 1.1, its fragments feature a strict subset of them (plus, possibly, the modal constant π). In the rest of this thesis, when we refer to a specific fragment of HS, we name it by its modal operators. The presence of the superscript $^{\pi}$ denotes that the modal constant π, when it is not already definable by the modalities, belongs to the language. For example, $\mathsf{A}\overline{\mathsf{A}}^{\pi}$ denotes the language featuring the modalities $\langle A\rangle$, $\langle \overline{A}\rangle$, and the modal constant π. For any given HS fragment $\mathcal{F} = \mathsf{X}_1\mathsf{X}_2\ldots\mathsf{X}_k$ and any given modal operator $\langle X\rangle$, we write $\langle X\rangle \in \mathcal{F}$ if $\langle X\rangle \in \{\langle X_1\rangle, \ldots, \langle X_k\rangle\}$. For any given pair of fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if $\langle X\rangle \in \mathcal{F}_1$ implies $\langle X\rangle \in \mathcal{F}_2$, for every modal operator $\langle X\rangle$. Finally, let $\mathcal{L}$ be a logic and p be a propositional letter. By $\mathcal{L}$ formulae we denote the set of syntactically well-formed formulae belonging to the language of $\mathcal{L}$ and by *p-interval* an interval in which p is true.

2

Expressiveness of HS fragments

The comparative analysis of the expressiveness of the variety of interval logics has been a major research problem in the area. In particular, the natural and important problem arises to analyze the mutual definabilities among the modal operators of the logic HS and to classify the fragments of HS with respect to their expressiveness.

The present chapter addresses and solves that problem with respect to the strict semantics (excluding point intervals) and over the class of all linear orders, by identifying a sound and complete set of inter-definability equations among the modal operators of HS and thus providing a complete classification of all fragments of HS with respect to their expressiveness [44]. Using that result, we find out that there are exactly 1347 expressively different such fragments out of $2^{12} = 4096$ sets of modal operators in HS.

The choice of strict semantics, excluding point intervals, instead of including them (non-strict semantics), conforms to the definition of interval adopted by Allen in [3]. It has at least two strong motivations. First, a number of representation paradoxes arise when the non-strict semantics is adopted, due to the presence of point intervals, as pointed out in [3]. Second, when point intervals are included, there seems to be no intuitive semantics for interval relations that makes them both pairwise disjoint and jointly exhaustive. However, the classification with respect to the non-strict semantics remains an open preblem that is worth to be addressed.

Definition 2.0.1. A modal operator $\langle X \rangle$ of HS is *definable* in an HS fragment $\mathcal{F}$, denoted $\langle X \rangle \lhd \mathcal{F}$, if $\langle X \rangle p \equiv \psi$ for some formula $\psi = \psi(p)$ of $\mathcal{F}$, for any fixed propositional variable p. In such a case, the equivalence $\langle X \rangle p \equiv \psi$ is called an *inter-definability equation for $\langle X \rangle$ in $\mathcal{F}$.*

It is known from [59] that, in the strict semantics, all modal operators in HS are definable in the fragment containing the modalities $\langle A \rangle$, $\langle B \rangle$, and $\langle E \rangle$, and their transposes $\langle \overline{A} \rangle$, $\langle \overline{B} \rangle$, and $\langle \overline{E} \rangle$ (In the non-strict semantics, the four modalities $\langle B \rangle$, $\langle E \rangle$, $\langle \overline{B} \rangle$, and $\langle \overline{E} \rangle$ suffice, as shown in [101]).

Here, we compare and classify the expressiveness of all fragments of HS on the class of all interval structures over linear orders. Formally, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be any pair of such fragments. We say that:

- $\mathcal{F}_2$ is *at least as expressive as* $\mathcal{F}_1$, denoted $\mathcal{F}_1 \preceq \mathcal{F}_2$, if every operator $\langle X \rangle \in \mathcal{F}_1$ is definable in $\mathcal{F}_2$.
- $\mathcal{F}_1$ is *strictly less expressive* than $\mathcal{F}_2$, denoted $\mathcal{F}_1 \prec \mathcal{F}_2$, if $\mathcal{F}_1 \preceq \mathcal{F}_2$ but not $\mathcal{F}_2 \preceq \mathcal{F}_1$.
- $\mathcal{F}_1$ and $\mathcal{F}_2$ are *equally expressive* (or, *expressively equivalent*), denoted $\mathcal{F}_1 \equiv \mathcal{F}_2$, if $\mathcal{F}_1 \preceq \mathcal{F}_2$ and $\mathcal{F}_2 \preceq \mathcal{F}_1$.
- $\mathcal{F}_1$ and $\mathcal{F}_2$ are *expressively incomparable*, denoted $\mathcal{F}_1 \not\equiv \mathcal{F}_2$, if neither $\mathcal{F}_1 \preceq \mathcal{F}_2$ nor $\mathcal{F}_2 \preceq \mathcal{F}_1$.

In order to show non-definability of a given modal operator in a given fragment, we use a standard technique in modal logic, based on the notion of *bisimulation* and the invariance of modal formulae with respect to bisimulations (see, e.g., [11]). More precisely, with every fragment $\mathcal{F}$ of HS we associate $\mathcal{F}$-bisimulations, preserving the truth of all formulae in $\mathcal{F}$. Thus, in order to prove that an operator $\langle X \rangle$ is not definable in $\mathcal{F}$, it suffices to construct a pair of interval models M and M' and an $\mathcal{F}$-bisimulation between them, relating a pair of intervals $[a, b] \in M$ and $[a', b'] \in M'$, such that $M, [a, b] \Vdash \langle X \rangle p$, while $M', [a', b'] \nVdash \langle X \rangle p$.

2.1 Comparing the expressiveness of the HS fragments

In order to classify all fragments of HS with respect to their expressiveness, it suffices to identify all definabilities of modal operators $\langle X \rangle$ in fragments $\mathcal{F}$, where $\langle X \rangle \notin \mathcal{F}$. A definability $\langle X \rangle \lhd \mathcal{F}$ is *optimal* if $\langle X \rangle \ntriangleleft \mathcal{F}'$ for any fragment $\mathcal{F}'$ such that $\mathcal{F}' \prec \mathcal{F}$. A set of such definabilities is *optimal* if it consists of optimal definabilities. The rest of the chapter is devoted to prove the following theorem.

Theorem 2.1.1 ([44])**.** *The set of inter-definability equations given in Table 2.1 is sound, complete, and optimal.*

Most of the equations in Table 2.1 are known from [59] and subsequent publications, but the definability $\langle L \rangle \lhd \overline{\mathsf{B}}\mathsf{E}$ and the symmetrical one, $\langle \overline{L} \rangle \lhd \mathsf{B}\overline{\mathsf{E}}$, are new. The soundness of the given set of inter-definability equations is proved by the following lemma.

Lemma 2.1.2. *The set of inter-definability equations given in Table 2.1 is sound.*

Proof. As we already pointed out, we only need to prove the soundness for the new inter-definability equation $\langle L \rangle p \equiv \langle \overline{B} \rangle [E] \langle \overline{B} \rangle \langle E \rangle p$ (the proof for the equation that defines the transposed modality $\langle \overline{L} \rangle$ is basically the same and thus omitted). First, we prove the left-to-right direction. Suppose that $M, [a, b] \Vdash \langle L \rangle p$ for some model

$\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$	$\langle L\rangle \lhd \mathsf{A}$
$\langle \overline{L}\rangle p \equiv \langle \overline{A}\rangle\langle \overline{A}\rangle p$	$\langle \overline{L}\rangle \lhd \overline{\mathsf{A}}$
$\langle O\rangle p \equiv \langle E\rangle\langle \overline{B}\rangle p$	$\langle O\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$
$\langle \overline{O}\rangle p \equiv \langle B\rangle\langle \overline{E}\rangle p$	$\langle \overline{O}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$
$\langle D\rangle p \equiv \langle E\rangle\langle B\rangle p$	$\langle D\rangle \lhd \mathsf{BE}$
$\langle \overline{D}\rangle p \equiv \langle \overline{E}\rangle\langle \overline{B}\rangle p$	$\langle \overline{D}\rangle \lhd \overline{\mathsf{BE}}$
$\langle L\rangle p \equiv \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$	$\langle L\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$
$\langle \overline{L}\rangle p \equiv \langle \overline{E}\rangle[B]\langle \overline{E}\rangle\langle B\rangle p$	$\langle \overline{L}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$

Table 2.1: The complete set of inter-definability equations

M and interval $[a, b]$. This means that there exists an interval $[c, d]$ such that $b < c$ and $M, [c, d] \Vdash p$ (see Fig. 2.1). We exhibit an interval $[a, y]$, with $y > b$ such that, for every x (strictly) in between a and y, the interval $[x, y]$ is such that there exist two points y' and x' such that $y' > y$, $x < x' < y'$, and $[x', y']$ satisfies p. Let y be equal to c. The interval $[a, c]$, which is started by $[a, b]$, is such that for any of its ending intervals, that is, for any interval of the form $[x, c]$, with $a < x$, we have that $x < c < d$ and $M, [c, d] \Vdash p$. As for the other direction, we must show that

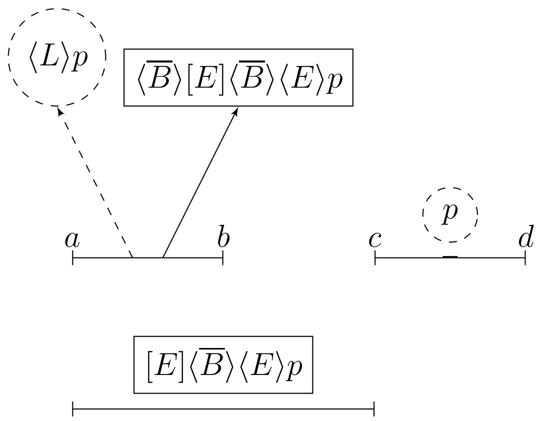


Figure 2.1: $\langle L\rangle p \equiv \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$

$\langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$ implies $\langle L\rangle p$. To this end, suppose that $M, [a, b] \Vdash \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$ for a model M and an interval $[a, b]$. Then, there exists an interval $[a, c]$, for some $c > b$ such that $[E]\langle \overline{B}\rangle\langle E\rangle p$ is true on $[a, c]$ (see Fig. 2.1). As a consequence, the interval $[b, c]$ must satisfy $\langle \overline{B}\rangle\langle E\rangle p$, that means that there are two points x and y such that $y > c$, $b < x < y$, and $[x, y]$ satisfies p. Since $x > b$, then $M, [a, b] \Vdash \langle L\rangle p$. $\square$

Proving completeness is the hard task; optimality will be established together with it. The completeness proof is organized as follows. For each HS operator $\langle X\rangle$, we show that $\langle X\rangle$ is not definable in any fragment of HS that does not contain $\langle X\rangle$ and does not contain as definable (according to Table 2.1) all operators of some of the fragments in which $\langle X\rangle$ is definable (according to Table 2.1). More formally, for each HS operator $\langle X\rangle$, the proof consists of the following steps:

1. using Table 2.1, identify all fragments $\mathcal{F}_i$ such that $\langle X\rangle \lhd \mathcal{F}_i$;
2. identify the list $\mathcal{M}_1, \ldots, \mathcal{M}_m$ of all $\subseteq$-maximal fragments of HS that contain neither the operator $\langle X\rangle$ nor any of the fragments $\mathcal{F}_i$ identified by the previous step;
3. for each fragment $\mathcal{M}_i$, with $i \in \{1, \ldots, m\}$, provide a bisimulation for $\mathcal{M}_i$ which is not a bisimulation for X.

The details of the completeness proof will be provided in a series of lemmas (of increasing complexity) in the next section.

2.2 The completeness proof

In this section, we will prove that, for each modal operator $\langle X\rangle$ of HS, the set of inter-definability equations for $\langle X\rangle$ in Table 2.1 is complete for that operator, that is, $\langle X\rangle$ is not definable in any fragment of HS (not containing $\langle X\rangle$) that does not contain (as definable) all operators of some of the fragments listed in Table 2.1 in which $\langle X\rangle$ is definable. From now on, we will denote by $\overline{\mathbb{Q}}$ the set $\mathbb{R} \setminus \mathbb{Q}$.

2.2.1 Completeness for $\langle L\rangle$ and $\langle \overline{L}\rangle$

Lemma 2.2.1. *The set of inter-definability equations for $\langle L\rangle$ and $\langle \overline{L}\rangle$ given in Table 2.1 is complete.*

Proof. According to Table 2.1, $\langle L\rangle$ is definable in terms of A and $\overline{\mathsf{B}}$E. Hence, the fragments BEDO$\overline{\mathsf{ALEDO}}$ and BDO$\overline{\mathsf{ALBEDO}}$ are the only $\subseteq$-maximal ones not featuring $\langle L\rangle$ and containing neither A nor $\overline{\mathsf{B}}$E. To prove the thesis, it suffices to exhibit a bisimulation for each one of these two fragments that does not preserve the relation induced by $\langle L\rangle$. Thanks to Lemma 2.1.2, BEDO$\overline{\mathsf{ALEDO}}$ and BDO$\overline{\mathsf{ALBEDO}}$ are expressively equivalent to BEO$\overline{\mathsf{AED}}$ and BDO$\overline{\mathsf{ABE}}$, respectively. Thus, to all our purposes, we can simply refer to the latter ones instead of the former ones.

As for the first fragment, let $M_1 = \langle \mathbb{I}(\mathbb{N}), V_1\rangle$ and $M_2 = \langle \mathbb{I}(\mathbb{N}), V_2\rangle$ be two models and let V_1 and V_2 be such that $V_1(p) = \{[2, 3]\}$ and $V_2(p) = \emptyset$, where p is the only propositional letter of the language. Moreover, let Z be a relation between (intervals of) M_1 and M_2 defined as $Z = \{[0, 1], [0, 1]\}$. It can be easily shown that Z is a BEO$\overline{\mathsf{AED}}$-bisimulation. The local property is trivially satisfied, since all Z-related intervals satisfy $\neg p$. As for the forward and backward conditions, it suffices to notice that, starting from the interval $[0, 1]$, it is not possible to reach any other interval using any of the modal operators of the fragment. At the same time, Z does not preserve the relation induced by the modality $\langle L\rangle$. Indeed, $([0, 1], [0, 1]) \in Z$

and $M_1, [0,1] \Vdash \langle L \rangle p$, but $M_2, [0,1] \Vdash \neg\langle L \rangle p$. Therefore, $\langle L \rangle$ is not definable in $\mathsf{BEDO}\overline{\mathsf{ALEDO}}$.

As for the second fragment, let $M_1 = \langle \mathbb{I}(\mathbb{Z}^-), V_1 \rangle$ and $M_2 = \langle \mathbb{I}(\mathbb{Z}^-), V_2 \rangle$ be two models based on the set $\mathbb{Z}^- = \{\ldots, -2, -1\}$, and let V_1 and V_2 be such that $V_1(p) = \{[-2,-1]\}$ and $V_2(p) = \emptyset$, where p is the only propositional letter of the language. Moreover, let Z be a relation between (intervals of) M_1 and M_2 defined as follows: $([x,y],[w,z]) \in Z \overset{def}{\Leftrightarrow} [x,y] = [w,z]$ and $[x,y] \neq [-2,-1]$. We prove that Z is a $\mathsf{BDO}\overline{\mathsf{ABE}}$-bisimulation. First, the local property is trivially satisfied, since all Z-related intervals satisfy $\neg p$. Moreover, starting from any interval, the only interval that satisfies p, that is, $[-2,-1]$, cannot be reached using the set of modal operators featured by the fragment. At the same time, Z does not preserve the relation induced by $\langle L \rangle$, as $([-4,-3],[-4,-3]) \in Z$ and $M_1, [-4,-3] \Vdash \langle L \rangle p$, but $M_2, [-4,-3] \Vdash \neg\langle L \rangle p$. Therefore, $\langle L \rangle$ is not definable in $\mathsf{BDO}\overline{\mathsf{ALBEDO}}$.

A completely symmetric argument can be applied for the completeness proof of $\langle \overline{L} \rangle$. □

2.2.2 Completeness for $\langle E \rangle$ and $\langle B \rangle$

Lemma 2.2.2. *The set of inter-definability equations for $\langle E \rangle$ and $\langle B \rangle$ given in Table 2.1 is complete.*

Proof. According to Table 2.1, $\langle E \rangle$ is not definable in terms of any other HS fragment. Thus, we will show that $\langle E \rangle$ is not definable in terms of the only $\subseteq$-maximal fragment not featuring it, namely, $\mathsf{ALBDO}\overline{\mathsf{ALBEDO}}$. (The symmetric modality $\langle B \rangle$ can be dealt with using similar arguments.) Thanks to Lemma 2.1.2, it actually suffices to provide a bisimulation for $\mathsf{ABDO}\overline{\mathsf{ABE}}$.

Let $M_1 = \langle \mathbb{I}(\mathbb{R}), V_1 \rangle$ and $M_2 = \langle \mathbb{I}(\mathbb{R}), V_2 \rangle$, where p is the only propositional letter of the language. The valuation function $V_1 : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ is defined as: $[x,y] \in V_1(p) \overset{def}{\Leftrightarrow} x \in \mathbb{Q}$ iff $y \in \mathbb{Q}$, and the valuation function $V_2 : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ as: $[w,z] \in V_2(p) \overset{def}{\Leftrightarrow} w \in \mathbb{Q}$ iff $z \in \mathbb{Q}$, and $([0,3],[w,z]) \notin R_E$. Moreover, let Z be a relation between (intervals of) M_1 and M_2 defined as follows: $([x,y],[w,z]) \in Z \overset{def}{\Leftrightarrow} [x,y] \in V_1(p)$ iff $[w,z] \in V_2(p)$.

We show that Z is an $\mathsf{ABDO}\overline{\mathsf{ABE}}$-bisimulation between M_1 and M_2. The satisfaction of the local condition immediately follows from the definition. The forward condition can be checked as follows. Let $[x,y]$ and $[w,z]$ be two Z-related intervals. For each modal operator $\langle X \rangle$ of the language, let us assume that $[x,y] R_X [x',y']$. We have to exhibit an interval $[w',z']$ such that $[x',y']$ and $[w',z']$ are Z-related, and $[w,z]$ and $[w',z']$ are R_X-related. We proceed case by case.

- If $\langle X \rangle = \langle A \rangle$ (and thus $y = x'$), then suppose that $[x',y'] \in V_1(p)$ (resp., $[x',y'] \notin V_1(p)$). We can always find a point $z' > z$ such that $[z,z'] \in V_2(p)$ (resp., $[z,z'] \notin V_2(p)$), independently from z belonging to $\mathbb{Q}$ or $\overline{\mathbb{Q}}$ (since both $\mathbb{Q}$

and $\overline{\mathbb{Q}}$ are right-unbounded). This implies that $[x', y']$ and $[z, z']$ are Z-related. Since $[w, z]$ and $[z, z']$ are obviously R_A-related, we have the thesis.

- If $\langle X \rangle = \langle B \rangle$, the argument is similar to the previous one, but, in this case, the density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ is exploited.
- If $\langle X \rangle = \langle D \rangle$, it suffices to choose two points w' and z' such that $w < w' < z' < z$, $z' \neq 3$, w' belongs to $\mathbb{Q}$ if and only if x' does, and z' belongs to $\mathbb{Q}$ if and only if y' does. As in the previous case, the existence of such points is guaranteed by the density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$.
- If $\langle X \rangle = \langle O \rangle$, w' and z' are required to be such that $w < w' < z < z'$, and both density and right-unboundedness of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ must be exploited.
- If $\langle X \rangle = \langle \overline{A} \rangle$, a symmetric argument to the one used for the modality $\langle A \rangle$ can be used. In this case, the left-unboundedness of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ is exploited.
- If $\langle X \rangle = \langle \overline{B} \rangle$, the argument is simular to the one used for the operator $\langle A \rangle$.
- If $\langle X \rangle = \langle \overline{E} \rangle$, a symmetric argument to the one used for the modality $\langle \overline{B} \rangle$ can be used. In this case, the left-unboundedness of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ is exploited.

The backward condition can be verified in a very similar way, thus proving that Z is an $\mathsf{ABDO\overline{ABE}}$-bisimulation between M_1 and M_2. On the other hand, Z does not preserve the relation induced by $\langle E \rangle$: we have that $([0,3],[0,3]) \in Z$, $M_1, [0,3] \Vdash \langle E \rangle p$, but $M_2, [0,3] \Vdash \neg\langle E \rangle p$. Therefore, $\langle E \rangle$ cannot be defined in the fragment $\mathsf{ALBDO\overline{ALBEDO}}$. □

2.2.3 Completeness for $\langle \overline{E} \rangle$ and $\langle \overline{B} \rangle$

Lemma 2.2.3. *The set of inter-definability equations for $\langle \overline{E} \rangle$ and $\langle \overline{B} \rangle$ given in Table 2.1 is complete.*

Proof. According to Table 2.1, $\langle \overline{E} \rangle$ is not definable in terms of any other HS fragment. Thus, we will show that $\langle \overline{E} \rangle$ is not definable in terms of the only $\subseteq$-maximal HS fragment not featuring it, namely, $\mathsf{ALBEDO\overline{ALBDO}}$. (The symmetric modality $\langle \overline{B} \rangle$ can be dealt with using similar arguments.) Thanks to Lemma 2.1.2, it actually suffices to provide a bisimulation for $\mathsf{ABE\overline{ABDO}}$.

Let $M_1 = \langle \mathbb{I}(\mathbb{R}), V_1 \rangle$ and $M_2 = \langle \mathbb{I}(\mathbb{R}), V_2 \rangle$ be two models over the set of propositional letters $\mathcal{AP} = \{p\}$, with valuation functions $V_1 : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ and $V_2 : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ defined, respectively, as follows: $[x, y] \in V_1(p) \overset{def}{\Leftrightarrow} x \in \mathbb{Q}$ iff $y \in \mathbb{Q}$ and $[w, z] \in V_2(p) \overset{def}{\Leftrightarrow} w \in \mathbb{Q}$ iff $z \in \mathbb{Q}$, and $([0,3],[w,z]) \notin R_{\overline{E}}$. Then, we define the relation Z between (intervals of) M_1 and M_2 as: $([x, y],[w, z]) \in Z \overset{def}{\Leftrightarrow} [x, y] \in V_1(p)$ iff $[w, z] \in V_2(p)$.

By exploiting a very similar argument to the one used for the bisimulation of the previous section, it is not difficult to see that Z is an $\mathsf{ABE\overline{ABDO}}$-bisimulation between M_1 and M_2. On the other hand, Z does not preserve the relation induced by $\langle \overline{E} \rangle$: we have that $([0,3],[0,3]) \in Z$, $M_1, [0,3] \Vdash \langle \overline{E} \rangle p$, but $M_2, [0,3] \Vdash \neg \langle \overline{E} \rangle p$. Therefore, $\langle \overline{E} \rangle$ cannot be defined in the fragment $\mathsf{ALBEDO\overline{ALBDO}}$. □

2.2.4 Completeness for $\langle A \rangle$ and $\langle \overline{A} \rangle$

In order to define the bisimulations for the proofs of the remaining cases, we need to exploit a well-known property of the set of real numbers $\mathbb{R}$: $\mathbb{R}$ (resp., $\mathbb{Q}$, $\overline{\mathbb{Q}}$) can be partitioned into a countable number of pairwise disjoint subsets, each one of which is dense in $\mathbb{R}$. More formally, there are countably many non-empty sets $\mathbb{R}_i$ (resp., $\mathbb{Q}_i$, $\overline{\mathbb{Q}}_i$), with $i \in \mathbb{N}$, such that, for each $i \in \mathbb{N}$, $\mathbb{R}_i$ (resp., $\mathbb{Q}_i$, $\overline{\mathbb{Q}}_i$) is dense in $\mathbb{R}$, $\mathbb{R} = \bigcup_{i \in \mathbb{N}} \mathbb{R}_i$ (resp., $\mathbb{Q} = \bigcup_{i \in \mathbb{N}} \mathbb{Q}_i$, $\overline{\mathbb{Q}} = \bigcup_{i \in \mathbb{N}} \overline{\mathbb{Q}}_i$), and $\mathbb{R}_i \cap \mathbb{R}_j = \emptyset$, (resp., $\mathbb{Q}_i \cap \mathbb{Q}_j = \emptyset$, $\overline{\mathbb{Q}}_i \cap \overline{\mathbb{Q}}_j = \emptyset$), for each $i, j \in \mathbb{N}$ with $i \neq j$.

As an example, in the following we define two partitions, for $\mathbb{Q}$ and $\overline{\mathbb{Q}}$, that fit the above requirements. First, let P_r be the set of all and only the prime numbers, we define the partition $\mathcal{P}(\mathbb{Q})$ as follows:

$$\mathcal{P}(\mathbb{Q}) = \{\mathbb{Q}_p \mid p \in P_r, p > 2\} \cup \{\mathbb{Q}_2 \cup \mathcal{Q}^-\}$$

where $\mathcal{Q}^- = \mathbb{Q} \setminus \bigcup_{p \in P_r} \mathbb{Q}_p$ and $\mathbb{Q}_p = \left\{\frac{a}{p^m} \mid p \nmid a, a \in \mathbb{Z}, a \neq 0, m \in \mathbb{N}^+\right\}$, for every $p \in P_r$. Then, we define the partition $\mathcal{P}(\overline{\mathbb{Q}})$ as follows:

$$\mathcal{P}(\overline{\mathbb{Q}}) = \{\sqrt{p}\,\mathbb{Q} \mid p \in P_r, p > 2\} \cup \{\sqrt{2}\,\mathbb{Q} \cup \overline{\mathcal{Q}^-}\}$$

where $\overline{\mathcal{Q}^-} = \overline{\mathbb{Q}} \setminus \bigcup_{p \in P_r} \sqrt{p}\,\mathbb{Q}$ and, for every $a \in \mathbb{R}$, $a\,\mathbb{Q} = \{a \cdot q \mid q \in \mathbb{Q}\}$. Thus, we have a partition of $\mathbb{Q}$ (resp., $\overline{\mathbb{Q}}$) in infinitely countably many subsets $\mathbb{Q}_i$ (resp., $\overline{\mathbb{Q}}_i$) with $i \in \mathbb{N}$. Moreover, it is possible to show that $\mathbb{Q}_i$ (resp., $\overline{\mathbb{Q}}_i$) is dense in $\mathbb{R}$. Now, let $\mathbb{R}_i = \mathbb{Q}_i \cup \overline{\mathbb{Q}}_i$ for each $i \in \mathbb{N}$. It is easy to verify that the set $\mathcal{P}(\mathbb{R}) = \{\mathbb{R}_i \mid i \in \mathbb{N}\}$ represents an infinite and countable partition of $\mathbb{R}$ such that $\mathbb{R}_i$ is dense in $\mathbb{R}$ for each $i \in \mathbb{N}$. Finally, it is simple to convince ourselves that it is also possible to partition $\mathbb{R}$, $\mathbb{Q}$, or $\overline{\mathbb{Q}}$ in finitely many subsets that are dense in $\mathbb{R}$.

Lemma 2.2.4. *The set of inter-definability equations for $\langle A \rangle$ and $\langle \overline{A} \rangle$ given in Table 2.1 is complete.*

Proof. According to Table 2.1, it suffices to show that $\langle A \rangle$ is not definable in the only $\subseteq$-maximal fragment not containing it, namely, $\mathsf{LBEDO\overline{ALBEDO}}$, which, by Lemma 2.1.2, turns out to be equivalent to $\mathsf{LBE\overline{ABE}}$.

Let $M_1 = \langle \mathbb{I}(\mathbb{R}), V_1 \rangle$ and $M_2 = \langle \mathbb{I}(\mathbb{R}), V_2 \rangle$ be two models built on the only propositional letter p. In order to define the valuation functions V_1 and V_2, we take advantage of two partitions of the set $\mathbb{R}$, one for M_1 and the other one for M_2, each of them consisting of exactly four sets that are dense in $\mathbb{R}$. Formally, for $j = 1, 2$

and $i = 1, \ldots, 4$, let $\mathbb{R}_j^i$ be dense in $\mathbb{R}$. Moreover, for $j = 1, 2$, let $\mathbb{R} = \bigcup_{i=1}^4 \mathbb{R}_j^i$ and $\mathbb{R}_j^i \cap \mathbb{R}_j^{i'} = \emptyset$ for each $i, i' \in \{1, 2, 3, 4\}$ with $i \neq i'$.

For $j = 1, 2$, we force points in $\mathbb{R}_j^1$ (resp., $\mathbb{R}_j^2$, $\mathbb{R}_j^3$, $\mathbb{R}_j^4$) to behave in the same way with respect to the truth of $p/\neg p$ over the intervals they initiate and terminate by imposing the following constraints:

$$\begin{array}{l}
\forall x, y(\text{ if } x \in \mathbb{R}_j^1, \text{ then } M_j, [x, y] \Vdash \neg p);\\
\forall x, y(\text{ if } x \in \mathbb{R}_j^2, \text{ then } M_j, [x, y] \Vdash \neg p);\\
\forall x, y(\text{ if } x \in \mathbb{R}_j^3, \text{ then } (M_j, [x, y] \Vdash p \text{ iff } y \in \mathbb{R}_j^1 \cup \mathbb{R}_j^3));\\
\forall x, y(\text{ if } x \in \mathbb{R}_j^4, \text{ then } (M_j, [x, y] \Vdash p \text{ iff } y \in \mathbb{R}_j^2 \cup \mathbb{R}_j^4)).
\end{array}$$

It can be easily shown that, from the given constraints, it immediately follows that:

$$\begin{array}{l}
\forall x, y(\text{ if } y \in \mathbb{R}_j^1, \text{ then } (M_j, [x, y] \Vdash p \text{ iff } x \in \mathbb{R}_j^3));\\
\forall x, y(\text{ if } y \in \mathbb{R}_j^2, \text{ then } (M_j, [x, y] \Vdash p \text{ iff } x \in \mathbb{R}_j^4));\\
\forall x, y(\text{ if } y \in \mathbb{R}_j^3, \text{ then } (M_j, [x, y] \Vdash p \text{ iff } x \in \mathbb{R}_j^3));\\
\forall x, y(\text{ if } y \in \mathbb{R}_j^4, \text{ then } (M_j, [x, y] \Vdash p \text{ iff } x \in \mathbb{R}_j^4)).
\end{array}$$

The above constraints univocally induces the following definition of the valuation functions $V_j(p) : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$:

$$[x, y] \in V_j(p) \overset{def}{\Leftrightarrow} \begin{cases} (x \in \mathbb{R}_j^3 \wedge y \in \mathbb{R}_j^1 \cup \mathbb{R}_j^3) \\ \vee(x \in \mathbb{R}_j^4 \wedge y \in \mathbb{R}_j^2 \cup \mathbb{R}_j^4). \end{cases}$$

Now, let Z be the relation between (intervals of) M_1 and M_2 defined as follows. Two intervals $[x, y]$ and $[w, z]$ are Z-related if and only if at least one of the following conditions holds:

1. $x \in \mathbb{R}_1^1 \cup \mathbb{R}_1^2$ and $w \in \mathbb{R}_2^1 \cup \mathbb{R}_2^2$;
2. $x \in \mathbb{R}_1^3$, $w \in \mathbb{R}_2^3$, and ($y \in \mathbb{R}_1^1 \cup \mathbb{R}_1^3$ iff $z \in \mathbb{R}_2^1 \cup \mathbb{R}_2^3$);
3. $x \in \mathbb{R}_1^3$, $w \in \mathbb{R}_2^4$, and ($y \in \mathbb{R}_1^1 \cup \mathbb{R}_1^3$ iff $z \in \mathbb{R}_2^2 \cup \mathbb{R}_2^4$);
4. $x \in \mathbb{R}_1^4$, $w \in \mathbb{R}_2^3$, and ($y \in \mathbb{R}_1^2 \cup \mathbb{R}_1^4$ iff $z \in \mathbb{R}_2^1 \cup \mathbb{R}_2^3$);
5. $x \in \mathbb{R}_1^4$, $w \in \mathbb{R}_2^4$, and ($y \in \mathbb{R}_1^2 \cup \mathbb{R}_1^4$ iff $z \in \mathbb{R}_2^2 \cup \mathbb{R}_2^4$).

We show that the relation Z is an $\mathsf{LBE}\overline{\mathsf{ABE}}$-bisimulation. It can be easily checked that every pair $([x, y], [w, z])$ of Z-related intervals is such that either $[x, y] \in V_1(p)$ and $[w, z] \in V_2(p)$ or $[x, y] \notin V_1(p)$ and $[w, z] \notin V_2(p)$. In order to verify the forward condition, let $[x, y]$ and $[w, z]$ be two Z-related intervals. For each modal operator $\langle X \rangle$ of the language and each interval $[x', y']$ such that $[x, y] R_X [x', y']$, we have to exhibit an interval $[w', z']$ such that $[x', y']$ and $[w', z']$ are Z-related, and $[w, z]$ and $[w', z']$ are R_X-related. We proceed case by case.

- If $\langle X\rangle = \langle L\rangle$, we must consider five sub-cases depending on the sets x' and y' belong to:

  (i) if $x' \in \mathbb{R}^1_1 \cup \mathbb{R}^2_1$, then for each $w' \in \mathbb{R}^1_2$ such that $w' > z$, we have that, for every $z' > w'$, $([x', y'], [w', z']) \in Z$ and $[w, z]R_L[w', z']$ (the existence of w' is guaranteed by right-unboundedness of $\mathbb{R}^1_2$);

  (ii) if $x' \in \mathbb{R}^3_1$ and $y' \in \mathbb{R}^1_1 \cup \mathbb{R}^3_1$, then for each w', z' such that $z < w' < z'$ and $w', z' \in \mathbb{R}^3_2$, we have that $([x', y'], [w', z']) \in Z$ and $[w, z]R_L[w', z']$ (right-unboundedness of $\mathbb{R}^3_2$);

  (iii) if $x' \in \mathbb{R}^3_1$ and $y' \in \mathbb{R}^2_1 \cup \mathbb{R}^4_1$, then for each w', z' such that $z < w' < z'$, $w' \in \mathbb{R}^3_2$, and $z' \in \mathbb{R}^4_2$, we have that $([x', y'], [w', z']) \in Z$ and $[w, z]R_L[w', z']$ (right-unboundedness of $\mathbb{R}^3_2$ and $\mathbb{R}^4_2$);

  (iv) if $x' \in \mathbb{R}^4_1$ and $y' \in \mathbb{R}^1_1 \cup \mathbb{R}^3_1$, then for each w', z' such that $z < w' < z'$, $w' \in \mathbb{R}^4_2$, and $z' \in \mathbb{R}^3_2$, we have that $([x', y'], [w', z']) \in Z$ and $[w, z]R_L[w', z']$ (right-unboundedness of $\mathbb{R}^3_2$ and $\mathbb{R}^4_2$);

  (v) if $x' \in \mathbb{R}^4_1$ and $y' \in \mathbb{R}^2_1 \cup \mathbb{R}^4_1$, then for each w', z' such that $z < w' < z'$ and $w', z' \in \mathbb{R}^4_2$, we have that $([x', y'], [w', z']) \in Z$ and $[w, z]R_L[w', z']$ (right-unboundedness of $\mathbb{R}^4_2$).

- If $\langle X\rangle = \langle B\rangle$, then:

  (i) if $x \in \mathbb{R}^1_1 \cup \mathbb{R}^2_1$, then, by definition of Z, it must be $w \in \mathbb{R}^1_2 \cup \mathbb{R}^2_2$. Then, for any $w < z' < z$, both $([x, y'], [w, z']) \in Z$ and $[w, z]R_B[w, z']$ hold;

  (ii) if $x \in \mathbb{R}^i_1$ and $w \in \mathbb{R}^i_2$, for some $i \in \{3, 4\}$, and $y' \in \mathbb{R}^k_1$, for some $k \in \{1, 2, 3, 4\}$, then for any $w < z' < z$ such that $z' \in \mathbb{R}^k_2$, it holds that $([x, y'], [w, z']) \in Z$ and $[w, z]R_B[w, z']$ (the existence of z' is guaranteed by density of $\mathbb{R}^k_2$ in $\mathbb{R}$);

  (iii) if $x \in \mathbb{R}^i_1$ and $w \in \mathbb{R}^{i'}_2$ for $i, i' \in \{3, 4\}$ with $i \neq i'$, then if $y' \in \mathbb{R}^1_1 \cup \mathbb{R}^3_1$ (resp., $y' \in \mathbb{R}^2_1 \cup \mathbb{R}^4_1$), then for any $w < z' < z$ such that $z' \in \mathbb{R}^2_2 \cup \mathbb{R}^4_2$ (resp., $z' \in \mathbb{R}^1_2 \cup \mathbb{R}^3_2$), it holds that $([x, y'], [w, z']) \in Z$ and $[w, z]R_B[w, z']$ (density of $\mathbb{R}^2_2$ and $\mathbb{R}^4_2$, resp., $\mathbb{R}^1_2$ and $\mathbb{R}^3_2$, in $\mathbb{R}$).

- If $\langle X\rangle = \langle\overline{B}\rangle$, then an argument similar to previous one can be exploited.

- If $\langle X\rangle = \langle E\rangle$, then let $[x', y]$ be a generic interval such that $x' > x$. We want to exhibit an interval $[w', z]$, for some $w' > w$, such that $([x', y], [w', z]) \in Z$. We distinguish the following cases:

  (i) if $x' \in \mathbb{R}^1_1 \cup \mathbb{R}^2_1$, then for any w' such that $w < w' < z$ and $w' \in \mathbb{R}^1_2 \cup \mathbb{R}^2_2$, it holds $([x', y], [w', z]) \in Z$;

  (ii) if $x' \in \mathbb{R}^3_1$ (resp., $x' \in \mathbb{R}^4_1$), then we have two further cases:

a) if either $y \in \mathbb{R}_1^1 \cup \mathbb{R}_1^3$ and $z \in \mathbb{R}_2^1 \cup \mathbb{R}_2^3$ or $y \in \mathbb{R}_1^2 \cup \mathbb{R}_1^4$ and $z \in \mathbb{R}_2^2 \cup \mathbb{R}_2^4$, then for any w' such that $w < w' < z$ and $w' \in \mathbb{R}_2^3$ (resp., $w' \in \mathbb{R}_2^4$), it holds $([x', y], [w', z]) \in Z$;

b) if either $y \in \mathbb{R}_1^1 \cup \mathbb{R}_1^3$ and $z \in \mathbb{R}_2^2 \cup \mathbb{R}_2^4$ or $y \in \mathbb{R}_1^2 \cup \mathbb{R}_1^4$ and $z \in \mathbb{R}_2^1 \cup \mathbb{R}_2^3$, then for any w' such that $w < w' < z$ and $w' \in \mathbb{R}_2^4$ (resp., $w' \in \mathbb{R}_2^3$), it holds $([x', y], [w', z]) \in Z$;

- If $\langle X \rangle = \langle \overline{E} \rangle$, then an argument similar to previous one can be exploited.
- If $\langle X \rangle = \langle \overline{A} \rangle$, then an argument similar to previous one can be exploited.

The backward condition can be verified in a very similar way, thus proving that Z is an $\mathsf{LBE\overline{A}\overline{B}\overline{E}}$-bisimulation between M_1 and M_2. Let us consider now two intervals $[x, y]$ and $[w, z]$ such that $x \in \mathbb{R}_1^1$, $w \in \mathbb{R}_2^1$, $y \in \mathbb{R}_1^3$, and $z \in \mathbb{R}_2^1$. By definition of Z, $[x, y]$ and $[w, z]$ are Z-related, and by definition of V_1 and V_2, there exists $y' > y$ such that $M_1, [y, y'] \Vdash p$, but there is no $z' > z$ such that $M_2, [z, z'] \Vdash p$. Thus, $M_1, [x, y] \Vdash \langle A \rangle p$, but $M_2, [w, z] \Vdash \neg\langle A \rangle p$. This allows us to conclude that Z does not preserve the relation induced by $\langle A \rangle$, and thus $\langle A \rangle$ is not definable in terms of $\mathsf{LBEDO\overline{A}\overline{L}\overline{B}\overline{E}\overline{D}\overline{O}}$.

A completely symmetric argument can be applied for the completeness proof of $\langle \overline{A} \rangle$. □

2.2.5 Completeness for $\langle D \rangle$

To deal with the modality $\langle D \rangle$, we proceed as follows. We first introduce the notion of f-model, that is, for any given function $f : \mathbb{R} \to \mathbb{Q}$, we define a model M_f, called f-model, whose valuation is based on f. Then, for any given pair of functions f_1 and f_2, we define a suitable relation $Z_{f_1}^{f_2}$ between the models M_{f_1} and M_{f_2} (from now on, we will simply write Z when there is no ambiguity about the involved models). Finally, we specify the requirements that f_1 and f_2 must satisfy to make Z the bisimulation we want.

Lemma 2.2.5. *The set of inter-definability equations for $\langle D \rangle$ given in Table 2.1 is complete.*

Proof. According to Table 2.1, $\langle D \rangle$ is definable in terms of BE. The fragments $\mathsf{ALBO\overline{A}\overline{L}\overline{B}\overline{E}\overline{D}\overline{O}}$ and $\mathsf{ALEO\overline{A}\overline{L}\overline{B}\overline{E}\overline{D}\overline{O}}$ are thus the only $\subseteq$-maximal ones not featuring $\langle D \rangle$ and not containing BE. We should provide a bisimulation, not preserving the relation induced by $\langle D \rangle$, for each of these fragments, but, thanks to the symmetry of the operators, i.e., of $\langle B \rangle$ and $\langle E \rangle$, it suffices to consider only one of them, say $\mathsf{ALBO\overline{A}\overline{L}\overline{B}\overline{E}\overline{D}\overline{O}}$. Thanks to Lemma 2.1.2, it actually suffices to provide a bisimulation for $\mathsf{ABO\overline{A}\overline{B}\overline{E}}$. Given a function $f : \mathbb{R} \to \mathbb{Q}$, we define the *$f$-model* M_f, over a language with one propositional letter p only, as the pair $\langle \mathbb{I}(\mathbb{R}), V_f \rangle$, where

$V_f : \mathcal{AP} \rightarrow 2^{\mathbb{I}(\mathbb{R})}$ is defined as follows: $[x, y] \in V_f(p) \overset{def}{\Leftrightarrow} y \geq f(x)$. For any given pair of functions f_1 and f_2 (from $\mathbb{R}$ to $\mathbb{Q}$), the relation Z is defined as follows:

$$([x, y], [w, z]) \in Z \overset{def}{\Leftrightarrow} x \equiv w,\ y \equiv z, \text{ and } [x, y] \equiv_l [w, z],$$

where $u \equiv v \overset{def}{\Leftrightarrow} u \in \mathbb{Q}$ iff $v \in \mathbb{Q}$ and $[u, u'] \equiv_l [v, v'] \overset{def}{\Leftrightarrow} u' \sim f_1(u)$ and $v' \sim f_2(v)$, for some $\sim \in \{<, =, >\}$. Finally, the following constraints are imposed on f:

(i) for every $x \in \mathbb{R}$, $f(x) > x$,

(ii) for every $x \in \mathbb{Q}$, both $f^{-1}(x) \cap \mathbb{Q}$ and $f^{-1}(x) \cap \overline{\mathbb{Q}}$ are left-unbounded (notice that surjectivity of f immediately follows), and

(iii) for every $x, y \in \mathbb{R}$, if $x < y$, then there exists $u_1 \in \mathbb{Q}$ (resp., $u_2 \in \overline{\mathbb{Q}}$) such that $x < u_1 < y$ (resp., $x < u_2 < y$) and $y < f(u_1)$ (resp., $y < f(u_2)$).

Now, we show that if both f_1 and f_2 satisfy the above conditions, then Z is an $\mathsf{ABO\overline{A}\overline{B}\overline{E}}$-bisimulation between M_{f_1} and M_{f_2}. Let $[x, y]$ and $[w, z]$ be two Z-related intervals. By definition, $y \sim f_1(x)$ and $z \sim f_2(w)$ for some $\sim \in \{<, =, >\}$. If $\sim \in \{=, >\}$, then both $[x, y]$ and $[w, z]$ satisfy p; otherwise, both of them satisfy $\neg p$. The local condition is thus satisfied. As for the forward condition, let $[x, y]$ and $[x', y']$ be two intervals in M_{f_1} and $[w, z]$ an interval in M_{f_2}. We have to prove that if $[x, y]$ and $[w, z]$ are Z-related, then, for each modal operator $\langle X \rangle$ of $\mathsf{ABO\overline{A}\overline{B}\overline{E}}$ such that $[x, y]R_X[x', y']$, there exists an interval $[w', z']$ such that $[x', y']$ and $[w', z']$ are Z-related and $[w, z]R_X[w', z']$. Once more, we proceed case by case.

- If $\langle X \rangle = \langle A \rangle$, then, by definition of $\langle A \rangle$, $x' = y$ and we are forced to choose $w' = z$. By $y \equiv z$, it immediately follows $x' \equiv w'$. We must find a point $z' > z$ such that $y' \equiv z'$ and both $y' \sim f_1(y)$ and $z' \sim f_2(z)$ for some $\sim \in \{<, =, >\}$. Let us suppose that $y' < f_1(y)$. In such a case, we choose a point z' such that $z < z' < f_2(z)$ and $y' \equiv z'$. The existence of such a point is guaranteed by condition (i) on f_2 and by the density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ in $\mathbb{R}$. Otherwise, if $y' = f_1(y)$, we choose $z' = f_2(z)$. Since the codomain of both f_1 and f_2 is $\mathbb{Q}$, y' and z' belong to $\mathbb{Q}$ and thus $y' \equiv z'$. Finally, if $y' > f_1(y)$, we choose $z' > f_2(z)$ such that $y' \equiv z'$. The existence of such a point is guaranteed by right-unboundedness of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$.

- If $\langle X \rangle = \langle B \rangle$, then an argument similar to the previous one can be used. First of all, notice that, in this case, we have $x' = x$ and we are forced to choose $w' = w$. By $x \equiv w$, it immediately follows $x' \equiv w'$. We must find a point z' such that $w < z' < z$, $y' \equiv z'$, and both $y' \sim f_1(x)$ and $z' \sim f_2(w)$ for some $\sim \in \{<, =, >\}$. Let us suppose that $y' < f_1(x)$. Then, we can choose any point z' such that $w < z' < min(z, f_2(w))$ and $y' \equiv z'$. The existence of such a point is guaranteed by condition (i), and by density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$. Otherwise, if $y' \geq f_1(x)$, notice that, by semantics of operator $\langle B \rangle$, it also holds $y > f_1(x)$, as well as, by definition of Z, $z > f_2(w)$. So, if $y' = f_1(x)$, we can choose $z' = f_2(w)$, and we are ensured that both y' and z' belong to $\mathbb{Q}$ ($y' \equiv z'$), since they are image of, respectively, x and w. Finally, if $y' > f_1(x)$, we can choose any z' such that $f_2(w) < z' < z$ and $y' \equiv z'$. The existence of such a point is guaranteed by density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$.

- If $\langle X \rangle = \langle O \rangle$, then, by condition (iii), there exists of a point w' such that $w < w' < z$, $x' \equiv w'$, and $f_2(w') > z$. Moreover, if $y' < f_1(x')$, then we choose any point z' such that $z < z' < f_2(w')$ and $y' \equiv z'$. The existence of such a point is guaranteed by density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$. Otherwise, if $y' = f_1(x')$, we choose $z' = f_2(w')$, and we are ensured that both y' and z' belong to $\mathbb{Q}$ ($y' \equiv z'$), since they are image of, respectively, x' and w'. Finally, if $y' > f_1(x')$, we choose any z' such that $z' > f_2(w')$ and $y' \equiv z'$. The existence of such a point is guaranteed by right-unboundedness of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$.

- If $\langle X \rangle = \langle \overline{A} \rangle$, then $y' = x$ and we are forced to choose $z' = w$. By $x \equiv w$, it immediately follows $y' \equiv z'$. We must find a point $w' < w$ such that $x' \equiv w'$ and both $x \sim f_1(x')$ and $w \sim f_2(w')$ for some $\sim \in \{<, =, >\}$. Let us suppose that $x < f_1(x')$. Then, consider any point $u \in \mathbb{Q}$ such that $u > w$. By condition (ii), there exists a point u' such that $u' < w$, $f_2(u') = u$, and $x' \equiv u'$. If we choose $w' = u'$, the interval $[w', w]$ is such that $[w, z]$ and $[w', w]$ are $R_{\overline{A}}$-related, and $[x', x]$ and $[w', w]$ are Z-related. Otherwise, if $x = f_1(x')$, we choose any point w' such that $w = f_2(w')$ and $x' \equiv w'$. The existence of such a point is guaranteed by condition (ii). Finally, if $x > f_1(x')$, then consider any point $u \in \mathbb{Q}$ such that $u < w$. By condition (ii), there exists a point u' such that $u' < w$, $f_2(u') = u$, and $x' \equiv u'$. If we choose $w' = u'$, the interval $[w', w]$ is such that $[w, z]$ and $[w', w]$ are $R_{\overline{A}}$-related, and $[x', x]$ and $[w', w]$ are Z-related.

- If $\langle X \rangle = \langle \overline{B} \rangle$, then $x' = x$ and we are forced to choose $w' = w$. By $x \equiv w$, it immediately follows $x' \equiv w'$. We must find a point z' such that $z' > z$, $y' \equiv z'$, and both $y' \sim f_1(x)$ and $z' \sim f_2(w)$ for some $\sim \in \{<, =, >\}$. First, notice that, by semantics of operator $\langle \overline{B} \rangle$, if $y' \leq f_1(x)$, it also holds $y < f_1(x)$, as well as, by definition of Z, $z < f_2(w)$. Let us suppose that $y' < f_1(x)$. Then, we choose any point z' such that $z < z' < f_2(w)$ and $y' \equiv z'$. The existence of such a point is guaranteed by density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$. Otherwise, if $y' = f_1(x)$, we choose $z' = f_2(w)$, and we are ensured that both y' and z' belong to $\mathbb{Q}$, since they are image of, respectively, x and w. Finally, if $y' > f_1(x)$, we choose any z' such that $z' > max(z, f_2(w))$ and $y' \equiv z'$. The existence of such a point is guaranteed by right-unboundedness of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$.

- If $\langle X \rangle = \langle \overline{E} \rangle$, then $y' = y$ and we are forced to choose $z' = z$. By $y \equiv z$, it immediately follows $y' \equiv z'$. We must find a point $w' < w$ such that $x' \equiv w'$ and both $y \sim f_1(x')$ and $z \sim f_2(w')$ for some $\sim \in \{<, =, >\}$. Let us suppose that $y < f_1(x')$. Then, consider any point $u \in \mathbb{Q}$ such that $u > z$. By condition (ii), there exists a point u' such that $u' < w$, $f_2(u') = u$, and $x' \equiv u'$. If we choose $w' = u'$, the interval $[w', z]$ is such that $[w, z]$ and $[w', z]$ are $R_{\overline{E}}$-related, and $[x', y]$ and $[w', z]$ are Z-related. Otherwise, if $y = f_1(x')$, we choose any point $w' < w$ such that $z = f_2(w')$ and $x' \equiv w'$. The existence of such a point is guaranteed by condition (ii). Finally, if $y > f_1(x')$, then consider any point $u \in \mathbb{Q}$ such that $u < w$. By condition (ii), there exists a point u' such that $u' < w$, $f_2(u') = u$, and $x' \equiv u'$. If we choose $w' = u'$, the interval $[w', z]$ is such that $[w, z]$ and $[w', z]$ are $R_{\overline{E}}$-related, and $[x', y]$ and $[w', z]$ are Z-related.

Satisfaction of the backward condition can be checked in a similar way.

To complete the proof, we exhibit two functions that meet the requirements we have imposed to f_1 and f_2, but do not preserve the relation induced by $\langle D \rangle$. Let $\mathcal{P}(\mathbb{Q}) = \{\mathbb{Q}_q \mid q \in \mathbb{Q}\}$ and $\mathcal{P}(\overline{\mathbb{Q}}) = \{\overline{\mathbb{Q}}_q \mid q \in \mathbb{Q}\}$ be infinite and countable partitions of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$, respectively, such that for every $q \in \mathbb{Q}$, both $\mathbb{Q}_q$ and $\overline{\mathbb{Q}}_q$ are dense in $\mathbb{R}$. For every $q \in \mathbb{Q}$, let $\mathbb{R}_q = \mathbb{Q}_q \cup \overline{\mathbb{Q}}_q$. We define a function $g : \mathbb{R} \to \mathbb{Q}$ that maps every real number x to the index q (a rational number) of the class $\mathbb{R}_q$ it belongs to. Formally, for every $x \in \mathbb{R}$, $g(x) = q$, where $q \in \mathbb{Q}$ is the unique rational number such that $x \in \mathbb{R}_q$. The two functions $f_1 : \mathbb{R} \to \mathbb{Q}$ and $f_2 : \mathbb{R} \to \mathbb{Q}$ are defined as follows:

$$f_1(x) = \begin{cases} g(x) & \text{if } x < g(x),\ x \neq 1, \text{ and } x \neq 0 \\ 2 & \text{if } x = 1 \\ \lceil x + 3 \rceil & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} g(x) & \text{if } x < g(x) \text{ and } x \notin [0, 3) \\ \lceil x + 3 \rceil & \text{otherwise} \end{cases}$$

It is not difficult to check that the above-defined functions meet the requirements for f_1 and f_2, and thus Z is an $\mathsf{ABO\overline{ABE}}$-bisimulation. On the other hand, Z does not preserve the relation induced by $\langle D \rangle$. Consider the interval $[0, 3]$ in M_{f_1} and the interval $[0, 3]$ in M_{f_2}. It is immediate to see that these two intervals are Z-related. However, $M_{f_1}, [0, 3] \Vdash \langle D \rangle p$ (as $M_{f_1}, [1, 2] \Vdash p$), but $M_{f_2}, [0, 3] \Vdash \neg \langle D \rangle p$. This allows us to conclude that Z does not preserve the relation induced by $\langle D \rangle$, and thus $\langle D \rangle$ is not definable in terms of the fragment $\mathsf{ALBO\overline{ALBEDO}}$. ☐

2.2.6 Completeness for $\langle \overline{D} \rangle$

To deal with the modality $\langle \overline{D} \rangle$, we proceed similarly to the case of fragment $\langle D \rangle$.

Lemma 2.2.6. *The set of inter-definability equations for $\langle \overline{D} \rangle$ given in Table 2.1 is complete.*

Proof. According to Table 2.1, $\langle \overline{D} \rangle$ is definable in terms of the HS fragment $\overline{\mathsf{BE}}$. The two fragments $\mathsf{ALBEDO}\overline{\mathsf{ALBO}}$ and $\mathsf{ALBEDO}\overline{\mathsf{ALEO}}$ are the only $\subseteq$-maximal ones not featuring $\langle \overline{D} \rangle$ and not containing $\overline{\mathsf{BE}}$. We should provide a bisimulation, not preserving the relation induced by $\langle \overline{D} \rangle$, for each of these fragments, but, thanks to the symmetry of the operators, it suffices to consider only one of them, say $\mathsf{ALBEDO}\overline{\mathsf{ALBO}}$. Thanks to Lemma 2.1.2, it actually suffices to provide a bisimulation for $\mathsf{ABE}\overline{\mathsf{ABO}}$. Given a function $f : \mathbb{R} \to \mathbb{Q}$, we define the f*-model* M_f, over a language with one propositional letter p only, as the pair $\langle \mathbb{I}(\mathbb{R}), V_f \rangle$, where the valuation function $V_f : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ is defined as follows: $[x, y] \in V_f(p) \overset{def}{\Leftrightarrow} y \leq f(x)$.

For any given pair of functions f_1 and f_2 (from $\mathbb{R}$ to $\mathbb{Q}$), the relation Z is defined as follows:

$$([x, y], [w, z]) \in Z \overset{def}{\Leftrightarrow} x \equiv w,\ y \equiv z, \text{ and } [x, y] \equiv_l [w, z],$$

where $u \equiv v \overset{def}{\Leftrightarrow} u \in \mathbb{Q}$ iff $v \in \mathbb{Q}$ and $[u, u'] \equiv_l [v, v'] \overset{def}{\Leftrightarrow} u' \sim f_1(u)$ and $v' \sim f_2(v)$, for some $\sim \in \{<, =, >\}$. Finally, the following constraints are imposed on f:

(i) $f(x) > x$ for every $x \in \mathbb{R}$,

(ii) for each $x \in \mathbb{Q}$ either:

- $x \leq 2$ and both $f^{-1}(x) \cap \mathbb{Q}$ and $f^{-1}(x) \cap \overline{\mathbb{Q}}$ are dense in $\mathbb{R}^{<x}$ or
- $x > 2$ and both $f^{-1}(x) \cap \mathbb{Q}$ and $f^{-1}(x) \cap \overline{\mathbb{Q}}$ are dense in $\mathbb{R}^{[1,x)}$,

where by $\mathbb{R}^{<x}$ and $\mathbb{R}^{[1,x)}$ we denote, respectively, the sets $\{y \in \mathbb{R} \mid y < x\}$ and $\{y \in \mathbb{R} \mid 1 \leq y < x\}$.

Now, we show that if both f_1 and f_2 satisfy the above conditions, then Z is an $\mathsf{ABE}\overline{\mathsf{ABO}}$-bisimulation between M_{f_1} and M_{f_2}. It is trivial to verify that the local condition is fulfilled. As for the forward condition, consider the three intervals $[x, y]$, $[x', y']$ (in M_{f_1}), and $[w, z]$ (in M_{f_2}) and suppose that $[x, y]$ and $[w, z]$ are Z-related, and that $[x, y]$ and $[x', y']$ are R_X-related for some modal operator $\langle X \rangle$ of the logic $\mathsf{ABE}\overline{\mathsf{ABO}}$. The proof proceeds case by case. The proofs for the cases corresponding to the operators $\langle A \rangle$, $\langle B \rangle$, and $\langle \overline{B} \rangle$ are very similar to that one of the respective cases in Lemma 2.2.5, and thus are omitted. On the other hand, proofs for operators $\langle \overline{A} \rangle$, $\langle E \rangle$, and $\langle \overline{O} \rangle$ are similar, so we only give the proof for the operator $\langle \overline{A} \rangle$. First of all, notice that $y' = x$ and we are forced to choose $z' = w$. By $x \equiv w$, it immediately follows $y' \equiv z'$. We must find a point $w' < w$ such that $x' \equiv w'$ and both $x \sim f_1(x')$ and $w \sim f_2(w')$ for some $\sim \in \{<, =, >\}$. Thus, let us distinguish three cases:

- if $x < f_1(x')$, then we distinguish two further cases:
  - if $w < 2$, then consider any point $u \in \mathbb{Q}$ such that $w < u \leq 2$. By condition (ii), there exists a point u' such that $u' < w$, $f_2(u') = u$, and $x' \equiv u'$. If we choose $w' = u'$, the interval $[w', w]$ is such that $[w, z]$ and $[w', w]$ are $R_{\overline{A}}$-related, and $[x', x]$ and $[w', w]$ are Z-related,

  - if $w \geq 2$, then consider any point $u \in \mathbb{Q}$ such that $u > w$. By condition (ii), there exists a point u' such that $1 < u' < w$, $f_2(u') = u$, and $x' \equiv u'$. If we choose $w' = u'$, the interval $[w', w]$ is such that $[w, z]$ and $[w', w]$ are $R_{\overline{A}}$-related, and $[x', x]$ and $[w', w]$ are Z-related,

- if $x = f_1(x')$, then we choose any point w' such that $w = f_2(w')$ and $x' \equiv w'$. The existence of such a point is guaranteed by condition (ii).

- if $x > f_1(x')$, then consider any point $u \in \mathbb{Q}$ such that $u < min(2, w)$. By condition (ii), there exists a point u' such that $f_2(u') = u$, and $x' \equiv u'$. If we choose $w' = u'$, the interval $[w', w]$ is such that $[w, z]$ and $[w', w]$ are $R_{\overline{A}}$-related, and $[x', x]$ and $[w', w]$ are Z-related.

The backward condition can be verified in the same way.

To complete the proof, we exhibit two functions that meet the requirements we have imposed to f_1 and f_2, but do not preserve the relation induced by $\langle \overline{D} \rangle$. To this end, we exploit the same partitioning of $\mathbb{Q}$ (resp., $\overline{\mathbb{Q}}$, $\mathbb{R}$) in infinitely countably many subsets introduced in the previous section for the completeness proof of $\langle D \rangle$. Also the function $g : \mathbb{R} \to \mathbb{Q}$ is defined as in the previous section. The two functions $f_1 : \mathbb{R} \to \mathbb{Q}$ and $f_2 : \mathbb{R} \to \mathbb{Q}$ are defined as follows:

$$f_1(x) = \begin{cases} g(x) & \text{if } x < g(x),\ x \neq 1, \text{ and } x \neq 0 \\ 3 & \text{if } x = 0 \\ \lceil x + 1 \rceil & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} g(x) & \text{if } x < g(x),\ x \neq 1, \\ & \quad \text{and } x \geq 1 \text{ or } g(x) \leq 2 \\ \lceil x + 1 \rceil & \text{otherwise} \end{cases}$$

It is not difficult to check that the above-defined functions meet the requirements for f_1 and f_2, and thus Z is an $\mathsf{ABE\overline{ABO}}$-bisimulation. On the other hand, Z does not preserve the relation induced by $\langle \overline{D} \rangle$. Consider the interval $[1, 2]$ in M_{f_1} and the interval $[1, 2]$ in M_{f_2}. It is immediate to see that these two intervals are Z-related. However, $M_{f_1}, [1, 2] \Vdash \langle \overline{D} \rangle p$ (as $M_{f_1}, [0, 3] \Vdash p$), but $M_{f_2}, [1, 2] \Vdash \neg \langle \overline{D} \rangle p$. This allows us to conclude that Z does not preserve the relation induced by $\langle \overline{D} \rangle$, and thus $\langle \overline{D} \rangle$ is not definable in terms of the fragment $\mathsf{ALBEDO\overline{ALBO}}$. □

2.2.7 Completeness for $\langle O \rangle$ and $\langle \overline{O} \rangle$

Lemma 2.2.7. *The set of inter-definability equations for $\langle O \rangle$ and $\langle \overline{O} \rangle$ given in Table 2.1 is complete.*

Proof. According to Table 2.1, $\langle O \rangle$ is definable in terms of the HS fragment $\mathsf{\overline{B}E}$. The two fragments $\mathsf{ALBED\overline{ALEDO}}$ and $\mathsf{ALBD\overline{ALBEDO}}$ are the only $\subseteq$-maximal ones not featuring $\langle O \rangle$ and not containing $\overline{\mathsf{B}}\mathsf{E}$. We have to provide a bisimulation, not preserving the relation induced by $\langle O \rangle$, for each one of these fragments. Thanks to Lemma 2.1.2, it actually suffices to provide a bisimulation for $\mathsf{ABE}\overline{\mathsf{AED}}$ and one for $\mathsf{ABD}\overline{\mathsf{ABE}}$.

$\mathsf{ABE}\overline{\mathsf{AED}}$-bisimulation. This bisimulation is very similar to those ones presented in Section 2.2.2 and Section 2.2.3.

Let $M_1 = \langle \mathbb{I}(\mathbb{R}), V_1 \rangle$ and $M_2 = \langle \mathbb{I}(\mathbb{R}), V_2 \rangle$ be two models over the set of propositional letters $\mathcal{AP} = \{p\}$, with valuation functions $V_1 : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ and $V_2 : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ defined, respectively, as follows: $[x, y] \in V_1(p) \overset{def}{\Leftrightarrow} x \in \mathbb{Q}$ iff $y \in \mathbb{Q}$ and $[w, z] \in V_2(p) \overset{def}{\Leftrightarrow} w \in \mathbb{Q}$ iff $z \in \mathbb{Q}$, and $([0, 3], [w, z]) \notin R_O$ (that is, it is not the case that $0 < w < 3 < z$). Then, we define the relation Z between (intervals of) M_1 and M_2 as: $([x, y], [w, z]) \in Z \overset{def}{\Leftrightarrow} [x, y] \in V_1(p)$ iff $[w, z] \in V_2(p)$. By exploiting an argument very similar to the one used in Section 2.2.2, it is possible to show that Z is an $\mathsf{ABE}\overline{\mathsf{AED}}$-bisimulation between M_1 and M_2. On the other hand, Z does not preserve the relation induced by $\langle O \rangle$. Consider the interval $[0, 3]$ in M_1 and the interval $[0, 3]$ in M_2. It is immediate to see that these two intervals are Z-related. However, $M_1, [0, 3] \Vdash \langle O \rangle p$, but $M_2, [0, 3] \Vdash \neg \langle O \rangle p$. This allows us to conclude that Z does not preserve the relation induced by $\langle O \rangle$, and thus $\langle O \rangle$ is not definable in terms of the fragment $\mathsf{ALBED}\overline{\mathsf{ALEDO}}$.

$\mathsf{ABD}\overline{\mathsf{ABE}}$-bisimulation. In order to define a bisimulation for the fragment $\mathsf{ABD}\overline{\mathsf{ABE}}$ but not for O, we exploit an argument similar to that one used in Section 2.2.5. Nevertheless, we must suitably modify the involved ingredients.

First of all, we need to define a new version of the valuation functions. To this end, we need to "rearrange" the previous partitions of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ used in Section 2.2.5. Actually, we still need two infinite and countable partitions $\mathcal{P}(\mathbb{Q})$ of $\mathbb{Q}$ and $\mathcal{P}(\overline{\mathbb{Q}})$ of $\overline{\mathbb{Q}}$ defined as in the previous section. Nevertheless, it is useful to provide a more suitable enumeration for both of them, as follows: $\mathcal{P}(\mathbb{Q}) = \{\mathbb{Q}_q^c \mid c \in \{a, b\}, q \in \mathbb{Q}\}$ and $\mathcal{P}(\overline{\mathbb{Q}}) = \{\overline{\mathbb{Q}}_q^c \mid c \in \{a, b\}, q \in \mathbb{Q}\}$. Analogously to Section 2.2.5, we require these partitions to be such that, for each $q \in \mathbb{Q}, c \in \{a, b\}$, sets $\mathbb{Q}_q^c$ and $\overline{\mathbb{Q}}_q^c$ are dense in $\mathbb{R}$. Now, we define the partition $\mathcal{P}(\mathbb{R})$ of $\mathbb{R}$ as: $\mathcal{P}(\mathbb{R}) = \{\mathbb{R}_q^c \mid c \in \{a, b\}, q \in \mathbb{Q}\}$, where $\mathbb{R}_q^c = \mathbb{Q}_q^c \cup \overline{\mathbb{Q}}_q^c$, for each $q \in \mathbb{Q}$, $c \in \{a, b\}$. For each $c \in \{a, b\}$, $\mathbb{Q}^c$ (resp., $\overline{\mathbb{Q}}^c$, $\mathbb{R}^c$) denotes the set $\bigcup_{q \in \mathbb{Q}} \mathbb{Q}_q^c$ (resp., $\bigcup_{q \in \mathbb{Q}} \overline{\mathbb{Q}}_q^c$, $\bigcup_{q \in \mathbb{Q}} \mathbb{R}_q^c$). In addition, we define $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{I}(\mathbb{D})$ as follows: $\mathcal{S}_1 = \{[x, y] \mid x, y \in \mathbb{R}^c, c \in \{a, b\}\}$ and $\mathcal{S}_2 = \{[w, z] \mid w, z \in \mathbb{R}^c, c \in \{a, b\}\} \setminus \{[w, z] \mid 0 < w < 3 < z\}$. Finally, for each $i \in \{1, 2\}$, we define the shorthand $\overline{\mathcal{S}}_i = \mathbb{I}(\mathbb{D}) \setminus \mathcal{S}_i$.

Then, let $i \in \{1, 2\}$. Given a function $f_i : \mathbb{R} \to \mathbb{Q}$, we define the f_i-model M_{f_i}, over a language with one propositional letter p only, as the pair $\langle \mathbb{I}(\mathbb{R}), V_{f_i} \rangle$, where the valuation function $V_{f_i} : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{R})}$ is defined as follows: $[x, y] \in V_{f_i}(p) \overset{def}{\Leftrightarrow}$ either $y = f_i(x)$ or $y < f_i(x)$ and $[x, y] \in \mathcal{S}_i$.

For any given pair of functions f_1 and f_2 (from $\mathbb{R}$ to $\mathbb{Q}$), the relation Z is defined as follows:

$$([x,y],[w,z]) \in Z \overset{def}{\Leftrightarrow} x \equiv w,\ y \equiv z, \text{ and } [x,y] \equiv_l [w,z],$$

where $u \equiv v \overset{def}{\Leftrightarrow} u \in \mathbb{Q}$ iff $v \in \mathbb{Q}$ and

$$[u,u'] \equiv_l [v,v'] \overset{def}{\Leftrightarrow} \begin{cases} \text{either } u' > f_1(u) \text{ and } v' > f_2(v) \\ \quad \text{or } u' = f_1(u) \text{ and } v' = f_2(v) \\ \quad \text{or } u' < f_1(u),\ v' < f_2(v), \\ \qquad \text{and } [u,u'] \in \mathcal{S}_1 \text{ iff } [v,v'] \in \mathcal{S}_2 \end{cases}$$

Finally, the following constraints are imposed on f:

(i) $f(x) > x$ for every $x \in \mathbb{R}$,

(ii) for each $x \in \mathbb{Q}$, $f^{-1}(x) \cap \mathbb{Q}^a$, $f^{-1}(x) \cap \mathbb{Q}^b$, $f^{-1}(x) \cap \overline{\mathbb{Q}}^a$, and $f^{-1}(x) \cap \overline{\mathbb{Q}}^b$ are unbounded to left,

(iii) for each $x, y \in \mathbb{R}$, if $x < y$, then there exist:

- $u_1 \in \mathbb{Q}^a$ such that $x < u_1 < y$ and $y > f(u_1)$,
- $u_2 \in \mathbb{Q}^b$ such that $x < u_2 < y$ and $y > f(u_2)$,
- $u_3 \in \overline{\mathbb{Q}}^a$ such that $x < u_3 < y$ and $y > f(u_3)$, and
- $u_4 \in \overline{\mathbb{Q}}^b$ such that $x < u_4 < y$ and $y > f(u_4)$.

Now, we show that if both f_1 and f_2 satisfy the above conditions and, additionally, f_2 is such that $f_2(w) \leq 3$ for each $0 < w < 3$, then Z is an $\mathsf{ABD\overline{ABE}}$-bisimulation between M_{f_1} and M_{f_2}. Let $[x,y]$ and $[w,z]$ be two Z-related intervals. It is easy to see that the local condition is verified. As for the forward condition, let $[x,y]$ and $[x',y']$ be two intervals in M_{f_1} and $[w,z]$ an interval in M_{f_2}. We have to prove that if $[x,y]$ and $[w,z]$ are Z-related, then, for each modal operator $\langle X \rangle$ of $\mathsf{ABD\overline{ABE}}$ such that $[x,y]R_X[x',y']$, there exists an interval $[w',z']$ such that $[x',y']$ and $[w',z']$ are Z-related and $[w,z]R_X[w',z']$. Once more, we proceed case by case.

- If $\langle X \rangle = \langle A \rangle$, then $x' = y$ and we are forced to choose $w' = z$. By $y \equiv z$, it immediately follows $x' \equiv w'$. We must find a point $z' > z$ such that $y' \equiv z'$ and either one of the following holds:

  a) $y' > f_1(y)$ and $z' > f_2(z)$,

  b) $y' = f_1(y)$ and $z' = f_2(z)$,

  c) $y' < f_1(y)$, $z' < f_2(z)$, and $[y,y'] \in \mathcal{S}_1$ iff $[z,z'] \in \mathcal{S}_2$.

  Thus, let us distinguish three cases:

- if $y' > f_1(y)$, then we choose a z' such that $z' > f_2(z)$ and $z' \equiv y'$. The existence of such a point is guaranteed by the right-unboundedness of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$;
- if $y' = f_1(y)$, then we choose $z' = f_2(z)$. Since the codomain of both f_1 and f_2 is $\mathbb{Q}$, y' and z' belong to $\mathbb{Q}$ and thus $y' \equiv z'$;
- if $y' < f_1(y)$, then we choose a point z' such that $z' < f_2(z)$, $y' \equiv z'$, and $[y, y'] \in S_1$ iff $[z, z'] \in S_2$. In order to fulfill this last condition ($[y, y'] \in S_1$ iff $[z, z'] \in S_2$), we choose z' depending on y and y': if $y \in \mathbb{R}^c$ and $y' \in \mathbb{R}^{c'}$ for some $c, c' \in \{a, b\}$, with $c \neq c'$, then we choose $z' \in \mathbb{R}^a$ iff $z \notin \mathbb{R}^a$, otherwise, if both y and y' belong to $\mathbb{R}^c$ for some $c \in \{a, b\}$, then we choose z' in such a way that $z' \in \mathbb{R}^a$ iff $z \in \mathbb{R}^a$ and $([0, 3], [z, z']) \notin R_O$. The existence of such a point is guaranteed by condition (i) and by the density of $\mathbb{Q}^a$, $\mathbb{Q}^b$, $\overline{\mathbb{Q}}^a$, and $\overline{\mathbb{Q}}^b$ (and thus the one of $\mathbb{R}^a$ and $\mathbb{R}^b$).

- If $\langle X \rangle = \langle B \rangle$, then an argument very similar to the previous one can be used.
- If $\langle X \rangle = \langle D \rangle$, then, by condition (iii), there exists of a point w' such that $w < w' < z$, $x' \equiv w'$, and $z > f_2(w')$. Without lost of generality, suppose $w' \in \mathbb{R}^a$. If $y' > f_1(x')$, then we choose any point z' such that $f_2(w') < z' < z$ and $y' \equiv z'$. The existence of such a point is guaranteed by density of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$. Otherwise, if $y' = f_1(x')$, we choose $z' = f_2(w')$. Since the codomain of both f_1 and f_2 is $\mathbb{Q}$, y' and z' belong to $\mathbb{Q}$ and thus $y' \equiv z'$. Finally, if $y' < f_1(x')$, we choose any z' such that $w' < z' < f_2(w')$, $y' \equiv z'$, and $[x', y'] \in S_1$ iff $[w', z'] \in S_2$. In order to fulfill this last condition ($[x', y'] \in S_1$ iff $[w', z'] \in S_2$), we choose z' depending on x' and y': if $x' \in \mathbb{R}^c$ and $y' \in \mathbb{R}^{c'}$ for some $c, c' \in \{a, b\}$, with $c \neq c'$, then we choose $z' \in \mathbb{R}^b$, otherwise, if both x' and y' belong to $\mathbb{R}^c$ for some $c \in \{a, b\}$, then we choose $z' \in \mathbb{R}^a$ in such a way that $([0, 3], [w', z']) \notin R_O$. The existence of such a point is guaranteed by condition (i) and by the density of $\mathbb{Q}^a$, $\mathbb{Q}^b$, $\overline{\mathbb{Q}}^a$, and $\overline{\mathbb{Q}}^b$ (and thus the one of $\mathbb{R}^a$ and $\mathbb{R}^b$).
- If $\langle X \rangle = \langle \overline{A} \rangle$, then $y' = x$ and we are forced to choose $z' = w$. By $x \equiv w$, it immediately follows $y' \equiv z'$. We must find a point $w' < w$ such that $x' \equiv w'$ and either one of the following holds:

  a) $x > f_1(x')$ and $w > f_2(w')$,

  b) $x = f_1(x')$ and $w = f_2(w')$,

  c) $x < f_1(x')$, $w < f_2(w')$, and $[x', x] \in \mathcal{S}_1$ iff $[w', w] \in \mathcal{S}_2$.

  Thus, let us distinguish three cases:

  - if $x > f_1(x')$, then consider any point $u \in \mathbb{Q}$ such that $u < w$. By condition (ii), there exists a point u' such that $u' < w$, $f_2(u') = u$, and

$x' \equiv u'$. If we choose $w' = u'$, the interval $[w', w]$ is such that $[w, z]$ and $[w', w]$ are $R_{\overline{A}}$-related, and $[x', x]$ and $[w', w]$ are Z-related;

- if $x = f_1(x')$, we choose any point w' such that $w = f_2(w')$ and $x' \equiv w'$. The existence of such a point is guaranteed by condition (ii);
- if $x < f_1(x')$, then consider any point $u \in \mathbb{Q}$ such that $u > w$. We distinguish two cases:
  * if $[x', x] \in \mathcal{S}_1$, then, by condition (ii), there exists a point u' such that $u' < min(0, w)$, $f_2(u') = u$, $x' \equiv u'$, and $u' \in \mathbb{R}^a$ iff $w \in \mathbb{R}^a$. If we choose $w' = u'$, the interval $[w', w]$ is such that $[w, z]$ and $[w', w]$ are $R_{\overline{A}}$-related, and $[x', x]$ and $[w', w]$ are Z-related;
  * if $[x', x] \notin \mathcal{S}_1$, then, by condition (ii), there exists a point u' such that $u' < w$, $f_2(u') = u$, $x' \equiv u'$, and $u' \in \mathbb{R}^a$ iff $w \notin \mathbb{R}^a$. If we choose $w' = u'$, the interval $[w', w]$ is such that $[w, z]$ and $[w', w]$ are $R_{\overline{A}}$-related, and $[x', x]$ and $[w', w]$ are Z-related.

- If $\langle X \rangle = \langle \overline{B} \rangle$, then $x' = x$ and we are forced to choose $w' = w$. By $x \equiv w$, it immediately follows $x' \equiv w'$. We must find a point z' such that $z' > z$, $y' \equiv z'$, and either one of the following holds:

  a) $y' > f_1(x)$ and $z' > f_2(w)$,

  b) $y' = f_1(x)$ and $z' = f_2(w)$,

  c) $y' < f_1(x)$, $z' < f_2(w)$, and $[x, y'] \in \mathcal{S}_1$ iff $[w, z'] \in \mathcal{S}_2$.

  Thus, let us distinguish three cases:

  - if $y' > f_1(x)$, then we choose a z' such that $z' > max(f_2(w), z)$ and $z' \equiv y'$. The existence of such a point is guaranteed by the right-unboundedness of $\mathbb{Q}$ and $\overline{\mathbb{Q}}$;
  - if $y' = f_1(x)$, then $y < f_1(x)$ and, by definition of Z, $z < f_2(w)$. Then, we choose $z' = f_2(w)$. Since the codomain of both f_1 and f_2 is $\mathbb{Q}$, y' and z' belong to $\mathbb{Q}$ and thus $y' \equiv z'$;
  - if $y' < f_1(x)$, then $y < f_1(x)$ and, by definition of Z, $z < f_2(w)$. Then, we choose a point z' such that $z < z' < f_2(w)$, $y' \equiv z'$, and $[x, y'] \in S_1$ iff $[w, z'] \in S_2$. In order to fulfill this last condition ($[x, y'] \in S_1$ iff $[w, z'] \in S_2$), we choose z' depending on x and y': if $x \in \mathbb{R}^c$ and $y' \in \mathbb{R}^{c'}$ for some $c, c' \in \{a, b\}$, with $c \neq c'$, then we choose $z' \in \mathbb{R}^a$ iff $w \notin \mathbb{R}^a$, otherwise, if both x and y' belong to $\mathbb{R}^c$ for some $c \in \{a, b\}$, then we choose z' in such a way that $z' \in \mathbb{R}^a$ iff $w \in \mathbb{R}^a$. Notice that we are guaranteed that $([0, 3], [w, z']) \notin R_O$ by the condition of f_2 that forces $f_2(w) \leq 3$ for each $0 < w < 3$ and by the fact that we choose $z' < f_2(w)$. The existence of such a point is guaranteed by the density of $\mathbb{Q}^a$, $\mathbb{Q}^b$, $\overline{\mathbb{Q}}^a$, and $\overline{\mathbb{Q}}^b$ (and thus the one of $\mathbb{R}^a$ and $\mathbb{R}^b$).

- If $\langle X \rangle = \langle \overline{E} \rangle$, then $y' = y$ and we are forced to choose $z' = z$. By $y \equiv z$, it immediately follows $y' \equiv z'$. We must find a point $w' < w$ such that $x' \equiv w'$ and either one of the following holds:

  a) $y > f_1(x')$ and $z > f_2(w')$,

  b) $y = f_1(x')$ and $z = f_2(w')$,

  c) $y < f_1(x')$, $z < f_2(w')$, and $[x', y] \in \mathcal{S}_1$ iff $[w', z] \in \mathcal{S}_2$.

  Thus, let us distinguish three cases:

  – if $y > f_1(x')$, then consider any point $u \in \mathbb{Q}$ such that $u < z$. By condition (ii), there exists a point u' such that $u' < w$, $f_2(u') = u$, and $x' \equiv u'$. If we choose $w' = u'$, the interval $[w', z]$ is such that $[w, z]$ and $[w', z]$ are $R_{\overline{E}}$-related, and $[x', y]$ and $[w', z]$ are Z-related;

  – if $y = f_1(x')$, we choose any point $w' < w$ such that $z = f_2(w')$ and $x' \equiv w'$. The existence of such a point is guaranteed by condition (ii);

  – if $y < f_1(x')$, then consider any point $u \in \mathbb{Q}$ such that $u > z$. We distinguish two cases:

    * if $[x', y] \in \mathcal{S}_1$, then, by condition (ii), there exists a point u' such that $u' < min(0, w)$, $f_2(u') = u$, $x' \equiv u'$, and $u' \in \mathbb{R}^a$ iff $z \in \mathbb{R}^a$. If we choose $w' = u'$, the interval $[w', z]$ is such that $[w, z]$ and $[w', z]$ are $R_{\overline{E}}$-related, and $[x', y]$ and $[w', z]$ are Z-related;
    * if $[x', y] \notin \mathcal{S}_1$, then, by condition (ii), there exists a point u' such that $u' < w$, $f_2(u') = u$, $x' \equiv u'$, and $u' \in \mathbb{R}^a$ iff $z \notin \mathbb{R}^a$. If we choose $w' = u'$, the interval $[w', z]$ is such that $[w, z]$ and $[w', z]$ are $R_{\overline{E}}$-related, and $[x', y]$ and $[w', z]$ are Z-related.

Satisfaction of the backward condition can be checked in a similar way.

To complete the proof, we exhibit two functions that meet the requirements we have imposed to f_1 and f_2 plus the additional condition on f_2 ($f_2(w) \leq 3$ for each $0 < w < 3$), but do not preserve the relation induced by $\langle O \rangle$. Let $g : \mathbb{R} \to \mathbb{Q}$ be a function defined as follows: for each $x \in \mathbb{R}$, $g(x) = q$, where $q \in \mathbb{Q}$ is the unique rational number such that $x \in \mathbb{R}^a_q \cup R^b_q$. The two functions $f_1 : \mathbb{R} \to \mathbb{Q}$ and $f_2 : \mathbb{R} \to \mathbb{Q}$ are defined as follows:

$$f_1(x) = \begin{cases} g(x) & \text{if } x < g(x) \\ \lceil x + 3 \rceil & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} g(x) & \text{if } x < g(x) \text{ and } ([0,3],[x,g(x)]) \notin R_O \\ \lceil x + 3 \rceil & \text{if } x \geq g(x) \text{ and } x \notin (0,3) \\ 3 & \text{otherwise} \end{cases}$$

It is not difficult to check that the above-defined functions meet the requirements for f_1 and f_2, as well as the additionally condition on f_2 ($f_2(w) \leq 3$ for each $0 < w < 3$).

Thus, Z is an $\mathsf{ABD}\overline{\mathsf{ABE}}$-bisimulation. On the other hand, Z does not preserve the relation induced by $\langle O \rangle$. Consider the interval $[0,3]$ in M_{f_1} and the interval $[0,3]$ in M_{f_2}. It is immediate to see that these two intervals are Z-related. However, $M_{f_1}, [0,3] \Vdash \langle O \rangle p$, but $M_{f_2}, [0,3] \Vdash \neg\langle O \rangle p$. This allows us to conclude that Z does not preserve the relation induced by $\langle O \rangle$, and thus $\langle O \rangle$ is not definable in terms of the fragment $\mathsf{ALBD}\overline{\mathsf{ALBEDO}}$.

A completely symmetric argument can be applied for the completeness proof of $\langle \overline{O} \rangle$. □

2.2.8 Harvest

The proof of Theorem 2.1.1 follows immediately from the previous lemmas.

We have used the equations in Table 2.1 as the basis of a simple program that identifies and counts all expressively different fragments of HS with respect to the strict semantics. Using that program, we have found that, under our assumptions (strict semantics, over the class of all linear orders) there are exactly 1347 genuine, that is, expressively different, fragments out of $2^{12} = 4096$ different subsets of HS operators.

2.3 Conclusions

In this chapter, we have obtained a sound, complete, and optimal set of inter-definability equations among all modal operators in HS, thus providing a characterization of the relative expressive power of all interval logics definable as fragments of HS. Such a classification has a number of important applications. As an example, it allows one to properly identify the (small) set of HS fragments for which the decidability of the satisfiability problem is still an open problem (see Chapter 3).

It should be emphasized that the set of inter-definability equations listed in Table 2.1 and the resulting classification do not apply if the non-strict semantics is considered. For instance, if the non-strict semantics is assumed, it is shown in [101] that $\langle A \rangle$ (resp., $\langle \overline{A} \rangle$) can be defined in $\overline{\mathsf{B}}\mathsf{E}$ (resp., $\mathsf{B}\overline{\mathsf{E}}$). Moreover, there is no guarantee about the completeness of the set of equations in Table 2.1 if the semantics is restricted to specific classes of linear orders. For instance, in discrete linear orders, $\langle A \rangle$ can be defined in $\overline{\mathsf{B}}\mathsf{E}$ as follows: $\langle A \rangle p \equiv \varphi(p) \vee \langle E \rangle \varphi(p)$, where $\varphi(p)$ is a shorthand for $[E]\bot \wedge \langle \overline{B} \rangle([E][E]\bot \wedge \langle E \rangle(p \vee \langle \overline{B} \rangle p))$; likewise, $\langle \overline{A} \rangle$ is definable in $\mathsf{B}\overline{\mathsf{E}}$. As another example, in dense linear orders, $\langle L \rangle$ can be defined in DO as follows: $\langle L \rangle p \equiv \langle O \rangle(\langle O \rangle \top \wedge [O](\langle O \rangle p \vee \langle D \rangle p \vee \langle D \rangle \langle O \rangle p))$; likewise, $\langle \overline{L} \rangle$ is definable in $\mathsf{D}\overline{\mathsf{O}}$. (In view of these last two inter-definabilities, Lemma 2.2.1 cannot be proved by defining a bisimulation between models over the reals.)

The classification of the expressiveness of HS fragments with respect to the non-strict semantics, as well as over specific classes of linear orders, is still missing.

3

Undecidable fragments of HS

A well-known result, included in the seminal work of Halpern and Shoham [59], concerns the strong undecidability of HS. More precisely, HS turns out to be undecidable under very weak assumptions on the class of interval structures: undecidability holds for any class of interval structures over linear orderings that contains at least one linear ordering with an infinite ascending (or descending) chain of points, thus including all natural numerical time-flows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

For a long time, such a sweeping undecidability result has discouraged attempts for practical applications and further research on interval logics. A renewed interest in the area has been recently stimulated by the discovery of some interesting decidable fragments of HS [23, 25, 27, 28, 31]. As an effect, the identification of expressive enough, decidable fragments of HS has been added to the current research agenda for (interval) temporal logic. While the algebra of Allen's relations, the so-called Allen's Interval Algebra, has been extensively studied and completely classified from the point of view of computational complexity [71] (tractability/intractability of the consistency problem of fragments of Interval Algebra), the characterization of decidable/undecidable fragments of HS is considerably harder.

This chapter aims to contribute to the identification of the boundary between decidability and undecidability of the satisfiability problem of HS fragments. It summarizes known positive and negative results, it presents the main techniques so far exploited in order to prove undecidability, and it establishes new undecidability results. A complete picture of the state of the art about the classification of HS fragments with respect to the satisfiability problem can be found in Appendix A. To the web page `http://itl.dimi.uniud.it/content/logic-hs`, it is also possible to run a collection of web tools, allowing one to verify the status (decidable/undecidable/unknown) of any specific fragment with respect to the satisfiability problem, over various classes of linear orders (all, dense, discrete, and finite) and considering both strict and non-strict semantics, as well as to compare relative expressive power of any pair of HS fragments.

3.1 State of the art: decidable and undecidable fragments

The very first decidability results about interval logics were based on severe restrictions of the interval-based semantics, essentially reducing it to a point-based one. Such restrictions include *locality*, according to which all atomic propositions are evaluated point-wise, meaning that their truth over an interval is defined as truth at its initial point, and *homogeneity*, according to which truth of a formula over an interval implies truth of that formula over every sub-interval. By imposing such constraints, decidability of interval logics can be proved by embedding it into linear temporal logic [87, 101]. Decidability can also be achieved by constraining the class of temporal structures over which the logic is interpreted. This is the case with *split-structures*, where any interval can be "chopped" in at most one way. The decidability of various interval logics, including HS, interpreted over split-structures, has been proved by embedding them into first-order decidable theories of time granularities [84].

For some simple fragments of HS, like $\mathsf{B}\overline{\mathsf{B}}$ and $\mathsf{E}\overline{\mathsf{E}}$, decidability has been obtained without any semantic restriction by means of direct translation to the point-based semantics and reduction to decidability of respective point-based temporal logics [54]. In any of these logics, one of the endpoints of every interval related to the current one remains fixed, thereby reducing the interval-based semantics to the point-based one by mapping every interval of the generated sub-model to its non-fixed endpoint. Consequently, these fragments can be polynomially translated to the linear time Temporal Logic with Future and Past $\mathsf{TL}[\mathsf{F},\mathsf{P}]$, thus proving that they are NP-complete when interpreted on the class of all linearly ordered sets or on any of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ [48, 54].

Decidability results for fragments of HS with unrestricted interval-based semantics, non-reducible to point-based one, have been recently obtained by means of a translation method. This is the case with $\mathsf{A}\overline{\mathsf{A}}$, also known as *Propositional Neighborhood Logic* (PNL) [53][1]. In [24, 25], decidability in NEXPTIME of PNL has been proved by translation to the two-variable fragment of first-order logic with binary relations over linear domains $\mathsf{FO}^2[=,<]$ and reference to the NEXPTIME-complete decidability result for $\mathsf{FO}^2[=,<]$ by Otto [90] (for proof details and for NEXPTIME-hardness, we refer the reader to [24, 25]). Otto's results, and consequently the decidability of PNL, apply not only to the class of all linear orders, but also to some natural sub-classes of it, such as the class of all well-founded linear orders, the class of all finite linear orders, and $\mathbb{N}$. On the basis of such results, in [27, 28, 31] optimal tableau-based decision procedures for such logics have been devised for a number of different classes of orderings. Very recent works extend the decidability of A to $\mathsf{AB}\overline{\mathsf{B}}\overline{\mathsf{L}}$ [32, 30] (the decidability of $\overline{\mathsf{A}}\mathsf{E}\overline{\mathsf{E}}\mathsf{L}$ immediately follows by symmetry) and,

[1]Since L and $\overline{\mathsf{L}}$ are definable in $\mathsf{A}\overline{\mathsf{A}}$, decidability of this fragment actually implies decidability of $\mathsf{A}\overline{\mathsf{A}}\mathsf{L}\overline{\mathsf{L}}$

whenever finite linear orderings are considered, the one of PNL to $\mathsf{AB\overline{B}\overline{A}}$ (and, by symmetry, to $\mathsf{AE\overline{E}\overline{A}}$) [83].

Finally, decidability of some fragments of HS has been demonstrated by taking advantage of the small model property with respect to suitable classes of truth-preserving *pseudo-models*. This method has been successfully applied to the logics of sub-intervals D, interpreted over dense linear orderings [21, 22, 23], subsequently extended to the maximal logic $\mathsf{B\overline{B}D\overline{D}L\overline{L}}$ (and, by symmetry, $\mathsf{E\overline{E}D\overline{D}L\overline{L}}$), interpreted over $\mathbb{Q}$ [81, 82].

The first undecidability result comes directly from the seminal work of Halpern and Shoham [59] and states the undecidability of whole HS, interpreted over almost all interesting classes of linearly ordered sets. In particular, it applies over all natural numerical time-flows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

Theorem 3.1.1 ([59]). *The satisfiability problem for* HS *is undecidable over any class of linear orderings that contains at least one linear ordering with an infinite ascending or descending sequence of points.*

In [73], Lodaya proved that the very small fragment BE is enough to yield undecidability. Recently, several fragments have been proved undecidable [17, 16, 19, 75, 78]. In the following, we will present a number of undecidability results, actually improving most of the previous ones (except for the undecidability result for the fragment D, given in [78]).

Let $\langle X \rangle$ and $\langle Y \rangle$ be two HS operators, we denote by $\mathsf{X}^*\mathsf{Y}^*$ the set of fragments $\mathsf{X}^*\mathsf{Y}^* = \{\mathsf{XY}, \mathsf{X\overline{Y}}, \mathsf{\overline{X}Y}, \mathsf{\overline{XY}}\}$. As a matter of fact, all known undecidability results can be derived from the following four main results:

- undecidability of $\mathsf{A}^*\mathsf{D}^*$,
- undecidability of $\mathsf{B}^*\mathsf{E}^*$,
- undecidability of O and its inverse $\overline{\mathsf{O}}$,
- undecidability of D and its inverse $\overline{\mathsf{D}}$, when interpreted over classes of discrete linear orderings.

In this chapter, we show the undecidability of the first three groups of fragments, when either strict or non-strict semantics is considered, by means of reductions from (suitable versions of) the tiling problem. As for the fragments D and $\overline{\mathsf{D}}$, its undecidability proof (over classes of discrete linear orderings) will be officially reported in a forthcoming publication (for the moment, we refer to the personal communication of Marcinkowski and Michaliszyn [78]).

In the literature, the undecidability proofs are usually given by assuming the existence of a linear ordering containing an infinite (ascending or descending) sequence of points. Another contribution of this chapter is showing how to relax also this (weak) assumption and to generalize such results to classes of finite linear orderings. As a consequence, our results hold over any class of linear orderings containing at least a linear order with an arbitrary large sequence of points. In particular, they hold over the class of all finite linear orderings as well as over the classical orderings based on $\mathbb{N}$, $\mathbb{Z}^-$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

3.1.1 Undecidability: the reduction technique

The usual way to show the undecidability of a logic is by means of reductions from undecidable problems. For our purposes, we will mainly exploit techniques based on a reduction from the *Octant Tiling Problem*. This is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile the second octant of the integer plane $\mathcal{O} = \{(i,j) : i,j \in \mathbb{N} \wedge 0 \leq i \leq j\}$. For every tile type $t_i \in \mathcal{T}$, let $right(t_i)$, $left(t_i)$, $up(t_i)$, and $down(t_i)$ be the colors of the corresponding sides of t_i. To solve the problem, one must find a function $f : \mathcal{O} \rightarrow \mathcal{T}$ such that

$$right(f(n,m)) = left(f(n+1,m))$$

and

$$up(f(n,m)) = down(f(n,m+1)).$$

In [13], the undecidability of the $\mathbb{N} \times \mathbb{N}$ Tiling Problem is proved from that of the $\mathbb{Z} \times \mathbb{Z}$ Tiling Problem (known to be co-r.e. complete by a reduction from the halting problem of a Turing machine), by a simple application of the König's Lemma:

Theorem 3.1.2 ([13]). *A set $\mathcal{T}$ of tile types tiles $\mathbb{Z} \times \mathbb{Z}$ if and only if it tiles $\mathbb{N} \times \mathbb{N}$.*

In the same way, it is possible to prove the following corollary:

Corollary 3.1.3. *A set $\mathcal{T}$ of tile types tiles $\mathbb{N} \times \mathbb{N}$ if and only if it tiles $\mathcal{O}$.*

Proof. The proof follows the one in [13] for Theorem 3.1.2 above. If $\mathcal{T}$ tiles $\mathbb{N} \times \mathbb{N}$, it is easy to verify that $\mathcal{T}$ also tiles $\mathcal{O}$. As for the inverse, suppose that $\tau : \mathcal{O} \rightarrow \mathcal{T}$ is a tiling of $\mathcal{O}$ by $\mathcal{T}$. There exists at least one tile type $t \in \mathcal{T}$ such that for each n there exists $i > n$ with $\tau(0,i) = t$. Now, for each $k \in \mathbb{N}$, let $S_k = \{0, \ldots, k\} \times \{0, \ldots, k\}$ be the square whose edge has length $k+1$ and whose bottom-left corner corresponds to the origin of the first quadrant. We define a finitely branching tree in which the nodes at depth k represent all the correct tilings τ_k of S_k by $\mathcal{T}$ such that $\tau(0,0) = t$. The root node represents the unique tiling of S_0 with such a requirement ($\tau_0(0,0) = t$) and the children of a tiling τ_k are the possible extensions to tilings τ_{k+1} of S_{k+1}. Notice that for each tiling t_k of S_k there are finitely many extensions τ_{k+1} for S_{k+1}, since $\mathcal{T}$ is a finite set. This tree contains paths of any finite length. By König's Lemma, it also contains an infinite path from the root, which means that $\mathcal{T}$ tiles $\mathbb{N} \times \mathbb{N}$. □

Given an instance of the Octant Tiling Problem $OTP(\mathcal{T})$, where $\mathcal{T}$ is the finite set of tiles types, a reduction from $OTP(\mathcal{T})$ to the satisfiability problem for a logic $\mathcal{L}$ consists of the construction of a formula $\Phi_{\mathcal{T}}$, parametric in $\mathcal{T}$ and belonging to the language of $\mathcal{L}$, such that $\Phi_{\mathcal{T}}$ is satisfiable if and only if $\mathcal{T}$ tiles $\mathcal{O}$. Let us fix an arbitrary finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ and assume that the set of atomic propositions $\mathcal{AP}$ is finite (but arbitrary) and contains, inter alia, the following propositional variables: u, $*$, Id, tile, $\mathsf{t_1}, \ldots, \mathsf{t_k}$, and up_rel. The generic structure of the proof is the following. First, for any given HS fragment $\mathcal{L}$ and starting interval $[a, b]$, we consider a (possibly infinite) set of intervals $\mathcal{G}_{[a,b]}$, containing a restricted set of intervals that can be reached by the modalities of $\mathcal{L}$ starting from $[a, b]$. Intuitively, $\mathcal{G}_{[a,b]}$ is the set of intervals on which we focus our attention. Even if it is not explictly clarified, when we talk about intervals we always mean intervals belonging to $\mathcal{G}_{[a,b]}$. In addition, we define, in terms of the modalities of $\mathcal{L}$, the (derived) modal operator $[G]$ (*global operator*), such that $[G]\varphi$ holds over the interval $[a, b]$ if and only if φ holds over each interval in $\mathcal{G}_{[a,b]}$. Then, the proof is based on the following main steps:

- definition of the u-*chain*: we set our framework by forcing the existence of a unique infinite chain of u-intervals (u-chain, for short) on the linear ordering. They will be used as cells to arrange the tiling. We also have to provide a way to step from an u-interval to its immediate successor in the chain;

- definition of the Id-*chain*: the octant is encoded by means of a unique infinite sequence of Id-intervals (Id-chain, for short), each of them representing a row of the octant. An Id-interval is composed by a sequence of u-intervals; each u-interval is used either to represent a part of the plane or to separate two rows. In the former case it is labelled with tile, while in the latter case it is labelled with $*$;

- encoding of the *above-neighbor* and *right-neighbor* relations, connecting each tile in the octant with, respectively, the one immediately above it and the one at its right, if any. The encoding of such relations must be done in such a way that the following *commutativity property* holds.

In the following, if two tiles t_1 and t_2 are connected through the above-neighbor (resp., right-neighbor) relation, then we will simply say that t_1 is *above-connected* (resp., *right-connected*) to t_2. We use the same expression when we refer to tile-intervals, that is, we say that two tile-intervals are above-connected (resp., right-connected) if they encode tiles of the octant that are above-connected (resp., right-connected).

Definition 3.1.4 (commutativity property)**.** Given two tile-intervals $[c, d]$ and $[e, f]$, if there exists a tile-interval $[d_1, e_1]$, such that $[c, d]$ is right-connected to $[d_1, e_1]$ and $[d_1, e_1]$ is above-connected to $[e, f]$, then there exists also a tile-interval $[d_2, e_2]$ such that $[c, d]$ is above-connected to $[d_2, e_2]$ and $[d_2, e_2]$ is right-connected to $[e, f]$.

Even if the frame of the proof is basically the same for most of our results, the nature of the different modalities featured by different logics substantially influences the technicalities of the reduction. As a consequence, for each different logic, we possibly exploit additional propositional letters besides the above mentioned ones. From now on in this chapter, for each formula of the kind $[G](\varphi \to \varphi_1 \wedge \ldots \wedge \varphi_n)$, identified by the number (k), we use the number identifying the formula followed by a Roman number in order to restrict the consequent of the implication to the conjunct pointed by the Roman numeral, that is, $(k\text{-I})$ denotes the formula $[G](\varphi \to \varphi_1)$, $(k\text{-II})$ denotes the formula $[G](\varphi \to \varphi_2)$, and so on.

3.2 The fragments AD, $\mathsf{A\overline{D}}$, $\mathsf{\overline{A}D}$, and $\mathsf{\overline{AD}}$

In this section, we present a reduction from the Octant Tiling Problem to the satisfiability problem for each fragment belonging to the set $\mathsf{A^*D^*}$. At the end of the section, we also show how to enforce the undecidability results to deal with classes of finite linear orders, by exploiting a reduction from the Finite Tiling Problem.

3.2.1 The fragment AD

Let $[a, b]$ be a generic interval. The set $\mathcal{G}_{[a,b]}$ contains the interval $[a, b]$ and all the intervals $[c, d]$, with $c \geq b$. The global operator $[G]$ is defined as:

$$[G]p \equiv p \wedge [A]p \wedge [A][A]p.$$

Definition of the u-chain. In order to build a chain of u-intervals, we need to chop each u-interval into a pair $\langle \mathsf{u}_1\text{-interval}, \mathsf{u}_2\text{-interval}\rangle$. The propositional letters u_1 and u_2 are instrumental to the construction of the u-chain. The following formulae define the u-chain:

$$\neg\mathsf{u} \wedge \neg\mathsf{u}_1 \wedge \neg\mathsf{u}_2 \wedge \langle A\rangle\mathsf{u} \wedge [G](\mathsf{u} \to \langle A\rangle\mathsf{u}) \quad (3.1)$$

$$[G](\langle A\rangle\mathsf{u} \leftrightarrow \langle A\rangle\mathsf{u}_1) \quad (3.2)$$

$$[G](\mathsf{u} \to \neg\mathsf{u}_1 \wedge \neg\langle D\rangle\langle A\rangle\mathsf{u} \wedge \neg\langle D\rangle\mathsf{u}_1 \wedge \neg\langle D\rangle\mathsf{u}_2 \wedge \langle D\rangle\top) \quad (3.3)$$

$$[G](\mathsf{u}_1 \to \langle A\rangle\mathsf{u}_2 \wedge \neg\langle D\rangle\langle A\rangle\mathsf{u}) \quad (3.4)$$

$$[G](\mathsf{u}_2 \to \langle A\rangle\mathsf{u} \wedge \neg\langle D\rangle\langle A\rangle\mathsf{u} \wedge \neg\langle D\rangle\mathsf{u}_1 \wedge \langle D\rangle\top) \quad (3.5)$$

$$[G](\langle A\rangle\mathsf{u}_1 \to \neg\langle A\rangle\mathsf{u}_2) \quad (3.6)$$

$$(3.1) \wedge \ldots \wedge (3.6) \quad (3.7)$$

It is worth pointing out that the different role of u_2 and u_1 reflects the asymmetry of the logic, which includes the future operator $\langle A\rangle$ but not the past operator $\langle \overline{A}\rangle$.

Lemma 3.2.1. *Let $M, [a, b] \Vdash (3.7)$. Then, there exists an infinite sequence of points $b = b_0 < b_1 < \ldots$ in M, such that for each $i \geq 0$:*

1. *$M, [b_i, b_{i+1}] \Vdash \mathsf{u}$;*

2. there exists c_i such that $b_i < c_i < b_{i+1}$, $M, [b_i, c_i] \Vdash \mathsf{u}_1$, and $M, [c_i, b_{i+1}] \Vdash \mathsf{u}_2$;

and no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies u, unless $c > b_i$ for each i.

Proof. The existence of such a sequence is guaranteed by (3.1). We show that for each interval $[b_i, b_{i+1}]$ in the sequence, there exists c_i such that $b_i < c_i < b_{i+1}$, $[b_i, c_i]$ satisfies u_1, and $[c_i, b_{i+1}]$ satisfies u_2.

Consider any interval $[b_i, b_{i+1}]$ of the above sequence. Since u holds, by (3.3-V) there is at least one point d_i such that $b_i < d_i < b_{i+1}$. By (3.2), for each $i \geq 0$ there exists a point $c_i > b_i$ such that $[b_i, c_i]$ satisfies u_1. By (3.3-I), we have that $c_i \neq b_{i+1}$. Suppose, by contradiction, that $c_i > b_{i+1}$. Then, $[d_i, b_{i+1}]$ is strictly contained in the u_1-interval $[b_i, c_i]$ and it meets the u-interval $[b_{i+1}, b_{i+2}]$, contradicting (3.4-II). So, it must be that $b_i < c_i < b_{i+1}$. Now, by (3.4-I), there exists $f_i > c_i$ such that $[c_i, f_i]$ satisfies u_2. We show that $f_i = b_{i+1}$. Suppose, by contradiction, that $f_i < b_{i+1}$. In this case, we have a contradiction with (3.3-IV). Let us suppose that $f_i > b_{i+1}$. By (3.5-IV), the u_2-interval $[c_i, f_i]$ must contain an interval $[g_i, h_i]$. We show that there is no way to properly locate $[g_i, h_i]$. If $g_i < b_{i+1}$, then $[g_i, b_{i+1}]$ is strictly contained in the u_2-interval $[c_i, f_i]$ and meets the u-interval $[b_{i+1}, b_{i+2}]$, which contradicts (3.5-II). If $g_i \geq b_{i+1}$, then $b_{i+1} < h_i < f_i$. To show that such an alternative is inconsistent, we compare the relative position of c_{i+1} and f_i as follows:

- if $c_{i+1} > f_i$, then the interval $[h_i, f_i]$ is strictly contained in the u_1-interval $[b_{i+1}, c_{i+1}]$, and since f_i starts a u-interval (by (3.5-I)), this contradicts (3.4-II);
- if $c_{i+1} = f_i$, then, by (3.5-I) and (3.2), f_i starts a u_1-interval, and by (3.4-I), it also starts a u_2-interval, which contradicts (3.6);
- if $c_{i+1} < f_i$, then we have that the u_1-interval $[b_{i+1}, c_{i+1}]$ is contained in the u_2-interval $[c_i.f_i]$, contradicting (3.5-III).

Hence, it is not possible $f_i > b_{i+1}$ and we can conclude that $f_i = b_{i+1}$.

To conclude the proof, we show that there exists no other u-interval $[c, d] \in \mathcal{G}_{[a,b]}$, unless $c > b_i$ for each i. Suppose, by contradiction, that there exists such an interval $[c, d]$ and let c_1 be the point such that $[c, c_1]$ satisfies u_1 and $[c_1, d]$ satisfies u_2. We distinguish the following cases:

- if $b_i < c < b_{i+1}$ for some i, then there are the following possibilities:
  - if $d \leq b_{i+1}$, then the u-interval $[b_i, b_{i+1}]$ contains the u_1-interval $[c, c_1]$, contradicting (3.3-III);
  - if $d > b_{i+1}$, then we distinguish the following cases:
    * if $d \geq b_{i+2}$, then we have that the u-interval $[c, d]$ contains the u_1-interval $[b_{i+1}, c_{i+1}]$, contradicting (3.3-III);
    * if $d < b_{i+2}$, then we distinguish the following cases:

- if $c_1 < b_{i+1}$, then we have that the u-interval $[b_i, b_{i+1}]$ contains the u_1-interval $[c, c_1]$, contradicting (3.3-III);
- if $c_1 = b_{i+1}$, then we have both a u_1- and a u_2-interval starting at c_1, contradicting (3.6);
- if $c_1 > b_{i+1}$, then we have that the u-interval $[b_{i+1}, b_{i+2}]$ contains the u_2-interval $[c_1, d]$, contradicting (3.3-IV);

- if $c = b_i$ for some i, then $d \neq b_{i+1}$. If $d < b_{i+1}$, then the u-interval starting at d leads to a contradiction by exploiting the argument of the previous item, since $b_i < d < b_{i+1}$. If $d > b_{i+1}$, then we can again exploit the argument of the previous item, since the u-interval $[b_{i+1}, b_{i+2}]$ is such that $c < b_{i+1} < d$.

This concludes the proof. □

Definition of the Id-chain. The following bunch of formulae defines the Id-chain:

$$[G]((\mathsf{u} \leftrightarrow (* \vee \mathsf{tile})) \wedge (* \rightarrow \neg\mathsf{tile})) \tag{3.8}$$

$$\neg\mathsf{Id} \wedge \langle A\rangle(* \wedge \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle(* \wedge [G](* \rightarrow \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle\mathsf{tile}))))) \tag{3.9}$$

$$[G](\langle A\rangle\mathsf{Id} \leftrightarrow \langle A\rangle*) \tag{3.10}$$

$$[G](\mathsf{Id} \rightarrow \langle A\rangle * \wedge \neg\langle D\rangle*) \tag{3.11}$$

$$(3.8) \wedge \ldots \wedge (3.11) \tag{3.12}$$

Lemma 3.2.2. *Let $M, [a, b] \Vdash (3.7) \wedge (3.12)$, and let $b = b_1^0 < b_1^1 < \ldots < b_1^{k_1} = b_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$ be the sequence of points defined by Lemma 3.2.1. Then, for each $j \geq 1$, we have:*

1. *$M, [b_j^0, b_j^{k_j}] \Vdash \mathsf{Id}$;*
2. *$M, [b_j^0, b_j^1] \Vdash *$;*
3. *$M, [b_j^i, b_j^{i+1}] \Vdash \mathsf{tile}$ for each $0 < i < k_j$.*

Furthermore, we have that $k_1 = 2$, $k_j > 2$ for each $j > 1$, and no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies Id (resp., $*$, tile), unless $c > b_j^i$ for each $i, j > 0$.

Proof. We first prove the item 1, that is $M, [b_j^0, b_j^{k_j}] \Vdash \mathsf{Id}$ for each $j \geq 1$. By (3.8), (3.10) (left-to-right direction) and (3.11-I), each Id-interval starts and finishes with u-intervals. The existence of the Id-chain, starting from $b_0 = b$, is guaranteed by (3.9), (3.10) (right-to-left direction), and (3.11-I). As for the structure of the Id-intervals:

- the first u-interval of each Id-interval ($[b_j^0, b_j^1]$) is a $*$-interval by (3.10) (left-to-right direction), proving item 2;
- as for item 3, suppose, by contradiction, that there is a u-interval $[b_j^i, b_j^{i+1}]$, with $1 \leq i \leq k_j - 1$ (i.e., the u-interval is not the first one of the j-th Id-interval), satisfying $*$. Then, we distinguish two cases:

– $i < k_j - 1$. In this case, $[b_j^i, b_j^{i+1}]$ is not the last u-interval of the considered ld-interval, which means that it is a $*$-interval strictly contained in the ld-interval $[b_j^0, b_j^{k_j}]$, and this contradicts (3.11-II);

– $i = k_j - 1$. In this case, the $*$-interval $[b_j^i, b_j^{i+1}]$ meets the $*$-interval $[b_{j+1}^0, b_{j+1}^1]$, contradicting (3.9).

It remains to be shown that no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ (unless $c > b_j^i$ for each $i, j > 0$) satisfies ld (resp., $*$, tile). Let us consider the case in which there exists an interval $[c, d] \in \mathcal{G}_{[a,b]}$ such that $[c, d] \neq [b_j^0, b_j^{k_j}]$ for each j and $[c, d]$ satisfies ld. By Lemma 3.2.1, (3.8), and (3.11-I) we have that $d = b_{j'}^{i'}$ for some $j' > 0$, $i' \geq 0$ and that $[b_{j'}^{i'}, b_{j'}^{i'+1}]$ satisfies $*$. We consider the following cases:

- if $c = b_j^0$ for some j, then $d \neq b_j^{k_j}$. If $d < b_j^{k_j}$, then we have that $d = b_j^i$ for some $i > 0$ and that $[b_j^i, b_j^{i+1}]$ satisfies both $*$ (by (3.11-I)) and tile (by item 3), contradicting (3.8). Otherwise, if $d > b_j^{k_j}$, then we distinguish two cases:
  – if $d = b_{j+1}^1$, then $[b_{j+1}^1, b_{j+1}^2]$ and $[b_{j+1}^0, b_{j+1}^1]$ satisfy $*$ by (3.11-I), contradicting (3.9);
  – if $d > b_{j+1}^1$, then the ld-interval $[c, d]$ contains the $*$-interval $[b_{j+1}^0, b_{j+1}^1]$, contradicting (3.11-II);
- if $c \neq b_j^0$ for each j, then we have, by Lemma 3.2.1, (3.8), and (3.10) (left-to-right direction), that $c = b_j^i$ for some $i, j > 0$ and that $[b_j^i, b_j^{i+1}]$ satisfies $*$. Since by item 3 we have that $[b_j^i, b_j^{i+1}]$ satisfies also tile, then we have a contradiction with (3.8).

Now, let us show that no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ (unless $c > b_j^i$ for each $i, j > 0$) satisfies $*$. Suppose that there exists an interval $[c, d] \in \mathcal{G}_{[a,b]}$ such that $[c, d] \neq [b_j^0, b_j^1]$ for each j and $[c, d]$ satisfies $*$. By (3.8) and Lemma 3.2.1 we have that $[c, d] = [b_j^i, b_j^{i+1}]$ for some $i, j > 0$. Since by item 3 we have that $[b_j^i, b_j^{i+1}]$ also satisfies tile, then we have a contradiction with (3.8). We can use a similar argument in order to show that no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ (unless $c > b_j^i$ for each $i, j > 0$) satisfies tile. Finally, we can conclude that $k_1 = 2$ and that $k_j > 2$ for each $j > 1$ by (3.9). □

Fig. 3.1 shows how the encoding of the octant plane is done exploiting the u-chain and ld-chain. As a matter of fact, so far we have only encoded the levels of the octant by means of the ld-intervals, the first one of which has exactly one tile, while the rest of them have at least two tiles. Now, we encode the neighbor relations, connecting each tile with its above-neighbor and with its right-neighbor, if any, in the octant. This will allow us to force the j-th ld-interval to contain exactly j tile-intervals.

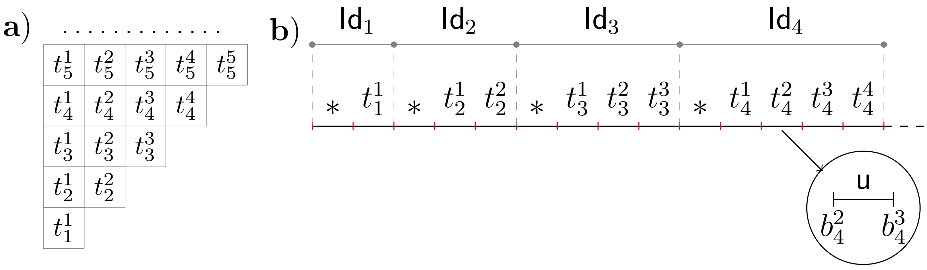


Figure 3.1: The encoding of the Octant Tiling Problem: **a)** cartesian representation; **b)** interval representation

Right-neighbor relation. The right-neighbor relation connects two consecutive tiles at the same level. We say that two tile-intervals $[b^i_j, b^{i+1}_j]$ and $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ are *right-connected* if and only if $j' = j$ and $i' = i + 1$. The encoding of the right-neighbor relation is trivial by exploiting the adjacency between two consecutive tiles belonging to the same level, that is, from a tile-interval $[b^i_j, b^{i+1}_j]$ it is possible to refer to the tile-interval $[b^{i+1}_j, b^{i+2}_j]$, if any, with which it is right-connected, simply by means of the operator $\langle A\rangle$.

Above-neighbor relation. The above-neighbor relation connects each tile with its above neighbor in the octant. If $[b^i_j, b^{i+1}_j]$ and $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ are, respectively, the i-th tile-interval of the j-th Id-interval and the i'-th tile-interval of the j'-th Id-interval, then we say that $[b^i_j, b^{i+1}_j]$ is *above-connected* to $[b^{i'}_{j'}, b^{i'+1}_{j'}]$ if and only if $j' = j+1$ and $i = i'$. In order to encode the above-neighbor relation, we use the propositional letter up_rel. More precisely, the up_rel-interval $[b^{i+1}_j, b^i_{j+1}]$ connects the tile-interval $[b^i_j, b^{i+1}_j]$ with the tile-interval $[b^i_{j+1}, b^{i+1}_{j+1}]$. Let $[b^i_j, b^{i+1}_j]$ be a tile-interval, we say that it is an *odd* (resp., *even*) tile-interval if i is odd (resp., even). The relation up_rel is encoded by means of the additional propositional letters $\mathsf{up\_rel_o}$ (connecting odd tile-intervals) and $\mathsf{up\_rel_e}$ (connecting even tile-intervals). We have that $\mathsf{up\_rel} \leftrightarrow \mathsf{up\_rel_o} \oplus \mathsf{up\_rel_e}$, where $\oplus$ denotes the "exclusive or". As shown in Fig. 3.2, the intervals $\mathsf{up\_rel_o}$ and $\mathsf{up\_rel_e}$ alternate (*strict interleaving property*), namely, if $[b^i_j, b^{i+1}_j]$ is a tile-interval such that b^{i+1}_j is the starting point of a $\mathsf{up\_rel_o}$-interval (resp., $\mathsf{up\_rel_e}$-interval), then

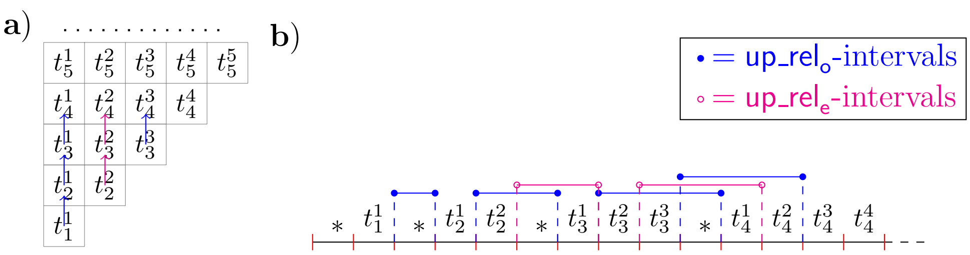


Figure 3.2: The encoding of the above-neighbor relation in the fragment AD: $\mathsf{up\_rel_o}$- and $\mathsf{up\_rel_e}$-intervals alternate

the next tile-interval $[b_j^{i+1}, b_j^{i+2}]$, if any, is connected to its above-neighbor by means of a $\mathsf{up\_rel_e}$-interval (resp., $\mathsf{up\_rel_o}$-interval). Moreover, we prevent any two up_rel-intervals to start or to end at the same point and to be contained one into the other. Finally, for any level, each tile-interval must be above-connected to some tile-interval of the next level and for each tile-interval, but the last one of the level, there must be some tile-interval of the previous level, if any, which is above-connected to it (formula (3.26) below). This guarantees that each level has exactly one tile-interval more than the previous one. Let $\alpha, \beta \in \{\text{o}, \text{e}\}$, with $\alpha \neq \beta$. The following formulae encode the properties of the above-neighbor relation:

$$\neg \mathsf{up\_rel} \wedge \neg\langle A\rangle \mathsf{up\_rel} \wedge \langle A\rangle(* \wedge \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle(* \wedge \mathsf{up\_rel_o}))) \tag{3.13}$$
$$[G](\mathsf{up\_rel} \leftrightarrow (\mathsf{up\_rel_o} \vee \mathsf{up\_rel_e})) \tag{3.14}$$
$$[G](\langle A\rangle \mathsf{up\_rel_o} \rightarrow \neg\langle A\rangle \mathsf{up\_rel_e}) \tag{3.15}$$
$$[G](\mathsf{tile} \rightarrow \langle A\rangle \mathsf{up\_rel}) \tag{3.16}$$
$$[G](\langle A\rangle \mathsf{up\_rel} \rightarrow \langle A\rangle \mathsf{u}) \tag{3.17}$$
$$[G](\mathsf{u} \wedge \langle A\rangle \mathsf{up\_rel} \rightarrow \mathsf{tile}) \tag{3.18}$$
$$[G](\mathsf{up\_rel} \rightarrow \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle \mathsf{tile})) \tag{3.19}$$
$$[G](\mathsf{up\_rel}_\alpha \rightarrow \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle \mathsf{up\_rel}_\alpha)) \tag{3.20}$$
$$[G](\langle A\rangle * \rightarrow [A](\mathsf{up\_rel} \rightarrow \neg\langle D\rangle\langle A\rangle *)) \tag{3.21}$$
$$[G]((\mathsf{up\_rel} \rightarrow \neg\langle D\rangle \mathsf{Id}) \wedge (\mathsf{Id} \rightarrow \neg\langle D\rangle \mathsf{up\_rel})) \tag{3.22}$$
$$[G](\langle A\rangle \mathsf{up\_rel}_\alpha \wedge \langle A\rangle \mathsf{tile} \rightarrow \langle A\rangle(\mathsf{tile} \wedge \langle A\rangle \mathsf{up\_rel}_\beta)) \tag{3.23}$$
$$[G](\mathsf{up\_rel} \rightarrow \neg\langle D\rangle \mathsf{up\_rel}) \tag{3.24}$$
$$[G](* \rightarrow \langle A\rangle(\mathsf{tile} \wedge [A](\mathsf{up\_rel} \rightarrow \neg\langle D\rangle *))) \tag{3.25}$$
$$\begin{aligned}[G](\langle A\rangle(\mathsf{u}_2 \wedge \langle A\rangle \mathsf{up\_rel}_\alpha) \rightarrow [A](\langle D\rangle \mathsf{up\_rel} \wedge \langle A\rangle(\mathsf{u}_2 \wedge \langle A\rangle \mathsf{tile})\\ \wedge \neg\langle D\rangle \mathsf{up\_rel}_\beta \rightarrow \neg\langle A\rangle(\mathsf{u}_2 \wedge \langle A\rangle \mathsf{up\_rel}_\beta)))\end{aligned} \tag{3.26}$$
$$(3.13) \wedge \ldots \wedge (3.26) \tag{3.27}$$

Lemma 3.2.3. *Let $M, [a, b] \Vdash (3.7) \wedge (3.12) \wedge (3.27)$, and let $b = b_1^0 < b_1^1 < \ldots b_1^{k_1} = b_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$ be the sequence of points defined in Lemma 3.2.2. Then, the following properties hold:*

1. *for each* up_rel*-interval $[c, d]$, there exist c', d' such that $[c', c]$ and $[d, d']$ are* tile*-intervals;*

2. *(strict interleaving property) for each* tile*-interval $[b_j^i, b_j^{i+1}]$, with $i < k_j - 1$, such that there exists an* $\mathsf{up\_rel_o}$*-interval (resp.,* $\mathsf{up\_rel_e}$*-interval) starting at b_j^{i+1}, there exists a* $\mathsf{up\_rel_e}$*-interval (resp.,* $\mathsf{up\_rel_o}$*-interval) starting at b_j^{i+2};*

3. *for every $j > 0$, $b_j^{k_j-1}$ is not the right endpoint of any* up_rel*-interval;*

4. *for each* up_rel*-interval $[b_j^i, b_{j'}^{i'}]$, with $1 < i \leq k_j$, we have that $j' = j + 1$.*

Proof. The proof proceeds point by point.

1. Let $[c,d]$ be an up_rel-interval. By (3.19), we have that there exists d' such that $[d,d']$ is a tile-interval. By (3.17), (3.18), and Lemma 3.2.1, there exists c' such that $[c',c]$ is a tile-interval as well.

2. Straightforwardly by (3.23).

3. Straightforwardly by (3.19).

4. Let $[b_j^i, b_{j'}^{i'}]$ be am up_rel-interval, with $1 < i \leq k_j$, and suppose, by contradiction, that $j' \neq j+1$. Two cases are possible:

   - $j' > j+1$. In this case, we have that:
     (a) if $i = k_j$, that is, $[b_j^{i-1}, b_j^i]$ is the last tile-interval of the j-th Id-interval, then $[b_j^{i-1}, b_j^i]$ satisfies $\langle A \rangle *$. So, (3.21) applies and we have a contradiction because the last tile-interval of the $(j+1)$-th Id-interval $[b_{j+1}^{k_{j+1}-1}, b_{j+1}^{k_{j+1}}]$ meets the $*$-interval $[b_{j+2}^0, b_{j+2}^1]$ and it is contained in the up_rel-interval $[b_j^i, b_{j'}^{i'}]$;
     (b) otherwise, we have that the up_rel-interval $[b_j^i, b_{j'}^{i'}]$ contains the Id-interval $[b_{j+1}^0, b_{j+1}^{k_{j+1}}]$, contradicting (3.22-I);
   - $j' = j$. In this case, it immediately follows that $i < i'$ and thus the up_rel-interval $[b_j^i, b_j^{i'}]$ is contained in the j-th Id-interval, contradicting (3.22-II).

□

Lemma 3.2.4. *Let* $M, [a,b] \Vdash (3.7) \wedge (3.12) \wedge (3.27)$*, and let* $b = b_1^0 < b_1^1 < \ldots b_1^{k_1} = b_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$ *be the sequence of points defined in Lemma 3.2.2. Each* tile*-interval* $[b_j^i, b_j^{i+1}]$ *is above-connected to exactly one* tile*-interval and, if* $i < k_j - 1$*, then there exists exactly one* tile*-interval which is above-connected to it.*

Proof. First of all, we observe that, by (3.16) and (3.19), each tile-interval is above-connected with at least one tile-interval.

As a second step, consider a tile-interval $[b_j^i, b_j^{i+1}]$ such that $i < k_j - 1$, which implies that $j > 1$ (since the first level contains only one tile-interval). Then, by Lemma 3.2.3, there exists $b_{j+1}^{i'}$ such that $[b_j^{i+1}, b_{j+1}^{i'}]$ satisfies up_rel. By (3.14) and (3.15), $[b_j^{i+1}, b_{j+1}^{i'}]$ satisfies exactly one between up_rel_o and up_rel_e, say up_rel_o (the other case is symmetric). We prove that there exists a point c such that $[c, b_j^i]$ satisfies up_rel. Suppose, by contradiction, that there is not such a point. The proof proceeds by induction on i:

- base case ($i = 1$): if $j = 2$, then by (3.13), $[b_1^2, b_2^1]$ satisfies up_rel (a contradiction). If $j > 2$, let us consider the interval $[b_{j-1}^0, b_{j-1}^1]$. By (3.16), (3.25), and Lemma 3.2.3, we have that $[b_{j-1}^2, b_j^1]$ satisfies up_rel (again a contradiction);

- inductive step ($i > 1$): by the strict interleaving property, there exists a point $b^{i''}_{j+1}$ such that $[b^i_j, b^{i''}_{j+1}]$ satisfies $\mathsf{up\_rel_e}$. Furthermore, by inductive hypothesis, there exists a point $b^{i'''}_{j-1}$ such that $[b^{i'''}_{j-1}, b^{i-1}_j]$ satisfies $\mathsf{up\_rel}$. In particular, $[b^{i'''}_{j-1}, b^{i-1}_j]$ satisfies $\mathsf{up\_rel_e}$, otherwise, by (3.20), there would be an $\mathsf{up\_rel_o}$- and an $\mathsf{up\_rel_e}$-interval starting at b^i_j, contradicting (3.15). Let $\overline{c}$ be the point such that $b^{i'''-1}_{j-1} < \overline{c} < b^{i'''}_{j-1}$ and that $[\overline{c}, b^{i'''}_{j-1}]$ satisfies $\mathsf{u_2}$. Similarly, let $\overline{d}$ be the point such that $b^i_j < \overline{d} < b^{i+1}_j$ and that $[\overline{d}, b^{i+1}_j]$ satisfies $\mathsf{u_2}$. We show that, by applying (3.26) to any interval ending in $\overline{c}$, we get a contradiction. Let us consider the interval $[\overline{c}, \overline{d}]$. It satisfies the following formulae:
  - $\langle D\rangle\mathsf{up\_rel}$: $[b^{i'''}_{j-1}, b^{i-1}_j]$ satisfies $\mathsf{up\_rel_e}$ and $\overline{c} < b^{i'''}_{j-1} < b^{i-1}_j < \overline{d}$;
  - $\langle A\rangle(\mathsf{u_2} \wedge \langle A\rangle\mathsf{tile})$: $[b^i_j, b^{i+1}_j]$ is not the last tile of the j-th Id-interval;
  - $\neg\langle D\rangle\mathsf{up\_rel_o}$. Suppose that there exists $[h, h']$ satisfying $\mathsf{up\_rel_o}$, with $\overline{c} < h < h' < \overline{d}$. We must distinguish among the following cases:
    - if $h = b^{i'''}_{j-1}$, then there are an $\mathsf{up\_rel_o}$- and an $\mathsf{up\_rel_e}$-interval starting both from h, contradicting (3.15);
    - if $h > b^{i'''}_{j-1}$ and $h' < b^{i-1}_j$, then the interval $[h, h']$ satisfies $\mathsf{up\_rel}$ and it is contained in the $\mathsf{up\_rel}$-interval $[b^{i'''}_{j-1}, b^{i-1}_j]$, contradicting (3.24);
    - if $h > b^{i'''}_{j-1}$ and $h' = b^{i-1}_j$, then we have that $[b^{i'''}_{j-1}, b^{i-1}_j]$ and $[h, b^{i-1}_j]$ are, respectively, an $\mathsf{up\_rel_e}$- and an $\mathsf{up\_rel_o}$-interval and, by (3.20), b^i_j starts both an $\mathsf{up\_rel_e}$- and an $\mathsf{up\_rel_o}$-interval, which is in contradiction with (3.15);
    - if $h > b^{i'''}_{j-1}$ and $h' = b^i_j$, then we have a contradiction with the hypothesis that there is no $\mathsf{up\_rel}$-interval ending at b^i_j.

    This allow us to conclude that there is no $\mathsf{up\_rel_o}$-interval contained in $[\overline{c}, \overline{d}]$.

  On the other hand, $[\overline{c}, \overline{d}]$ does not satisfy the formula $\neg\langle A\rangle(\mathsf{u_2} \wedge \langle A\rangle\mathsf{up\_rel_o})$, because $[\overline{d}, b^{i+1}_j]$ satisfies $\mathsf{u_2}$ and $[b^{i+1}_j, b^{i'}_{j+1}]$ is a $\mathsf{up\_rel_o}$-interval, contradicting (3.26).

So we can conclude that for each interval $[b^i_j, b^{i+1}_j]$ satisfying tile and such that $[b^i_j, b^{i+1}_j]$ is not the last tile-interval of the j-th Id-interval, there exists a point c such that $[c, b^i_j]$ satisfies $\mathsf{up\_rel}$.

Finally, we show that each tile-interval is above-connected to at most one tile-interval and there is at most one tile-interval above-connected to it. Suppose, by contradiction, that for some $[b^i_j, b^{i'}_{j+1}]$ and $[b^i_j, b^{i''}_{j+1}]$, with $b^{i'}_{j+1} < b^{i''}_{j+1}$ (the case in which $b^{i'}_{j+1} > b^{i''}_{j+1}$ is symmetric), we have that both $[b^i_j, b^{i'}_{j+1}]$ and $[b^i_j, b^{i''}_{j+1}]$ are $\mathsf{up\_rel}$-intervals. If $[b^i_j, b^{i'}_{j+1}]$ satisfies $\mathsf{up\_rel_o}$ and $[b^i_j, b^{i''}_{j+1}]$ satisfies $\mathsf{up\_rel_e}$ (or vice versa), then (3.15) is contradicted. Then, let us suppose that $[b^i_j, b^{i'}_{j+1}]$ and $[b^i_j, b^{i''}_{j+1}]$ satisfy $\mathsf{up\_rel_o}$ (the case in which both of them satisfy $\mathsf{up\_rel_e}$ is symmetric).

By (3.20), both $b_{j+1}^{i'+1}$ and $b_{j+1}^{i''+1}$ start an $\mathsf{up\_rel_o}$-interval. By the strict interleaving property, an $\mathsf{up\_rel_e}$-interval starts at the point $b_{j+1}^{i'+2}$. Since $[b_{j+1}^{i'+1}, b_{j+1}^{i'+2}]$ is not the last tile of the $(j+1)$-th Id-interval, then, as we have already shown, there exists a point c such that $[c, b_{j+1}^{i'+1}]$ is an $\mathsf{up\_rel}$-interval. By (3.20) and (3.15), we have that $[c, b_{j+1}^{i'+1}]$ is a $\mathsf{up\_rel_e}$-interval. We show that the existence of such an interval leads to a contradiction:

- if $c < b_j^i$, then the $\mathsf{up\_rel}$-interval $[c, b_{j+1}^{i'+1}]$ contains the $\mathsf{up\_rel}$-interval $[b_j^i, b_{j+1}^{i'}]$, contradicting (3.24);
- if $c = b_j^i$, then b_j^i starts both an $\mathsf{up\_rel_o}$- and an $\mathsf{up\_rel_e}$-interval, contradicting (3.15);
- if $c > b_j^i$, then the $\mathsf{up\_rel}$-interval $[b_j^i, b_{j+1}^{i''}]$ contains the $\mathsf{up\_rel}$-interval $[c, b_{j+1}^{i'+1}]$, contradicting (3.24).

In a similar way, we can prove that it cannot happen that two distinct $\mathsf{up\_rel}$-intervals end at the same point. □

Commutativity property. We prove now that the right- and above-neighbor relations commute, as formally stated by the following lemma.

Lemma 3.2.5 (commutativity property). *Let $M, [a, b] \Vdash (3.7) \wedge (3.12) \wedge (3.27)$, and let $b = b_1^0 < b_1^1 < \ldots b_1^{k_1} = b_2^0 < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$ be the sequence of points defined in Lemma 3.2.2. Then, the commutativity property holds.*

Proof. Let $[b_j^i, b_j^{i+1}]$ and $[b_{j'}^{i'}, b_{j'}^{i'+1}]$ be two tile-intervals and suppose that there exists a tile-interval $[c, d]$ such that $[b_j^i, b_j^{i+1}]$ is right-connected to $[c, d]$ and $[c, d]$ is above-connected to $[b_{j'}^{i'}, b_{j'}^{i'+1}]$. Then, we have that $[c, d] = [b_j^{i+1}, b_j^{i+2}]$ and that $[b_j^{i+2}, b_{j'}^{i'}]$ is an $\mathsf{up\_rel}$-interval. As a consequence, we have that $j' = j + 1$. Since $[b_j^i, b_j^{i+1}]$ satisfies tile, it is above-connected with exactly one tile-interval, say it $[b_{j+1}^{i''}, b_{j+1}^{i''+1}]$. Thus, $[b_j^{i+1}, b_{j+1}^{i''}]$ is an $\mathsf{up\_rel}$-interval (by Lemma 3.2.4). We want to show that $[b_{j+1}^{i''}, b_{j+1}^{i''+1}]$ is right connected to $[b_{j+1}^{i'}, b_{j+1}^{i'+1}]$. Since the only interval that is right-connected to $[b_{j+1}^{i'}, b_{j+1}^{i'+1}]$, if any, is the interval $[b_{j+1}^{i'-1}, b_{j+1}^{i'}]$, then it suffices to show that $[b_{j+1}^{i''}, b_{j+1}^{i''+1}] = [b_{j+1}^{i'-1}, b_{j+1}^{i'}]$, that is, $b_{j+1}^{i''} = b_{j+1}^{i'-1}$. Suppose by contradiction that this is not the case. We have two possibilities:

- let $b_{j+1}^{i''} > b_{j+1}^{i'-1}$. Two cases are possible:
  - if $b_{j+1}^{i''} = b_{j+1}^{i'}$, then there are two intervals, $[b_j^{i+1}, b_{j+1}^{i'}]$ and $[b_j^{i+2}, b_{j+1}^{i'}]$, ending at the same point and satisfying $\mathsf{up\_rel}$, contradicting Lemma 3.2.4;
  - if $b_{j+1}^{i''} > b_{j+1}^{i'}$, then the $\mathsf{up\_rel}$-interval $[b_j^{i+1}, b_{j+1}^{i''}]$ contains the $\mathsf{up\_rel}$-interval $[b_j^{i+2}, b_{j+1}^{i'}]$, contradicting (3.24);

- let $b^{i''}_{j+1} < b^{i'-1}_{j+1}$. By Lemma 3.2.4, there exists a point $b^{i'''}_{j}$ such that $[b^{i'''}_{j}, b^{i'-1}_{j+1}]$ satisfies up_rel. We must consider the following cases:
  - if $b^{i'''}_{j} > b^{i+2}_{j}$, then the up_rel-interval $[b^{i+2}_{j}, b^{i'}_{j+1}]$ contains the up_rel-interval $[b^{i'''}_{j}, b^{i'-1}_{j+1}]$, contradicting (3.24);
  - if $b^{i'''}_{j} = b^{i+2}_{j}$, then the two intervals $[b^{i+2}_{j}, b^{i'}_{j+1}]$ and $[b^{i+2}_{j}, b^{i'-1}_{j+1}]$ start from the same point and satisfy up_rel, thus contradicting Lemma 3.2.4;
  - if $b^{i'''}_{j} = b^{i+1}_{j}$, then the two intervals $[b^{i+1}_{j}, b^{i''}_{j+1}]$ and $[b^{i+1}_{j}, b^{i'-1}_{j+1}]$ start from the same point and satisfy up_rel, thus contradicting Lemma 3.2.4;
  - if $b^{i'''}_{j} < b^{i+1}_{j}$, then the up_rel-interval $[b^{i'''}_{j}, b^{i'-1}_{j+1}]$ contains the up_rel-interval $[b^{i+1}_{j}, b^{i''}_{j+1}]$, contradicting (3.24).

Hence, we can conclude that $b^{i''}_{j+1} = b^{i'-1}_{j+1}$, which implies $[b^{i}_{j}, b^{i+1}_{j}]$ is above-connected to $[b^{i'-1}_{j+1}, b^{i'}_{j+1}]$, whence the thesis. □

As an immediate consequence of the previous lemma we have the following corollary.

Corollary 3.2.6. *The i-th* tile*-interval of the j-th level (*ld*-interval) is above-connected with the i-th* tile*-interval of the $(j+1)$-th level.*

Tiling the plane. The tile-intervals represent the elements of the octant to be tiled. The tiles for the elements of the octant are the elements of the set $\mathcal{T} = \{t_1, t_2, \ldots, t_k\}$, each of which is encoded by the respective element in the set of propositional letters $\mathtt{T} = \{\mathtt{t_1}, \mathtt{t_2}, \ldots, \mathtt{t_k}\}$. With the first formula of the following ones, we constrain each tile-interval (and no other interval) to be tiled by exactly one tile. The last two formulae constrain the tiles that are right- or above-connected to respect the color constraints and complete the encoding of the Octant Tiling Problem:

$$[G]((\bigvee_{i=1}^{k} \mathtt{t_i} \leftrightarrow \mathsf{tile}) \wedge (\bigwedge_{i,j=1, i\neq j}^{k} \neg(\mathtt{t_i} \wedge \mathtt{t_j}))) \tag{3.28}$$

$$[G](\mathsf{tile} \rightarrow \bigvee_{up(t_i)=down(t_j)} (\mathtt{t_i} \wedge \langle A\rangle(\mathsf{up\_rel} \wedge \langle A\rangle \mathtt{t_j}))) \tag{3.29}$$

$$[G](\mathsf{tile} \wedge \langle A\rangle \mathsf{tile} \rightarrow \bigvee_{right(t_i)=left(t_j)} (\mathtt{t_i} \wedge \langle A\rangle \mathtt{t_j})) \tag{3.30}$$

$$(3.28) \wedge \ldots \wedge (3.30) \tag{3.31}$$

Given the set of tile types $\mathcal{T} = \{t_1, t_2, \ldots, t_k\}$, we define the formula:

$$\Phi_{\mathcal{T}} = (3.7) \wedge (3.12) \wedge (3.27) \wedge (3.31)$$

Now, we can conclude the following lemma:

Lemma 3.2.7. *Given any finite set of tile types $\mathcal{T} = \{t_1, t_2, \ldots, t_k\}$, the formula $\Phi_{\mathcal{T}}$ is satisfiable if and only if $\mathcal{T}$ can tile the second octant $\mathcal{O}$.*

Proof. As for the "only if" direction of the implication, suppose $M, [a, b] \Vdash \Phi_{\mathcal{T}}$, for some model M and some interval $[a, b] \in M$. Let $b = b_1^0 < b_1^1 < b_1^2 = b_2^0 < \ldots < b_2^{k_2} = b_3^0 < \ldots < b_j^0 < b_j^1 < \ldots < b_j^{k_j} = b_{j+1}^0 < \ldots$ be the sequence of points defined by Lemma 3.2.2. Furthermore, for each $j > 0$ and $0 < i < k_j$, we have that $[b_j^i, b_j^{i+1}]$ is a tile-interval and this implies that $M, [b_j^i, b_j^{i+1}] \Vdash \mathtt{t_k}$, for an unique k. Then, for each i, j, with $0 \leq i \leq j$, we put $f(i, j) = t_k$, where $\mathtt{t_k}$ is the unique propositional letter of the set $\mathtt{T} = \{\mathtt{t_1}, \mathtt{t_2}, \ldots, \mathtt{t_k}\}$ such that $M, [b_{j+1}^{i+1}, b_{j+1}^{i+2}] \Vdash \mathtt{t_k}$. By Lemma 3.2.4 and 3.2.5, Corollary 3.2.6, and formula (3.31), the function $f : \mathcal{O} \mapsto \mathcal{T}$ defines a correct tiling of $\mathcal{O}$.

On the other hand, as for the "if" direction, let $f : \mathcal{O} \mapsto \mathcal{T}$ be a correct tiling of $\mathcal{O}$. We provide a model M and an interval $[a, b]$ such that $M, [a, b] \Vdash \Phi_{\mathcal{T}}$ (see Fig. 3.3). Let $M = \langle \mathbb{I}(\mathbb{N}), V \rangle$ be a model whose valuation function $V : \mathbb{I}(\mathbb{N}) \mapsto 2^{\mathcal{AP}}$ is defined as follows:

- $\mathsf{u} \in V([i, j])$ if and only if $j = i + 2$ and $i = 2n$ for some $n > 0$;
- $\mathsf{u_1} \in V([i, j])$ if and only if $j = i + 1$ and $\mathsf{u} \in V([i, i + 2])$;
- $\mathsf{u_2} \in V([i, j])$ if and only if $j = i + 1$ and $\mathsf{u} \in V([i - 1, i + 1])$;
- $* \in V([i, j])$ if and only if $\mathsf{u} \in V([i, j])$ and $i = g(n)$ for some $n \geq 0$, where the function $g : \mathbb{N} \to \mathbb{N}^+$ is defined as: $g(n) = (n + 1)(n + 2)$ for each $n \in \mathbb{N}$;
- $\mathsf{Id} \in V([i, j])$ if and only if $* \in V([i, i + 2])$, $* \in V([j, j + 2])$, $i = g(n)$, and $j = g(n + 1)$ for some $n > 0$;
- $\mathsf{tile} \in V([i, j])$ if and only if $\mathsf{u} \in V([i, j])$ and $* \notin V([i, j])$;
- for each $h \in \{1, \ldots, k\}$, $\mathtt{t_h} \in V([i, j])$ if and only if $\mathsf{tile} \in V([i, j])$, $f(l, m) = t_h$, and $i = g(m) + 2 \cdot l + 2$, for some l, m, with $0 \leq l \leq m$;
- $\mathsf{up\_rel_o} \in V([i, j])$ if and only if $\mathsf{tile} \in V([i - 2, i])$, $\mathsf{tile} \in V([j, j + 2])$, $i - 2 = g(m) + 2 \cdot l + 2$, and $j = g(m + 1) + 2 \cdot l + 2$ for some $0 \leq l \leq m$ such that $l = 2 \cdot n$ for some $n \geq 0$;
- $\mathsf{up\_rel_e} \in V([i, j])$ if and only if $\mathsf{tile} \in V([i - 2, i])$, $\mathsf{tile} \in V([j, j + 2])$, $i - 2 = g(m) + 2 \cdot l + 2$, and $j = g(m + 1) + 2 \cdot l + 2$ for some $0 \leq l \leq m$ such that $l = 2 \cdot n + 1$ for some $n \geq 0$;
- $\mathsf{up\_rel} \in V([i, j])$ if and only if $\mathsf{up\_rel_o} \in V([i, j])$ or $\mathsf{up\_rel_e} \in V([i, j])$.

It is straightforward to check that $\Phi_{\mathcal{T}}$ is satisfied over the model $M = \langle \mathbb{I}(\mathbb{N}), V \rangle$ and the interval $[0, 2]$, whence the thesis.

□

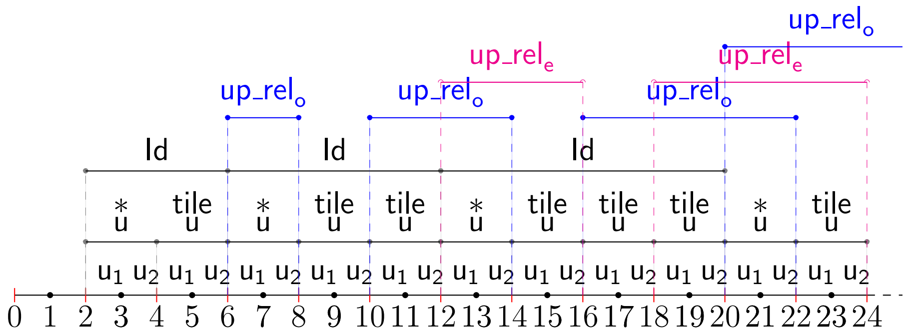


Figure 3.3: A model satisfying the formula $\Phi_{\mathcal{T}}$

Theorem 3.2.8. *The satisfiability problem for the fragment* AD *of* HS *is undecidable in any class of linear orderings that contains at least one linear ordering with an infinite ascending sequence of points.*

3.2.2 The fragments A$\overline{\text{D}}$, $\overline{\text{A}}$D and $\overline{\text{AD}}$

In order to adapt the construction above to the logic A$\overline{\text{D}}$, it suffices to replace each formula containing the operator D with an equivalent formula belonging to the language of A$\overline{\text{D}}$. So, we will substitute the formulae (3.3), (3.4), (3.5), (3.11), (3.21), (3.22), (3.24), (3.25), and (3.26). Most of them can be replaced with minimum effort, but some of them need to be re-thought. To this end, we will exploit three new propositional letters, namely, k_1, k_2, and first, besides the ones used in the previous section.

Let us denote by $\Phi'_{\mathcal{T}}$ the conjunction of the formulae from (3.1) to (3.30) involving neither $\langle D \rangle$ nor $\langle \overline{D} \rangle$. As a preliminary step, we have to restrict the truth of propositional letters to intervals of $\mathcal{G}_{[a,b]}$. Indeed, due to the semantics of $\langle \overline{D} \rangle$ and $[\overline{D}]$, formulae involving these operators might refer to intervals not belonging to the set $\mathcal{G}_{[a,b]}$. As an example, let us consider the formula $[G](\langle \overline{D} \rangle p \rightarrow q)$. Intuitively, such a formula forces each interval of $\mathcal{G}_{[a,b]}$ contained in a p-interval to be a q-interval. The problem is that the p-interval containing the current interval could be an interval not belonging to $\mathcal{G}_{[a,b]}$, while we are interested in restricting our attention to the intervals of $\mathcal{G}_{[a,b]}$. In other words, what we want to be able to say is something like "each interval of $\mathcal{G}_{[a,b]}$, that is contained in some interval of $\mathcal{G}_{[a,b]}$ satisying p, must satisfies q". Thus, with the following formula, evaluated over the initial interval $[a, b]$, we force intervals not belonging to $\mathcal{G}_{[a,b]}$ not to satisfy any propositional letter, except for the new propositional letters k and k_1:

$$[\overline{D}]\neg p \wedge [A][\overline{D}]\neg p, \text{ for each } p \in \mathcal{AP} \setminus \{\mathsf{k}, \mathsf{k}_1\} \tag{3.32}$$

It is immediate to verify that $\Phi'_{\mathcal{T}}$ also forces p to be false over $[a, b]$, for each $p \in \mathcal{AP} \setminus \{\mathsf{k}_1, \mathsf{k}_2\}$. For example, formula (3.1) (resp., (3.9), (3.13)) expressly force u, u_1, and u_2 (resp., Id, $\mathsf{up\_rel}$) to be false over the initial interval $[a, b]$. Then, most of the above formulae can be replaced by using the following translation schema, whose correctness depend on the previous formula (3.32):

Lemma 3.2.9. *Let $M, [a, b] \Vdash$ (3.32), and let $p, q \in \mathcal{AP} \backslash \{\mathsf{k}, \mathsf{k}_1\}$. Then, the following equivalences hold:*

$$M, [a, b] \Vdash [G](p \rightarrow \neg\langle D\rangle\langle A\rangle q) \Leftrightarrow M, [a, b] \Vdash [G](\langle\overline{D}\rangle p \rightarrow \neg\langle A\rangle q) \tag{3.33}$$

$$M, [a, b] \Vdash [G](p \rightarrow \neg\langle D\rangle q) \Leftrightarrow M, [a, b] \Vdash [G](q \rightarrow \neg\langle\overline{D}\rangle p) \tag{3.34}$$

Proof. First, we prove the equation (3.33).

$\Rightarrow$ Suppose that $M, [a, b] \Vdash [G](p \rightarrow \neg\langle D\rangle\langle A\rangle q)$. This means that, for each $[c, d] \in \mathcal{G}_{[a,b]}$ satisfying p and for each $[e, f]$ such that $c < e < f < d$, there is no $g > f$ with $[f, g]$ satisfying q. Now, suppose by contradiction that $M, [a, b] \nVdash [G](\langle\overline{D}\rangle p \rightarrow \neg\langle A\rangle q)$. Then, there exists some interval $[e, f] \in \mathcal{G}_{[a,b]}$ such that, for some $[c, d]$ with $c < e < f < d$, it is the case that $[c, d]$ satisfies p and, for some $g > f$, $[f, g]$ satisfies q. By (3.32), since $[c, d]$ satisfies the propositional letter p, it belongs to $\mathcal{G}_{[a,b]}$. Then, by hypothesis, we have that $\neg\langle D\rangle\langle A\rangle q$ must be true over $[c, d]$, too; but, due to $[f, g]$, we have a contradiction, hence the thesis.

$\Leftarrow$ Suppose that $M, [a, b] \Vdash [G](\langle\overline{D}\rangle p \rightarrow \neg\langle A\rangle q)$. This means that if $[c, d] \in \mathcal{G}_{[a,b]}$ is such that there exists $[e, f]$ satisfying p, with $e < c < d < f$, then $[d, d']$ does not satisfy q for any $d' > d$. Now, suppose by contradiction that $M, [a, b] \nVdash [G](p \rightarrow \neg\langle D\rangle\langle A\rangle q)$. Then, there exists an interval $[e, f]$ in $\mathcal{G}_{[a,b]}$ satisfying p and such that there is an interval $[c, d]$, with $e < c < d < f$, and a q-interval $[d, d']$, for some $d' > d$. Notice that $[e, f]$ satisfies the propositional letter p. As a consequence, by $\Phi'_{\mathcal{T}}$, it must be different from $[a, b]$. It is easy to see that, since $[c, d]$ is a sub-interval of $[e, f]$, which belongs to $\mathcal{G}_{[a,b]} \setminus \{[a, b]\}$, it must, in turn, belong to $\mathcal{G}_{[a,b]}$. But, since $M, [a, b] \Vdash [G](\langle\overline{D}\rangle p \rightarrow \neg\langle A\rangle q)$, we have that $M, [c, d] \Vdash \neg\langle A\rangle q$, which is a contradiction, hence the thesis.

The proof for the equivalence (3.34) is similar. □

We can use the previous lemma in order to translate, into the language of $\mathsf{A\overline{D}}$, the formulae (3.3) (except for (3.3-V)), (3.4), (3.5) (except for (3.5-IV)), (3.11), (3.22), and (3.24). Let $\Phi''_{\mathcal{T}}$ be the conjunction among $\Phi'_{\mathcal{T}}$, (3.32), and the translations of the above formulae.

Next, let us consider (3.3-V) and (3.5-IV). In order to capture the properties expressed by these formulae, we take advantage of the two new propositional letters k and k_1. Consider the following set of formulae:

$$\langle\overline{D}\rangle\mathsf{k} \wedge [G][\overline{D}](\mathsf{k} \rightarrow \neg\langle A\rangle\mathsf{u}) \tag{3.35}$$
$$[G](\mathsf{u} \rightarrow \neg\langle\overline{D}\rangle\mathsf{k}) \tag{3.36}$$
$$[G][\overline{D}](\mathsf{k} \rightarrow \langle A\rangle(\langle A\rangle\mathsf{u} \wedge \langle\overline{D}\rangle\mathsf{k})) \tag{3.37}$$
$$[G](\mathsf{u}_1 \rightarrow \langle\overline{D}\rangle\mathsf{k}_1) \tag{3.38}$$
$$[G][\overline{D}](\mathsf{k}_1 \rightarrow \neg\langle A\rangle\mathsf{u}) \tag{3.39}$$
$$[G](\mathsf{u} \rightarrow \neg\langle\overline{D}\rangle\mathsf{k}_1) \tag{3.40}$$

The first three formulae (from (3.35) to (3.37)) guarantee the existence of a point inside each u-interval $[b_i, b_{i+1}]$, that is, for each $i \geq 0$, there exists c_i such that $b_i < c_i < b_{i+1}$. This is not exactly the same statement of (3.3-V), but it is enough to force u_1-intervals to begin u-intervals (and not vice versa).

Lemma 3.2.10. *Let $M, [a, b] \Vdash \Phi''_{\mathcal{T}} \wedge (3.35) \wedge \ldots \wedge (3.37)$. Then, there exists a sequence of points $b = b_0 < b_1 < \ldots$ such that $[b_i, b_{i+1}]$ satisfies* u*. Moreover, for each $i \geq 0$, there exists a point c'_i such that $b_i < c'_i < b_{i+1}$.*

Proof. First of all, notice that $\Phi''_{\mathcal{T}}$ forces the existence of a sequence $b_0 < b_1 < \ldots$ of points such that $b = b_0$ and $[b_i, b_{i+1}]$ satisfies u for each $i \geq 0$. The rest of the proof is by induction on i. We will show that, for each $i \geq 0$, there exists a point c'_i ending a k-interval and such that $b_i < c'_i < b_{i+1}$.
Base case ($i = 0$). By the first conjunct of (3.35), there exists an interval $[c, d]$ satisfying k and such that $c < a$ and $d > b$. Consider the relative position of d and b_1. If $d = b_1$, then the k-interval $[c, d]$ meets the u-interval $[b_1, b_2]$, contradicting the second conjunct of (3.35); otherwise, if $d > b_1$, then the k-interval $[c, d]$ contains the u-interval $[b_0, b_1]$, contradicting (3.36). Then, it must be $d < b_1$, hence it must exist a point in between b_0 and b_1 ending a k-interval.
Inductive step ($i > 0$). By inductive hypothesis, we have that, for each $j < i$, there exists a point c'_j ending a k-interval and such that $b_j < c'_j < b_{j+1}$. We have to show the existence of a k-interval ending in c'_i, with $b_i < c'_i < b_{i+1}$. Consider the k-interval $[c, c'_{i-1}]$. By (3.37), it must exist an interval $[c'_{i-1}, d]$ such that d starts an u-interval and $[c'_{i-1}, d]$ is contained in a k-interval, say it $[c', d']$. If $d < b_i$, then the u-interval $[b_{i-1}, b_i]$ contains the interval $[c'_{i-1}, d]$ that, in turn, meets the u-interval starting at the point d, contradicting the translation of (3.3-II). Then, it must be $d \geq b_i$. We have to show that the k-interval $[c', d']$ must be such that $b_i < d' < b_{i+1}$. From $d \geq b_i$ and the fact that $[c'_i, d]$ is contained in $[c', d']$, it immediately follows that $d' > b_i$. In order to show that $d' < b_{i+1}$ we proceed by contradiction. If $d' = b_{i+1}$, then the k-interval $[c', d']$ meets the u-interval $[b_{i+1}, b_{i+2}]$, contradicting the second conjunct of (3.35); otherwise, if $d' > b_{i+1}$, then the k-interval $[c', d']$ contains the u-interval $[b_i, b_{i+1}]$, contradicting (3.36). Then, it must be $d' < b_{i+1}$, hence it must exist a point in between b_i and b_{i+1} ending a k-interval. □

Exploiting the previous lemma, the proof of Lemma 3.2.1 can be adapted in order to show that, for each $i \geq 0$, there exists a point c_i such that $b_i < c_i < b_{i+1}$ and $[b_i, c_i]$ satisfies u_1. In a similar way, the formulae from (3.38) to (3.40) guarantee that, for each $i \geq 0$, there exists a point d_i such that $c_i < d_i < b_{i+1}$. The existence of such a point is exploited by the proof of Lemma 3.2.1 for showing that $[c_i, b_{i+1}]$ satisfies u_2 for each $i \geq 0$, which means that (3.3-V) and (3.5-IV) can be replaced by (3.35) $\wedge \ldots \wedge$ (3.40).

As for the remaining formulae, (3.21) is replaced by:

$$[G](* \to [A](\langle A \rangle * \to \neg\langle \overline{D} \rangle \mathsf{up\_rel})) \tag{3.41}$$

In order to replace (3.25), we use the new propositional letter first and the following formulae:

$$[G](* \to \langle A \rangle(\mathsf{tile} \wedge [A](\mathsf{up\_rel} \to \mathsf{first}))) \tag{3.42}$$

$$[G](* \to \neg\langle \overline{D} \rangle \mathsf{first}) \tag{3.43}$$

Finally, (3.26) is replaced by:

$$[G](\mathsf{tile} \wedge \langle A \rangle * \to \langle A \rangle(\mathsf{up\_rel} \wedge \langle A \rangle(\mathsf{tile} \wedge \langle A \rangle(\mathsf{tile} \wedge \langle A \rangle *)))) \tag{3.44}$$

$$[G]((\langle \overline{D} \rangle \mathsf{up\_rel}_\alpha \wedge \langle A \rangle(\mathsf{u}_2 \wedge \langle A \rangle(\mathsf{tile} \wedge \langle A \rangle \mathsf{up\_rel}_\beta \wedge \neg\langle A \rangle *)) \to \langle \overline{D} \rangle \mathsf{up\_rel}_\beta) \tag{3.45}$$

The following lemma shows that (3.26) is correctly replaced by the above formulae.

Lemma 3.2.11. *Lemma 3.2.4 holds with respect to the fragment* A$\overline{\text{D}}$ *by replacing the* AD *formula (3.26) with the* A$\overline{\text{D}}$ *formulae (3.44) and (3.45).*

Proof. Notice that, as far as the proof of Lemma 3.2.4 is concerned, the formula (3.26) is only necessary to prove that for each tile-interval of a level, but the last one, there is at least one tile-interval that is above-connected to it. The rest of the lemma can be proved in the same way. Thus, we only need to show, using (3.44) and (3.45) instead of (3.26), that for each tile-interval $[b_j^i, b_j^{i+1}]$, if $i < k_j - 1$, then there exists at least one tile-interval which is above-connected to it.

Consider a tile-interval $[b_j^i, b_j^{i+1}]$ for some i such that $0 < i < k_j - 1$ (it is not the last tile-interval of the j-th Id-interval; this also implies $j > 1$). Then, by Lemma 3.2.4, there exists $d > b_j^{i+1}$ such that $[b_j^{i+1}, d]$ satisfies up_rel. By (3.14) and (3.15), it satisfies exactly one between $\mathsf{up\_rel_o}$ and $\mathsf{up\_rel_e}$; suppose that $[b_j^{i+1}, d]$ satisfies $\mathsf{up\_rel_o}$ (the other case is symmetric). We will prove that $[c, b_j^i]$ satisfies up_rel for some $c < b_j^i$. Notice that, if $[c, b_j^i]$ satisfies up_rel for some $c < b_j^i$, then it must satisfy $\mathsf{up\_rel_o}$, otherwise, if it satisfied $\mathsf{up\_rel_e}$, by (3.20), there would be an $\mathsf{up\_rel_o}$- and an $\mathsf{up\_rel_e}$-interval starting at b_j^{i+1}, contradicting (3.15). Now, suppose, by contradiction, that there is not $c < b_j^i$ such that $[c, b_j^i]$ satisfies up_rel. The proof proceeds by induction on i.

Base case ($i = 1$). The proof is identical to the one of Lemma 3.2.4;

Inductive step ($i > 1$). Consider the interval $[b_j^{i-1}, b_j^i]$; for the strict interleaving property, there exists $d' > b_j^i$ such that $[b_j^i, d']$ satisfies $\mathsf{up\_rel_e}$. Furthermore, by inductive hypothesis, there exists $c' < b_j^{i-1}$ such that $[c', b_j^{i-1}]$ satisfies $\mathsf{up\_rel}$. In particular, $[c', b_j^{i-1}]$ satisfies $\mathsf{up\_rel_e}$, otherwise, by (3.20), there would be an $\mathsf{up\_rel_o}$- and an $\mathsf{up\_rel_e}$-interval starting at b_j^i, contradicting (3.15). Let $[c'', c']$ be the tile-interval above-connected to $[b_j^{i-1}, b_j^i]$ by means of the $\mathsf{up\_rel_e}$-interval $[c', b_j^{i-1}]$. Notice that $[c'', c']$ is not the last tile-interval of the $(j-1)$-th ld-interval: if this was the case, then, by (3.44), the interval $[b_j^{i-1}, b_j^i]$ would be the second last tile of the j-th ld-interval, which is a contradiction, since $[b_j^i, b_j^{i+1}]$ is not the last tile-interval of its level. Let us call $[c', e]$ the tile-interval which immediately follows the tile-interval $[c'', c']$ and which is above-connected to some tile-interval $[f, g]$ by means of the $\mathsf{up\_rel}$-interval $[e, f]$. By the strict interleaving property, $[e, f]$ satisfies $\mathsf{up\_rel_o}$, and by (3.20), $[g, g']$ satisfies $\mathsf{up\_rel_o}$ for some $g' > g$. Notice that $f > b_j^{i+1}$: suppose that $f < b_j^{i-1}$, then the $\mathsf{up\_rel}$-interval $[c', b_j^{i-1}]$ contains the $\mathsf{up\_rel}$-interval $[e, f]$, contradicting the A$\overline{\text{D}}$ "version" of (3.24). Moreover, if $f = b_j^{i-1}$, by (3.20), there are an $\mathsf{up\_rel_o}$- and an $\mathsf{up\_rel_e}$-interval starting at b_j^i, contradicting (3.15); if $f = b_j^i$, we are contradicting the hypothesis per absurdum that there are not $\mathsf{up\_rel}$-intervals ending at b_j^i; finally, if $f = b_j^{i+1}$, then the strict interleaving property is contradicted ($[b_j^i, b_j^{i+1}]$ and $[f, g]$ are consecutive tile-intervals starting both an $\mathsf{up\_rel_o}$-interval). So, we can state $f > b_j^{i+1}$. Now, consider the tile-interval $[h, f]$, immediately before the tile-interval $[f, g]$. By the strict interleaving property, there is an $\mathsf{up\_rel_e}$-interval starting at f. Now, we distinguish two cases, both of them leading to contradiction.

- There is h' such that $[h', h]$ satisfies $\mathsf{up\_rel}$, then $[h', h]$ satisfies $\mathsf{up\_rel_e}$ (again, by (3.20) and (3.15)). We distinguish the following cases:

  1. $h' > e$, then the $\mathsf{up\_rel}$-interval $[e, f]$ contains the $\mathsf{up\_rel}$-interval $[h', h]$, contradicting the A$\overline{\text{D}}$ "version" of (3.24);
  2. $h' = e$, then there are an $\mathsf{up\_rel_o}$- and an $\mathsf{up\_rel_e}$-intervals starting both at e, contradicting (3.15);
  3. $h' = c'$, then (3.45) is contradicted, indeed the interval $[e, s]$, where s is the point splitting the u-interval $[b_j^{i-1}, b_j^i]$, satisfies the hypothesis of the implication (since $[h', h]$ satisfies $\mathsf{up\_rel_e}$, $[s, b_j^i]$ satisfies $\mathsf{u_2}$, $[b_j^i, b_j^{i+1}]$ satisfies tile, $[b_j^{i+1}, d]$ satisfies $\mathsf{up\_rel_o}$, and $[b_j^i, b_j^{i+1}]$ is not the last tile-interval of the j-th ld-interval, so $[b_j^{i+1}, d_1]$ satisfies $\neg *$ for any $d_1 > b_j^{i+1}$), but not the thesis: suppose that there is an interval $[l, m]$ satisfying $\mathsf{up\_rel_o}$, with $l < e < s < m$. Then, if $l = c'$, then there are an $\mathsf{up\_rel_o}$- and an $\mathsf{up\_rel_e}$-interval starting both from c', contradicting (3.15); instead, if $l < c'$, then the $\mathsf{up\_rel}$-interval $[l, m]$ contains the $\mathsf{up\_rel}$-interval $[c', b_j^{i-1}]$, contradicting the A$\overline{\text{D}}$ "version" of (3.24);
  4. $h' < c'$, then the $\mathsf{up\_rel}$-interval $[h', h]$ contains the $\mathsf{up\_rel}$-interval $[c', b_j^{i-1}]$, contradicting the A$\overline{\text{D}}$ "version" of (3.24).

- There is no h' such that $[h', h]$ satisfies up_rel, then (3.45) is contradicted again: the interval $[s', s'']$, where s' is the point which chops the u-interval right after the tile-interval $[c', e]$ and s'' is the point chopping the u-interval right before the tile-interval $[h, f]$, satisfies the hypothesis of the implication (since $[e, f]$ satisfies $\mathsf{up\_rel}_\mathsf{o}$, $[s'', h]$ satisfies u_2, $[h, f]$ satisfies tile, f starts an $\mathsf{up\_rel}_\mathsf{e}$-interval, and $[h, f]$ is not the last tile of the j-th Id-interval, so $[f, f_1]$ satisfies $\neg *$ for any $f_1 > f$), but not the thesis: suppose that there is an interval $[l, m]$ satisfying $\mathsf{up\_rel}_\mathsf{e}$, with $l < s' < s'' < m$, then we distinguish the following cases:

   1. $l = e$, then there are an $\mathsf{up\_rel}_\mathsf{o}$- and an $\mathsf{up\_rel}_\mathsf{e}$-intervals starting both at e, contradicting (3.15);
   2. $m = h$, then the hypothesis that there is no h' such that $[h', h]$ satisfies up_rel is contradicted;
   3. $l < e$ and $m = f$, then, by (3.20), there are an $\mathsf{up\_rel}_\mathsf{o}$- and an $\mathsf{up\_rel}_\mathsf{e}$-intervals starting both at g, contradicting (3.15);
   4. $l < e$ and $m > f$, then the up_rel-interval $[l, m]$ contains the up_rel-interval $[e, f]$, contradicting the $\mathsf{A\overline{D}}$ "version" of (3.24).

Thus, there must be a point c such that $[c, b^i_j]$ satisfies up_rel. □

Theorem 3.2.12. *The satisfiability problem for the fragment* $\mathsf{A\overline{D}}$ *of* HS *is undecidable in any class of linear orderings that contains at least one linear ordering with an infinite ascending sequence of points.*

The previous reduction can easily be extended, by symmetry, to the fragments $\overline{\mathsf{A}}\mathsf{D}$ and $\overline{\mathsf{A}}\overline{\mathsf{D}}$, provided that there is an infinite descending sequence of points.

Theorem 3.2.13. *The satisfiability problems for the fragments* $\overline{\mathsf{A}}\mathsf{D}$ *and* $\overline{\mathsf{A}}\overline{\mathsf{D}}$ *of* HS *are undecidable in any class of linear orderings that contains at least one linear ordering with an infinite descending sequence of points.*

3.2.3 Extending undecidability to classes of finite linear orders

The above results hold for classes of linear orderings containing at least a linear ordering with an infinite ascending sequence of points. In this section, we show how to adapt them in order to deal with classes of finite linear orders. In particular, we show how to modify the construction used for the logic AD in order to encode the *Finite Plane Tiling Problem.* This is the problem of establishing if a given set of tile types $\mathcal{T}$ can tile a rectangular area whose edges are colored by the same distinguished color \$. Such a problem has been introduced and shown to be undecidable in [66].

In the following, we analyse the formulae used in Section 3.2.1. Some of them will be unchanged, others will be adapted to encode the Finite Plane Tiling Problem.

Consider the set of formulae from (3.1) to (3.6). Formulae (3.2), (3.3), (3.4), (3.6) are unchanged. Formulae (3.1), (3.5) are changed in, respectively:

$$\neg \mathsf{u} \wedge \neg \mathsf{u}_1 \wedge \neg \mathsf{u}_2 \wedge \langle A \rangle \mathsf{u} \wedge [G](\mathsf{u} \wedge \langle A \rangle \top \rightarrow \langle A \rangle \mathsf{u}) \tag{3.46}$$

$$[G](\mathsf{u}_2 \rightarrow (\langle A \rangle \top \rightarrow \langle A \rangle \mathsf{u}) \wedge \neg \langle D \rangle \langle A \rangle \mathsf{u} \wedge \neg \langle D \rangle \mathsf{u}_1 \wedge \langle D \rangle \top) \tag{3.47}$$

Moreover, we introduce the following:

$$\neg \mathsf{end} \wedge \neg \langle A \rangle \mathsf{end} \wedge \langle A \rangle \langle A \rangle \mathsf{end} \tag{3.48}$$

$$[G](\mathsf{end} \rightarrow \mathsf{u} \wedge [A] \neg \mathsf{end} \wedge [A][A] \neg \mathsf{end}) \tag{3.49}$$

$$(3.46) \wedge \ldots \wedge (3.49) \tag{3.50}$$

Let $\Phi_1 = (3.2) \wedge (3.3) \wedge (3.4) \wedge (3.6) \wedge (3.50)$.

Lemma 3.2.14. *Let $M, [a, b] \Vdash \Phi_1$. Then, there exists a finite sequence of points $b = b_0 < b_1 < \ldots < b_r$ in M, such that $M, [b_i, b_{i+1}] \Vdash \mathsf{u}$ for each $0 \leq i < r - 1$, $M, [b_{r-1}, b_r] \Vdash \mathsf{end}$, and no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies u or end, unless $c > b_r$.*

Now, consider the set of formulae from (3.8) to (3.11). Formula (3.10) is unchanged. Formulae (3.8), (3.9), and (3.11) are changed into, respectively:

$$[G]((\mathsf{u} \wedge (\langle A \rangle \mathsf{end} \vee \langle A \rangle \langle A \rangle \mathsf{end}) \leftrightarrow (* \vee \mathsf{tile})) \wedge (* \rightarrow \neg \mathsf{tile})) \tag{3.51}$$

$$\neg \mathsf{Id} \wedge \langle A \rangle * \wedge [G](* \rightarrow \neg \langle A \rangle *) \tag{3.52}$$

$$[G](\mathsf{Id} \rightarrow (\langle A \rangle \mathsf{end} \vee \langle A \rangle *) \wedge \neg \langle D \rangle *) \tag{3.53}$$

Moreover, we introduce the following:

$$[G](\mathsf{Id} \wedge \langle A \rangle \mathsf{end} \leftrightarrow \mathsf{Id\_end}) \tag{3.54}$$

$$[G](\langle A \rangle \mathsf{Id\_end} \rightarrow \langle A \rangle (* \wedge [G](\mathsf{tile} \rightarrow \mathsf{tile\_end}))) \tag{3.55}$$

$$[G](\mathsf{tile\_end} \rightarrow \mathsf{tile} \wedge \neg \langle A \rangle *) \tag{3.56}$$

$$\langle A \rangle \mathsf{Id\_bgn} \wedge \langle A \rangle (* \wedge \langle A \rangle \mathsf{tile\_bgn}) \tag{3.57}$$

$$[G](\mathsf{Id\_bgn} \rightarrow \mathsf{Id} \wedge [A] \neg \mathsf{Id\_bgn} \wedge [A][A] \neg \mathsf{Id\_bgn} \wedge [A][A] \neg \mathsf{tile\_bgn}) \tag{3.58}$$

$$[G](\mathsf{tile\_bgn} \rightarrow \mathsf{tile} \wedge [A](\mathsf{tile} \rightarrow \mathsf{tile\_bgn})) \tag{3.59}$$

$$(3.51) \wedge \ldots \wedge (3.59) \tag{3.60}$$

Let $\Phi_2 = \Phi_1 \wedge (3.10) \wedge (3.60)$.

Lemma 3.2.15. *Let $M, [a, b] \Vdash \Phi_2$. Then, there exists a finite sequence of points $b = b_1^0 < b_1^1 < \ldots < b_1^{k_1} = b_2^0 < \ldots < b_2^{k_2} = b_3^0 < \ldots < b_m^0 < \ldots < b_m^{k_m} = b_{r-1} < b_r$ in M, such that:*

1. $M, [b_j^0, b_j^1] \Vdash *$,

2. $M, [b_j^i, b_j^{i+1}] \Vdash \mathsf{tile}$ *for each* $0 < j \leq m, 0 < i < k_j$,

3. $M, [b_j^0, b_j^{k_j}] \Vdash \mathsf{Id}$ *for each* $0 < j \leq m$,

4. $M, [b_1^i, b_1^{i+1}] \Vdash \mathsf{tile\_bgn}$ *for each* $0 < i < k_j$,

5. $M, [b_m^i, b_m^{i+1}] \Vdash \mathsf{tile\_end}$ *for each* $0 < i < k_j$.

Moreover, no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ *satisfies* $*$ *(resp.,* tile, Id, tile_bgn, tile_end*).*

As for the set of formulae from (3.13) to (3.26). Formulae (3.14), (3.15), (3.17), (3.18), (3.21), (3.22), (3.23), (3.24), and (3.25) are unchanged. Formulae (3.13), (3.16), (3.19), (3.20), and (3.26) are changed in, respectively:

$$\neg\langle A\rangle \mathsf{up\_rel} \tag{3.61}$$

$$[G](\mathsf{tile} \wedge \neg\mathsf{tile\_end} \rightarrow \langle A\rangle \mathsf{up\_rel}) \tag{3.62}$$

$$[G](\mathsf{up\_rel} \rightarrow \langle A\rangle \mathsf{tile}) \tag{3.63}$$

$$[G](\mathsf{up\_rel}_\alpha \rightarrow \langle A\rangle(\mathsf{tile} \wedge (\neg\mathsf{tile\_end} \rightarrow \langle A\rangle \mathsf{up\_rel}_\alpha))) \tag{3.64}$$

$$\begin{aligned}&[G](\langle A\rangle(\mathsf{u}_2 \wedge \langle A\rangle \mathsf{up\_rel}_\alpha) \rightarrow [A](\langle D\rangle \mathsf{up\_rel} \wedge \neg\langle D\rangle \mathsf{up\_rel}_\beta \\ &\quad \rightarrow \neg\langle A\rangle(\mathsf{u}_2 \wedge \langle A\rangle(\mathsf{up\_rel}_\beta \vee \mathsf{up\_rel\_end}_\beta))))\end{aligned} \tag{3.65}$$

Moreover, we introduce the following:

$$[G](\mathsf{tile\_end} \rightarrow \langle A\rangle \mathsf{up\_rel\_end}) \tag{3.66}$$

$$[G](\langle A\rangle \mathsf{up\_rel\_end} \rightarrow \langle A\rangle \mathsf{tile\_end} \vee \langle A\rangle \mathsf{end}) \tag{3.67}$$

$$[G](* \rightarrow \neg\langle A\rangle \mathsf{up\_rel\_end}) \tag{3.68}$$

$$[G](\mathsf{up\_rel\_end} \leftrightarrow \mathsf{up\_rel\_end}_\mathsf{o} \vee \mathsf{up\_rel\_end}_\mathsf{e}) \tag{3.69}$$

$$[G](\langle A\rangle \mathsf{up\_rel\_end}_\alpha \wedge \langle A\rangle \mathsf{tile\_end} \rightarrow \langle A\rangle \mathsf{tile\_end} \wedge \langle A\rangle \mathsf{up\_rel\_end}_\beta) \tag{3.70}$$

$$[G](\mathsf{up\_rel}_\alpha \wedge \langle A\rangle \mathsf{tile\_end} \rightarrow \langle A\rangle(\mathsf{tile\_end} \wedge \langle A\rangle \mathsf{up\_rel\_end}_\alpha)) \tag{3.71}$$

$$[G](\langle A\rangle \mathsf{up\_rel\_end}_\alpha \rightarrow \neg\langle A\rangle \mathsf{up\_rel\_end}_\beta) \tag{3.72}$$

$$(3.61) \wedge \ldots \wedge (3.72) \tag{3.73}$$

Let $\Phi_3 = \Phi_2 \wedge (3.14) \wedge (3.15) \wedge (3.17) \wedge (3.18) \wedge (3.21) \wedge (3.22) \wedge (3.23) \wedge (3.24) \wedge (3.25) \wedge (3.73)$.

Lemma 3.2.16. *Let* $M, [a, b] \Vdash \Phi_2$ *and let* $b = b_1^0 < b_1^1 < \ldots < b_1^{k_1} = b_2^0 < \ldots < b_2^{k_2} = b_3^0 < \ldots < b_m^0 < \ldots < b_m^{k_m} = b_{r-1} < b_r$ *be the finite sequence of points of Lemma 3.2.15. Then,* $M, [b_j^{i+1}, b_{j+1}^i] \Vdash \mathsf{up\_rel}$ *for each* $0 < j \leq m, 0 < i < k_j$*, and* $k_j = k_{j'}$ *for each* j, j' *such that* $0 < j, j' \leq m$*, with* $j \neq j'$*.*

Finally, consider the set of formulae from (3.28) to (3.30). Formula (3.28) is unchanged. Formulae (3.29) and (3.30) are changed in, respectively:

$$[G](\mathsf{tile} \wedge \neg \mathsf{tile\_end} \rightarrow \bigvee_{up(t_i)=down(t_j)} (\mathtt{t_i} \wedge \langle A\rangle(\mathsf{up\_rel} \wedge \langle A\rangle \mathtt{t_j}))) \tag{3.74}$$

$$[G](\mathsf{tile} \wedge \langle A\rangle \mathsf{tile} \rightarrow \bigvee_{right(t_i)=left(t_j)} (\mathtt{t_i} \wedge \langle A\rangle \mathtt{t_j})) \tag{3.75}$$

Moreover, we introduce the following:

$$[G](\mathsf{tile\_end} \rightarrow \bigvee_{up(t_i)=\$} \mathtt{t_i}) \tag{3.76}$$

$$[G](\mathsf{tile\_bgn} \rightarrow \bigvee_{down(t_i)=\$} \mathtt{t_i}) \tag{3.77}$$

$$[G](\mathsf{tile} \wedge \langle A\rangle(* \vee \mathsf{end}) \rightarrow \bigvee_{right(t_i)=\$} \mathtt{t_i}) \tag{3.78}$$

$$[G](* \rightarrow \bigvee_{left(t_i)=\$} \langle A\rangle \mathtt{t_i}) \tag{3.79}$$

An analogous construction can also be done for the fragments $\mathsf{A\overline{D}}$, $\mathsf{\overline{A}D}$, and $\mathsf{\overline{A}\,\overline{D}}$. Thus, we can state the following theorem.

Theorem 3.2.17. *The satisfiability problem for the fragment* AD *(resp.,* $\mathsf{A\overline{D}}$, $\mathsf{\overline{A}D}$, $\mathsf{\overline{A}\,\overline{D}}$*) of* HS *is undecidable in any class of linear orderings that contains, for each* $n > 0$*, at least one linear ordering with length greater than* n*.*

3.3 The fragments BE, $\mathsf{B\overline{E}}$, $\mathsf{\overline{B}E}$, and $\mathsf{\overline{B}\,\overline{E}}$

In this section, we show the undecidability of the HS fragments BE, $\mathsf{B\overline{E}}$, $\mathsf{\overline{B}E}$, and $\mathsf{\overline{B}\,\overline{E}}$. We give a detailed proof for the fragments $\mathsf{\overline{B}E}$ and $\mathsf{\overline{B}\,\overline{E}}$, based on a reduction from the Octant Tiling Problem. The undecidability of $\mathsf{B\overline{E}}$ immediately follows, by symmetry, from the undecidability of $\mathsf{\overline{B}E}$. As for the fragment BE, it must be faced in a slightly different way. Since its operators $\langle B\rangle$ and $\langle E\rangle$ allows one to only refer to sub-intervals of the current one, there is no way of encoding an infinite plane, such as the second octant of the integer plane, unless assuming denseness. Thus, the undecidability of BE is achieved by means of a reduction from the Finite Tiling Problem, analogously to Section 3.2.3.

An useful way to facilitate the reading of the proofs is to use the interpretation proposed by Marx and Reynolds in [76]. According to such an interpretation, intervals are viewed as point of the semi-plane identified by the set $\{(x, y) \mid x < y\}$ (if the strict semantics is considered) or by the set $\{(x, y) \mid x \leq y\}$ (if the non-strict semantics is considered), and the modal operators $\langle B\rangle$, $\langle \overline{B}\rangle$, $\langle E\rangle$, and $\langle \overline{E}\rangle$ have, respectively, the following intuitive semantics:

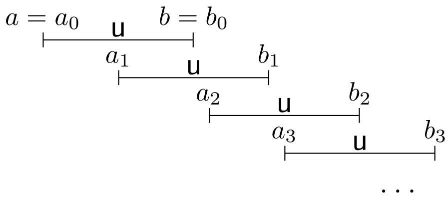


Figure 3.4: The construction of the u-chain in the fragment $\overline{\mathsf{B}}\mathsf{E}$

- $\langle B \rangle \varphi$ iff there exists a point *below* the current one in which φ holds,
- $\langle \overline{B} \rangle \varphi$ iff there exists a point *above* the current one in which φ holds,
- $\langle E \rangle \varphi$ iff there exists a point *on the right of* the current one in which φ holds,
- $\langle \overline{E} \rangle \varphi$ iff there exists a point *on the left of* the current one in which φ holds.

3.3.1 The fragment $\overline{\mathsf{B}}\mathsf{E}$

Let $[a, b]$ be a generic interval. The set $\mathcal{G}_{[a,b]}$ contains the interval $[a, b]$ and all the intervals $[c, d]$, with $c \geq a$ and $d \geq b$. The global operator $[G]$ is defined as:

$$[G]p = p \wedge [E]p \wedge [\overline{B}](p \wedge [E]p)$$

Definition of the u-chain. The construction of the u-chain, shown in Fig. 3.4, is done by means of the following formulae:

$$\neg \mathsf{u} \wedge [E]\neg \mathsf{u} \wedge [\overline{B}]\neg \mathsf{u} \wedge \langle E \rangle \langle \overline{B} \rangle \mathsf{u} \wedge [E](\langle \overline{B} \rangle \mathsf{u} \rightarrow [E][\overline{B}]\neg \mathsf{u}) \tag{3.80}$$

$$[G](\mathsf{u} \rightarrow [E]\neg \mathsf{u} \wedge [\overline{B}]\neg \mathsf{u} \wedge \langle E \rangle \langle \overline{B} \rangle \mathsf{u} \wedge [E](\langle \overline{B} \rangle \mathsf{u} \rightarrow [E][\overline{B}]\neg \mathsf{u})) \tag{3.81}$$

$$[G](\langle \overline{B} \rangle \mathsf{u} \rightarrow \neg \langle E \rangle \mathsf{u}) \tag{3.82}$$

$$(3.80) \wedge \ldots \wedge (3.82) \tag{3.83}$$

Lemma 3.3.1. *Let $M, [a, b] \Vdash$ (3.83) and let $[a, b] = [a_0, b_0]$. Then, there exists an infinite sequence of intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_i, b_i], \ldots$ belonging to $\mathcal{G}_{[a,b]}$, with $a_{i-1} < a_i < b_{i-1} < b_i$ and $b_{i-1} \leq a_{i+1}$ for each $i > 0$, and such that $M, [a_i, b_i] \Vdash \mathsf{u}$ and no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies u, unless $c > b_i$ for each $i > 0$.*

In order to refer to the next u-interval of the sequence, we define the abbreviation:

$$\langle X_{\mathsf{u}} \rangle \varphi = \langle E \rangle \langle \overline{B} \rangle (\mathsf{u} \wedge \varphi)$$

Definition of the Id**-chain.**

$$[G]((\mathsf{u} \leftrightarrow (* \vee \mathsf{tile})) \wedge (* \rightarrow \neg \mathsf{tile})) \tag{3.84}$$
$$\langle X_{\mathsf{u}} \rangle (* \wedge \langle X_{\mathsf{u}} \rangle (\mathsf{tile} \wedge \langle X_{\mathsf{u}} \rangle (* \wedge [G](* \rightarrow \langle X_{\mathsf{u}} \rangle (\mathsf{tile} \wedge \langle X_{\mathsf{u}} \rangle \mathsf{tile}))))) \tag{3.85}$$
$$\neg \mathsf{Id} \wedge [E]\neg \mathsf{Id} \wedge [\overline{B}]\neg \mathsf{Id} \wedge [E](\langle \overline{B} \rangle \mathsf{Id} \rightarrow \langle \overline{B} \rangle \mathsf{u}) \tag{3.86}$$
$$[G](\mathsf{u} \rightarrow [E]\neg \mathsf{Id} \wedge [E](\langle \overline{B} \rangle \mathsf{Id} \rightarrow \langle \overline{B} \rangle \mathsf{u})) \tag{3.87}$$
$$[G](\mathsf{Id} \rightarrow \neg \mathsf{u} \wedge [E]\neg \mathsf{Id} \wedge [\overline{B}]\neg \mathsf{Id} \wedge [\overline{B}]\neg \mathsf{u}) \tag{3.88}$$
$$[G](* \rightarrow \langle \overline{B} \rangle \mathsf{Id}) \tag{3.89}$$
$$[G](\mathsf{u} \wedge \langle \overline{B} \rangle \mathsf{Id} \rightarrow *) \tag{3.90}$$
$$[G](\mathsf{Id} \rightarrow \langle E \rangle *) \tag{3.91}$$
$$\langle X_{\mathsf{u}} \rangle ([\overline{B}](\langle E \rangle * \rightarrow \langle E \rangle \mathsf{Id})) \tag{3.92}$$
$$[G](\langle \overline{B} \rangle \mathsf{Id} \rightarrow \neg \langle E \rangle *) \tag{3.93}$$
$$(3.84) \wedge \ldots \wedge (3.93) \tag{3.94}$$

Lemma 3.3.2. *Let $[a, b]$ such that $M, [a, b] \Vdash (3.83) \wedge (3.94)$ and let $[a_1^0, b_1^0], [a_1^1, b_1^1], \ldots, [a_1^{k_1}, b_1^{k_1}], [a_2^0, b_2^0], \ldots, [a_2^{k_2}, b_2^{k_2}], \ldots, [a_j^0, b_j^0], \ldots, [a_j^{k_j}, b_j^{k_j}], \ldots$ be the sequence of intervals defined by Lemma 3.3.1. Then, $k_1 = 1$, $k_j > 1$ for each $j > 1$ and for each $j \geq 1$ we have that:*

- *$M, [a_j^0, b_j^0] \Vdash *$;*
- *$M, [a_j^i, b_j^i] \Vdash$ tile for each $1 \leq i \leq k_j$;*
- *$M, [a_j^0, b_{j+1}^0] \Vdash$ Id.*

Furthermore, no other interval $[c, d]$ belonging to $\mathcal{G}_{[a,b]}$ satisfies $*$, tile*, or* Id*, unless $c > b_j^i$ for each $i, j > 0$.*

Right-neighbor Relation. The encoding of the right-neighbor relation is trivial, since, from a tile-interval $[a_j^i, b_j^i]$, it is possible to refer to the tile-interval $[a_j^{i+1}, b_j^{i+1}]$, to which it is right connected, simply by exploiting the $\langle X_{\mathsf{u}} \rangle$ operator.

Above-neighbor Relation. The encoding of the above-neighbor relation is simpler than the ones for the previous fragments. To this end we only need the propositional letter up_rel. In particular, if $[a_j^i, b_j^i]$ and $[a_{j'}^{i'}, b_{j'}^{i'}]$ are tile-intervals, then we say that $[a_j^i, b_j^i]$ is above connected to $[a_{j'}^{i'}, b_{j'}^{i'}]$ if and only if $[a_j^i, b_{j'}^{i'}]$ is a up_rel-interval. The encoding is done exploiting the following formulae:

$$\neg\mathsf{up\_rel} \wedge [E]\neg\mathsf{up\_rel} \wedge [\overline{B}]\neg\mathsf{up\_rel} \wedge [E](\langle\overline{B}\rangle\mathsf{up\_rel} \rightarrow \langle\overline{B}\rangle\mathsf{u}) \tag{3.95}$$

$$[G](\mathsf{u} \rightarrow [E]\neg\mathsf{up\_rel} \wedge [E](\langle\overline{B}\rangle\mathsf{up\_rel} \rightarrow \langle\overline{B}\rangle\mathsf{u})) \tag{3.96}$$

$$[G](\mathsf{up\_rel} \rightarrow \neg\mathsf{u} \wedge [E]\neg\mathsf{up\_rel} \wedge [\overline{B}]\neg\mathsf{up\_rel} \wedge [\overline{B}]\neg\mathsf{u})) \tag{3.97}$$

$$[G](\mathsf{tile} \rightarrow \langle\overline{B}\rangle\mathsf{up\_rel}) \tag{3.98}$$

$$[G](\mathsf{u} \wedge \langle\overline{B}\rangle\mathsf{up\_rel} \rightarrow \mathsf{tile}) \tag{3.99}$$

$$[G](\mathsf{up\_rel} \rightarrow \langle E\rangle(\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle\mathsf{tile})) \tag{3.100}$$

$$[\overline{B}](\langle E\rangle(\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle\mathsf{tile}) \rightarrow \langle E\rangle\mathsf{up\_rel}) \tag{3.101}$$

$$[G](\langle\overline{B}\rangle\mathsf{Id} \rightarrow \neg\langle E\rangle\mathsf{up\_rel}) \tag{3.102}$$

$$[G](\langle\overline{B}\rangle\mathsf{up\_rel} \rightarrow \neg\langle E\rangle\mathsf{Id}) \tag{3.103}$$

$$[G](\langle\overline{B}\rangle\mathsf{up\_rel} \rightarrow \neg\langle E\rangle\mathsf{up\_rel}) \tag{3.104}$$

$$(3.95) \wedge \ldots \wedge (3.104) \tag{3.105}$$

Lemma 3.3.3. *Let $M, [a, b] \Vdash (3.83) \wedge (3.94) \wedge (3.105)$ and consider the sequence of points guaranteed by Lemma 3.3.1 and 3.3.2. Then, the following properties hold:*

- *each* tile*-interval $[a^i_j, b^i_j]$ is above connected to some* tile*-interval $[a^{i'}_{j'}, b^{i'}_{j'}]$, where $j \leq j'$;*
- *if $[a^i_j, b^{i'}_{j'}]$ is an* up_rel*-interval, with $1 \leq i \leq k_j$, then both $[a^i_j, b^i_j]$ and $[a^{i'}_{j'}, b^{i'}_{j'}]$ are* tile*-intervals, and $j' = j + 1$;*
- *each* tile*-interval $[a^i_j, b^i_j]$ such that $i < k_j$ has some* tile*-interval $[a^{i'}_{j-1}, b^{i'}_{j-1}]$ above connected to it;*
- *for any $j > 0$, $b^{k_j}_j$ is not the endpoint of any* up_rel*-interval, that is, the last* tile*-interval of each* Id*-interval has no* tile*-interval above connected to it;*
- *(uniqueness property) if $[a^i_j, b^i_j]$ is a* tile*-interval, then there exists at most one* up_rel*-interval starting at a^i_j, and at most one* up_rel*-interval ending at b^i_j, that is, each* tile*-interval is above connected to at most one* tile*-interval and there exists at most one* tile*-interval above connected to it.*

Finally, no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies up_rel*, unless $c > b^i_j$ for each $i, j > 0$.*

Lemma 3.3.4 (commutativity property)**.** *Let $M, [a, b] \Vdash (3.83) \wedge (3.94) \wedge (3.105)$ and consider the sequence of points of Lemma 3.3.1 and 3.3.2. Then, the commutativity property holds.*

Tiling the plane. In order to force how to tile the plane and in order to express the color constraints, we use the following formulae:

$$[G](\bigvee_{i=1}^{k} \mathtt{t_i} \leftrightarrow \mathsf{tile}) \tag{3.106}$$

$$[G](\mathsf{tile} \rightarrow \bigwedge_{i,j=1, i\neq j}^{k} \neg(\mathtt{t_i} \wedge \mathtt{t_j})) \tag{3.107}$$

$$[G](\mathsf{tile} \rightarrow \bigvee_{up(t_i)=down(t_j)} (\mathtt{t_i} \wedge \langle\overline{B}\rangle(\mathsf{up\_rel} \wedge \langle E\rangle \mathtt{t_j}))) \tag{3.108}$$

$$[G](\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle \mathsf{tile} \rightarrow \bigvee_{right(t_i)=left(t_j)} (\mathtt{t_i} \wedge \langle X_{\mathsf{u}}\rangle \mathtt{t_j})) \tag{3.109}$$

$$(3.106) \wedge \ldots \wedge (3.109) \tag{3.110}$$

Given the set of tiles $T = \{t_1, t_2, \ldots, t_k\}$, we define the formula:

$$\Phi_{\mathcal{T}} = (3.83) \wedge (3.94) \wedge (3.105) \wedge (3.110)$$

Lemma 3.3.5. *Given any finite set of tiles $\mathcal{T} = \{t_1, t_2, \ldots, t_k\}$, the formula $\Phi_{\mathcal{T}}$ is satisfiable if and only if $\mathcal{T}$ can tile the second octant $\mathcal{O}$.*

Theorem 3.3.6. *The satisfiability problem for the fragment* $\overline{\text{B}}$E *of* HS *is undecidable in any class of linear orderings that contains at least one linear ordering with an infinite ascending sequence of points.*

3.3.2 The fragment B$\overline{\text{E}}$

The previous reductions built for the fragment $\overline{\text{B}}$E can easily be extended, by symmetry, to the fragments B$\overline{\text{E}}$, provided that there exists an infinite descending sequence of points.

Theorem 3.3.7. *The satisfiability problem for the fragment* B$\overline{\text{E}}$ *of the Strict Modal Logic of Allen's Relations is undecidable in any class of linear orderings that contains at least one linear ordering with an infinite descending sequence of points.*

3.3.3 The fragment $\overline{\text{BE}}$

Let $[a, b]$ be a generic interval. The set $\mathcal{G}_{[a,b]}$ contains the interval $[a, b]$ and all its super-intervals, that is, all intervals $[c, d]$, with $c \leq a$ and $d \geq b$. The global operator $[G]$ is defined as:

$$[G]p = p \wedge [\overline{E}]p \wedge [\overline{B}]p \wedge [\overline{B}][\overline{E}]p$$

Definition of the u-chain.

$$\neg\mathsf{u} \wedge \neg\mathsf{k} \wedge \langle\overline{B}\rangle\mathsf{k} \wedge [\overline{B}]\neg\mathsf{u} \wedge [\overline{E}](\neg\mathsf{u} \wedge \neg\mathsf{k}) \tag{3.111}$$
$$[G](\mathsf{k} \rightarrow \neg\mathsf{u} \wedge [\overline{B}](\neg\mathsf{u} \wedge \neg\mathsf{k}) \wedge [\overline{E}]\neg\mathsf{k} \wedge \langle\overline{E}\rangle\mathsf{u}) \tag{3.112}$$
$$[\overline{E}](\langle\overline{B}\rangle\mathsf{k} \rightarrow \langle\overline{B}\rangle\mathsf{u}) \tag{3.113}$$
$$[G](\mathsf{u} \rightarrow [\overline{E}](\neg\mathsf{u} \wedge \neg\mathsf{k}) \wedge [\overline{B}]\neg\mathsf{u} \wedge \langle\overline{B}\rangle\mathsf{k}) \tag{3.114}$$
$$[\overline{B}](\langle\overline{E}\rangle\mathsf{u} \rightarrow \mathsf{k} \vee \langle\overline{E}\rangle\mathsf{k}) \tag{3.115}$$
$$[G]((\langle\overline{B}\rangle\mathsf{u} \rightarrow \neg\langle\overline{E}\rangle\mathsf{u}) \wedge (\langle\overline{B}\rangle\mathsf{k} \rightarrow \neg\langle\overline{E}\rangle\mathsf{k})) \tag{3.116}$$
$$(3.111) \wedge \ldots \wedge (3.116) \tag{3.117}$$

Lemma 3.3.8. *Let $M, [a, b] \Vdash (3.117)$ and let $[a, b] = [a_0, b_0]$. Then, there exists an infinite sequence of intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_i, b_i], \ldots$ belonging to $\mathcal{G}_{[a,b]}$, with $a_i < a_{i-1} < b_{i-1} < b_i$ for each $i > 0$, and such that $M, [a_i, b_i] \Vdash \mathsf{u}$ and no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies u, unless $c > b_i$ for each $i > 0$.*

In order to refer to the next u-interval of the sequence, we define the abbreviation:

$$\langle X_{\mathsf{u}}\rangle\varphi = \langle\overline{B}\rangle(\mathsf{k} \wedge \langle\overline{E}\rangle(\mathsf{u} \wedge \varphi))$$

Definition of the ld-chain.

$$[G]((\mathsf{u} \leftrightarrow (* \vee \mathsf{tile})) \wedge (* \rightarrow \neg\mathsf{tile})) \tag{3.118}$$
$$\langle X_{\mathsf{u}}\rangle(* \wedge \langle X_{\mathsf{u}}\rangle(\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle(* \wedge [G](* \rightarrow \langle X_{\mathsf{u}}\rangle(\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle\mathsf{tile}))))) \tag{3.119}$$
$$\neg\mathsf{ld} \wedge [\overline{E}]\neg\mathsf{ld} \wedge [\overline{B}]\neg\mathsf{ld} \wedge [\overline{E}](\langle\overline{B}\rangle\mathsf{ld} \rightarrow \langle\overline{B}\rangle*) \tag{3.120}$$
$$[G](* \rightarrow \langle\overline{B}\rangle\mathsf{ld}) \tag{3.121}$$
$$[G](\mathsf{ld} \rightarrow \neg\mathsf{u} \wedge [\overline{E}]\neg\mathsf{ld} \wedge [\overline{B}]\neg\mathsf{ld}) \tag{3.122}$$
$$[G](\mathsf{ld} \rightarrow \langle\overline{E}\rangle*) \tag{3.123}$$
$$[\overline{B}](\langle\overline{E}\rangle * \wedge \langle\overline{E}\rangle\mathsf{k} \rightarrow \langle\overline{E}\rangle\mathsf{ld}) \tag{3.124}$$
$$[G](\langle\overline{E}\rangle* \rightarrow \neg\langle\overline{B}\rangle\mathsf{ld}) \tag{3.125}$$
$$(3.118) \wedge \ldots \wedge (3.125) \tag{3.126}$$

Lemma 3.3.9. *Let $[a, b]$ such that $M, [a, b] \Vdash (3.117) \wedge (3.126)$ and let $[a_1^0, b_1^0], [a_1^1, b_1^1], \ldots, [a_1^{k_1}, b_1^{k_1}], [a_2^0, b_2^0], \ldots, [a_2^{k_2}, b_2^{k_2}], \ldots, [a_j^0, b_j^0], \ldots, [a_j^{k_j}, b_j^{k_j}], \ldots$ be the sequence of intervals defined by Lemma 3.3.8, then $k_1 = 1$, $k_j > 1$ for each $j > 1$ and for each $j \geq 1$ we have that:*

- *$M, [a_j^0, b_j^0] \Vdash *$;*
- *$M, [a_j^i, b_j^i] \Vdash \mathsf{tile}$ for each $1 \leq i \leq k_j$;*
- *$M, [a_j^0, b_{j+1}^0] \Vdash \mathsf{ld}$.*

Furthermore, no other interval $[c, d]$ belonging to $\mathcal{G}_{[a,b]}$ satisfies $*$, tile, or Id, unless $c > b^i_j$ for each $i, j > 0$.

Right-neighbor relation. The encoding of the right-neighbor relation is trivial, since, from a tile-interval $[a^i_j, b^i_j]$, it is possible to refer to the tile-interval $[a^{i+1}_j, b^{i+1}_j]$, to which it is right connected, simply by exploiting the $\langle X_\mathsf{u} \rangle$ operator.

Above-neighbor relation. The encoding of the above-neighbor relation is very similar to that one for the fragment $\mathsf{\overline{B}E}$. In particular, if $[a^i_j, b^i_j]$ and $[a^{i'}_{j'}, b^{i'}_{j'}]$ are tile-intervals, then we say that $[a^i_j, b^i_j]$ is above connected to $[a^{i'}_{j'}, b^{i'}_{j'}]$ if and only if $[a^i_j, b^{i'}_{j'}]$ is an $\mathsf{up\_rel}$-interval. The encoding is done exploiting the following formulae:

$$\neg\mathsf{up\_rel} \wedge [\overline{E}]\neg\mathsf{up\_rel} \wedge [\overline{B}]\neg\mathsf{up\_rel} \wedge [\overline{E}](\langle\overline{B}\rangle\mathsf{up\_rel} \rightarrow \langle\overline{B}\rangle\mathsf{tile}) \tag{3.127}$$

$$[G](\mathsf{tile} \rightarrow \langle\overline{B}\rangle\mathsf{up\_rel}) \tag{3.128}$$

$$[G](\mathsf{up\_rel} \rightarrow \neg\mathsf{u} \wedge [E]\neg\mathsf{up\_rel} \wedge [\overline{B}]\neg\mathsf{up\_rel}) \tag{3.129}$$

$$[G](\mathsf{up\_rel} \rightarrow \langle E\rangle(\mathsf{tile} \wedge \langle X_\mathsf{u}\rangle\mathsf{tile})) \tag{3.130}$$

$$[\overline{B}](\langle E\rangle(\mathsf{tile} \wedge \langle X_\mathsf{u}\rangle\mathsf{tile}) \rightarrow \langle E\rangle\mathsf{up\_rel}) \tag{3.131}$$

$$[G](\langle\overline{B}\rangle\mathsf{Id} \rightarrow \neg\langle\overline{E}\rangle\mathsf{up\_rel}) \tag{3.132}$$

$$[G](\langle\overline{B}\rangle\mathsf{up\_rel} \rightarrow \neg\langle\overline{E}\rangle\mathsf{Id}) \tag{3.133}$$

$$[G](\langle\overline{B}\rangle\mathsf{up\_rel} \rightarrow \neg\langle\overline{E}\rangle\mathsf{up\_rel}) \tag{3.134}$$

$$(3.127) \wedge \ldots \wedge (3.134) \tag{3.135}$$

Lemma 3.3.10. *Let $M, [a, b] \Vdash (3.117) \wedge (3.126) \wedge (3.135)$ and consider the sequence of points guaranteed by Lemma 3.3.8 and 3.3.9. Then, the following properties hold:*

- *each tile-interval $[a^i_j, b^i_j]$ is above connected to some tile-interval $[a^{i'}_{j'}, b^{i'}_{j'}]$, where $j \leq j'$;*
- *if $[a^i_j, b^{i'}_{j'}]$ is an $\mathsf{up\_rel}$-interval, with $1 \leq i \leq k_j$, then both $[a^i_j, b^i_j]$ and $[a^{i'}_{j'}, b^{i'}_{j'}]$ are tile-intervals, and $j' = j + 1$;*
- *each tile-interval $[a^i_j, b^i_j]$ such that $i < k_j$ has some tile-interval $[a^{i'}_{j-1}, b^{i'}_{j-1}]$ above connected to it;*
- *for any $j > 0$, $b^{k_j}_j$ is not the endpoint of any $\mathsf{up\_rel}$-interval, that is, the last tile-interval of each Id-interval has no tile-interval above connected to it;*
- *(uniqueness property) if $[a^i_j, b^i_j]$ is a tile-interval, then there exists at most one $\mathsf{up\_rel}$-interval starting at a^i_j, and at most one $\mathsf{up\_rel}$-interval ending at b^i_j, that is, each tile-interval is above connected to at most one tile-interval and there exists at most one tile-interval above connected to it.*

Finally, no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies $\mathsf{up\_rel}$, unless $c > b^i_j$ for each $i, j > 0$.

Lemma 3.3.11 (commutativity property)**.** *Let* $M, [a,b] \Vdash$ (3.117)$\wedge$(3.126)$\wedge$(3.135) *and consider the sequence of points of Lemma 3.3.8 and 3.3.9. Then, the commutativity property holds.*

Tiling the plane. In order to force how to tile the plane and in order to express the color constraints, we use the following formulae:

$$[G](\bigvee_{i=1}^{k} \mathtt{t_i} \leftrightarrow \mathsf{tile}) \tag{3.136}$$

$$[G](\mathsf{tile} \rightarrow \bigwedge_{i,j=1, i\neq j}^{k} \neg(\mathtt{t_i} \wedge \mathtt{t_j})) \tag{3.137}$$

$$[G](\mathsf{tile} \rightarrow \bigvee_{up(t_i)=down(t_j)} (\mathtt{t_i} \wedge \langle \overline{B} \rangle(\mathsf{up\_rel} \wedge \langle \overline{E} \rangle \mathtt{t_j}))) \tag{3.138}$$

$$[G](\mathsf{tile} \wedge \langle X_{\mathsf{u}} \rangle \mathsf{tile} \rightarrow \bigvee_{right(t_i)=left(t_j)} (\mathtt{t_i} \wedge \langle X_{\mathsf{u}} \rangle \mathtt{t_j})) \tag{3.139}$$

$$(3.136) \wedge \ldots \wedge (3.139) \tag{3.140}$$

Given the set of tiles $T = \{t_1, t_2, \ldots, t_k\}$, we define the formula:

$$\Phi_{\mathcal{T}} = (3.117) \wedge (3.126) \wedge (3.135) \wedge (3.140)$$

Lemma 3.3.12. *Given any finite set of tiles* $\mathcal{T} = \{t_1, t_2, \ldots, t_k\}$*, the formula* $\Phi_{\mathcal{T}}$ *is satisfiable if and only if* $\mathcal{T}$ *can tile the second octant* $\mathcal{O}$*.*

Theorem 3.3.13. *The satisfiability problem for the fragment* $\overline{\mathsf{BE}}$ *of* HS *is undecidable in any class of linear orderings that contains at least one linear ordering with an infinite ascending and descending sequence of points.*

3.3.4 The fragment BE

In this section, we give a reduction from the Finite Tiling Problem to the satisfiability problem for the fragment BE of HS.

Let $[a,b]$ be a generic interval. The set $\mathcal{G}_{[a,b]}$ contains the interval $[a,b]$ and all its sub-intervals, that is, all intervals $[c,d]$, with $c \geq a$ and $d \leq b$. The global operator $[G]$ is defined as:

$$[G]p = p \wedge [E]p \wedge [B]p \wedge [B][E]p$$

Definition of the u**-chain.**

$$\neg\mathsf{u} \wedge \neg\mathsf{k} \wedge \langle B\rangle\mathsf{k} \wedge [B]\neg\mathsf{u} \wedge [E](\neg\mathsf{u} \wedge \neg\mathsf{k}) \wedge \langle B\rangle\langle E\rangle([B]\bot \wedge \mathsf{last}) \tag{3.141}$$
$$[G](\mathsf{last} \rightarrow \mathsf{u} \wedge [B]\bot) \tag{3.142}$$
$$[G](\mathsf{k} \rightarrow \neg\mathsf{u} \wedge [B](\neg\mathsf{u} \wedge \neg\mathsf{k}) \wedge [E]\neg\mathsf{k} \wedge \langle E\rangle\mathsf{u}) \tag{3.143}$$
$$[E](\langle B\rangle\mathsf{k} \rightarrow \langle B\rangle\mathsf{u}) \tag{3.144}$$
$$[G](\mathsf{u} \rightarrow [E](\neg\mathsf{u} \wedge \neg\mathsf{k}) \wedge [B]\neg\mathsf{u} \wedge (\neg\mathsf{last} \rightarrow \langle B\rangle\mathsf{k})) \tag{3.145}$$
$$[B](\langle E\rangle\mathsf{u} \rightarrow \mathsf{k} \vee \langle E\rangle\mathsf{k}) \tag{3.146}$$
$$[G]((\langle B\rangle\mathsf{u} \rightarrow \neg\langle E\rangle\mathsf{u}) \wedge (\langle B\rangle\mathsf{k} \rightarrow \neg\langle E\rangle\mathsf{k})) \tag{3.147}$$
$$(3.141) \wedge \ldots \wedge (3.147) \tag{3.148}$$

Lemma 3.3.14. *Let $M, [a, b] \Vdash$ (3.148) and let $[a, b] = [a_0, b_0]$. Then, there exists a finite sequence of intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_i, b_i], \ldots, [a_r, b_r]$ belonging to $\mathcal{G}_{[a,b]}$, with $a_{i-1} < a_i < b_i < b_{i-1}$ for each $0 < i \le r$, and such that $M, [a_i, b_i] \Vdash \mathsf{u}$ and no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies u.*

In order to refer to the next u-interval of the sequence, we define the abbreviation:

$$\langle X_{\mathsf{u}}\rangle\varphi = \neg\mathsf{last} \rightarrow \langle B\rangle(\mathsf{k} \wedge \langle E\rangle(\mathsf{u} \wedge \varphi))$$

Definition of the ld**-chain.**

$$[G]((\mathsf{u} \leftrightarrow (* \vee \mathsf{tile})) \wedge (* \rightarrow \neg\mathsf{tile}) \wedge (\mathsf{last} \rightarrow *)) \tag{3.149}$$
$$\langle X_{\mathsf{u}}\rangle * \wedge [G](* \rightarrow \langle X_{\mathsf{u}}\rangle\mathsf{tile}) \tag{3.150}$$
$$\neg\mathsf{ld} \wedge [E]\neg\mathsf{ld} \wedge [B]\neg\mathsf{ld} \wedge [E](\langle B\rangle\mathsf{ld} \rightarrow \langle B\rangle *) \tag{3.151}$$
$$[G](* \wedge \neg\mathsf{last} \rightarrow \langle B\rangle\mathsf{ld}) \tag{3.152}$$
$$[G](\mathsf{ld} \rightarrow \neg\mathsf{u} \wedge [E]\neg\mathsf{ld} \wedge [B]\neg\mathsf{ld}) \tag{3.153}$$
$$[G](\mathsf{ld} \rightarrow \langle E\rangle *) \tag{3.154}$$
$$[B](\langle E\rangle * \wedge \langle E\rangle\mathsf{k} \rightarrow \langle E\rangle\mathsf{ld}) \tag{3.155}$$
$$[G](\langle E\rangle * \rightarrow \neg\langle B\rangle\mathsf{ld}) \tag{3.156}$$
$$(3.149) \wedge \ldots \wedge (3.156) \tag{3.157}$$

Lemma 3.3.15. *Let $[a, b]$ such that $M, [a, b] \Vdash$ (3.148) $\wedge$ (3.157) and let $[a_1^0, b_1^0]$, $[a_1^1, b_1^1], \ldots, [a_1^{k_1}, b_1^{k_1}], [a_2^0, b_2^0], \ldots, [a_2^{k_2}, b_2^{k_2}], \ldots, [a_q^0, b_q^0], \ldots, [a_q^{k_q}, b_q^{k_q}], [a_{q+1}^0, b_{q+1}^0]$ be the sequence of intervals defined by Lemma 3.3.14. Then, we have that:*

- *$k_j \ge 1$ for each $1 \le j \le q$;*
- *$M, [a_j^0, b_j^0] \Vdash *$ for each $1 \le j \le q + 1$;*
- *$M, [a_j^i, b_j^i] \Vdash \mathsf{tile}$ for each $1 \le i \le k_j$, $1 \le j \le q$;*
- *$M, [a_j^0, b_{j+1}^0] \Vdash \mathsf{ld}$ for each $1 \le j \le q$.*

Furthermore, no other interval $[c,d]$ *belonging to* $\mathcal{G}_{[a,b]}$ *satisfies* $*$, tile, *or* Id.

Right-neighbor relation. The encoding of the right-neighbor relation is trivial, since, from a tile-interval $[a^i_j, b^i_j]$, it is possible to refer to the tile-interval $[a^{i+1}_j, b^{i+1}_j]$, to which it is right connected, simply by exploiting the $\langle X_{\mathsf{u}}\rangle$ operator.

Above-neighbor relation. The encoding of the above-neighbor relation is very similar to the previous ones. In particular, if $[a^i_j, b^i_j]$ and $[a^{i'}_{j'}, b^{i'}_{j'}]$ are tile-intervals, then we say that $[a^i_j, b^i_j]$ is above connected to $[a^{i'}_{j'}, b^{i'}_{j'}]$ if and only if $[a^i_j, b^{i'}_{j'}]$ is an up_rel-interval. The encoding is done exploiting the following formulae:

$$\neg\mathsf{up\_rel} \wedge [E]\neg\mathsf{up\_rel} \wedge [B]\neg\mathsf{up\_rel} \wedge [E](\langle B\rangle\mathsf{up\_rel} \rightarrow \langle B\rangle\mathsf{tile}) \tag{3.158}$$

$$[G](\mathsf{tile} \wedge [B][E](* \rightarrow \mathsf{last}) \leftrightarrow \mathsf{tile\_end}) \tag{3.159}$$

$$\langle X_{\mathsf{u}}\rangle(* \wedge \langle X_{\mathsf{u}}\rangle\mathsf{tile\_bgn} \wedge \langle B\rangle(\mathsf{Id} \wedge \langle E\rangle(* \wedge [B][E]\neg\mathsf{tile\_bgn}))) \tag{3.160}$$

$$[G]((\mathsf{tile\_bgn} \rightarrow \mathsf{tile}) \wedge (\mathsf{tile\_bgn} \wedge \langle X_{\mathsf{u}}\rangle\mathsf{tile} \rightarrow \langle X_{\mathsf{u}}\rangle\mathsf{tile\_bgn})) \tag{3.161}$$

$$[G](\mathsf{tile} \rightarrow (\neg\mathsf{tile\_end} \leftrightarrow \langle B\rangle\mathsf{up\_rel})) \tag{3.162}$$

$$[G](\mathsf{up\_rel} \rightarrow \neg\mathsf{u} \wedge [E]\neg\mathsf{up\_rel} \wedge [B]\neg\mathsf{up\_rel} \wedge [B]\neg\mathsf{tile}) \tag{3.163}$$

$$[G](\mathsf{up\_rel} \rightarrow \langle E\rangle\mathsf{tile}) \tag{3.164}$$

$$[B](\langle E\rangle(\mathsf{tile} \wedge \neg\mathsf{tile\_bgn}) \rightarrow \langle E\rangle\mathsf{up\_rel}) \tag{3.165}$$

$$[G](\langle B\rangle\mathsf{Id} \rightarrow \neg\langle E\rangle\mathsf{up\_rel}) \tag{3.166}$$

$$[G](\langle B\rangle\mathsf{up\_rel} \rightarrow \neg\langle E\rangle\mathsf{Id}) \tag{3.167}$$

$$[G](\langle B\rangle\mathsf{up\_rel} \rightarrow \neg\langle E\rangle\mathsf{up\_rel}) \tag{3.168}$$

$$(3.158) \wedge \ldots \wedge (3.168) \tag{3.169}$$

Lemma 3.3.16. *Let* $M, [a,b] \Vdash (3.148) \wedge (3.157) \wedge (3.169)$ *and consider the sequence of points guaranteed by Lemma 3.3.14 and 3.3.15. Then, the following properties hold:*

- *each* tile*-interval* $[a^i_j, b^i_j]$ *not belonging to the last* Id*-interval is above connected to some* tile*-interval* $[a^{i'}_{j'}, b^{i'}_{j'}]$, *where* $j \leq j'$;
- *if* $[a^i_j, b^{i'}_{j'}]$ *is an* up_rel*-interval, with* $1 \leq i \leq k_j$, *then both* $[a^i_j, b^i_j]$ *and* $[a^{i'}_{j'}, b^{i'}_{j'}]$ *are* tile*-intervals, and* $j' = j + 1$;
- *each* tile*-interval* $[a^i_j, b^i_j]$ *has some* tile*-interval* $[a^{i'}_{j-1}, b^{i'}_{j-1}]$ *above connected to it;*
- *(uniqueness property) if* $[a^i_j, b^i_j]$ *is a* tile*-interval, then there exists at most one* up_rel*-interval starting at* a^i_j, *and at most one* up_rel*-interval ending at* b^i_j, *that is, each* tile*-interval is above connected to at most one* tile*-interval and there exists at most one* tile*-interval above connected to it.*

Finally, no other interval $[c,d] \in \mathcal{G}_{[a,b]}$ *satisfies* up_rel.

Lemma 3.3.17 (commutativity property)**.** *Let $M, [a, b] \Vdash$ (3.148)$\wedge$(3.157)$\wedge$(3.169) and consider the sequence of points of Lemma 3.3.14 and 3.3.15. Then, the commutativity property holds.*

Tiling the plane. In order to force how to tile the plane and in order to express the color constraints, we use the following formulae:

$$[G](\bigvee_{i=1}^{k} \mathtt{t_i} \leftrightarrow \mathsf{tile}) \tag{3.170}$$

$$[G](\mathsf{tile} \to \bigwedge_{i,j=1, i\neq j}^{k} \neg(\mathtt{t_i} \wedge \mathtt{t_j})) \tag{3.171}$$

$$[G](\mathsf{tile} \wedge \neg\mathsf{tile\_end} \to \bigvee_{up(t_i)=down(t_j)} (\mathtt{t_i} \wedge \langle B\rangle(\mathsf{up\_rel} \wedge \langle E\rangle \mathtt{t_j}))) \tag{3.172}$$

$$[G](\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle\mathsf{tile} \to \bigvee_{right(t_i)=left(t_j)} (\mathtt{t_i} \wedge \langle X_{\mathsf{u}}\rangle \mathtt{t_j})) \tag{3.173}$$

$$[G](\mathsf{tile\_end} \to \bigvee_{up(t_i)=\$} \mathtt{t_i}) \tag{3.174}$$

$$[G](\mathsf{tile\_bgn} \to \bigvee_{down(t_i)=\$} \mathtt{t_i}) \tag{3.175}$$

$$[G](\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle * \to \bigvee_{right(t_i)=\$} \mathtt{t_i}) \tag{3.176}$$

$$[G](* \wedge \neg\mathsf{last} \to \bigvee_{left(t_i)=\$} \langle X_{\mathsf{u}}\rangle \mathtt{t_i}) \tag{3.177}$$

$$(3.170) \wedge \ldots \wedge (3.177) \tag{3.178}$$

Given the set of tiles $T = \{t_1, t_2, \ldots, t_k\}$, we define the formula:

$$\Phi_{\mathcal{T}} = (3.148) \wedge (3.157) \wedge (3.169) \wedge (3.178)$$

Lemma 3.3.18. *Given any finite set of tiles $\mathcal{T} = \{t_1, t_2, \ldots, t_k\}$, the formula $\Phi_{\mathcal{T}}$ is satisfiable if and only if $\mathcal{T}$ can tile a rectangular area whose edges are colored by the same distinguished color \$.*

Theorem 3.3.19. *The satisfiability problem for the fragment* BE *of* HS *is undecidable in any class of linear orderings that contains, for each $n > 0$, at least one linear ordering with length greater than n.*

3.4 The fragment O

Let $[a, b]$ be an interval such that there exists at least a point in between a and b. The set $\mathcal{G}_{[a,b]}$ contains the interval $[a, b]$ and all the intervals $[c, d]$ such that $c > a$,

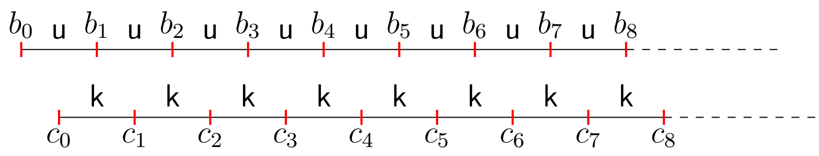


Figure 3.5: The encoding of the u-chain in the fragment O: u-intervals are adjacent and each pair of consecutive u-intervals is connected by a k-interval

$d > b$, and there exists at least a point in between c and d. The global operator $[G]$ is defined as:

$$[G]p \equiv p \wedge [O]p \wedge [O][O]p.$$

Definition of the u-chain. The main problem we must solve when dealing with the logic O is the construction of the u-chain. This task can be summarized in the three following step.

1. To force the existence of a unique chain of u-intervals. Formally, we force the existence of an infinite sequence of u-intervals $[b_0, b'_0], [b_1, b'_1], \ldots, [b_i, b'_i], \ldots$, with $b \leq b_0$ and $b'_i = b_{i+1}$ for each $i \in \mathbb{N}$.

2. To specify how to reach, from a given u-interval, the next one by using only the operator $\langle O \rangle$. To this end, we exploit the propositional letter k, connecting two consecutive u-intervals. Thus, we also force the existence of an infinite sequence of k-intervals $[c_0, c'_0], [c_1, c'_1], \ldots, [c_i, c'_i], \ldots$ such that $b_i < c_i < b'_i$, $b_{i+1} < c'_i < b'_{i+1}$, and $c'_i = c_{i+1}$ for each $i \in \mathbb{N}$. In such a way we guarantee the existence of an infinite sequence of k-intervals interleaving the sequence of u-intervals (Fig. 3.5).

3. To guarantee that there are no other u- or k-intervals belonging to $\mathcal{G}_{[a,b]}$ (unicity of the u- and k-chains).

The third step is the most difficult. In order to guarantee the unicity of the u- and k-chains, we will show, in the following, that it is possible, with the only operator $\langle O \rangle$, to somehow express properties of (proper) sub-intervals of a given interval. In particular, for a particular kind of propositional letters p, that we call *disjointly-bounded* (see Definition 3.4.2), it is possible to express properties like "for each interval $[a, b]$, if $[a, b]$ satisfies p then any sub-interval of $[a, b]$ does not satisfy p".

Let M be a model over the set $\mathcal{AP}$ of propositional letters and $[a, b]$ an interval over M.

Definition 3.4.1. The propositional letters p and q belonging to $\mathcal{AP}$ are said *disjoint* if, for each pair of intervals $\langle [c, d], [e, f] \rangle$ such that $[c, d]$ satisfies p and $[e, f]$ satisfies q, either $d \leq e$ or $f \leq c$. The propositional letter q is said *disjoint consequent* of p if p and q are disjoint and any p-interval is followed by a q-interval, that is, for each interval $[c, d]$ $(\in \mathcal{G}_{[a,b]})$ satisfying p, there exists an interval $[e, f]$ $(\in \mathcal{G}_{[a,b]})$, with $e \geq d$, satisfying q.

Definition 3.4.2. The propositional letter $p \in \mathcal{AP}$ is said to be *disjointly-bounded in* $\langle M, [a,b] \rangle$ if it fits the following requirements:

1. $[a,b]$ neither satisfies p nor overlaps a p-interval, that is, p holds only over interval $[c,d]$, with $c \geq b$,
2. p-intervals do not overlap each other, that is, there are not two intervals $[c,d]$ and $[e,f]$ satisfying p and such that $c < e < d < f$,
3. p has a disjoint consequent.

In the following, we see how to exploit the above definitions in order to state properties about sub-intervals of a given interval. First of all, for each propositional letter p that is disjointly-bounded in $\langle M, [a,b] \rangle$, it is possible to suitably define the propositional letter inside_p in such a way that it will turn out to be true over all sub-intervals of p-intervals. Then, by simply saying that inside_p-intervals and p-intervals cannot overlap, we are done. In order to suitably define inside_p, we exploit the fact that p is disjointly-bounded (and thus there exists a propositional letter, say it q, that is a disjoint consequent of p) and the auxiliary propositional letter $\overrightarrow{p}$, that is true over a suitable subset of interval starting inside a p-interval and ending outside it.

$$[G](p \rightarrow [O](\langle O \rangle q \rightarrow \overrightarrow{p})) \tag{3.179}$$

$$[G](\neg p \wedge [O](\langle O \rangle q \rightarrow \overrightarrow{p}) \rightarrow \mathsf{inside}_p) \tag{3.180}$$

$$[G]((\mathsf{inside}_p \rightarrow \neg \langle O \rangle p) \wedge (p \rightarrow \neg \langle O \rangle \mathsf{inside}_p)) \tag{3.181}$$

Lemma 3.4.3. *Let M be a model, $[a,b]$ be an interval over M, and $p, q \in \mathcal{AP}$ be two propositional letters such that p is disjointly-bounded in $\langle M, [a,b] \rangle$ and q is a disjoint consequent of p. If $M, [a,b] \Vdash$ (3.179) $\wedge \ldots \wedge$ (3.181), then there is no p-interval (belonging to $\mathcal{G}_{[a,b]}$) that is sub-interval of another p-interval.*

Proof. Suppose, by contradiction, that there exist two intervals $[c,d]$ and $[e,f]$ (belonging to $\mathcal{G}_{[a,b]}$) satisfying p and such that $[e,f]$ is sub-interval of $[c,d]$. By definition of sub-interval, we have that $c < e$ or $f < d$. Without loss of generality, let us suppose that $c < e$ (the other case is analogous). Since $[e,f] \in \mathcal{G}_{[a,b]}$, then there exists a point in between e and f, say it e'. The interval $[c,e']$ is a sub-interval of $[c,d]$. Moreover, it cannot satisfy p, since it overlaps the p-interval $[e,f]$ (and p is a propositional letter disjointly-bounded in $\langle M, [a,b] \rangle$). By (3.179) and by the fact that q is a disjoint consequent of p, each interval starting in between c and d, and ending inside a q-interval, satisfies $\overrightarrow{p}$. Thus, $[c,e']$ satisfies $\neg p$ and $[O](\langle O \rangle q \rightarrow \overrightarrow{p})$. By (3.180), it must also satisfy inside_p. But this contradicts (3.181), hence the thesis. ☐

From now on, for a given propositional letter p that is disjointly-bounded in $\langle M, [a,b] \rangle$, we will use *not_subint*(p) to denote the property that there is no p-interval that is a sub-interval of another p-interval. The following formulae force u_1,

u_2, k_1, and k_2 to be disjointly-bounded propositional letters.

$$\neg \mathsf{u} \wedge \neg \mathsf{k} \wedge [O](\neg \mathsf{u} \wedge \neg \mathsf{k}) \tag{3.182}$$
$$[G]((\mathsf{u} \leftrightarrow \mathsf{u}_1 \vee \mathsf{u}_2) \wedge (\mathsf{k} \leftrightarrow \mathsf{k}_1 \vee \mathsf{k}_2) \wedge (\mathsf{u}_1 \rightarrow \neg \mathsf{u}_2) \wedge (\mathsf{k}_1 \rightarrow \neg \mathsf{k}_2)) \tag{3.183}$$
$$[G]((\mathsf{u}_1 \rightarrow [O](\neg \mathsf{u} \wedge \neg \mathsf{k}_2)) \wedge (\mathsf{u}_2 \rightarrow [O](\neg \mathsf{u} \wedge \neg \mathsf{k}_1))) \tag{3.184}$$
$$[G]((\mathsf{k}_1 \rightarrow [O](\neg \mathsf{k} \wedge \neg \mathsf{u}_1)) \wedge (\mathsf{k}_2 \rightarrow [O](\neg \mathsf{k} \wedge \neg \mathsf{u}_2))) \tag{3.185}$$
$$[G]((\langle O\rangle \mathsf{u}_1 \rightarrow \neg \langle O\rangle \mathsf{u}_2) \wedge (\langle O\rangle \mathsf{k}_1 \wedge \neg \langle O\rangle \mathsf{k}_2)) \tag{3.186}$$
$$[G]((\mathsf{u}_1 \rightarrow \langle O\rangle \mathsf{k}_1) \wedge (\mathsf{k}_1 \rightarrow \langle O\rangle \mathsf{u}_2) \wedge (\mathsf{u}_2 \rightarrow \langle O\rangle \mathsf{k}_2) \wedge (\mathsf{k}_2 \rightarrow \langle O\rangle \mathsf{u}_1)) \tag{3.187}$$
$$(3.182) \wedge \ldots \wedge (3.187) \tag{3.188}$$

Lemma 3.4.4. *Let M be a model and $[a, b]$ and interval over M such that $M, [a, b] \Vdash$ (3.188). Then u_1, u_2, k_1, and k_2 are disjointly-bounded propositional letters.*

Proof. We only give the proof for the propositional letter u_1, since the other cases are analogous. We have to show that u_1 meets the three requirements of Definition 3.4.2. By (3.182) and (3.183), $[a, b]$ neither satisfies u_1 nor overlaps a u_1-interval (requirement 1). By (3.183) and (3.184), u_1-intervals do not overlap each other (requirement 2). We have to show that u_2 is a disjoint consequent of u_1 (requirement 3). First, we have to prove that u_1 and u_2 are disjoint. To this end, suppose, by contradiction, that there are two intervals $[c, d]$ and $[e, f]$ such that $[c, d]$ satisfies u_1, $[e, f]$ satisfies u_2, $e < d$, and $f > c$. Thus, we distinguish three cases:

- if $e < c$, then if $c < f < d$, then (3.184) is contradicted, if $f \geq d$, then (3.186) is contradicted;
- if $e = c$, then (3.186) is contradicted;
- if $e > c$, then if $f \leq d$, then (3.186) is contradicted, if $f > d$, then (3.184) is contradicted.

Then, we have to prove that, for each u_1-interval $[c, d]$, there exists an u_2-interval $[e, f]$, with $e \geq d$. This immediately follows from (3.187) and from the fact that u_1 and u_2 are disjoint. ☐

As a consequence of the previous lemma, we can force u_1-intervals (resp., u_2-intervals, k_1-intervals, k_2-intervals) not to be sub-intervals of other u_1-intervals (resp., u_2-intervals, k_1-intervals, k_2-intervals) by means of the formula $\mathit{not\_subint}(\mathsf{u}_1)$ (resp., $\mathit{not\_subint}(\mathsf{u}_2)$, $\mathit{not\_subint}(\mathsf{k}_1)$, $\mathit{not\_subint}(\mathsf{k}_2)$). In order to build the u-chain, we exploit the following formulae.

$$\langle O\rangle\langle O\rangle(\mathsf{u}_1 \wedge \mathsf{first}) \tag{3.189}$$
$$[G](\mathsf{u} \vee \mathsf{k} \rightarrow [O]\neg\mathsf{first} \wedge [O][O]\neg\mathsf{first}) \tag{3.190}$$
$$[G]((\mathsf{first} \rightarrow \mathsf{u}_1) \wedge (\mathsf{first} \rightarrow [O][O]\neg\mathsf{first})) \tag{3.191}$$
$$not\_subint(\mathsf{u}_1) \wedge not\_subint(\mathsf{u}_2) \wedge not\_subint(\mathsf{k}_1) \wedge not\_subint(\mathsf{k}_2) \tag{3.192}$$
$$[G](\mathsf{u} \vee \mathsf{k} \rightarrow [O]\langle O\rangle(\mathsf{u} \vee \mathsf{k})) \tag{3.193}$$
$$(3.189) \wedge \ldots \wedge (3.193) \tag{3.194}$$

Lemma 3.4.5. *Let M be a model and $[a, b]$ and interval over M such that $M, [a, b] \Vdash$ (3.188) $\wedge$ (3.194). Then:*

(a) there exists an infinite sequence of u*-intervals $[b_0, b'_0], [b_1, b'_1], \ldots, [b_i, b'_i], \ldots$, with $b \leq b_0$, $b'_i = b_{i+1}$ for each $i \in \mathbb{N}$, and such that $M, [b_0, b'_0] \Vdash$* first*,*

(b) there exists an infinite sequence of k*-intervals $[c_0, c'_0], [c_1, c'_1], \ldots, [c_i, c'_i], \ldots$ such that $b_i < c_i < b'_i$, $b_{i+1} < c'_i < b'_{i+1}$, and $c'_i = c_{i+1}$ for each $i \in \mathbb{N}$, and*

(c) each other interval $[c, d] \in \mathcal{G}_{[a,b]}$ does satisfy neither u*,* k*, nor* first*, unless $c > b_i$ for each $i \in \mathbb{N}$.*

Proof. For the sake of simplicity, we will first prove a variant of points (a) and (b), that is, respectively,

(a') there exists an infinite sequence of u-intervals $[b_0, b'_0], [b_1, b'_1], \ldots, [b_i, b'_i], \ldots$, with $b \leq b_0$, $b'_i \leq b_{i+1}$ for each $i \in \mathbb{N}$, and such that $M, [b_0, b'_0] \Vdash$ first,

(b') there exists an infinite sequence of k-intervals $[c_0, c'_0], [c_1, c'_1], \ldots, [c_i, c'_i], \ldots$ such that $b_i < c_i < b'_i$, $b_{i+1} < c'_i < b'_{i+1}$, and $c'_i \leq c_{i+1}$ for each $i \in \mathbb{N}$.

Then, we will prove point (c). Finally, we will force $b'_i = b_{i+1}$ and $c'_i = c_{i+1}$ for each $i \in \mathbb{N}$, actually proving the original version of points (a) and (b).

As for the proof of points (a') and (b'), it is simple to see that formulae (3.182), (3.183), (3.184), (3.185), (3.187), and (3.189) are enough to guarantee the existence of the u- and k-chains with the desired properties. We must show, now, that each other interval satisfies neither u nor k. As a preliminary step, it is useful to show that an u-interval (resp., k-interval) belonging to $\mathcal{G}_{[a,b]}$ cannot be sub-interval of u-intervals or k-intervals. Formula (3.186) guarantees that it cannot exist an u_1-interval (resp., k_1-interval) that is sub-interval of an u_2-interval (resp., k_2-interval) or, vice versa, an u_2-interval (resp., k_2-interval) that is sub-interval of an u_1-interval (resp., k_1-interval). Moreover, since, by Lemma 3.4.4, u_1, u_2, k_1, and k_2 are disjointly bounded, then (3.192) guarantees that no u_1-interval (resp., u_2-interval, k_1-interval, k_2-interval) can be sub-interval of another u_1-interval (resp., u_2-interval, k_1-interval, k_2-interval). So far, we have shown that no u-interval (resp., k-interval) can be sub-interval of any u-interval (resp., k-interval). It remains to show that no u-interval can be sub-interval of any k-interval, and vice versa. Suppose, by contradiction, that the u-interval $[c', d']$ is sub-interval of the k-interval $[c'', d'']$. By (3.187), there must exist a k-interval, say it $[c''', d''']$, starting in between c' and d'. Then, we either have (i) $d''' \leq d''$ and the k-interval $[c''', d''']$ is sub-interval of the k-interval $[c'', d'']$, contradicting the previous statement, or (ii) $d''' > d''$ and the k-interval $[c'', d'']$ overlaps the k-interval $[c''', d''']$, contradicting (3.185). With a similar argument, one can show that no k-interval can be sub-interval of a u-interval. Thus, we can state that u-intervals (resp., k-intervals) cannot be sub-intervals of u- or k-intervals. Now, let us focus on the point (c) of the lemma. Suppose, by contradiction, the existence of the u-interval $[c, d]$, belonging to $\mathcal{G}_{[a,b]}$ and such that $[c, d] \neq [b_i, b'_i]$ for any $i \in \mathbb{N}$. By (3.182), it must be $c \geq b$. Now, let us distinguish the following cases:

- if $b \leq c < b_0$, then one of the following:
  - if $d < b'_0$, then (3.190) is contradicted,
  - if $d \geq b'_0$, then the u-interval $[b_0, b'_0]$ is sub-interval of the u-interval $[c, d]$,
- if $c = b_i$ for some $i \in \mathbb{N}$, then one of the following:
  - if $d < b'_i$, then the u-interval $[c, d]$ is sub-interval of the u-interval $[b_i, b'_i]$,
  - if $d = b'_i$, then we are contradicting the hypothesis "per absurdum" that $[c, d] \neq [b_i, b'_i]$ for any $i \in \mathbb{N}$,
  - if $d > b'_i$, then the u-interval $[b_i, b'_i]$ is sub-interval of the u-interval $[c, d]$,
- if $b_i < c < b'_i$ for some $i \in \mathbb{N}$, then one of the following:
  - if $d \leq b'_i$, then the u-interval $[c, d]$ is sub-interval of the u-interval $[b_i, b'_i]$,
  - if $d > b'_i$, then the u-interval $[b_i, b'_i]$ overlaps the u-interval $[c, d]$, contradicting (3.184),
- if $b'_i \leq c < b_{i+1}$ for some $i \in \mathbb{N}$, then one of the following:
  - if $d \leq b_{i+1}$, then the u-interval $[c, d]$ is sub-interval of the k-interval $[c_i, c'_i]$,
  - if $b_{i+1} < d < b'_{i+1}$, then the u-interval $[c, d]$ overlaps the u-interval $[b_{i+1}, b'_{i+1}]$, contradicting (3.184),
  - if $d \geq b'_{i+1}$, then the u-interval $[b_{i+1}, b'_{i+1}]$ is sub-interval of the u-interval $[c, d]$.

Thus, there cannot exist an u-interval $[c, d] \in \mathcal{G}_{[a,b]}$ such that $[c, d] \neq [b_i, b'_i]$ for any $i \in \mathbb{N}$. A similar argument can be exploited to prove that there cannot exist a k-interval $[c, d] \in \mathcal{G}_{[a,b]}$ such that $[c, d] \neq [c_i, c'_i]$ for any $i \in \mathbb{N}$. In addition, suppose, by contradiction, the existence of the interval $[c, d]$, belonging to $\mathcal{G}_{[a,b]}$, satisfying first, and such that $[c, d] \neq [b_0, b'_0]$. By the first conjunct of (3.191), it must be

$[c, d] = [b_i, b'_i]$ for some $i \in \mathbb{N}$, with $i \neq 0$. Thus, the second conjunct of (3.191) is contradicted.

Finally, suppose, by contradiction, that it is the case that $b'_i < b_{i+1}$ for some $i \in \mathbb{N}$. By the previous argument, there must be b_i, c_i, c'_i, b'_{i+1} such that $b_i < c_i < b'_i$, $b_{i+1} < c'_i < b'_{i+1}$, and $[c_i, c'_i]$ satisfies k. By point (c), there cannot exist an u- or k-interval starting in between c_i and b_{i+1}. Then, the interval $[b_i, b'_i]$ contradicts (3.193), since it overlaps the interval $[c_i, b_{i+1}]$ that, in turn, does not overlap any u- or k-interval. Thus, it must be $b'_i = b_{i+1}$ for each $i \in \mathbb{N}$. In a very similar way, it is possible to show that it must also be $c'_i = c_{i+1}$ for each $i \in \mathbb{N}$. □

Finally, we define the operator $\langle X_\mathsf{u} \rangle$ to step from one u-interval to the next one in the sequence. More precisely, whenever it is evaluated over the starting interval $[a, b]$, the operator $\langle X_\mathsf{u} \rangle$ allows one to express properties concerning the first u-interval of the sequence, namely $[b_0, b'_0]$; instead, when it is evaluated over an u-interval $[b_i, b'_i]$, $\langle X_\mathsf{u} \rangle$ allows one to express properties of the next u-interval of the sequence, namely $[b_{i+1}, b'_{i+1}]$.

$$\langle X_\mathsf{u} \rangle \varphi \equiv (\neg \mathsf{u} \wedge \langle O \rangle \langle O \rangle (\mathsf{first} \wedge \varphi)) \vee (\mathsf{u} \wedge \langle O \rangle (\mathsf{k} \wedge \langle O \rangle (\mathsf{u} \wedge \varphi)))$$

Definition of the Id-chain. In order to define the Id-chain, we take advantage of the following set of formulae:

$$\neg \mathsf{Id} \wedge \neg \langle O \rangle \mathsf{Id} \wedge [G](\mathsf{Id} \rightarrow \neg \langle O \rangle \mathsf{Id}) \tag{3.195}$$
$$\langle X_\mathsf{u} \rangle (* \wedge \langle X_\mathsf{u} \rangle (\mathsf{tile} \wedge \mathsf{Id} \wedge \langle X_\mathsf{u} \rangle * \wedge [G](* \rightarrow \langle X_\mathsf{u} \rangle (\mathsf{tile} \wedge \langle X_\mathsf{u} \rangle \mathsf{tile})))) \tag{3.196}$$
$$[G]((\mathsf{u} \leftrightarrow * \vee \mathsf{tile}) \wedge (* \rightarrow \neg \mathsf{tile})) \tag{3.197}$$
$$[G](* \rightarrow \langle O \rangle (\mathsf{k} \wedge \langle O \rangle \mathsf{Id})) \tag{3.198}$$
$$[G](\mathsf{Id} \rightarrow \langle O \rangle (\mathsf{k} \wedge \langle O \rangle *)) \tag{3.199}$$
$$[G]((\mathsf{u} \rightarrow \neg \langle O \rangle \mathsf{Id}) \wedge (\mathsf{Id} \rightarrow \neg \langle O \rangle \mathsf{u})) \tag{3.200}$$
$$[G](\langle O \rangle * \rightarrow \neg \langle O \rangle \mathsf{Id}) \tag{3.201}$$
$$not\_subint(\mathsf{Id}) \tag{3.202}$$
$$(3.195) \wedge \ldots \wedge (3.202) \tag{3.203}$$

Lemma 3.4.6. *Let* $M, [a, b] \Vdash (3.188) \wedge (3.194) \wedge (3.203)$ *and let* $b \leq b_1^0 < c_1^0 < b_1^1 < \ldots < b_1^{k_1-1} < c_1^{k_1-1} < b_1^{k_1} = b_2^0 < c_2^0 = c_1^{k_1} < b_2^1 < \ldots < b_2^{k_2} = b_3^0 < \ldots$ *be the sequence of points, defined by Lemma 3.4.5, such that* $[b_j^i, b_j^{i+1}]$ *satisfies* u *and* $[c_j^i, c_j^{i+1}]$ *satisfies* k *for each* $j \geq 1, 0 \leq i < k_j$. *Then, for each* $j \geq 1$, *we have:*

(a) $M, [b_j^0, b_j^1] \Vdash *$;

(b) $M, [b_j^i, b_j^{i+1}] \Vdash \mathsf{tile}$ *for each* $0 < i < k_j$;

(c) $M, [b_j^1, b_{j+1}^0] \Vdash \mathsf{Id}$;

(d) $k_1 = 2$, $k_l > 2$ *for each* $l > 1$;

and no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ *satisfies* $*$ *(resp.,* tile*,* ld*), unless* $c > b_j^i$ *for each* $i, j > 0$.

Proof. First of all, we show that ld is a disjointly-bounded propositional letter. By (3.195), it is easy to see that ld meets the first two requirements of Definition 3.4.2. By (3.200) and (3.201), $*$ and ld are disjoint, and, by (3.199), $*$ is a disjoint consequent of ld. Thus, ld is a disjointly-bounded propositional letter. The proof proceeds case by case.

(a) Observe that there exists an infinite sequence of $*$-intervals, thanks to (3.196), (3.198), and (3.199). Let us denote by $[b_1^0, b_1^1], [b_2^0, b_2^1], \ldots, [b_j^0, b_j^1], \ldots$ such a sequence. By the first conjunct of (3.197), we can assume that, for each $j > 0$, there is no $*$-interval between $[b_j^0, b_j^1]$ and $[b_{j+1}^0, b_{j+1}^1]$.

(b) By (3.197), each interval satisfying $*$ or tile is an u-interval and each u-interval satisfies either $*$ or tile. Then, the u-intervals between two consecutive $*$-intervals (if any) must be tile-intervals.

(c) By (3.198), for each k-interval $[c_j^0, c_j^1]$ overlapped by a $*$-interval, there exists an ld-interval $[c, d]$, with $c_j^0 < c < c_j^1 < d$. We show that $c = b_j^1$ and $d = b_{j+1}^0$. Suppose that $c < b_j^1$. Then, the u-interval $[b_j^0, b_j^1]$ overlaps the ld-interval $[c, d]$, contradicting (3.200). On the other hand, if $c > b_1^1$, then we distinguish two cases.

   - $j = 1$. In this case, by (3.196), we have that $[b_j^1, b_j^2]$ is the ld-interval representing the first level of the octant. Now, if $d > b_1^2$, then the u-interval $[b_1^1, b_1^2]$ overlaps the ld-interval $[c, d]$, contradicting (3.200); otherwise, if $d \leq b_1^2$, then the ld-interval $[c, d]$ is a sub-interval of the ld-interval $[b_1^1, b_1^2]$, contradicting (3.202) (recall that ld is a disjointly-bounded propositional letter).
   - $j > 1$ ($[b_j^1, b_j^2]$ is not the last tile-interval of the jth level). In this case, the k-interval $[c_j^1, c_j^2]$ does not overlap a $*$-interval (since $[b_j^2, b_j^3]$ is a tile-interval). Thus, due to (3.199), it must be $d > c_j^2$, and the u-interval $[b_j^1, b_j^2]$ overlaps the ld-interval $[c, d]$, contradicting (3.200).

   Hence, it must be $c = b_j^1$. Now, we have to show that $d = b_{j+1}^0$, that is, the ld-interval starting immediately after the $*$-interval $[b_j^0, b_j^1]$ ends at the point in which the next $*$-interval starts. Suppose, by contradiction, that $d \neq b_{j+1}^0$. Suppose that $j = 1$. In this case, if $d < b_2^0$ (resp., $d > b_2^0$), then the ld-interval $[c, d]$ (resp., $[b_1^1, b_1^2]$) is a sub-interval of the ld-interval $[b_1^1, b_1^2]$ (resp., $[c, d]$), contradicting (3.202). So, let us suppose $j > 1$, and consider the following cases:

   - if $d \leq c_j^{k_j - 1}$, then (3.199) is contradicted, since either $[c, d]$ does not overlap any k-interval or it overlaps a k-interval that does not overlap any $*$-interval;

- if $c_j^{k_j-1} < d < b_{j+1}^0$, then the ld-interval $[c, d]$ overlaps the u-interval $[b_j^{k_j-1}, b_j^{k_j}]$, contradicting (3.200);
- if $b_{j+1}^0 < d < b_{j+1}^1$, then the ld-interval $[c, d]$ overlaps the u-interval $[b_{j+1}^0, b_{j+1}^1]$, contradicting (3.200);
- if $d \geq b_{j+1}^1$, then (3.201) is contradicted, since the interval $[a', c_{j+1}^0]$, where a' is a generic point in between a and b, overlaps both the $*$-interval $[b_{j+1}^0, b_{j+1}^1]$ and the $\mathsf{up\_rel}$-interval $[c, d]$.

Hence, it must be $d = b_{j+1}^0$.

(d) By (3.196), it immediately follows that $k_1 = 2$ and $k_l > 2$ when $l > 1$.

Finally, suppose, by contradiction, that there exists an ld-interval $[c, d] \in \mathcal{G}_{[a,b]}$ such that $[c, d] \neq [b_j^1, b_{j+1}^0]$ for each $j > 0$ and that $c \leq b_j^i$ for some $i, j > 0$. By (3.195), the interval $[a, b]$ neither satisfies ld nor overlaps an interval that satisfies ld, thus $c \geq b$, and one of the following cases arise.

1. If $b \leq c < b_1^0$, then, by (3.199), it must be $d > c_1^0$, and (3.201) is contradicted.
2. If $b_j^0 \leq c < c_j^0$ for some $j > 0$, then (3.201) is contradicted.
3. If $c_j^0 \leq c < b_j^1$ for some $j > 0$, then, due to (3.199), it must be $d > c_j^1$ and the u-interval $[b_j^0, b_j^1]$ overlaps the ld-interval $[c, d]$, contradicting (3.200).
4. If $c = b_j^1$ for some $j > 0$, then we have already shown that it must be $d = b_{j+1}^0$.
5. If $b_j^1 < c < b_{j+1}^0$ for some $j > 0$, then:
   (a) if $d \leq b_{j+1}^0$, then the ld-interval $[c, d]$ is sub-interval of the ld-interval $[b_j^1, b_{j+1}^0]$, contradicting (3.202),
   (b) if $d > b_{j+1}^0$, then the ld-interval $[b_j^1, b_{j+1}^0]$ overlaps the ld-interval $[c, d]$, contradicting (3.195).

The fact that no other interval $[c, d] \in \mathcal{G}_{[a,b]}$ satisfies $*$ or tile, unless $c > b_j^i$ for each $i, j > 0$ can be proved by a similar argument. □

Above-neighbor relation. We proceed now with the encoding of the above-neighbor relation (Fig. 3.6). Intuitively, the above-neighbor relation connects each tile-interval with its vertical neighbor in the octant (e.g., t_2^2 with t_2^3 in Fig. 3.6). To model such a relation, we use intervals labeled by $\mathsf{up\_rel}$ as follows: $\mathsf{up\_rel}$-intervals connect pairs of tile-intervals encoding pairs of above-connected tiles of the octant.

We distinguish between *backward* and *forward* rows of $\mathcal{O}$ using the propositional letters b and f: we label each u-interval with b (resp., f) if it belongs to a backward (resp., forward) row (formulae (3.204)-(3.205)). Intuitively, the tiles belonging to forward rows of $\mathcal{O}$ are encoded in ascending order, while those belonging to backward

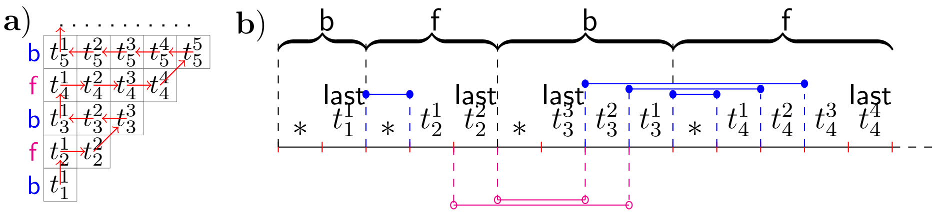


Figure 3.6: The encoding of the above-neighbor relation in the fragment O: up_rel-intervals starting from backward (resp., forward) rows of the octant do not overlap

rows are encoded in descending order (the tiling is encoded in a zig-zag manner). In particular, this means that the left-most tile-interval of a backward level encodes the last tile of that row (and not the first one) in $\mathcal{O}$. Let $\alpha, \beta \in \{\mathsf{b}, \mathsf{f}\}$, with $\alpha \neq \beta$:

$$\langle X_{\mathsf{u}} \rangle \mathsf{b} \wedge [G]((\mathsf{u} \leftrightarrow \mathsf{b} \vee \mathsf{f}) \wedge (\mathsf{b} \rightarrow \neg \mathsf{f})) \tag{3.204}$$

$$[G]((\alpha \wedge \neg \langle X_{\mathsf{u}} \rangle * \rightarrow \langle X_{\mathsf{u}} \rangle \alpha) \wedge (\alpha \wedge \langle X_{\mathsf{u}} \rangle * \rightarrow \langle X_{\mathsf{u}} \rangle \beta)) \tag{3.205}$$

$$(3.204) \wedge \ldots \wedge (3.205) \tag{3.206}$$

Lemma 3.4.7. *If $M, [a,b] \Vdash$ (3.188) $\wedge$ (3.194) $\wedge$ (3.203) $\wedge$ (3.206), then there exists a sequence of points like that defined in Lemma 3.4.6 such that $M, [b^i_j, b^{i+1}_j] \Vdash \mathsf{b}$ if and only if j is an odd number and $M, [b^i_j, b^{i+1}_j] \Vdash \mathsf{f}$ if and only j is an even number. Furthermore, we have that no other interval $[c,d] \in \mathcal{G}_{[a,b]}$ satisfies b or f, unless $c > b^i_j$ for each $i, j > 0$.*

We make use of such an alternation between backward and forward rows to use the operator $\langle O \rangle$ for correctly enconding the above-neighbor relation. We constrain each up_rel-interval starting from a backward (resp., forward) row not to overlap any other up_rel-interval starting from a backward (resp., forward) row. The structure of the encoding is shown in Fig. 3.6, where up_rel-intervals starting inside forward (resp., backward) rows are placed one inside the other. Consider, for instance, how the 3rd and 4th level of the octant are encoded in Fig. 3.6b. The 1st tile-interval of the 3rd level (t^3_3) is connected to the second from last tile-interval of the 4th level (t^3_4), the 2nd tile-interval of the 3rd level (t^2_3) is connected to the third from last tile-interval of the 4th level (t^2_4), and so on. Notice that, in forward (resp., backward) level, the last (resp., first) tile-interval has no tile-intervals above-connected to it, in order to constrain each level to have exactly one tile-interval more than the previous one (these tile-intervals are labeled with last).

Formally, we define the above-neighbor relation as follows. If $[b^i_j, b^{i+1}_j]$ is a tile-interval belonging to a forward (resp., backward) row, then we say that it is above-connected to the tile-interval $[b^{j+2-i}_{j+1}, b^{j+2-i+1}_{j+1}]$ (resp., $[b^{j+2-i-1}_{j+1}, b^{j+2-i}_{j+1}]$). We capture this situation by labelling with up_rel the interval $[c^i_j, c^{j+2-i}_{j+1}]$ (resp., $[c^i_j, c^{j+2-i-1}_{j+1}]$). Moreover, we distinguish between up_rel-intervals starting from forward and backward rows and, for each one of these cases, between those starting from odd and even tile-intervals. To this end, we use a new propositional letter, namely, $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$ (resp., $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$, $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$, $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$) to label up_rel-intervals starting from an odd tile-interval of a backward row (resp., even tile-interval/backward row, odd/forward, even/forward). Moreover, to ease the reading of the formulae, we group $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$ and $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$ in $\mathsf{up\_rel}^{\mathsf{b}}$ ($\mathsf{up\_rel}^{\mathsf{b}} \leftrightarrow \mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}} \oplus \mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$), and similarly for $\mathsf{up\_rel}^{\mathsf{f}}$. Finally, up_rel is exactly one among $\mathsf{up\_rel}^{\mathsf{b}}$ and $\mathsf{up\_rel}^{\mathsf{f}}$ ($\mathsf{up\_rel} \leftrightarrow \mathsf{up\_rel}^{\mathsf{b}} \oplus \mathsf{up\_rel}^{\mathsf{f}}$). In such a way, we encode the correspondence between tiles of consecutive rows of the plane induced by the above-neighbor relation. Let $\alpha, \beta \in \{\mathsf{b}, \mathsf{f}\}$ and $\gamma, \delta \in \{\mathsf{o}, \mathsf{e}\}$, with $\alpha \neq \beta$ and $\gamma \neq \delta$:

$$\neg\mathsf{up\_rel} \wedge \neg\langle O\rangle\mathsf{up\_rel} \tag{3.207}$$
$$[G]((\mathsf{up\_rel} \leftrightarrow \mathsf{up\_rel}^{\mathsf{b}} \vee \mathsf{up\_rel}^{\mathsf{f}}) \wedge (\mathsf{up\_rel}^{\alpha} \leftrightarrow \mathsf{up\_rel}^{\alpha}_{\mathsf{o}} \vee \mathsf{up\_rel}^{\alpha}_{\mathsf{e}})) \tag{3.208}$$
$$[G]((\mathsf{k} \vee * \rightarrow \neg\langle O\rangle\mathsf{up\_rel}) \wedge (\mathsf{up\_rel} \rightarrow \neg\langle O\rangle\mathsf{k})) \tag{3.209}$$
$$[G](\mathsf{u} \wedge \langle O\rangle\mathsf{up\_rel}^{\alpha}_{\gamma} \rightarrow \neg\langle O\rangle\mathsf{up\_rel}^{\alpha}_{\delta} \wedge \neg\langle O\rangle\mathsf{up\_rel}^{\beta}) \tag{3.210}$$
$$[G](\mathsf{up\_rel}^{\alpha} \rightarrow \neg\langle O\rangle\mathsf{up\_rel}^{\alpha}) \tag{3.211}$$
$$[G](\mathsf{up\_rel} \rightarrow \langle O\rangle\mathsf{Id}) \tag{3.212}$$
$$[G](\langle O\rangle\mathsf{up\_rel} \rightarrow \neg\langle O\rangle\mathsf{first}) \tag{3.213}$$
$$[G](\mathsf{up\_rel}^{\alpha}_{\gamma} \rightarrow \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle\mathsf{up\_rel}^{\beta}_{\gamma})) \tag{3.214}$$
$$(3.207) \wedge \ldots \wedge (3.214) \tag{3.215}$$

Lemma 3.4.8. *If* $M, [a, b] \Vdash (3.188) \wedge (3.194) \wedge (3.203) \wedge (3.206) \wedge (3.215)$, *then there exists a sequence of points like that defined in Lemma 3.4.6 such that, for each* $i \geq 0$, $j > 0$, *the following properties hold:*

a) if $[c, d]$ *satisfies* up_rel, *then* $c = c^i_j$ *and* $d = c^{i'}_{j'}$ *for some* $i, i', j, j' > 0$, *that is, each* up_rel*-interval starts and ends inside a* tile*-interval. More precisely, it starts (resp., ends) at the same point in which a* k*-interval starts (resp., ends);*

b) $[c^i_j, c^{i'}_{j'}]$ *satisfies* up_rel *if and only if it satisfies exactly one between* $\mathsf{up\_rel}^{\mathsf{b}}$ *and* $\mathsf{up\_rel}^{\mathsf{f}}$ *and* $[c^i_j, c^{i'}_{j'}]$ *satisfies* $\mathsf{up\_rel}^{\mathsf{b}}$ *(resp.,* $\mathsf{up\_rel}^{\mathsf{f}}$*) if and only if it satisfies exactly one between* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$ *and* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$ *(resp., between* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$ *and* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$*);*

c) for each $\alpha, \beta \in \{\mathsf{b}, \mathsf{f}\}$ *and* $\gamma, \delta \in \{\mathsf{o}, \mathsf{e}\}$, *if* $[c^i_j, c^{i'}_{j'}]$ *satisfies* $\mathsf{up\_rel}^{\alpha}_{\gamma}$, *then there is no other interval starting at* c^i_j *satisfying* $\mathsf{up\_rel}^{\beta}_{\delta}$ *such that* $\mathsf{up\_rel}^{\alpha}_{\gamma} \neq \mathsf{up\_rel}^{\beta}_{\delta}$*;*

d) each $\mathsf{up\_rel}^{\mathsf{b}}$*-interval (resp.,* $\mathsf{up\_rel}^{\mathsf{f}}$*-interval) does not overlap any other* $\mathsf{up\_rel}^{\mathsf{b}}$*-interval (resp.,* $\mathsf{up\_rel}^{\mathsf{f}}$*-interval);*

e) if $[c^i_j, c^{i'}_{j'}]$ *satisfies* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$ *(resp.,* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$, $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$, $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$*), then there exists an* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{o}}$*-interval (resp.,* $\mathsf{up\_rel}^{\mathsf{f}}_{\mathsf{e}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{o}}$*-interval,* $\mathsf{up\_rel}^{\mathsf{b}}_{\mathsf{e}}$*-interval) starting at* $c^{i'}_{j'}$.

Proof. We only proof point a), that is the less intuitive. Let $[c,d]$ be an up_rel-interval. First, we show that it must be $c = c^i_j$, for some $i, j > 0$. Then, we prove that $d = c^{i'}_{j'}$, for some $i', j' > 0$. Notice that we want to exclude also the case in which $c = c^0_j$ (resp., $d = c^0_{j'}$) for some $j > 0$ (resp., $j' > 0$), since this would imply the existence of an up_rel-interval starting (resp., ending) inside a $*$-interval. This is done by means of (3.209) (first conjunct) and (3.214). Now, we show that $c = c^i_j$, for some $i, j > 0$. By (3.207), it must be $c \geq b$ and, by (3.213) and (3.214), it follows $c \geq c^0_1$. Moreover, by (3.209) and (3.214), it cannot be the case that $b^i_j \leq c < c^i_j$ for any $i \geq 0, j > 0$. It only remains to exclude the case in which $c^i_j < c < b^{i+1}_j$ for some $i \geq 0, j > 0$. Thus, suppose, by contradiction, that $c^i_j < c < b^{i+1}_j$ for some $i \geq 0, j > 0$. If $d > c^{i+1}_j$, then (3.209) is contradicted; otherwise, if $d \leq c^{i+1}_j$, then, by (3.212), $[c,d]$ overlaps an Id-interval. As a consequence, there should be an Id-interval starting at b^{i+1}_j, that means that $[b^i_j, b^{i+1}_j]$ is a $*$-interval. This lead to a contradiction with (3.209), since the $*$-interval $[b^i_j, b^{i+1}_j]$ overlaps the up_rel-interval $[c,d]$. Thus, we have that $c = c^i_j$ for some $i, j > 0$. Now, we want to prove that $d = c^{i'}_{j'}$ for some $i', j' > 0$. It is easy to see that, if $d \neq c^{i'}_{j'}$ for any $j', i' > 0$, then there would be an up_rel-interval overlapping a k-interval, contradicting (3.209), hence the thesis. □

Now, we constrain each tile-interval, apart from the ones representing the last tile of some level, to have a tile-interval above-connected to it. To this end, we label each tile-interval representing the last tile of some row of the octant with the new propositional letter last (formulae (3.221)-(3.223)). Next, we force all and only those tile-intervals not labelled with last to have a tile-interval above-connected to them (formulae (3.224)-(3.227)):

$$[G](\mathsf{tile} \rightarrow \langle O\rangle \mathsf{up\_rel}) \tag{3.216}$$
$$[G](\alpha \rightarrow [O](\mathsf{up\_rel} \rightarrow \mathsf{up\_rel}^{\alpha})) \tag{3.217}$$
$$[G](\mathsf{up\_rel}^{\alpha} \rightarrow \langle O\rangle \beta) \tag{3.218}$$
$$[G]((\langle O\rangle * \rightarrow \neg(\langle O\rangle \mathsf{up\_rel}^{\mathsf{b}} \wedge \langle O\rangle \mathsf{up\_rel}^{\mathsf{f}})) \tag{3.219}$$
$$[G](\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\gamma} \wedge \langle X_{\mathsf{u}}\rangle \mathsf{tile} \rightarrow \langle X_{\mathsf{u}}\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\delta})) \tag{3.220}$$
$$[G](\mathsf{last} \rightarrow \mathsf{tile}) \tag{3.221}$$
$$[G]((* \wedge \mathsf{b} \rightarrow \langle X_{\mathsf{u}}\rangle \mathsf{last}) \wedge (\mathsf{f} \wedge \langle X_{\mathsf{u}}\rangle * \rightarrow \mathsf{last})) \tag{3.222}$$
$$[G]((\mathsf{last} \wedge \mathsf{f} \rightarrow \langle X_{\mathsf{u}}\rangle *) \wedge (\mathsf{b} \wedge \langle X_{\mathsf{u}}\rangle \mathsf{last} \rightarrow *)) \tag{3.223}$$
$$[G](* \wedge \mathsf{f} \rightarrow \langle X_{\mathsf{u}}\rangle(\mathsf{tile} \wedge \langle O\rangle(\mathsf{up\_rel} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle *)))) \tag{3.224}$$
$$[G](\mathsf{last} \wedge \mathsf{b} \rightarrow \langle O\rangle(\mathsf{up\_rel} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle(\mathsf{tile} \wedge \langle X_{\mathsf{u}}\rangle *)))) \tag{3.225}$$
$$[G](\mathsf{k} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\gamma}) \rightarrow [O]((\langle O\rangle \mathsf{up\_rel}^{\alpha}_{\gamma} \wedge \langle O\rangle(\mathsf{k} \wedge \langle O\rangle(\mathsf{tile} \wedge \langle O\rangle \mathsf{up\_rel}^{\beta}_{\delta} \wedge \neg \mathsf{last})) \rightarrow \langle O\rangle \mathsf{up\_rel}^{\alpha}_{\delta})) \tag{3.226}$$
$$[G](\mathsf{up\_rel} \rightarrow \neg\langle O\rangle \mathsf{last}) \tag{3.227}$$
$$(3.216) \wedge \ldots \wedge (3.227) \tag{3.228}$$

Lemma 3.4.9. *If* $M, [a, b] \Vdash$ (3.188) $\wedge$ (3.194) $\wedge$ (3.203) $\wedge$ (3.206) $\wedge$ (3.215) $\wedge$ (3.228)*, then there exists a sequence of points like that defined in Lemma 3.4.6 such that the following properties hold:*

a) for each up_rel*-interval* $[c_j^i, c_{j'}^{i'}]$*, connecting the* tile*-interval* $[b_j^i, b_j^{i+1}]$ *to the* tile*-interval* $[b_{j'}^{i'}, b_{j'}^{i'+1}]$*, if* $[c, d]$ *satisfies* $\mathsf{up\_rel}^{\mathsf{b}}$ *(resp.,* $\mathsf{up\_rel}^{\mathsf{f}}$*), then* $[b_j^i, b_j^{i+1}]$ *satisfies* b *(resp.,* f*) and* $[b_{j'}^{i'}, b_{j'}^{i'+1}]$ *satisfies* f *(resp.,* b*);*

b) (strict alternation property) for each tile*-interval* $[b_j^i,\ b_j^{i+1}]$*, with* $i < k_j - 1$*, such that there exists an* $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{b}}$*-interval (resp.,* $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{b}}$*-interval,* $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{f}}$*-interval,* $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{f}}$*-interval) starting at* c_j^i*, there exists an* $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{b}}$*-interval (resp.,* $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{b}}$*-interval,* $\mathsf{up\_rel}_{\mathsf{e}}^{\mathsf{f}}$*-interval,* $\mathsf{up\_rel}_{\mathsf{o}}^{\mathsf{f}}$*-interval) starting at* c_j^{i+1}*;*

c) for every tile*-interval* $[b_j^i, b_j^{i+1}]$ *satisfying* last*, there is no* up_rel*-interval ending at* c_j^i*;*

d) for each up_rel*-interval* $[c_j^i, c_{j'}^{i'}]$*, with* $0 < i < k_j$*, we have that* $j' = j + 1$*.*

Proof. a) Let $[c_j^i, c_{j'}^{i'}]$ be an up_rel-interval connecting the tile-interval $[b_j^i, b_j^{i+1}]$ to the tile-interval $[b_{j'}^{i'}, b_{j'}^{i'+1}]$. Suppose that $[c_j^i, c_{j'}^{i'}]$ satisfies $\mathsf{up\_rel}^{\mathsf{b}}$ (the other case is symmetric) and that $[b_j^i, b_j^{i+1}]$ satisfies f. Then, (3.217) is contradicted. Similarly, if $[b_{j'}^{i'}, b_{j'}^{i'+1}]$ satisfies b, then (3.218) is contradicted.

b) Straightforward, by (3.220);

c) Straightforward, by (3.227);

d) Let $[c_j^i, c_{j'}^{i'}]$ be an up_rel-interval, with $0 < i < k_j$, and suppose, by contradiction, that $j' \neq j + 1$. Suppose that $[c_j^i, c_{j'}^{i'}]$ is an $\mathsf{up\_rel}^{\mathsf{b}}$-interval (the other case is symmetric). By point a) of this lemma, we have that $[b_j^i, b_j^{i+1}]$ satisfies b and that $[b_{j'}^{i'}, b_{j'}^{i'+1}]$ satisfies f. Two cases are possible:

(i) if $j' = j$, then $[b_j^i, b_j^{i+1}]$ and $[b_{j'}^{i'}, b_{j'}^{i'+1}]$ belong to the same Id-interval. By Lemma 3.4.7, they must be both labelled with b or f, against the hypothesis;

(ii) if $j' > j+1$, then consider a tile-interval $[b_{j+1}^h, b_{j+1}^{h+1}]$ belonging to the $(j+1)$-th level. By Lemma 3.4.7, we have that $[b_{j+1}^h, b_{j+1}^{h+1}]$ satisfies f (since $[b_j^i, b_j^{i+1}]$ satisfies b) and, by (3.216) and (3.217), we have that there is an $\mathsf{up\_rel}^{\mathsf{f}}$-interval starting at c_{j+1}^h and ending at some point $c_{j''}^{h'}$ for some $j'' > j+1$, (by point (i)). Consider the $*$-interval $[b_{j+2}^0, b_{j+2}^1]$. We have that the interval $[a', c_{j+2}^0]$, where a' is a generic point in between a and b, overlaps the $*$-interval $[b_{j+2}^0, b_{j+2}^1]$, the $\mathsf{up\_rel}^{\mathsf{f}}$-interval $[c_{j+1}^h, c_{j''}^{h'}]$, and the $\mathsf{up\_rel}^{\mathsf{b}}$-interval $[c_j^i, c_{j'}^{i'}]$, contradicting (3.219).

Hence, the only possibility is $j' = j + 1$. $\square$

Lemma 3.4.10. *Each* tile*-interval* $[b_j^i, b_j^{i+1}]$ *is above-connected to exactly one* tile*-interval and, if it does not satisfy* last*, then there exists exactly one* tile*-interval which is above-connected to it.*

Proof. First of all, we observe that each tile-interval is above-connected with at least one tile-interval, by (3.216) and by Lemma 3.4.8, item a). Now, suppose, by contradiction, that there exists a tile-interval $[b_j^i, b_j^{i+1}]$ not satisfying last and such that there is no tile-interval above-connected to it. The proof proceeds by induction.
Base case. If $[b_j^i, b_j^{i+1}]$ is the rightmost interval of the j-th Id-interval not satisfying last and it satisfies f (resp., b), then we have that $i = k_j - 2$ (resp., $i = k_j - 1$). Formula (3.225) (resp., (3.224)) guarantees the existence of an $\mathsf{up\_rel}$-interval ending at c_j^i, leading to a contradiction.
Inductive step. Otherwise, if $[b_j^i, b_j^{i+1}]$ is not the rightmost interval of the j-th Id-interval not satisfying last, then the inductive case applies. So, we can assume the inductive hypothesis, that is, there is an $\mathsf{up\_rel}$-interval ending at c_j^{i+1} and starting at some point $c_{j-1}^{i'}$. We want to show that there exists also an $\mathsf{up\_rel}$-interval ending at c_j^i. Without loss of generality, suppose that $[c_{j-1}^{i'}, c_j^{i+1}]$ satisfies $\mathsf{up\_rel}_\mathsf{o}^\mathsf{f}$. Then, by Lemma 3.4.8, item e), there exists an $\mathsf{up\_rel}_\mathsf{o}^\mathsf{b}$-interval starting at c_j^{i+1} and, by the strict alternation property (Lemma 3.4.9, item b)), there exists an $\mathsf{up\_rel}_\mathsf{e}^\mathsf{b}$-interval starting at c_j^i. We show that, by applying (3.226) to the k-interval $[c_{j-1}^{i'-1}, c_{j-1}^{i'}]$, we get a contradiction. Indeed, $[c_{j-1}^{i'-1}, c_{j-1}^{i'}]$ satisfies $\mathsf{k} \wedge \langle O \rangle(\mathsf{tile} \wedge \langle O \rangle \mathsf{up\_rel}_\mathsf{o}^\mathsf{f})$ and it overlaps $[b_{j-1}^{i'}, b_j^i]$, which satisfies the following formulae:

- $\langle O \rangle \mathsf{up\_rel}_\mathsf{o}^\mathsf{f}$: $[c_{j-1}^{i'}, c_j^{i+1}]$ satisfies $\mathsf{up\_rel}_\mathsf{o}^\mathsf{f}$;
- $\langle O \rangle(\mathsf{k} \wedge \langle O \rangle(\mathsf{tile} \wedge \langle O \rangle \mathsf{up\_rel}_\mathsf{e}^\mathsf{b} \wedge \neg \mathsf{last}))$: the interval $[c_j^{i-1}, c_j^i]$ satisfies k and overlaps the tile-interval $[b_j^i, b_j^{i+1}]$, which does not satisfy last (by hypothesis) and overlaps an $\mathsf{up\_rel}_\mathsf{e}^\mathsf{b}$-interval (that one starting at c_j^i).

We show that $[b_{j-1}^{i'}, b_j^i]$ does not satisfy the formula $\langle O \rangle \mathsf{up\_rel}_\mathsf{e}^\mathsf{f}$, getting a contradiction with (3.226). Suppose that there exists an interval $[e, f]$ satisfying $\mathsf{up\_rel}_\mathsf{e}^\mathsf{f}$ and such that $b_{j-1}^{i'} < e < b_j^i < f$. We distinguish the following cases:

- if $f > c_j^{i+1}$ and $e > c_{j-1}^{i'}$, then the $\mathsf{up\_rel}_\mathsf{o}^\mathsf{f}$-interval $[c_{j-1}^{i'}, c_j^{i+1}]$ overlaps the $\mathsf{up\_rel}_\mathsf{e}^\mathsf{f}$-interval $[e, f]$, contradicting Lemma 3.4.8, item d);
- if $f > c_j^{i+1}$ and $e = c_{j-1}^{i'}$, then there are an $\mathsf{up\_rel}_\mathsf{o}^\mathsf{f}$- and an $\mathsf{up\_rel}_\mathsf{e}^\mathsf{f}$-interval starting at $c_{j-1}^{i'}$, contradicting Lemma 3.4.8, item c);
- if $f = c_j^{i+1}$, then there are an $\mathsf{up\_rel}_\mathsf{o}^\mathsf{f}$- and an $\mathsf{up\_rel}_\mathsf{e}^\mathsf{f}$-interval ending at c_j^{i+1} and, by Lemma 3.4.8, item e), there are an $\mathsf{up\_rel}_\mathsf{o}^\mathsf{b}$- and an $\mathsf{up\_rel}_\mathsf{e}^\mathsf{b}$-interval starting at c_j^{i+1}, contradicting Lemma 3.4.8, item c);

- finally, if $f = c_j^i$, we have a contradiction with the hypothesis.

Thus, there exists no such an interval, contradicting (3.226).

This proves that each tile-interval is above-connected to at least one tile-interval and, if it does not satisfy last, then there exists at least one tile-interval above-connected to it. Now, we show that such connections are unique. Suppose, by contradiction, that for some $[c_j^i, c_{j+1}^{i'}]$ and $[c_j^i, c_{j+1}^{i''}]$, with $c_{j+1}^{i'} < c_{j+1}^{i''}$ (the case $c_{j+1}^{i'} > c_{j+1}^{i''}$ is symmetric), we have that both $[c_j^i, c_{j+1}^{i'}]$ and $[c_j^i, c_{j+1}^{i''}]$ are up_rel-intervals. By Lemma 3.4.8, we have that they both satisfy the same propositional letter among $\mathsf{up\_rel}_\mathsf{o}^\mathsf{f}$, $\mathsf{up\_rel}_\mathsf{e}^\mathsf{f}$, $\mathsf{up\_rel}_\mathsf{o}^\mathsf{b}$, and $\mathsf{up\_rel}_\mathsf{e}^\mathsf{b}$, say $\mathsf{up\_rel}_\mathsf{o}^\mathsf{f}$ (the other cases are symmetric). Then, both $c_{j+1}^{i'}$ and $c_{j+1}^{i''}$ start an $\mathsf{up\_rel}_\mathsf{o}^\mathsf{b}$-interval by Lemma 3.4.8, item e). By the strict alternation property, an $\mathsf{up\_rel}_\mathsf{e}^\mathsf{b}$-interval starts at the point $c_{j+1}^{i'+1}$. Since $[b_{j+1}^{i'+1}, b_{j+1}^{i'+2}]$ does not satisfy last (it is neither the rightmost nor the leftmost tile-interval of the $(j+1)$-th Id-interval), then, as we have already shown, there exists a point c such that $[c, c_{j+1}^{i'+1}]$ is an up_rel-interval. By Lemma 3.4.8, items e) and c), we have that $[c, c_{j+1}^{i'+1}]$ is an $\mathsf{up\_rel}_\mathsf{e}^\mathsf{f}$-interval. We show that the existence of such an interval leads to a contradiction:

- if $c < c_j^i$, then the $\mathsf{up\_rel}_\mathsf{e}^\mathsf{f}$-interval $[c, c_{j+1}^{i'+1}]$ overlaps the $\mathsf{up\_rel}_\mathsf{o}^\mathsf{f}$-interval $[c_j^i, c_{j+1}^{i''}]$, contradicting Lemma 3.4.8, item d);

- if $c = c_j^i$, then c_j^i starts both an $\mathsf{up\_rel}_\mathsf{o}^\mathsf{f}$- and an $\mathsf{up\_rel}_\mathsf{e}^\mathsf{f}$-interval, contradicting Lemma 3.4.8, item c);

- if $c > c_j^i$, then the $\mathsf{up\_rel}_\mathsf{o}^\mathsf{f}$-interval $[c_j^i, c_{j+1}^{i'}]$ overlaps the $\mathsf{up\_rel}_\mathsf{e}^\mathsf{f}$-interval $[c, c_{j+1}^{i'+1}]$, contradicting Lemma 3.4.8, item d).

In a similar way, we can prove that two distinct up_rel-intervals cannot end at the same point. □

Right-neighbor relation. Intuitively, the right-neighbor relation connects each tile with its horizontal neighbor in the octant, if any (e.g., t_3^2 with t_3^3 in Fig. 3.6).

Again, in order to encode the right-neighbor relation, we must distinguish between forward and backward levels: a tile-interval belonging to a forward (resp., backward) level is right-connected to the tile-interval immediately to the right (resp., left), if any. For example, in Fig. 3.6b, the 2nd tile-interval of the 4th level (t_4^2) is right-connected to the tile-interval immediately to the right (t_4^3), since the 4th level is a forward one, while the 2nd tile-interval of the 3rd level (t_3^2) is right-connected to the tile-interval immediately to the left (t_3^3), since the 3rd level is a backward one.

As a consequence, we define the right-neighbor relation as follows. If $[b_j^i, b_j^{i+1}]$ is a tile-interval belonging to a forward (resp., backward) Id-interval, with $i \neq k_j - 1$ (resp., $i \neq 1$), then we say that it is right-connected to the tile-interval $[b_j^{i+1}, b_j^{i+2}]$ (resp., $[b_j^{i-1}, b_j^i]$).

Lemma 3.4.11 (Commutativity property). *If* $M, [a, b] \Vdash$ (3.188)∧(3.194)∧(3.203)∧(3.206) ∧ (3.215) ∧ (3.228)*, then there exists a sequence of points like the one defined in Lemma 3.4.6 such that the commutativity property holds.*

Tiling the plane The following formulae constrain each tile-interval (and no other interval) to be tiled by exactly one tile (formula (3.229)) and constrain the tiles that are right- or above-connected to respect the color constraints (from (3.230) to (3.232)):

$$[G]((\bigvee_{i=1}^{k} \mathtt{t_i} \leftrightarrow \mathsf{tile}) \wedge (\bigwedge_{i,j=1, i\neq j}^{k} \neg(\mathtt{t_i} \wedge \mathtt{t_j})) \tag{3.229}$$

$$[G](\mathsf{tile} \rightarrow \bigvee_{up(t_i)=down(t_j)} (\mathtt{t_i} \wedge \langle O\rangle(\mathsf{up\_rel} \wedge \langle O\rangle \mathtt{t_j}))) \tag{3.230}$$

$$[G](\mathsf{tile} \wedge \mathsf{f} \wedge \langle X_{\mathsf{u}}\rangle \mathsf{tile} \rightarrow \bigvee_{right(t_i)=left(t_j)} (\mathtt{t_i} \wedge \langle X_{\mathsf{u}}\rangle \mathtt{t_j})) \tag{3.231}$$

$$[G](\mathsf{tile} \wedge \mathsf{b} \wedge \langle X_{\mathsf{u}}\rangle \mathsf{tile} \rightarrow \bigvee_{left(t_i)=right(t_j)} (\mathtt{t_i} \wedge \langle X_{\mathsf{u}}\rangle \mathtt{t_j})) \tag{3.232}$$

$$(3.229) \wedge \ldots \wedge (3.232) \tag{3.233}$$

Given the set of tile types $\mathcal{T} = \{t_1, t_2, \ldots, t_k\}$, let $\Phi_{\mathcal{T}}$ be the formula

$$(3.188) \wedge (3.194) \wedge (3.203) \wedge (3.206) \wedge (3.215) \wedge (3.228) \wedge (3.233).$$

Lemma 3.4.12. *Given any finite set of tile types* $\mathcal{T} = \{t_1, t_2, \ldots, t_k\}$*, the formula* $\Phi_{\mathcal{T}}$ *is satisfiable if and only if* $\mathcal{T}$ *can tile the second octant* $\mathcal{O}$*.*

Since the above construction can be carried out on any linear ordering containing an infinite ascending chain of points, such as, for instance, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, the following theorem holds.

Theorem 3.4.13. *The satisfiability problem for the fragment* O *(resp.,* $\overline{\mathsf{O}}$*) of* HS *is undecidable in any class of linear orderings that contains at least one linear ordering with an infinite ascending (resp., descending) sequence of points.*

3.5 The current picture

The state of the art is summarized in Table 3.1 and Table 3.2. Almost all one-modality fragments have been classified with respect to their decidability/undecidability status (Table 3.1). The only exceptions are the logics D and its inverse $\overline{\mathsf{D}}$. Both of them are decidable when denseness is assumed [21, 22, 23, 81, 82]. and undecidable when discreteness is assumed [78]. Unfortunately, pairing these

Logics	No assumptions	Assuming denseness	Assuming discreteness
A (and $\overline{\mathsf{A}}$)	d	d	d
L (and $\overline{\mathsf{L}}$)	d	d	d
B (and $\overline{\mathsf{B}}$)	d	d	d
E (and $\overline{\mathsf{E}}$)	d	d	d
O (and $\overline{\mathsf{O}}$)	u	u	u
D (and $\overline{\mathsf{D}}$)	?	d	u
Legend: **d**: decidable fragments; **u**: undecidable fragments; **?**: open problems			

Table 3.1: Decidability/undecidability status for one-modality fragments

two results does not help to classify the fragment when no assumptions on the linear orderings are made, e.g., with respect to the class of all linear orderings. As for fragments with two modalities, the situation is a little bit more complicated. Even if we have the classification for most fragments, there are several cases for which the problem is not completely solved. Indeed, the fragments AE and A$\overline{\mathsf{E}}$ and, symmetrically, $\overline{\mathsf{A}}$B and $\overline{\mathsf{AB}}$, are decidable over the class of all finite linear orderings but their behaviour is unknown when interpreted over other classes of linear orderings. As for the fragments D$\overline{\mathsf{D}}$, DB, D$\overline{\mathsf{B}}$, DE, D$\overline{\mathsf{E}}$, DL, D$\overline{\mathsf{L}}$, $\overline{\mathsf{D}}$B, $\overline{\mathsf{DB}}$, $\overline{\mathsf{D}}$E, $\overline{\mathsf{DE}}$, $\overline{\mathsf{D}}$L, and $\overline{\mathsf{DL}}$, we only know that they are decidable over $\mathbb{Q}$ and undecidable

	A	L	B	E	O	D	$\overline{\mathsf{A}}$	$\overline{\mathsf{L}}$	$\overline{\mathsf{B}}$	$\overline{\mathsf{E}}$	$\overline{\mathsf{O}}$	$\overline{\mathsf{D}}$
A		≡ A	d	*	u	u	d	d	d	*	u	u
L			d	d	u	**	d	d	d	d	u	**
B				u	u	**	*	d	d	u	u	**
E					u	**	d	d	u	d	u	**
O						u	u	u	u	u	u	u
D							u	**	**	**	u	**
$\overline{\mathsf{A}}$								≡ $\overline{\mathsf{A}}$	*	d	u	u
$\overline{\mathsf{L}}$									d	d	u	**
$\overline{\mathsf{B}}$										u	u	**
$\overline{\mathsf{E}}$											u	**
$\overline{\mathsf{O}}$												u
$\overline{\mathsf{D}}$												

Legend:
* Decidable assuming finiteness of linear orderings, unknown in the other cases
** Undecidable assuming discreteness (or finiteness) of linear orderings, decidable over $\mathbb{Q}$, unknown in the other cases

Table 3.2: Decidability/undecidability status for two-modalities fragments

when discreteness is assumed, but their classification is an open problem in the other cases. All we know about fragments with more than two modalities directly comes from the above mentioned (un)decidability results (see Section 3.1).

To sum up, when no assumptions are made on the class of linear orderings (the class of all linear orderings), it turns out that the only one-modality and two-modalities fragments that remain unclassified are D, $\overline{\mathsf{D}}$, $\mathsf{D}\overline{\mathsf{D}}$, AE, $\mathsf{A}\overline{\mathsf{E}}$, $\overline{\mathsf{A}}\mathsf{B}$, $\overline{\mathsf{AB}}$, LD, $\mathsf{L}\overline{\mathsf{D}}$, $\overline{\mathsf{L}}\mathsf{D}$, $\overline{\mathsf{LD}}$, BD, $\mathsf{B}\overline{\mathsf{D}}$, $\overline{\mathsf{B}}\mathsf{D}$, $\overline{\mathsf{BD}}$, ED, $\mathsf{E}\overline{\mathsf{D}}$, $\overline{\mathsf{E}}\mathsf{D}$, $\overline{\mathsf{ED}}$ (19 out of 76). As for fragments with more than two modalities, the open cases are represented by fragments that neither contain one of the undecidable fragments listed above nor are contained in one of the two decidable fragments $\mathsf{AB}\overline{\mathsf{B}}\overline{\mathsf{L}}$ and $\overline{\mathsf{A}}\mathsf{E}\overline{\mathsf{B}}\mathsf{L}$. Among such open problems, the most interesting and challenging ones are surely those about the fragments D and $\mathsf{D}\overline{\mathsf{D}}$. The interest for these fragments has several reasons. First of all, they feature very natural relations, which, apparently, do not present strong conceptual difficulties (models for these two fragments are quite simple to figure out). In [73], Lodaya conjectured decidability of D and undecidability of $\mathsf{D}\overline{\mathsf{D}}$. After more than 10 years and several attempts, such problems, in their full generality, are still open, even if both the fragments have been classified when either discreteness or denseness is assumed. In particular, they represent two of the few cases in which the status of a fragments depends on the class of linear orderings on which it is interpreted (both of them are decidable over dense linear orderings and undecidable over discrete linear ones). It is worth to point out that D is the only one-modality fragment that is still unclassified, and proving its undecidability would mean to solve almost all the open cases.

Finally, a more analytic picture of the state of the art about the classification of HS fragments with respect to the satisfiability problem can be found in Appendix A. To the web page `http://itl.dimi.uniud.it/content/logic-hs`, it is also possible to run a collection of web tools, allowing one to verify the status (decidable/undecidable/unknown) of any specific fragment with respect to the satisfiability problem, over various classes of linear orders (all, dense, discrete, and finite) and considering both strict and non-strict semantics, as well as to compare relative expressive power of any pair of HS fragments.

4

Decidable extensions of PNL: metric PNL

Various metric extensions of point-based temporal logics have been proposed and studied in the literature. They include Alur and Henzinger's Timed Propositional Temporal Logic (TPTL) [4], two-sorted metric temporal logics, developed by by Montanari et al. in [79, 80], Quantitative Monadic Logic of Order, proposed by Hirshfeld and Rabinovich [62], and Owakine and Worrell's Metric Temporal Logic [91], which refines and extends Koymans' Metric Temporal Logic [69]. Little work in that respect has been done in the interval logic setting. Among the few contributions, we mention the extension of Allen's Interval Algebra with a notion of distance developed by Kautz and Ladkin in [67]. The most important quantitative interval temporal logic is definitely Duration Calculus (DC) [35, 60], an interval logic for real-time systems originally developed by Zhou Chaochen, C.A.R. Hoare, and A.P. Ravn [37], based on Moszkowski's ITL [87], which is quite expressive, but generally undecidable. A number of variants and fragments of DC have been proposed to model and to reason about real-time processes and systems [12, 35, 36, 39]. Many of them recover decidability by imposing semantic restrictions, such as the *locality* principle, that essentially reduce the interval logical system to a point-based one.

In this chapter, we present a family of non-conservative metric extensions of PNL, which allow one to express *metric properties* of interval structures over natural numbers. We mainly focus our attention on the most expressive language in this class, called *Metric* PNL (MPNL, for short). MPNL features a family of special atomic propositions representing integer constraints (equalities and inequalities) on the length of the intervals over which they are evaluated. MPNL is particularly suitable for quantitative interval reasoning, and thus it emerges as a viable alternative to existing logical systems for quantitative temporal reasoning. The future fragment of MPNL, also known as RPNL+INT, has been introduced and studied in [26]. Full MPNL has been considered in [18, 15]. The main contributions of this chapter are:

(i) proposal of a number of extensions of PNL with metric modalities and with interval length constraints, which turn out to be very useful to reason about interval structures over natural numbers;

(ii) expressive completeness of MPNL with respect to $\mathsf{FO}^2_r[\mathbb{N}, =, <, s]$, a proper fragment of the two-variable fragment $\mathsf{FO}^2[\mathbb{N}, =, <, s]$ of FO with equality, order, successor, and any family of binary relations, interpreted on natural numbers. We also show how to extend MPNL to obtain an interval logic, MPNL^+, which is expressively complete with respect to full $\mathsf{FO}^2[\mathbb{N}, =, <, s]$;

(iii) decidability and complexity of the satisfiability problem for MPNL, and undecidability of the satisfiability problem for $\mathsf{FO}^2[\mathbb{N}, =, <, s]$, and thus for MPNL^+;

(iv) analysis and classification of all the proposed metric extensions of PNL with respect to their expressive power;

(v) introduction and study of the Directional Area Calculus (DAC), along with its weakened version, called Weak Directional Area Calculus (WDAC), that are spatial generalizations of the future fragment RPNL+INT of MPNL.

The results given here can be compared with analogous results for PNL and $\mathsf{FO}^2[=, <]$ (the two-variable fragment of FO with equality on linear orders with a family of uninterpreted binary relations) [24, 25]. Unlike $\mathsf{FO}^2[=, <]$, which was already known to be decidable [90], the decidability of $\mathsf{FO}^2_r[\mathbb{N}, =, <, s]$ is a consequence of the decidability and expressive completeness results for MPNL. At the best of our knowledge, this result is new and of independent interest.

4.1 PNL and MPNL

4.1.1 Propositional Neighborhood Logics: PNL

In this section, we recall syntax and semantics definitions of PNL. Notice that, we use the modal operators $\Diamond_r$ and $\Diamond_l$ in place of, respectively, $\langle A \rangle$ and $\langle \overline{A} \rangle$ (used in the previous section). As a matter of fact, to make it easier to distinguish between the two semantics (strict and non-strict) from the syntax, it is a common practice in the literature to use the latter pair of operators when strict semantics is considered and the former one when non-strict sematincs is assumed. In this chapter, we focus on the non-strict semantics.

The language of the PNL consists of a set $\mathcal{AP}$ of atomic propositions, the propositional connectives $\neg, \vee$, and the modal operators $\Diamond_r$ and $\Diamond_l$, corresponding to the Allen's relation *meets* and its inverse *met-by* [3]. The other propositional connectives, as well as the logical constants $\top$ (*true*) and $\bot$ (*false*), and the dual modal operators $\Box_r$ and $\Box_l$, are defined as usual. In this chapter, we also *Formulae*, denoted by $\varphi, \psi, \ldots$, are generated by the following grammar:

$$\varphi ::= \pi \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond_r\varphi \mid \Diamond_l\varphi.$$

We recursively define the *truth* relation $\Vdash$ as follows:

- $M, [i, j] \Vdash \pi$ iff $i = j$;
- $M, [i, j] \Vdash p$ iff $p \in V([i, j])$, for any $p \in \mathcal{AP}$;
- $M, [i, j] \Vdash \neg\varphi$ iff it is not the case that $M, [i, j] \Vdash \varphi$;
- $M, [i, j] \Vdash \varphi \vee \psi$ iff $M, [i, j] \Vdash \varphi$ or $M, [i, j] \Vdash \psi$;
- $M, [i, j] \Vdash \Diamond_r\varphi$ iff there exists $h \geq j$ such that $M, [j, h] \Vdash \varphi$;
- $M, [i, j] \Vdash \Diamond_l\varphi$ iff there exists $h \leq i$ such that $M, [h, i] \Vdash \varphi$.

4.1.2 Metric PNL: MPNL

In this section, we introduce metric extensions of PNL interpreted over $\mathbb{N}$. Depending on the choice of the metric operators, a hierarchy of languages can be built. In Section 4.5, we will study the relative expressive power of these languages.

From now on, we denote by $\delta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ the *distance* function on $\mathbb{N}$, defined as $\delta(i, j) = |i - j|$. The results presented here may be suitably rephrased for any function δ satisfying the standard properties of distance over a linear order. The most expressive metric extension of PNL is based on *atomic propositions for length constraints*. These are pre-interpreted (atomic) propositional letters referring to the length of the current interval. Such propositions can be seen as the metric generalizations of the modal constant π. For each $\sim \in \{<, \leq, =, \geq, >\}$, we introduce the length constraint $\ell_{\sim \mathsf{k}}$, with the following semantics:

$$M, [i, j] \Vdash \ell_{\sim \mathsf{k}} \text{ iff } \delta(i, j) \sim k.$$

As a matter of fact, equality and inequality constraints are mutually definable. Indeed, constraints of the form $\ell_{=\mathsf{k}}$ are definable in terms of all the other ones:

$$
\begin{aligned}
&M, [i, j] \Vdash \ell_{=\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \neg\ell_{>\mathsf{k}}, && \text{for } k = 0\\
&M, [i, j] \Vdash \ell_{=\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \ell_{>\mathsf{k-1}} \wedge \neg\ell_{>\mathsf{k}}, && \text{for } k > 0\\
&M, [i, j] \Vdash \ell_{=\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \ell_{\geq\mathsf{k}} \wedge \neg\ell_{\geq\mathsf{k+1}} &&\\
&M, [i, j] \Vdash \ell_{=\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \ell_{<\mathsf{k+1}} \wedge \neg\ell_{<\mathsf{k}} &&\\
&M, [i, j] \Vdash \ell_{=\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \ell_{\leq\mathsf{k}}, && \text{for } k = 0\\
&M, [i, j] \Vdash \ell_{=\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \ell_{\leq\mathsf{k}} \wedge \neg\ell_{\leq\mathsf{k-1}}, && \text{for } k > 0
\end{aligned}
$$

and vice versa:

$$
\begin{aligned}
&M, [i, j] \Vdash \ell_{<\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \bot, && \text{for } k = 0\\
&M, [i, j] \Vdash \ell_{<\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \ell_{=\mathsf{0}} \vee \ldots \vee \ell_{=\mathsf{k-1}}, && \text{for } k > 0\\
&M, [i, j] \Vdash \ell_{\leq\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \ell_{=\mathsf{0}} \vee \ldots \vee \ell_{=\mathsf{k}} &&\\
&M, [i, j] \Vdash \ell_{>\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \neg\ell_{\leq\mathsf{k}} &&\\
&M, [i, j] \Vdash \ell_{\geq\mathsf{k}} \Leftrightarrow M, [i, j] \Vdash \neg\ell_{<\mathsf{k}} &&
\end{aligned}
$$

In Section 4.3, we will limit our attention to constraints of type $\ell_{=k}$, without taking into account the increase in length of formulae due to the above encoding.

4.2 MPNL at work

Finding an optimal balance between expressive power and computational complexity is a challenge for every knowledge representation and reasoning formalism. Interval temporal logics are not an exception in this respect. We believe that MPNL offers a good compromise between these two requirements. In Section 4.2.1, we show that MPNL makes it possible to encode (*metric versions* of) basic operators of point-based linear temporal logic (LTL) as well as interval modalities corresponding to Allen's relations. In addition, we show that it allows one to express limited forms of fuzziness. In Section 4.2.2, we show how to apply MPNL to model the distinctive features of some well-known applications (specification of real-time systems, medical guidelines, ambient intelligence).

4.2.1 Expressing basic temporal properties in MPNL

First, MPNL is expressive enough to encode the strict *sometimes in the future* (resp., *sometimes in the past*) operator of LTL:

$$\Diamond_r(\ell_{>0} \wedge \Diamond_r(\ell_{=0} \wedge p)).$$

Moreover, length constraints allow one to define a metric version of the *until* (resp., *since*) operator. For instance, the condition: '*p is true at a point in the future at distance k from the current interval and, until that point, q is true (pointwise)*' can be expressed as follows:

$$\Diamond_r(\ell_{=k} \wedge \Diamond_r(\ell_{=0} \wedge p)) \wedge \Box_r(\ell_{<k} \rightarrow \Diamond_r(\ell_{=0} \wedge q)).$$

MPNL can also be used to constrain interval length and to express metric versions of basic interval relations. First, we can constrain the length of the intervals over which a given property holds to be at least (resp., at most, exactly) k. As an example, the following formula constrains p to hold only over intervals of length l, with $k \leq l \leq k'$:

$$[G](p \rightarrow \ell_{\geq k} \wedge \ell_{\leq k'}), \qquad (*)$$

where the *universal modality* $[G]$ (*for all intervals*) is a shorthand for the formula:

$$[G]p \equiv \Box_l\Box_r\Box_r p \wedge \Box_l\Box_l\Box_r p.$$

By exploiting such a capability, metric versions of almost all Allen's relations can be expressed (the only exception is the *during* relation). As an example, we can state that: '*p holds only over intervals of length l, with $k \leq l \leq k'$, and any p-interval begins a q-interval*' as follows:

$$(*) \wedge [G] \bigwedge_{i=k}^{k'} (p \wedge \ell_{=\mathsf{i}} \rightarrow \Diamond_l \Diamond_r (\ell_{>\mathsf{i}} \wedge q)).$$

As another example, a metric version of Allen's relation *contains* (the inverse of the *during* relation) can be expressed by:

$$(*) \wedge [G] \bigwedge_{i=k}^{k'} (p \wedge \ell_{=\mathsf{i}} \rightarrow \bigvee_{j \neq 0, j+j' < i} (\Diamond_l \Diamond_r (\ell_{=\mathsf{j}} \wedge \Diamond_r (\ell_{=\mathsf{j}'} \wedge q)))).$$

The general picture is as follows. Allen's relations *meets*, *met-by*, *before*, and *later* can be captured (in their full generality) by PNL. Metric versions of the other relations can be given provided that the number of possible positions of at least one endpoint of the target interval is bounded by the length of the current interval. This is the case of all of them but the relation *during*, whose left and right endpoints can be arbitrarily located respectively before and after the current interval.

The relationships between the satisfiability problem for PNL and the consistency problem for Allen's Interval Networks have been studied in [95][1]. In general, the satisfiability problem for an expressive enough interval temporal logic is much harder than the problem of checking the consistency of a constraint network. The higher complexity of the former is balanced by the expressiveness of the interval logic that allows one to deal with, for instance, negative and disjunctive constraints. As an example, in [95], the author exploits the difference operator to simulate *nominals*, which are then used to force two specific intervals to satisfy a given Allen's relation (the difference operator can be defined in PNL, and thus in MPNL; its definition closely resembles that of the universal modality). Notice that there is no contradiction between the limits to PNL expressive power and its ability to encode (the consistency problem for) constraint networks: PNL allows one to capture Allen's relations among a *finite* number of intervals only (you need a nominal for each interval). The addition of a metric dimension makes it possible to avoid the use of nominals, but it forces one to assign a finite set of possible values for the length of the involved intervals (possibly infinitely many). Whenever there exist some natural bounds for the given finite set of intervals, constraint networks involving all but one Allen's relations can be easily encoded in MPNL (the resulting encoding turns out to be much more natural than the one using nominals).

Finally, MPNL makes it possible to express some forms of 'fuzziness'. As an example, the condition: '*p is true over the current interval and q is true over some interval close to it*', where by 'close' we mean that the right endpoint of the p-interval

[1] Spatial generalizations of the problem to (metric versions of) Weak Spatial PNL and Rectangle Algebra have been investigated in [20, 29] and will be discussed in Section 4.6.

is at distance at most k from the left endpoint of the q-interval, can be expressed as follows:

$$p \wedge (\Diamond_r \Diamond_l (\ell_{<\mathsf{k}} \wedge \Diamond_l \Diamond_r q) \vee \Diamond_r (\ell_{<\mathsf{k}} \wedge \Diamond_r q)).$$

4.2.2 Some applications of MPNL

In the following, we show that MPNL expressive power suffices to capture meaningful requirements of various application domains. To start with, we consider some basic safety conditions that characterize the behavior of a *gas-burner*. This is a classical example commonly used to illustrate the modeling capabilities of a specification formalism. For instance, a formalization of such an example in Duration Calculus can be found in [35].

Let the atomic proposition Gas (resp., $Flame$, $Leak$) be used to state that gas is flowing (resp., burning, leaking), e.g., $M, [i, j] \Vdash Gas$ means that gas is flowing over the interval $[i, j]$. The formula

$$[G](Leak \leftrightarrow Gas \wedge \neg Flame)$$

states that $Leak$ holds over an interval if and only if gas is flowing and not burning over that interval. The condition: '*it never happens that gas is leaking for more than k time units*' can be expressed as:

$$[G](\neg(\ell_{>\mathsf{k}} \wedge Leak))$$

Similarly, the condition: '*the gas burner will not leak uninterruptedly for k time units after the last leakage*' can be formalized as:

$$[G](Leak \rightarrow \neg \Diamond_r (\ell_{<\mathsf{k}} \wedge \Diamond_r Leak))$$

As another example, let us consider the case of a railway signaling system. A systematic analysis of such a case study, together with its formalization in Duration Calculus, has been done by Veludis and Nissanke in [100]. One of the distinctive features of this system is the large set of safety requirements it involves. Here, we choose one of them and we show how to encode it in MPNL. Most of the other requirements can be dealt with in a very similar way. The specification basically constrains the relationships between the controlling system and the controlled system, which is equipped with both *sensors* and *activators*. More precisely, let the atomic proposition $ReqToRed_i$ (resp., $ReqToYellow_i$, $ReqToGreen_i$) denote the fact that the controlling system has sent to the *signal* (semaphore) i the request to change the color to red (resp., yellow, green). Similarly, let $SignalOp_i$ denote the fact that the i-th signal is operative, that is, not broken. A typical (functional) requirement of the railway signaling system imposes that, when a request to change its color is sent to a signal, either the signal actually changes it within a fixed amount of time or the signal is declared non operative. Such a requirement can be formalized in MPNL as follows:

$$[G](ReqToRed_i \wedge SignalOp_i \rightarrow \Diamond_r(\ell_{\leq k} \wedge \neg\Diamond_r ProceedAspect_i) \vee \Diamond_r\Diamond_r\neg SignalOp_i),$$

where $ProceedAspect_i$ denotes the fact that the signal i is either yellow or green. Notice that MPNL allows one to possible bound the duration of the time period during which a signal is non operative.

Finally, let us consider the application of MPNL to the fields of *medical guidelines* and *ambient intelligence*. In the former (see [96]), events with duration, e.g., '*running a fever*', possibly paired with metric constraints, e.g., '*if a patient is running a fever for more than k time units, then administrate him/her drug D*', are quite common. Medical requirements of this kind can be easily encoded in MPNL. As an example, the above condition can be expressed in MPNL as follows:

$$[G]((Fever \wedge \ell_{>k}) \rightarrow \Diamond_r DrugD)$$

In general, many relevant conditions in medical guidelines are inherently interval-based as there are no general rules to deduce their occurrence from point-based data. The use of temporal logic in ambient intelligence, specifically in the area of Smart Homes [5, 52], has been advocated by Combi et al. in [42]. MPNL can be successfully used to express safety requirements referring to situations that can be properly modeled only in terms of time intervals, e.g., '*being in the kitchen*'.

4.3 Decidability of MPNL

In this section, we use a model-theoretic argument to show that the satisfiability problem for MPNL has the bounded-model property with respect to finitely presentable ultimately periodic models, and it is therefore decidable. From now on, let φ be any MPNL formula and let $\mathcal{AP}$ be the set of propositional letters of the language.

Definition 4.3.1. The *closure* of φ is the set $CL(\varphi)$ of all sub-formulae of $\Diamond_r\varphi$ and their negations (we identify $\neg\neg\psi$ with ψ, $\neg\Diamond_r\psi$ with $\Box_r\neg\psi$, and $\neg\Diamond_l\psi$ with $\Box_l\neg\psi$). Let $\bigodot \in \{\Diamond_r, \Diamond_l, \Box_r, \Box_l\}$. The set of *temporal requests* from $CL(\varphi)$ is the set $TF(\varphi) = \{\bigodot\psi \mid \bigodot\psi \in CL(\varphi)\}$.

Definition 4.3.2. A φ-*atom* is a set $A \subseteq CL(\varphi)$ such that for every $\psi \in CL(\varphi)$, $\psi \in A$ iff $\neg\psi \notin A$ and for every $\psi_1 \vee \psi_2 \in CL(\varphi)$, $\psi_1 \vee \psi_2 \in A$ iff $\psi_1 \in A$ or $\psi_2 \in A$.

We denote the set of all φ-atoms by A_φ. One can easily prove that $|CL(\varphi)| \leq 2(|\varphi| + 1)$, $|TF(\varphi)| \leq 2|\varphi|$, and $|A_\varphi| \leq 2^{|\varphi|+1}$. We now introduce a suitable labeling of interval structures based on φ-atoms.

Definition 4.3.3. A *(φ-)labeled interval structure* (LIS for short) is a structure $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$, where $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ is the interval structure over natural numbers (or over a finite subset of it) and $\mathcal{L} : \mathbb{I}(\mathbb{D}) \to A_\varphi$ is a *labeling function* such that for every pair of neighboring intervals $[i, j]$, $[j, h] \in \mathbb{I}(\mathbb{D})$, if $\Box_r \psi \in \mathcal{L}([i, j])$, then $\psi \in \mathcal{L}([j, h])$, and if $\Box_l \psi \in \mathcal{L}([j, h])$, then $\psi \in \mathcal{L}([i, j])$.

Notice that every interval model M induces a LIS, whose labeling function is the valuation function:

$$\psi \in \mathcal{L}([i, j]) \text{ iff } M, [i, j] \Vdash \psi.$$

Thus, LIS can be thought of as *quasi-models* for φ, in which the truth of formulae containing neither $\Diamond_r$, $\Diamond_l$ nor length constraints is determined by the labeling (due to the definitions of φ-atom and LIS). To obtain a model, we must also guarantee that the truth of the other formulae is in accordance with the labeling. To this end, we introduce the notion of fulfilling LIS.

Definition 4.3.4. A LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ is *fulfilling* iff:

- for every length constraint $\ell_{=\mathsf{k}} \in CL(\varphi)$ and interval $[i, j] \in \mathbb{I}(\mathbb{D})$, $\ell_{=\mathsf{k}} \in \mathcal{L}([i, j])$ iff $\delta(i, j) = k$;
- for every temporal formula $\Diamond_r \psi$ (resp., $\Diamond_l \psi$) in $TF(\varphi)$ and interval $[i, j] \in \mathbb{I}(\mathbb{D})$, if $\Diamond_r \psi \in \mathcal{L}([i, j])$ (resp., $\Diamond_l \psi \in \mathcal{L}([i, j])$), then there exists $h \geq j$ (resp., $h \leq i$) such that $\psi \in \mathcal{L}([j, h])$ (resp., $\psi \in \mathcal{L}([h, i])$).

Clearly, every interval model is a fulfilling LIS. Conversely, every fulfilling LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ can be transformed into a model $M(\mathbf{L})$ by defining the valuation in accordance with the labeling. Then, one can prove that for every $\psi \in CL(\varphi)$ and interval $[i, j] \in \mathbb{I}(\mathbb{D})$,

$$\psi \in \mathcal{L}([i, j]) \text{ iff } M(\mathbf{L}), [i, j] \models \psi$$

by a routine induction on ψ. Therefore, a formula φ is *satisfied* by a fulfilling LIS if and only if there exists an interval such that its label contains φ.

Let m be $\frac{|TF(\varphi)|}{2}$ and k be the maximum among the natural numbers occurring in the length constraints in φ. For example, if $\varphi = \Diamond_r(\ell_{>3} \wedge p \to \Diamond_l(\ell_{>5} \wedge q))$, then $m = 3$ and $k = 5$. We now introduce the fundamental notions of left and right temporal requests at a given point.

Definition 4.3.5. Given a LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ and a point $i \in D$, the set of *left* (resp., *right*) *temporal requests* at i, denoted by $REQ^L(i)$ (resp., $REQ^R(i)$), is the set of pairs of the type (τ, s), where τ is a temporal formula of the forms $\Diamond_l \psi$, $\Box_l \psi$ (resp., $\Diamond_r \psi$, $\Box_r \psi$) in $TF(\varphi)$ belonging to the labeling of any interval beginning (resp., ending) at i, and $s = +$, if there exists an interval $[j, i]$ (resp., $[i, j]$) such that $\tau \in \mathcal{L}([j, i])$ (resp., $\tau \in \mathcal{L}([i, j])$) and $\delta(j, i) > k$ (resp., $\delta(i, j) > k$), and $s = -$ otherwise.

For any $i \in D$, we write $REQ(i)$ for $REQ^L(i) \cup REQ^R(i)$. We denote by $REQ(\varphi)$ the set of all possible sets of temporal requests from $CL(\varphi)$; moreover, for the sake of brevity, we write $\tau \in REQ(i)$ when there exists a pair $(\tau, s) \in REQ(i)$. It is easy to show that $|REQ(\varphi)| = 2^{2 \cdot m}$. Moreover, by definition, any set of temporal requests $REQ^R(j)$ (resp., $REQ^L(i)$) can be entirely satisfied using at most m different points greater than j (resp., less than i).

Now, consider any MPNL formula φ such that φ is satisfiable on a finite model. We have to show that we can restrict our attention to models with a bounded size.

Definition 4.3.6. Given any LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$, we say that a *$k$-sequence in* $\mathbf{L}$ is a sequence of k consecutive points in D. Given a k-sequence σ in $\mathbf{L}$, its *sequence of requests* $REQ(\sigma)$ is defined as the k-sequence of temporal requests at the points in σ. We say that $i \in D$ *starts a k-sequence* σ if the temporal requests at $i, \ldots, i+k-1$ form an occurrence of $REQ(\sigma)$. Furthermore, the sequence of requests $REQ(\sigma)$ is said to be *abundant* in $\mathbf{L}$ iff it has at least $2 \cdot (m^2 + m) \cdot |REQ(\varphi)| + 1$ disjoint occurrences in D.

Intuitively, when a model for a given formula φ presents an abundant sequence, then the model can be shortened without affecting satisfiability of φ.

Lemma 4.3.7. *Let $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ be any LIS such that the sequence $REQ(\sigma)$ is abundant in it. Then, there exists an index q such that for each element $\mathcal{R} \in \{REQ(d) \mid i_q < d < i_{q+1}\}$, where i_q and i_{q+1} begin the q-th and the $q+1$-th occurrence of σ, respectively, $\mathcal{R}$ occurs at least $m^2 + m$ times before i_q and at least $m^2 + m$ times after $i_{q+1} + k - 1$.*

Proof. To prove this property, we proceed by contradiction. Suppose that $REQ(\sigma)$ is abundant, that is, it occurs $n > 2 \cdot (m^2 + m) \cdot |REQ(\varphi)|$ times in D and, for each q with $1 \leq q \leq n$, there exists a point $d(q)$ with $i_q < d(q) < i_{q+1}$, such that $REQ(d(q))$ occurs less than $(m^2 + m)$ times before i_q or less than $(m^2 + m)$ times after $j_{q+1} + k - 1$. Let $\Delta = \{d(q) | 1 \leq q \leq n\}$ the set of all such points. By hypothesis, there cannot be any $\mathcal{R} \in REQ(\varphi)$ such that $\mathcal{R}$ occurs more than $2 \cdot (m^2 + m)$ times in Δ. Then $|\Delta| \leq 2 \cdot (m^2 + m) \cdot |REQ(\varphi)|$, which is a contradiction. □

Lemma 4.3.8. *Let $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ be a fulfilling LIS that satisfies φ. Suppose that there exists an abundant k-sequence of requests $REQ(\sigma)$ and let q be the index whose existence is guaranteed by Lemma 4.3.7. Then, there exists a fulfilling LIS $\mathbf{L}^* = \langle \mathbb{D}^*, \mathbb{I}(\mathbb{D}^*), \mathcal{L}^* \rangle$ that satisfies φ such that $D^* = D \setminus \{i_q, \ldots, i_{q+1} - 1\}$.*

Proof. Let us fix a fulfilling LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ satisfying φ at some $[i, j]$, an abundant k-sequence $REQ(\sigma)$ in $\mathbf{L}$, and the index q identified by Lemma 4.3.7. Moreover, let $D^- = \{i_q, \ldots, i_{q+1} - 1\}$ and $D' = D \setminus D^-$. For the sake of readability, the points in D' will be denoted by the same numbers as in D. We now show how to suitably redefine the evaluation of the intervals in $\mathbb{I}(\mathbb{D}')$ to preserve satisfiability of φ (as a matter of fact, the temporal requests at all points in D' are preserved as well).

First, we consider all points $d < i_q$ and for each of them, for all p such that $0 \leq p \leq k-1$, we put $\mathcal{L}'([d, i_{q+1}+p]) = \mathcal{L}([d, i_q+p])$. Then, for all p, p' such that $0 \leq p \leq p' \leq k-1$, we put $\mathcal{L}'([i_{q+1}+p, i_{q+1}+p']) = \mathcal{L}([i_q+p, i_q+p'])$. In such a way, we guarantee that the intervals whose length has been shortened as an effect of the elimination of the points in D^- have a correct labeling in terms of all length constraints of the forms $\ell_{=\mathsf{k}'}$ and $\neg\ell_{=\mathsf{k}'}$. Moreover, since the requests (in both directions) in $\mathbf{L}$ at $i_{q+1}+p$ are equal to the requests at i_q+p, this operation is safe with respect to universal and existential requirements. Finally, since the lengths of intervals beginning before i_q and ending after $i_{q+1}+k-1$ are greater than k both in $\mathbf{L}$ and in $\mathbf{L}'$, there is no need to change their labeling. (Notice that, in D', i_{q+1} turns out to be the immediate successor of $i_q - 1$.)

The structure $\mathbf{L}' = \langle \mathbb{D}', \mathbb{I}(\mathbb{D}'), \mathcal{L}' \rangle$ defined so far is obviously a LIS, but it is not necessarily a fulfilling one. The removal of the points in the set D^- and the relabeling needed to guarantee correctness with respect to the length constraints may generate *defects*, that is, situations in which there exists a point $d < i_q$ (resp., $d \geq i_{q+1}+k$) and a formula of the type $\Diamond_r\psi$ (resp., $\Diamond_l\psi$) belonging to $REQ(d)$ such that ψ was satisfied in $\mathbf{L}$ by some interval $[d, d']$ (resp., $[d', d]$) and it is not satisfied in $\mathbf{L}'$, either because $d' \in D^-$ or because the labeling of $[d, d']$ (resp., $[d', d]$) has changed due to the above relabeling. We have to show how to repair such defects.

First, we collect and order the set of defects (assume that we have r of them). Suppose that the first one concerns the existence of a point $d < i_q$ and a formula $\Diamond_r\psi \in REQ(d)$, which is not satisfied anymore in $\mathbf{L}'$ (the case in which $d \geq i_{q+1}+k$ can be dealt with in a similar way). Since $\mathbf{L}$ is a fulfilling LIS, then there exists an interval $[d, d']$ such that $\psi \in \mathcal{L}([d, d'])$ and either $d' \in D^-$ or $i_{q+1} \leq d' < i_{q+1}+k$ and $\psi \notin \mathcal{L}'([d, d'])$. Moreover, for this to be the case, $\delta(d', d) > k$ in $\mathbf{L}$, and thus the defect necessarily involves a pair of the form $(\Diamond_r\psi, +) \in REQ(d)$. By Lemma 4.3.7, there exist at least $n = m^2 + m$ points $\{\bar{d}_1, \ldots, \bar{d}_n\}$ after $i_{q+1}+k-1$ such that $REQ(\bar{d}_i) = REQ(d')$, for $i = 1, \ldots, n$. We will choose one of these points, say $\bar{d}_i$, to satisfy the request. In general, this may require a change in the labeling of the interval $[d, \bar{d}_i]$, and to prevent such a change to make one or more requests in $REQ^L(\bar{d}_i)$ no longer satisfied, we will possibly have to redefine the labeling of more than one interval.

To start with, we take a point $d'' < i_q$ such that $REQ(d'') = REQ(d')$ (the existence of such a point is guaranteed by Lemma 4.3.7) and a minimal set of points $P^{d''} \subset D'$ such that, for each $(\Diamond_l\tau, +) \in REQ^L(d'')$, there exists a point $e \in P^{d''}$ such that $\tau \in \mathcal{L}([e, d''])$ and $\delta(e, d'') > k$. Now, for each point $e \in P^{d''}$, let $P^{d''}_e$ be a minimal set of points such that, for each $\Diamond_r\xi \in REQ^R(e)$, there exists a point $f \in P^{d''}_e$ such that $\xi \in \mathcal{L}([e, f])$. Finally, let $Q = \bigcup_{e \in P^{d''}} P^{d''}_e$. By the minimality requirements, we have that $|Q| \leq m^2$, since requests in $REQ^L(d'')$ need at most m points to be satisfied and, for each $e \in P^{d''}$, $REQ^R(e)$ can be satisfied using at most m points.

Consider the set $H = \{\bar{d}_1, \ldots, \bar{d}_n\} \setminus Q$. Since, by construction, $|H| \geq (m^2 + m) - m^2 = m$, there must be some point $\bar{d}_i \in H$ such that in $\mathbf{L}'$ the interval

$[d, \bar{d}_i]$ satisfies only those $\Diamond_r$-formulae of $REQ(d)$, if any, that are satisfied over some other interval beginning at d. Then, we create a new LIS $\mathbf{L}'_1$, and we put $\mathcal{L}'_1([d, \bar{d}_i]) = \mathcal{L}([d, d'])$. Since $REQ^R(\bar{d}_i) = REQ^R(d')$, such a change has no impact on the right-neighboring intervals of $[d, \bar{d}_i]$. On the contrary, there may exist one or more $\Diamond_l$-formulae in $REQ^L(\bar{d}_i)$ which, due to the change in the labeling of $[d, \bar{d}_i]$, are not satisfied anymore. In such a case, however, we can recover satisfiability, without introducing any new defect, by putting $\mathcal{L}'_1([e, \bar{d}_i]) = \mathcal{L}([e, d''])$ for all $e \in P^{d''}$. Notice that the intervals $[e, d'']$ cannot be shorter than k by definition of $P^{d''}$, and thus this relabeling is safe with respect to length constraints. The labeling of all other intervals is the one defined by $\mathbf{L}'$.

In this way, we have fixed the first defect without introducing any new defect. If we repeat the above procedure for each of the defects, according to their ordering, we obtain a finite sequence of LISs $\mathbf{L}'_1, \mathbf{L}'_2, \ldots, \mathbf{L}'_r$, where the last one is the LIS $\mathbf{L}^*$ we were looking for.

To conclude the proof, we have to show that $\mathbf{L}^*$ is still a LIS for φ. Let $[d, d']$ be the interval of $\mathbf{L}$ satisfying the formula φ. Since $\Diamond_r\varphi \in CL(\varphi)$, we have that $\Diamond_r\varphi \in REQ(d)$. If d is still present in $\mathbf{L}^*$, then, since the final LIS is fulfilling, we have that there must exist an interval $[d, d'']$ labelled with φ. If d is not a point of $\mathbf{L}^*$ anymore, then Lemma 4.3.7 guarantees that there exists another point d'' in $\mathbf{L}^*$ such that $REQ(d'') = REQ(d)$. Again, since $\mathbf{L}^*$ is fulfilling, we have that there must exist an interval $[d'', d''']$ labelled with φ. □

Lemma 4.3.8 guarantees that we can eliminate sequences of requests that occur 'sufficiently many' times in a LIS, without 'spoiling' the LIS. This eventually allows us to prove the following small-model theorem for finite satisfiability of MPNL.

Theorem 4.3.9 (Small-Model Theorem)**.** *If φ is any finitely satisfiable formula of* MPNL*, then there exists a fulfilling, finite LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ that satisfies φ such that $|D| \leq |REQ(\varphi)|^k \cdot (2 \cdot (m^2 + m) \cdot |REQ(\varphi)| + 1) \cdot k + k - 1$.*

Proof. Let $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ be any finite fulfilling LIS that satisfies φ. If $|D| \leq |REQ(\varphi)|^k \cdot (2 \cdot (m^2 + m) \cdot |REQ(\varphi)| + 1) \cdot k + k - 1$, then we are done. Otherwise, by an application of the pigeonhole principle, for at least one sequence $REQ(\sigma)$ of length k, we have that $REQ(\sigma)$ is abundant. In this case, we apply Lemma 4.3.8 sufficiently many times to get the requested maximum length. □

To deal with formulae that are satisfiable only over infinite models, we provide these models with a finite periodic representation, and we bound the lengths of their prefix and period.

Definition 4.3.10. A LIS $\mathbf{L} = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L} \rangle$ is *ultimately periodic*, with *prefix* L, *period* P, and *threshold* $k < P$ if, for every interval $[i, j]$,

- if $i \geq L$, then $\mathcal{L}([i, j]) = \mathcal{L}([i + P, j + P])$;
- if $j \geq L$ and $\delta(i, j) > k$, then $\mathcal{L}([i, j]) = \mathcal{L}([i, j + P])$.

It is worth noticing that, in every ultimately periodic LIS, $REQ(i) = REQ(i+P)$, for $i \geq L$, and that every ultimately periodic LIS is finitely presentable: it suffices to define its labeling only on the intervals $[i,j]$ such that $j < L+2\cdot P+k$; thereafter, it can be uniquely extended by periodicity. It can be easily shown that a finite LIS can be recovered as a special case of ultimately periodic LIS.

Lemma 4.3.11. *Let φ be an* MPNL *formula and $\mathbf{L} = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), \mathcal{L}\rangle$ be an infinite fulfilling LIS over $\mathbb{N}$ that satisfies φ. Then, there exists an infinite ultimately periodic fulfilling LIS $\mathbf{L}^* = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), \mathcal{L}^*\rangle$ over $\mathbb{N}$ that satisfies φ.*

Proof. First of all, let $[b,e]$ be the interval satisfying φ in $\mathbf{L}$. We define the set $REQ_{inf}(\varphi)$ as the subset of $REQ(\varphi)$ containing all and only the sets of requests that occur infinitely often in $\mathbf{L}$. Let $L, M \in \mathbb{N}$ be such that the following conditions are met: (i) $L \geq e$; (ii) for each point $r \geq L$, $REQ(r) \in REQ_{inf}(\varphi)$; (iii) each set of requests $\mathcal{R} \in REQ_{inf}(\varphi)$ occurs at least m^2+m times before L and at least m^2+m times between $L+k$ and M; (iv) for each point $i < L$ and any formula $\Diamond_r \tau \in REQ(i)$, τ is satisfied on some interval $[i,j]$, with $j < M$; and (v) the k-sequences of requests starting at L and at M are the same.

We put $P = M - L$. By condition (iii), $P > k$. We build an infinite ultimately periodic structure $\overline{\mathbf{L}}$ over the domain $\mathbb{N}$ with prefix L, period P, and threshold k. As a first step, for all points $d < M$, we put $\overline{REQ}(d) = REQ(d)$. Then, for all points $M+n$, with $0 \leq n < P$, we put $\overline{REQ}(M+n) = REQ(L+n)$ (by condition (v), this is already the case for $0 \leq n < k$), and, for all points $M+P+n$, with $0 \leq n < k$, we put $\overline{REQ}(M+P+n) = REQ(L+n)$.

The labeling is defined as follows. For all intervals $[i,j]$ such that $j < M$, we put $\overline{\mathcal{L}}([i,j]) = \mathcal{L}([i,j])$. As for intervals $[i,j]$, with $M \leq j < M+P$, we must distinguish different cases:

(a) if $i \geq M$, we put $\overline{\mathcal{L}}([i,j]) = \mathcal{L}([i-P, j-P])$;

(b) if $i < M$ (and thus $\overline{REQ}(i) = REQ(i)$), we must distinguish three sub-cases:

- (b1) if $\delta(i,j) \leq k$ (and thus, by condition (v), $\overline{REQ}(j) = REQ(j)$), then we put $\overline{\mathcal{L}}([i,j]) = \mathcal{L}([i,j])$;
- (b2) if $k < \delta(i,j) \leq k+P$, we put $\overline{\mathcal{L}}([i,j]) = \mathcal{L}([i,h])$ for some h such that $\overline{REQ}(j) = REQ(h)(= REQ(j-P))$ and $\delta(i,h) > k$ (the existence of such an h is guaranteed by conditions (ii) and (iii); if $M \leq j < M+k$, we can take $h = j$);
- (b3) if $\delta(i,j) > k+P$, we put $\overline{\mathcal{L}}([i,j]) = \mathcal{L}([i, j-P])$.

As for intervals $[i,j]$, with $M+P \leq j < M+P+k$, we must distinguish three cases:

(1) if $i \geq M$, we put $\overline{\mathcal{L}}([i,j]) = \overline{\mathcal{L}}([i-P, j-P])$;

(2) if $i < M$ and $\delta(j,i) > P+k$, then $\overline{\mathcal{L}}([i,j]) = \overline{\mathcal{L}}([i, j-P])$

(3) if $i < M$ and $\delta(j, i) \leq P + k$, then $\overline{\mathcal{L}}([i, j]) = \overline{\mathcal{L}}([i', j])$, for some i' such that $i' < L$ and $\overline{REQ}(i')(= REQ(i')) = \overline{REQ}(i)$ (the existence of such an i' is guaranteed by condition (iii)).

The above construction labels all sub-intervals of $[0, M + P + k]$ in a way that is consistent with the definition of LIS, but that is not necessarily fulfilling. The labels of intervals $[i, j]$, with $j < M$, remain unchanged and thus requests of points $i < L$ are not critical, as, by condition (iv), every request of every such point is satisfied on some interval $[i, j]$, with $j < M$. This is not the case with $L \leq i \leq M$. Indeed, it may happen that, for some point $L \leq i \leq M$ and some formula $\Diamond_r \psi \in \overline{REQ}(i)$, there is no interval satisfying ψ in $\overline{\mathbf{L}}$ (the only intervals satisfying it in $\mathbf{L}$ being of the form $[i, j]$, with $j > M+k$). We fix such *defects* as follows. Since $\overline{REQ}(i) = REQ(i)$, there exists a point $j > i$ such that $\psi \in \mathcal{L}([i, j])$ in $\mathbf{L}$. By condition (iii), there exist at least $m^2 + m$ points between $M + k$ and $M + P$ with the same set of requests as j. We proceed exactly as in the proof of Lemma 4.3.8: we fix the defect by choosing a point d' in between $M+k$ and $M+P$ and building a new LIS $\mathbf{L}_1$, which is identical to $\overline{\mathbf{L}}$ but for the labeling of the interval $[i, d']$ (we put $\mathcal{L}_1([i, d']) = \mathcal{L}([i, d])$). By applying such a repair procedure to each defect in a systematic manner, e.g., starting from the defect closest to the origin and then moving from left to right, we generate a finite sequence of LISs $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_r$, the last of which is such that every request of every point $i \leq M$ is fulfilled before $M + P$.

The ultimately periodic fulfilling LIS $\mathbf{L}^*$ is obtained from $\mathbf{L}_r$ by completing the specification of the labeling of the intervals in $\mathbb{I}(\mathbb{N})$ in such a way that the conditions of Definition 4.3.10 for an ultimately periodic LIS with prefix L, period P, and threshold k are satisfied. Formally, for every $i \geq M + P + k$ we put $REQ^*(i) = REQ^*(i - n \cdot P)$, where n is the least non-negative integer such that $i - n \cdot P < M + P + k$. Then, for every interval $[i, j]$ such that $j \geq M + P + k$, if $\delta(i, j) \leq P + k$, then we put $\mathcal{L}^*([i, j]) = \mathcal{L}^*([i - n \cdot P, j - n \cdot P])$, where n is the least non-negative integers such that $j - n \cdot P < M + P + k$ (notice that, since $\delta(i, j) \leq P + k$, it also holds $i - n \cdot P \geq L$); otherwise, we put $\mathcal{L}^*([i, j]) = \mathcal{L}^*([i - n \cdot P, j - q \cdot P])$, where n and q are respectively the least non-negative integers such that $L \leq i - n \cdot P < M$ and $M + k \leq j - q \cdot P < M + P + k$ (notice that $\delta([i - n \cdot P, j - q \cdot P]) > k$). It is straightforward to check that the labeling $\mathcal{L}^*$ respects all length constraints, and that the resulting structure $\mathbf{L}^* = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), \mathcal{L}^* \rangle$ is an ultimately periodic fulfilling LIS satisfying φ on $[b, e]$. □

Theorem 4.3.12 (Small Periodic Model Theorem). *If φ is any satisfiable formula of* MPNL*, then there exists a fulfilling, ultimately periodic LIS satisfying φ such that both the length L of the prefix and the length P of the period are less or equal to $|REQ(\varphi)|^k \cdot (2 \cdot (m^2 + m) \cdot |REQ(\varphi)| + 1) \cdot k + k - 1$.*

Proof. Existence of an ultimately periodic fulfilling LIS is guaranteed by Lemma 4.3.11. The bound on the prefix and of the period can be proved by exploiting Lemma 4.3.8. □

Corollary 4.3.13. *The satisfiability problem for* MPNL*, interpreted over* $\mathbb{N}$*, is decidable.*

The results of this section immediately give a double exponential time nondeterministic procedure for checking the satisfiability of any MPNL formula φ. Such a procedure nondeterministically checks models whose size is in general $O(2^{k\cdot|\varphi|})$, where $|\varphi|$ is the length of the formula to be checked for satisfiability. It has been shown in [26] that, in the case in which k is represented in binary, the right-neighborhood fragment of MPNL is complete for the class EXPSPACE. This means that, in the general case, the complexity for MPNL is located somewhere in between EXPSPACE and 2NEXPTIME (the exact complexity is still an open problem). It is worth noticing that, whenever k is a constant, it does not influence the complexity class and thus, since we have an NTIME($2^{|\varphi|}$) procedure for satisfiability and a NEXPTIME-hardness result [31], we can conclude that MPNL is NEXPTIME-complete. Similarly, when k is expressed in unary, the value of k increases linearly with the length of the formula and thus NTIME($2^{k\cdot|\varphi|}$)=NTIME($2^{|\varphi|^2}$); therefore, as in the previous case, MPNL is NEXPTIME-complete.

4.4 MPNL and two-variable fragments of First-Order logic for $(\mathbb{N}, <, s)$

4.4.1 PNL and two-variable fragments of First-Order logic

We start with a summary of results from [25], which will then be extended to MPNL. Let us denote by $\mathsf{FO}^2[=]$ the fragment of first-order logic with equality whose language contains only two distinct variables. Moreover, we denote the formulae of $\mathsf{FO}^2[=]$ by $\alpha, \beta, \ldots$. For example, the formula $\forall x(P(x) \rightarrow \forall y \exists x Q(x,y))$ belongs to $\mathsf{FO}^2[=]$, while the formula $\forall x(P(x) \rightarrow \forall y \exists z(Q(z,y) \wedge Q(z,x)))$ does not. We first focus our attention on the extension $\mathsf{FO}^2[=,<]$ of $\mathsf{FO}^2[=]$ over a purely relational vocabulary $\{=,<,P,Q,\ldots\}$ including equality and a distinguished binary relation $<$ interpreted as a linear order. Since atoms in two-variable fragments may involve at most two distinct variables, we can further assume, without loss of generality, that the arity of every relation in the considered vocabulary is exactly 2. Let x and y be the two variables of the language. The formulae of $\mathsf{FO}^2[=,<]$ can be defined recursively as follows:

$$\begin{array}{rcl} \alpha & ::= & A_0 \mid A_1 \mid \neg\alpha \mid \alpha \vee \beta \mid \exists x \alpha \mid \exists y \alpha \\ A_0 & ::= & x = x \mid x = y \mid y = x \mid y = y \mid x < y \mid y < x \\ A_1 & ::= & P(x,x) \mid P(x,y) \mid P(y,x) \mid P(y,y), \end{array}$$

where A_1 deals with (uninterpreted) binary predicates. For technical convenience, we assume that both variables x and y occur as (possibly vacuous) free variables in every formula $\alpha \in \mathsf{FO}^2[=,<]$, that is, $\alpha = \alpha(x,y)$.

Formulae of $\mathsf{FO}^2[=,<]$ are interpreted over *relational models* of the form $\mathbf{M} = \langle \mathbb{D}, V \rangle$, where $\mathbb{D} = \langle D, < \rangle$ is a linearly ordered set and V is a *valuation function* that assigns to every binary relation P a subset of $D \times D$. When we evaluate a formula $\alpha(x,y)$ on a pair of elements a, b, we write $\alpha(a,b)$ for $\alpha[x := a, y := b]$.

The decidability of the satisfiability problem for FO^2 without equality has been proved by Scott [97] by means of satisfiability-preserving reduction of any FO^2 formula to a formula of the form $\forall x \forall y \psi_0 \wedge \bigwedge_{i=1}^{m} \forall x \exists y \psi_i$, which belongs to the Gödel's prefix-defined class of first-order formulae, whose satisfiability problem was shown to be decidable by Gödel [13].

Later on, Mortimer extended this decidability result by including equality in the language [86]. Mortimer's result has been improved by Grädel, Kolaitis, and Vardi who lowered the complexity [56]. Finally, by building on techniques from [56] and taking advantage of an in-depth analysis of the basic 1-types and 2-types in $\mathsf{FO}^2[=,<]$ models, Otto proved the decidability of $\mathsf{FO}^2[=,<]$ over various classes of orders, including $\mathbb{N}$. In [25], Bresolin et al. show that $\mathsf{FO}^2[=,<]$ is expressively complete with respect to PNL^{π}. In the following, we extend this expressive completeness result (in the case of natural numbers) to MPNL.

4.4.2 Comparing the expressive power of interval and First-Order logics

There are various ways to compare the expressive power of different logics. The one we use here is comparing logics with respect to properties they can express. In doing this, we distinguish two different cases: the case in which we compare two interval logics on the same class of models, e.g., different fragments of MPNL, and the case in which we compare an interval logic with a first-order logic, e.g., MPNL and a suitable extension of $\mathsf{FO}^2[=,<]$.

Given two interval logics L and L' interpreted in the same class of models $\mathcal{C}$, we say that L' is *at least as expressive as* L (with respect to $\mathcal{C}$), denoted by $\mathrm{L} \preceq_{\mathcal{C}} \mathrm{L'}$, if there exists an effective translation τ from L to L' (inductively defined on the structure of formulae) such that for every model M in $\mathcal{C}$, any interval $[i,j]$ in the model, and any formula φ of L, we have $M,[i,j] \Vdash \varphi$ if and only if $M,[i,j] \Vdash \tau(\varphi)$. Furthermore, we say that L is *as expressive as* L', denoted by $\mathrm{L} \equiv_{\mathcal{C}} \mathrm{L'}$, if both $\mathrm{L} \preceq_{\mathcal{C}} \mathrm{L'}$ and $\mathrm{L'} \preceq_{\mathcal{C}} \mathrm{L}$, while we say that L' is *strictly more expressive than* L, denoted by $\mathrm{L} \prec_{\mathcal{C}} \mathrm{L'}$, if $\mathrm{L} \preceq_{\mathcal{C}} \mathrm{L'}$ and $\mathrm{L'} \not\preceq_{\mathcal{C}} \mathrm{L}$. Finally, we say that two logics are incomparable if no one of the above cases applies. In the following, we will omit the $\mathcal{C}$ subscript when it will be clear from the context.

When we compare interval logics with first-order logics interpreted in relational models, the above criteria are no longer adequate, since we need to compare logics which are interpreted in different types of model (interval models and relational models). We deal with this complication by following the approach outlined by

Venema in [102]. We first define suitable model transformations (from interval models to relational models and vice versa) and then we compare the expressiveness of interval and first-order logics modulo these transformations. In order to define the mapping from interval models to relational models, we associate a binary relation P with every propositional variable $p \in \mathcal{AP}$ of the considered interval logic, as in the following definition.

Definition 4.4.1 ([25]). Let $M = \langle I(\mathbb{D}), V_M \rangle$ be an interval model. The corresponding relational model $\eta(M)$ is a pair of the type $\langle \mathbb{D}, V_{\eta(M)} \rangle$, where for all $p \in \mathcal{AP}$, $V_{\eta(M)}(P) = \{(i,j) \in D \times D : [i,j] \in V_M(p)\}$.

To define the mapping from relational models to interval ones, we have to solve a technical problem: the truth of formulae in interval models is evaluated only on ordered pairs $[i,j]$, with $i \leq j$, while in relational models there is no such constraint. To deal with this problem, we associate two propositional letters $p^{\leq}$ and $p^{\geq}$ of the interval logic with every binary relation P.

Definition 4.4.2 ([25]). Let $\mathbf{M} = \langle \mathbb{D}, V_{\mathbf{M}} \rangle$ be a relational model. The corresponding interval model $\zeta(\mathbf{M})$ is a pair $\langle \mathbb{I}(\mathbb{D}), V_{\zeta(\mathbf{M})} \rangle$ such that for any binary relation P and any interval $[i,j]$, we have that $[i,j] \in V_{\zeta(\mathbf{M})}(p^{\leq})$ iff $(i,j) \in V_{\mathbf{M}}(P)$ and that $[i,j] \in V_{\zeta(\mathbf{M})}(p^{\geq})$ iff $(j,i) \in V_{\mathbf{M}}(P)$.

Therefore, given an interval logic L_I and a first-order logic L_{FO}, we say that L_{FO} is *at least as expressive as* L_I, denoted by $\mathrm{L}_I \preceq \mathrm{L}_{FO}$, if there exists an effective translation τ from L_I to L_{FO} such that for any interval model M, any interval $[i,j]$, and any formula φ of L_I, $M, [i,j] \Vdash \varphi$ if and only if $\eta(M) \models \tau(\varphi)(i,j)$. Conversely, we say that L_I is *at least as expressive as* L_{FO}, denoted by $\mathrm{L}_{FO} \preceq \mathrm{L}_I$, if there exists an effective translation τ' from L_{FO} to L_I such that for any relational model $\mathbf{M}$, any pair (i,j) of elements, and any formula φ of L_{FO}, $\mathbf{M} \models \varphi(i,j)$ if and only if $\zeta(\mathbf{M}), [i,j] \Vdash \tau'(\varphi)$ if $i \leq j$ or $\zeta(\mathbf{M}), [j,i] \Vdash \tau'(\varphi)$ otherwise. We say that L_I is *as expressive as* L_{FO}, denoted by $\mathrm{L}_I \equiv \mathrm{L}_{FO}$, if $\mathrm{L}_I \preceq \mathrm{L}_{FO}$ and $\mathrm{L}_{FO} \preceq \mathrm{L}_I$. $\mathrm{L}_I \prec \mathrm{L}_{FO}$ and $\mathrm{L}_{FO} \prec \mathrm{L}_I$ are defined as expected.

It should be clear from the context which one of the above notions we use each time: in the rest of this section, we will compare first-order logics with interval ones, while, in Section 4.5, we will compare different interval logics to each other.

4.4.3 The logic $\mathsf{FO}^2[\mathbb{N}, =, <, s]$

As we already pointed out, the relationships between PNL^{π} and $\mathsf{FO}^2[=, <]$ have been investigated by Bresolin et al. in [25].

Theorem 4.4.3. $\mathsf{PNL}^{\pi} \equiv \mathsf{FO}^2[=, <]$, *when interpreted over any class of linearly ordered sets.*

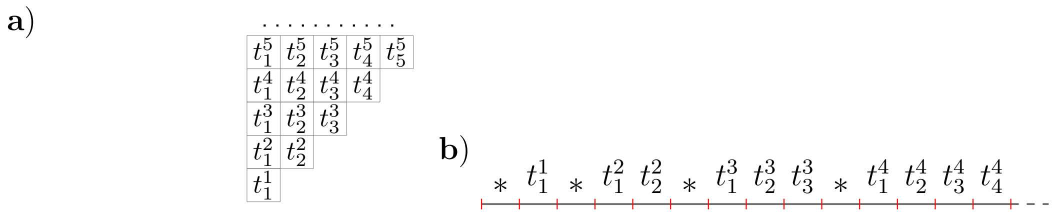


Figure 4.1: The encoding of the Octant Tiling Problem: **a)** cartesian representation; **b)** interval representation

We consider now the extension of $\mathsf{FO}^2[=,<]$ over $\mathbb{N}$ with the successor function s, denoted by $\mathsf{FO}^2[\mathbb{N},=,<,s]$. The terms of the language $\mathsf{FO}^2[\mathbb{N},=,<,s]$ are of the type $s^k(z)$, where $z \in \{x,y\}$ and $s^k(z)$ denotes z when $k=0$ and $\underbrace{s(s(\ldots s}_{k}(z)\ldots))$ when $k>0$. Formulae of $\mathsf{FO}^2[\mathbb{N},=,<,s]$ can be defined as in the case of the logic $\mathsf{FO}^2[=,<]$, mutatis mutandis. Using 2-pebble games and a standard model-theoretic argument, it is possible to prove that $\mathsf{FO}^2[\mathbb{N},=,<,s]$ is strictly more expressive than $\mathsf{FO}^2[=,<]$. That result, however, is also a direct consequence of the decidability and expressive completeness results given in [18, 25] and in this thesis.

Theorem 4.4.4. *The satisfiability problem for* $\mathsf{FO}^2[\mathbb{N},=,<,s]$ *is undecidable.*

Proof. The proof is based on a reduction from the Octant Tiling Problem. The reduction from the tiling problem for $\mathcal{O}$ to the satisfiability problem for $\mathsf{FO}^2[\mathbb{N},=,<,s]$ takes advantage of some special relational symbols, namely, those in the set $Let = \{*, Tile, Id, Id_e, Id_b, Id_d, Corr, T_1, T_2, \ldots, T_k\}$. The reduction consists of three main steps: (i) the encoding of an infinite chain that will be used to represent the tiles, (ii) the encoding of the above-neighbor relation by means of the relation $Corr$, and (iii) the encoding of the right-neighbor relation, which will make use of the successor function. The resulting schema is shown in Fig. 4.1. Pairs of successive points (*unit intervals*) are used as cells to arrange the tiling, while the relation Id is exploited to represent a row of the octant. Any Id consists of a sequence of unit intervals, each one of which is used either to represent a point of the plane or to separate two Ids. In the former case, it is labeled with the relation $Tile$, while, in the latter case, it is labeled with the relation $*$. Consider now the following formulae:

$$y = s(x) \wedge *(x,y) \tag{4.1}$$

$$\forall x,y \bigwedge_{P \in Let} (P(x,y) \rightarrow x < y) \tag{4.2}$$

$$\forall x,y(y = s(x) \leftrightarrow *(x,y) \vee Tile(x,y)) \tag{4.3}$$

$$\forall x,y(*(x,y) \rightarrow \neg Tile(x,y)) \tag{4.4}$$

$$\exists x(x = s(y) \wedge Tile(y,x) \wedge *(x,s(x))) \tag{4.5}$$

The conjunction α_1 of formulae (4.1), ..., (4.5) guarantees that there exists an infinite sequence of consecutive points $x_0, x_1, x_2 \ldots$. Formula (4.1) is used to start the chain: it is evaluated over two free variables x and y, that must correspond to two consecutive points, and it forces the predicate $*$ to be true when evaluated over the pair (x, y). Formula (4.2) forces the relational symbols in Let to hold only over ordered pairs (x, y) such that $x < y$. Formulae (4.3) and (4.4) guarantee that each pair x_i, x_{i+1} is labeled either by $*$ or by $Tile$. Finally, formula (4.5) states that $*(x_0, x_1)$, $Tile(x_1, x_2)$, and $*(x_2, x_3)$. Now, consider the conjunction α_2 of α_1 and the following formulae:

$$\exists y(y = s^2(x) \wedge Id(x, y)) \tag{4.6}$$
$$\forall x, y(Id(x, y) \rightarrow *(y, s(y))) \tag{4.7}$$
$$\forall x, y(Id(x, y) \rightarrow *(x, s(x))) \tag{4.8}$$
$$\forall x, y(*(x, y) \rightarrow \exists y(s(x) < y \wedge Id(x, y))) \tag{4.9}$$
$$\forall x, y(Id(x, y) \rightarrow Id_e(s(x), y)) \tag{4.10}$$
$$\forall x, y(Id_e(x, y) \wedge s(x) < y \rightarrow Id_e(s(x), y)) \tag{4.11}$$
$$\forall x, y(Id(x, s(y)) \rightarrow Id_b(x, y)) \tag{4.12}$$
$$\forall x, y(Id_b(x, s(y)) \wedge x < y \rightarrow Id_b(x, y)) \tag{4.13}$$
$$\forall x, y((Id_e(x, s(y)) \vee Id_d(x, s(y))) \wedge x < y \rightarrow Id_d(x, y)) \tag{4.14}$$
$$\forall x, y((Id_b(x, y) \vee Id_e(x, y) \vee Id_d(x, y)) \rightarrow \neg Id(x, y)) \tag{4.15}$$
$$\forall x, y \bigwedge_{\nu, \mu \in \{b, d, e\}, \nu \neq \mu} (Id_\nu(x, y) \rightarrow \neg Id_\mu(x, y)) \tag{4.16}$$

The formula α_2 builds a chain of Id, in such a way that (i) $Id(x_0, x_2)$ holds, (ii) each Id is followed by another Id, (iii) for each pair $x_i < x_j$, if $Id(x_i, x_j)$, then $*(x_i, x_{i+1})$, (iv) if $Id(x_i, x_j)$ then $\neg Id(x_h, x_k)$, for all $x_i \leq x_h \leq x_k \leq x_j$, with $(x_i, x_j) \neq (x_h, x_k)$, and (v) no pair of points is labeled by both Id_ν and Id_μ, with $\nu, \mu \in \{e, b, d\}$ and $\nu \neq \mu$. For any pair x_i, x_j such that $Id(x_i, x_j)$, the relation Id_e (resp., Id_b, Id_d) holds over all pairs x_k, x_j, with $x_i < x_k < x_j$ (resp., x_i, x_k, with $x_i < x_k < x_j$, x_h, x_k, with $x_i < x_h < x_k < x_j$). Condition (iv) prevents two Ids from holding over two pairs of points x_i, x_j and x_h, x_k such that either $x_i < x_h < x_k = x_j$ ((x_h, x_k) *ends* (x_i, x_j)) or $x_i = x_h < x_k < x_j$ ((x_h, x_k) *begins* (x_i, x_j)) or $x_i < x_h < x_k < x_j$ ((x_h, x_k) *is included in* (x_i, x_j)). Condition (v) excludes the existence of two pairs of points x_i, x_j and x_h, x_k such that $x_h < x_i < x_k < x_j$ ((x_h, x_k) *overlaps* (x_i, x_j)) and both $Id(x_i, x_j)$ and $Id(x_h, x_k)$ hold. Suppose, by contradiction, that there exist two pairs of points x_i, x_j and x_h, x_k such that (x_h, x_k) overlaps (x_i, x_j) and both $Id(x_i, x_j)$ and $Id(x_h, x_k)$ hold. By (4.10) and (4.11), we have that $Id_e(x_i, x_k)$ holds. Moreover, by (4.12) and (4.13), we have that $Id_b(x_i, x_k)$ holds as well. Thus, formula (4.16) is not satisfied (contradiction). As a third step, let α_3 be the conjunction of α_2 with the following formulae:

$$\forall x, y(Id(x, y) \rightarrow Corr(s(x), s(y))) \tag{4.17}$$

$$\forall x, y(Corr(x, y) \rightarrow Tile(x, s(x)) \wedge Tile(y, s(y))) \tag{4.18}$$

$$\begin{aligned} &\forall x, y(Corr(x, y) \wedge *(s(x), s^2(x)) \rightarrow \\ &\quad Tile(y, s(y)) \wedge Tile(s(y), s^2(y)) \wedge *(s^2(y), s^3(y))) \end{aligned} \tag{4.19}$$

$$\forall x, y(Corr(x, y) \wedge \neg * (s(x), s^2(x)) \rightarrow Corr(s(x), s(y))) \tag{4.20}$$

$$\forall x, y(Id(x, y) \rightarrow \neg Corr(x, y)) \tag{4.21}$$

Let $Tile(x_i, x_j)$ and $Tile(x_h, x_k)$ hold, and let $x_j < x_h$. We say that the two tiles are *above connected* if and only if $Corr(x_i, x_h)$. From α_3, it follows that the first tile of each Id is above connected to the first tile of the successive Id (formula (4.17)). Moreover, by taking advantage of the successor function, we extend such a property to the other tiles of any Id, that is, the i-th tile of an Id is above connected to the i-th tile of the successive Id (formula (4.20)). Finally, formulae (4.18) and (4.19) force each Id to have exactly one tile less than the next one. It can be easily shown that if α_3 holds, then the j-th Id provides an encoding of the j-th layer of the octant. Now, let $\alpha_{\mathcal{T}}$ be the conjunction of α_3 and the following formulae:

$$\forall x, y(Tile(x, y) \rightarrow \bigvee_{T \in \mathcal{T}} T(x, y) \wedge \bigwedge_{T, T' \in \mathcal{T},\ T \neq T'} \neg(T(x, y) \wedge T'(x, y)) \tag{4.22}$$

$$\forall x, y(T(x, y) \wedge Tile(s(x), s(y)) \rightarrow \bigvee_{T' \in \mathcal{T},\ right(T) = left(T')} T'(s(x), s(y))) \tag{4.23}$$

$$\forall x, y(Corr(x, y) \wedge T(x, s(x))) \rightarrow \bigvee_{T' \in \mathcal{T},\ up(T) = down(T')} T'(y, s(y))) \tag{4.24}$$

In view of the above steps, it is straightforward to check that, given any set of tile types $\mathcal{T}$, the formula $\alpha_{\mathcal{T}}$ is satisfiable if and only if $\mathcal{T}$ can tile $\mathcal{O}$. Thus, the satisfiability problem of $\mathsf{FO}^2[\mathbb{N}, =, <, s]$ is undecidable. $\square$

4.4.4 Expressive completeness of MPNL for a fragment of $\mathsf{FO}^2[\mathbb{N}, =, <, s]$

Let $\mathsf{FO}^2_r[\mathbb{N}, =, <, s]$ be the fragment of $\mathsf{FO}^2[\mathbb{N}, =, <, s]$ obtained by imposing the following restriction: if both variables x and y occur in the scope of (an occurrence of) a binary relation other than $=$ and $<$, then the successor function s cannot occur in the scope of that occurrence. As an example, each of the formulae $P(s^k(x), s^m(x)), P(x, y), s^k(x) = s^m(y)$, and $s^k(x) < s^m(y)$ belongs to $\mathsf{FO}^2_r[\mathbb{N}, =, <, s]$, but none of $P(x, s(y))$, $P(s(x), y)$, and, in general, $P(s^n(x), s^m(y))$ and $P(s^n(y), s^m(x))$, with $n + m > 0$, does. It is easy to check that the encoding used to show that $\mathsf{FO}^2[\mathbb{N}, =, <, s]$ is undecidable makes an essential use of formulae of the type that we have excluded from the fragment $\mathsf{FO}^2_r[\mathbb{N}, =, <, s]$ (e.g., see formula

4.10). By using 2-pebble games and a standard model-theoretic argument, one can show that:

$$\mathsf{FO}^2[=,<] \prec \mathsf{FO}^2_r[\mathbb{N},=,<,s] \prec \mathsf{FO}^2[\mathbb{N},=,<,s].$$

To prove that MPNL and $\mathsf{FO}^2_r[\mathbb{N},=,<,s]$ are expressively equivalent, we first define the standard translation $ST_{x,y}$ of the former into the latter as

$$ST_{x,y}(\varphi) = x \leq y \wedge ST'_{x,y}(\varphi),$$

where x, y are the two first-order variables in $\mathsf{FO}^2_r[\mathbb{N},=,<,s]$, and

$$\begin{array}{lcl} ST'_{x,y}(p) & = & P(x,y); \\ ST'_{x,y}(\ell_{=\mathsf{k}}) & = & s^k(x) = y; \\ ST'_{x,y}(\varphi \vee \psi) & = & ST'_{x,y}(\varphi) \vee ST'_{x,y}(\psi); \\ ST'_{x,y}(\neg\varphi) & = & \neg ST'_{x,y}(\varphi); \\ ST'_{x,y}(\Diamond_l \varphi) & = & \exists y(y \leq x \wedge ST'_{y,x}(\varphi)); \\ ST'_{x,y}(\Diamond_r \varphi) & = & \exists x(y \leq x \wedge ST'_{y,x}(\varphi)). \end{array}$$

Lemma 4.4.5. *For any* MPNL *formula* φ*, any interval model* $M = \langle \mathbb{I}(\mathbb{N}), V \rangle$*, and interval* $[a, b]$ *in* M*:*

$$M, [a, b] \Vdash \varphi \textit{ iff } \eta(M) \models ST_{x,y}(\varphi)[x := a, y := b].$$

Proof. Routine structural induction on φ. □

It is worth noticing that, given an MPNL formula φ, the length of the standard translation $ST_{x,y}(\varphi)$ depends not only on $|\varphi|$, but also on the maximum constant k appearing in length constraints, as atomic propositions of the form $\ell_{=\mathsf{k}}$ are translated by nesting k times the successor function s. Hence, the exact complexity of the translation depends on how metric constraints are encoded. When k is constant or encoded in unary, the standard translation is polynomial in the length of $|\varphi|$; when k is encoded in binary, we have that $k = O(2^{|\varphi|})$, and thus the standard translation is exponential in $|\varphi|$.

The inverse translation τ from $\mathsf{FO}^2_r[\mathbb{N},=,<,s]$ to MPNL is given in Table 4.1. In this case, the choice on the way in which metric constraints are encoded does not affect the complexity: the translation is always exponential in the size of the input formula, due to the clauses for the existential quantifier. The following lemma proves that it is correct.

Lemma 4.4.6. *For any formula* $\alpha(x, y)$ *of* $\mathsf{FO}^2_r[\mathbb{N},=,<,s]$*, any* $\mathsf{FO}^2_r[\mathbb{N},=,<,s]$ *model* $\mathbf{M} = \langle \mathbb{N}, V_{\mathbf{M}} \rangle$ *and any pair* $i, j \in \mathbb{N}$*, with* $i \leq j$*:*
(i) $\mathbf{M} \models \alpha(i, j)$ *if and only if* $\zeta(\mathbf{M}), [i, j] \Vdash \tau[x, y](\alpha)$*, and*
(ii) $\mathbf{M} \models \alpha(j, i)$ *if and only if* $\zeta(\mathbf{M}), [i, j] \Vdash \tau[y, x](\alpha)$*.*

Proof. The proof is by induction on the structural complexity of α (for the sake of simplicity, we only prove claim (i); claim (ii) can be proved similarly):

$$
\begin{aligned}
\tau[x,y](s^k(z) = s^m(z)) &= \top \quad (z \in \{x,y\}), && \text{if } k = m \\
&= \bot \quad (z \in \{x,y\}), && \text{if } k \neq m \\
\tau[x,y](s^k(z) < s^m(z)) &= \bot \quad (z \in \{x,y\}), && \text{if } k \geq m \\
&= \top \quad (z \in \{x,y\}), && \text{if } k < m \\
\tau[x,y](s^k(x) = s^m(y)) &= \bot, && \text{if } k < m \\
&= \ell_{=\mathsf{k-m}}, && \text{if } k \geq m \\
\tau[x,y](s^k(x) < s^m(y)) &= \top, && \text{if } k < m \\
&= \ell_{>\mathsf{k-m}}, && \text{if } k \geq m \\
\tau[x,y](s^m(y) < s^k(x)) &= \bot, && \text{if } k < m \\
&= \ell_{<\mathsf{k-m}}, && \text{if } k \geq m \\
\tau[x,y](\neg\alpha) &= \neg\tau[x,y](\alpha) \\
\tau[x,y](\alpha \vee \beta) &= \tau[x,y](\alpha) \vee \tau[x,y](\beta) \\
\tau[x,y](\exists x\beta) &= \Diamond_r(\tau[y,x](\beta)) \vee \Box_r\Diamond_l(\tau[x,y](\beta)) \\
\tau[x,y](\exists y\beta) &= \Diamond_l(\tau[y,x](\beta)) \vee \Box_l\Diamond_r(\tau[x,y](\beta)) \\
\tau[x,y](P(s^k(x), s^m(x))) &= \Diamond_l\Diamond_r(\ell_{=\mathsf{k}} \wedge \Diamond_r(\ell_{=\mathsf{m-k}} \wedge p^{\leq})), && \text{if } k < m \\
&= \Diamond_l\Diamond_r(\ell_{=\mathsf{k}} \wedge \Diamond_r(\ell_{=\mathsf{0}} \wedge p^{\leq} \wedge p^{\geq})), && \text{if } k = m \\
&= \Diamond_l\Diamond_r(\ell_{=\mathsf{m}} \wedge \Diamond_r(\ell_{=\mathsf{k-m}} \wedge p^{\geq})), && \text{if } k > m \\
\tau[x,y](P(s^k(y), s^m(y))) &= \Diamond_r(\ell_{=\mathsf{k}} \wedge \Diamond_r(\ell_{=\mathsf{m-k}} \wedge p^{\leq})), && \text{if } k < m \\
&= \Diamond_r(\ell_{=\mathsf{k}} \wedge \Diamond_r(\ell_{=\mathsf{0}} \wedge p^{\leq} \wedge p^{\geq})), && \text{if } k = m \\
&= \Diamond_r(\ell_{=\mathsf{m}} \wedge \Diamond_r(\ell_{=\mathsf{k-m}} \wedge p^{\geq})), && \text{if } k > m \\
\tau[x,y](P(x,y)) &= p^{\leq} \\
\tau[x,y](P(y,x)) &= p^{\geq}
\end{aligned}
$$

Table 4.1: Translation clauses from $\mathsf{FO}^2_r[\mathbb{N}, =, <, s]$ to MPNL

- $\alpha = (s^k(x) = s^m(x))$. If $k = m$, then both α and its translation $\tau[x,y](\alpha) = \top$ are true, while if $k \neq m$, then α and $\tau[x,y](\alpha) = \bot$ are both false; the same applies when y is used instead of x;

- $\alpha = (s^k(x) < s^m(x))$. If $k \geq m$, then both α and its translation $\tau[x,y](\alpha) = \bot$ are false, while if $k < m$, then α and $\tau[x,y](\alpha) = \top$ are both true; the same applies when y is used instead of x;

- $\alpha = (s^k(x) = s^m(y))$. Let $i < j$. If $k < m$, then $s^k(i) < s^m(j)$, and, since $\mathbf{M} \models \alpha(i,j)$ if and only if $s^k(i) = s^m(j)$, we have that $\mathbf{M} \not\models \alpha(i,j)$. On the other hand, it is immediate to see that $\tau[x,y](\alpha) = \bot$. If $m \leq k$, $s^k(i) = s^m(j)$ if and only if $j - i = k - m$, that is, $\mathbf{M} \models \alpha(i,j)$ if and only if $\zeta(\mathbf{M}), [i,j] \models \ell_{=\mathsf{k-m}}$. Likewise for the cases $\alpha = (s^m(y) = s^k(x))$, $\alpha = (s^k(x) < s^m(y))$, and $\alpha = (s^m(y) < s^k(x))$;

- $\alpha = (P(s^k(x), s^m(x)))$. Let $i < j$. If $k < m$, then $s^m(i) - s^k(i) = m - k$ and $s^k(i) - i = k$. Thus, $\mathbf{M} \models \alpha(i,j)$ if and only if P is true over the pair

$(s^k(i),\ s^{m-k}(s^k(i)))$, that is, $\mathbf{M} \models \alpha(i,j)$ if and only if $\zeta(\mathbf{M}), [i,j] \Vdash \Diamond_l \Diamond_r$ $(\ell_{=\mathsf{k}} \wedge \Diamond_r(\ell_{=\mathsf{m-k}} \wedge p^{\leq}))$. A similar reasoning path can be followed for the case of $m < k$. If $k = m$, then $s^k(i) = s^m(i)$, and thus P must be true over a point-interval, specifically, identified by the pair $(s^k(i), s^k(i))$. Hence, we have that $\mathbf{M} \models \alpha(i,j)$ if and only if $\zeta(\mathbf{M}), [i,j] \Vdash \Diamond_l \Diamond_r(\ell_{=\mathsf{k}} \wedge \Diamond_r(\ell_{=\mathsf{0}} \wedge p^{\leq} \wedge p^{\geq}))$. Likewise, when y substitutes x;

- $\alpha = P(x,y)$ or $\alpha = P(y,x)$. The claim follows from the valuation of $p^{\leq}$ and $p^{\geq}$;

- The Boolean cases are straightforward;

- $\alpha = \exists x \beta$. Suppose that $\mathbf{M} \models \alpha(i,j)$. Then, there is $l \in \mathbf{M}$ such that $\mathbf{M} \models \beta(l,j)$. There are two (non-exclusive) cases: $j \leq l$ and $l \leq j$. If $j \leq l$, by the inductive hypothesis, we have that $\zeta(\mathbf{M}), [j,l] \Vdash \tau[y,x](\beta)$ and thus $\zeta(\mathbf{M}), [i,j] \Vdash \Diamond_r(\tau[y,x](\beta))$. Likewise, if $l \leq j$, by the inductive hypothesis, we have that $\zeta(\mathbf{M}), [l,j] \Vdash \tau[x,y](\beta)$ and thus for every r such that $j \leq r$, $\zeta(\mathbf{M}), [j,r] \Vdash \Diamond_l(\tau[x,y](\beta))$, that is, $\zeta(\mathbf{M}), [i,j] \Vdash \Box_r \Diamond_l(\tau[x,y](\beta))$. Hence we have that $\zeta(\mathbf{M}),\ [i,j] \Vdash \Diamond_r(\tau[y,x](\beta)) \vee \Box_r \Diamond_l(\tau[x,y](\beta))$, that is, $\zeta(\mathbf{M}), [i,j] \Vdash \tau[x,y](\alpha)$. For the converse direction, it suffices to note that the interval $[i,j]$ has at least one right neighbor, viz. $[j,j]$, and thus the above argument can be reversed;

- $\alpha = \exists y \beta$. Analogous to the previous case.

$\square$

Theorem 4.4.7. *For any formula $\alpha(x,y)$ of $\mathsf{FO}^2_r[\mathbb{N},=,<,s]$ and any $\mathsf{FO}^2_r[\mathbb{N},=,<,s]$ model $\mathbf{M} = \langle \mathbb{N}, V_{\mathbf{M}} \rangle$, $\mathbf{M} \models \forall x \forall y \alpha(x,y)$ if and only if $\zeta(\mathbf{M}) \Vdash \tau[x,y](\alpha) \wedge \tau[y,x](\alpha)$. As a consequence, $\mathsf{FO}^2_r[\mathbb{N},=,<,s] \equiv \mathsf{MPNL}$.*

From Theorem 4.4.7, decidability of $\mathsf{FO}^2_r[\mathbb{N},=,<,s]$ immediately follows. A decision procedure for it can be obtained by first translating the input formula to MPNL and then applying the decision procedure for MPNL described in Section 4.3. Since the length of the translated formula is exponential, no matter how we encode the metric constants in MPNL, the lowest complexity of the procedure is obtained when we choose to use the unary encoding: the satisfiability problem for $\mathsf{FO}^2_r[\mathbb{N},=,<,s]$ is thus in 2NEXPTIME. A lower bound on the complexity can be given by observing that $\mathsf{FO}^2[=,<]$ is NEXPTIME-hard (the EXPSPACE-hardness result given for MPNL in Section 4.3 cannot be transferred to $\mathsf{FO}^2_r[\mathbb{N},=,<,s]$, since it relies on the binary encoding).

$$
\begin{array}{lll}
\tau[x,y](P(s^k(x), s^m(y))) = & \Diamond_{be}^{+k}\Diamond_{e}^{+(m-k)} p^{\leq}, & \text{if } k < m \\
& (\ell_{>0} \wedge \Diamond_{be}^{+k} p^{\leq}) \vee (\ell_{=0} \wedge \Diamond_{be}^{+k}(p^{\leq} \wedge p^{\geq})), & \text{if } k = m \\
& (\ell_{>k-m} \wedge \Diamond_{be}^{+m}\Diamond_{b}^{+(k-m)} p^{\leq}) \vee & \\
& \vee(\ell_{=k-m} \wedge \Diamond_{be}^{+m}\Diamond_{b}^{+(k-m)}(p^{\leq} \wedge p^{\geq})) \vee & \\
& \vee(\ell_{<k-m} \wedge \Diamond_{be}^{+m}\Diamond_{b}^{+(k-m)} p^{\geq}), & \text{if } k > m
\end{array}
$$

Table 4.2: The translation from $\mathsf{FO}^2[\mathbb{N}, =, <, s]$ to MPNL$^+$: the additional clause for $\tau[x,y](P(s^k(x), s^m(y)))$

4.4.5 Extension of MPNL expressively complete for $\mathsf{FO}^2[\mathbb{N}, =, <, s]$

To cover full $\mathsf{FO}^2[\mathbb{N}, =, <, s]$, MPNL can be extended with additional diamond modalities that shift respectively the beginning, the end, and both endpoints of the current interval to the right by a prescribed distance:

- $M, [i, j] \Vdash \Diamond_e^{+k}\psi$ iff $M, [i, j+k] \Vdash \psi$;
- $M, [i, j] \Vdash \Diamond_b^{+k}\psi$ iff $(i + k \leq j$ and $M, [i+k, j] \Vdash \psi)$ or $(i + k > j$ and $M, [j, i+k] \Vdash \psi)$;
- $M, [i, j] \Vdash \Diamond_{be}^{+k}\psi$ iff $M, [i+k, j+k] \Vdash \psi$.

Let MPNL$^+$ be the resulting language. The standard translation $ST'_{x,y}$ of MPNL formulae into $\mathsf{FO}^2[\mathbb{N}, =, <, s]$ can be extended to MPNL$^+$ as follows:

$$
\begin{array}{lll}
ST'_{x,y}(\Diamond_e^{+k}\psi) & = & ST'_{x,y}(\psi)[s^k(y)/y]; \\
ST'_{x,y}(\Diamond_b^{+k}\psi) & = & ST'_{x,y}(\psi)[s^k(x)/x]; \\
ST'_{x,y}(\Diamond_{be}^{+k}\psi) & = & ST'_{x,y}(\psi)[s^k(x)/x, s^k(y)/y],
\end{array}
$$

where $\alpha[t/z]$ denotes the result of the simultaneous substitution of the term t for all free occurrences of z in α.

It is immediate to see that if $ST'_{x,y}(\psi) \in \mathsf{FO}^2[\mathbb{N}, =, <, s]$, then $ST'_{x,y}(\psi)[s^k(x)/x, s^m(y)/y] \in \mathsf{FO}^2[\mathbb{N}, =, <, s]$ for any $k, m \in \mathbb{N}$, and thus the translation of all formulae of MPNL$^+$ remains within $\mathsf{FO}^2[\mathbb{N}, =, <, s]$. Conversely, we can extend the translation τ from $\mathsf{FO}^2_r[\mathbb{N}, =, <, s]$ to MPNL to a translation from $\mathsf{FO}^2[\mathbb{N}, =, <, s]$ to MPNL$^+$ by adding the clauses for the atomic formulae in Table 4.2. The extensions of the expressive completeness results are routine.

To conclude this section, we recall that Venema [102] has shown in a similar way that the interval temporal logic CDT, involving binary modalities based on the ternary interval relation 'chop' and its residuals (denoted respectively by C, D, and T) is expressively complete for the fragment of first-order logic with equality with three variables of which at most two are free, denoted by $\mathsf{FO}^3_2[=, <]$. Note that, when interpreted in $\mathbb{N}$, the successor function is definable in this fragment, which therefore strictly extends $\mathsf{FO}^2[\mathbb{N}, =, <, s]$. The resulting hierarchy of expressive completeness results is depicted in Fig. 4.2. Notice also that all the proposed translations from the first-order languages into interval ones are exponential in the size of the input formula[2].

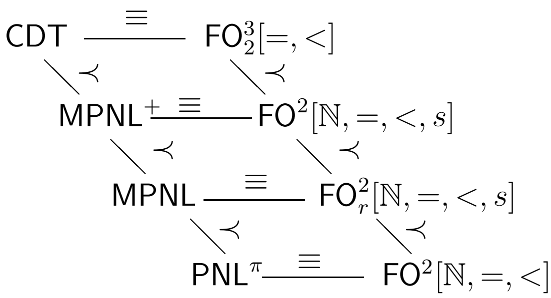


Figure 4.2: Expressive completeness results for interval logics

4.5 Classifying the expressive power of MPNL

In the previous sections, we have studied the expressiveness and the computational properties of MPNL. A natural question is whether there exist other interesting variants of PNL that deserve to be analyzed. In this section, we define a family of metric languages, and we compare their expressive power (see also [94]). As it will be proved in the following, MPNL is able to encode all the languages in the family, thus being the most expressive metric extension of PNL.

Let $\sim \in \{<, \leq, =, \geq, >\}$, $k \in \mathbb{N}$, and $k' \in \mathbb{N} \cup \{\infty\}$. We consider a set of *metric modalities* of the form $\Diamond_r^{\sim k}$, $\Diamond_r^{[k,k']}$, $\Diamond_r^{(k,k')}$, $\Diamond_r^{[k,k')}$, and $\Diamond_r^{(k,k']}$, as well as their inverses $\Diamond_l^{\sim k}$, $\Diamond_l^{[k,k']}$, $\Diamond_l^{(k,k')}$, $\Diamond_l^{[k,k')}$, and $\Diamond_l^{(k,k']}$, with the following semantics:

- $M, [i,j] \Vdash \Diamond_r^{\sim k}\psi$ iff there exists $m \geq j$ such that $\delta(j,m) \sim k$ and $M, [j,m] \Vdash \psi$;
- $M, [i,j] \Vdash \Diamond_r^{[k,k']}\psi$ iff there exists $m \geq j$ such that $k \leq \delta(j,m) \leq k'$ and $M, [j,m] \Vdash \psi$;
- $M, [i,j] \Vdash \Diamond_r^{(k,k')}\psi$ iff there exists $m \geq j$ such that $k < \delta(j,m) < k'$ and $M, [j,m] \Vdash \psi$.

The semantic clauses for $\Diamond_r^{[k,k')}$ and $\Diamond_r^{[k,k')}$, as well as those for the inverse modalities, are defined likewise. It is easy to show that all metric modalities are definable by

[2] At present, we do not know whether a polynomial translation for any of these cases exists.

exploiting the length constraints, e.g.:

$$\Diamond_r^{\sim k}\psi \equiv \Diamond_r(\ell_{\sim \mathsf{k}} \wedge \psi),$$
$$\Diamond_r^{[k,k']}\psi \equiv \Diamond_r(\ell_{=\mathsf{k}} \wedge \psi) \vee \Diamond_r(\ell_{=\mathsf{k+1}} \wedge \psi) \vee \ldots \vee \Diamond_r(\ell_{=\mathsf{k'}} \wedge \psi) \vee \bot,$$

and thus that all languages in the family are fragments of MPNL. Let $\kappa \in \{<k, \leq k, = k, \geq k, > k, [k,k'], (k,k'), [k,k'), (k,k']\}$, and let $\Diamond_o^\kappa$ be any of the two modal operators $\Diamond_l^\kappa$ and $\Diamond_r^\kappa$. The dual modalities are defined as usual, that is, $\Box_o^\kappa\psi = \neg\Diamond_o^\kappa\neg\psi$. Let ϵ be a special symbol such that $\Diamond_r^{\epsilon k} = \Diamond_r$ and $\Diamond_l^{\epsilon k} = \Diamond_l$, for any k, and let $S \subseteq \{\epsilon, <, \leq, =, \geq, >, [], (), [), (]\}$. We will denote by MPNL^S the language that features:

(i) the modal operators $\Diamond_l^{\sim k}$ and $\Diamond_r^{\sim k}$ for each $k \in \mathbb{N}$ and $\sim \in S \cap \{\epsilon, <, \leq, =, \geq, >\}$;

(ii) the modal operators $\Diamond_l^{[k,k']}$ and $\Diamond_r^{[k,k']}$ (resp., $\Diamond_l^{(k,k')}$ and $\Diamond_r^{(k,k')}$, $\Diamond_l^{[k,k')}$ and $\Diamond_r^{[k,k')}$, $\Diamond_l^{(k,k']}$ and $\Diamond_r^{(k,k']}$), for each $k \in \mathbb{N}$, $k' \in \mathbb{N} \cup \{\infty\}$, if $[] \in S$ (resp., $() \in S$, $[) \in S$, $(] \in S$).

We will denote by MPNL_l^S the extension of MPNL^S with the length constraints (this means that MPNL_l^ϵ is exactly the language MPNL of the previous sections). For the sake of simplicity, we will omit the curly brackets in the superscript; for example, if $S = \{<, >\}$, we will write simply $\mathsf{MPNL}^{<,>}$ instead of $\mathsf{MPNL}^{\{<,>\}}$. Thus, we have that $\mathsf{MPNL}^\epsilon \equiv \mathsf{PNL}$ and $\mathsf{MPNL}_l^\epsilon \equiv \mathsf{MPNL}$. Moreover, by the following lemma, we can reduce the number of interesting fragments:

Lemma 4.5.1. *If $o \in \{r, l\}$, whenever $\Diamond_o^{<k}$ (resp., $\Diamond_o^{[k,k']}$, $\Diamond_o^{(k,k']}$) is included in the language, then $\Diamond_o^{\leq k}$ (resp., $\Diamond_o^{[k,k')}$, $\Diamond_o^{(k,k')}$) can be defined, and the other way around.*

$\Diamond_o^{<k}\psi \Leftrightarrow$	$\bot$ $(k = 0)$	$\Diamond_o\psi \Leftrightarrow$	$\Diamond_o^{\geq 0}\psi$
	$\Diamond_o^{\leq k-1}\psi$ $(k > 0)$		$\Diamond_o^{[0,\infty]}\psi$
$\Diamond_o^{[k,k']}\psi \Leftrightarrow$	$\Diamond_o^{[k,k')}\psi$ $(k' = \infty)$	$\Diamond_o^{<k}\psi \Leftrightarrow$	$\Diamond_o^{=0}\psi \vee \ldots \vee \Diamond_o^{=k-1}\psi \vee \bot$
	$\Diamond_o^{[k,k'+1)}\psi$ $(k' \neq \infty)$	$\Diamond_o^{=k}\psi \Leftrightarrow$	$\Diamond_o^{[k,k]}\psi$
$\Diamond_o^{[k,k')}\psi \Leftrightarrow$	$\Diamond_o^{[k,k']}\psi$ $(k' = \infty)$	$\Diamond_o^{>k}\psi \Leftrightarrow$	$\Diamond_o^{\geq k+1}\psi$
	$\bot$ $(k' = 0)$		$\Diamond_o^{(k,\infty)}\psi$
	$\Diamond_o^{[k,k'-1]}$ $(k' > 0)$	$\Diamond_o^{\geq k}\psi \Leftrightarrow$	$\Diamond_o^{[k,\infty]}\psi$
$\Diamond_o^{(k,k']}\psi \Leftrightarrow$	$\Diamond_o^{(k,k')}\psi$ $(k' = \infty)$	$\Diamond_o^{(k,k')}\psi \Leftrightarrow$	$\Diamond_o^{[k+1,k']}\psi$ $(k' = \infty)$
	$\Diamond_o^{(k,k'+1)}\psi$ $(k' \neq \infty)$		$\bot$ $(k' = 0)$
$\Diamond_o^{(k,k')}\psi \Leftrightarrow$	$\Diamond_o^{(k,k']}\psi$ $(k' = \infty)$		$\Diamond_o^{[k+1,k'-1]}\psi$ $(k' > 0)$
	$\bot$ $(k' = 0)$		
	$\Diamond_o^{(k,k'-1]}$ $(k' > 0)$		

Table 4.3: Equivalences between metric operators, $o \in \{r, l\}$

Proof. See Table 4.3, left column. □

Thus, without loss of generality, from now on we can focus our attention on languages characterized by subsets of the set $\{\epsilon, <, =, >, \geq, [], ()\}$. As we will see, some languages will be expressive enough to embed non-metric PNL, while some others will not. We will use the expression *weak Metric Propositional Neighborhood Logics* (wMPNL) to denote the latter.

In order to compare the expressive power of interval languages, we use two standard techniques in modal logic, based on bisimulation [11] and bisimulation games [55].

Given an interval logic L, for each modality $\Diamond$ in the language of L, we denote by $R_\Diamond$ the (interval) relation on which $\Diamond$ is based. Now, given a pair of L models M, M', with $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}'), V'\rangle$, we say that a relation $Z \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ is a *bisimulation* if $([a,b],[a',b']) \in Z$ implies that *(i)* $[a,b]$ and $[a',b']$ satisfy the same atomic propositions, *(ii)* for every relation $R_\Diamond$ and every interval $[c,d]$ such that $[a,b]\ R_\Diamond\ [c,d]$, there exists an interval $[c',d']$ such that $[a',b']\ R_\Diamond\ [c',d']$ and $([c,d],[c',d']) \in Z$, and *(iii)* for every relation $R_\Diamond$ and every interval $[c',d']$ such that $[a',b']\ R_\Diamond\ [c',d']$, there exists an interval $[c,d]$ such that $[a,b]\ R_\Diamond\ [c,d]$ and $([c,d],[c',d']) \in Z$. Interval logics are invariant under bisimulation, as it is the case with modal logic [11].

Proposition 4.5.2. *Let L be a language for interval logics, M and M' be two L models, and $Z \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ be a bisimulation for L. Then, every pair $([a,b],[a',b']) \in Z$ is such that $[a,b]$ and $[a',b']$ satisfy the same L formulae.*

The above proposition can be proved by induction on the structural complexity of formulae.

The notion of bisimulation game can be viewed as a generalization of the notion of bisimulation. In the context of interval logics, we define the notion of a *N-moves bisimulation game* (for the interval logic L) to be played by two players, Player I and Player II, on a pair of L models M, M', with $M = \langle \mathbb{I}(\mathbb{D}), V\rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}'), V'\rangle$. The game starts from a given *initial configuration*, where a *configuration* is a pair of intervals $([a,b],[a',b'])$, with $[a,b] \in \mathbb{I}(\mathbb{D})$ and $[a',b'] \in \mathbb{I}(\mathbb{D}')$. A configuration $([a,b],[a',b'])$ is *matching* if $[a,b]$ and $[a',b']$ satisfy the same atomic propositions in their respective models. The *moves* of the game depend on the modal operators of L: for each modality $\Diamond$ in the language of L, Player I can play the corresponding move: choose M (resp., M'), and an interval $[c,d] \in \mathbb{I}(\mathbb{D})$ (resp., $[c',d'] \in \mathbb{I}(\mathbb{D}')$) such that $[a,b]\ R_\Diamond\ [c,d]$ (resp., $[a',b']\ R_\Diamond\ [c',d']$). Player II must reply by choosing an interval $[c',d'] \in \mathbb{I}(\mathbb{D}')$ (resp., $[c,d] \in \mathbb{I}(\mathbb{D})$), which leads to the new configuration $([c,d],[c',d'])$. If after any given round the current configuration is not matching, Player I wins the game; otherwise, after N rounds, Player II wins the game. Intuitively, Player II has a *winning strategy* in the N-moves bisimulation game on the models M and M' with a given initial configuration if she can win regardless of the moves played by Player I; otherwise, Player I has a winning strategy. A formal definition of winning strategy can be found in [55]. The following key property of N-move bisimulation games can be proved routinely, in analogy with similar results about bisimulation games in modal logic [55][3].

Proposition 4.5.3. *Let L be a language for interval logic with finitely many atomic propositions. For all $N \geq 0$, Player II has a winning strategy in the N-move L-bisimulation game on M and M' with initial configuration $([a, b], [a', b'])$ if and only if $[a, b]$ and $[a', b']$ satisfy the same L formulae with modal depth at most N.*

In order to prove that a modal operator $\bigcirc$ is not definable in L, it suffices to construct a pair of interval models M and M' and a bisimulation (resp., a bisimulation game such that Player II has a winning strategy) between them, relating a pair of intervals $[a, b] \in M$ and $[a', b'] \in M'$, such that $M, [a, b] \Vdash \bigcirc p$, but $M', [a', b'] \nVdash \bigcirc p$.

In the following, we will characterize points of $\mathbb{D}'$ with the symbol $'$, in order to distinguish between points (and intervals) of M and M'. As an example, consider two models M and M' based on $\mathbb{N}$. $[0, 1]$ will denote an interval of M, while $[0', 1']$ will denote an interval of M'. Analogously, if $i = 0$ and $j = 1$, the interval $[i, j]$ will refer to the interval $[0, 1] \in M$, while $[i', j']$ to the interval $[0', 1'] \in M'$.

4.5.1 The class of wMPNL

Here, we analyze the set of languages in wMPNL. Formally, wMPNL is the subset of metric extensions (notice that it contains fragments of MPNL) defined as follows:

$$w\mathsf{MPNL} = \{L \mid L \preceq \mathsf{MPNL} \text{ and } \mathsf{PNL} \not\preceq L\}.$$

The following lemma states some basic results that we will use to classify languages in wMPNL.

Lemma 4.5.4. *If $o \in \{r, l\}$, whenever any of the modalities in $\{\Diamond_o^{\geq k}, \Diamond_o^{[k,k']}\}$ (resp., $\{\Diamond_o^{=k}, \Diamond_o^{[k,k']}\}$, $\{\Diamond_o^{\geq k}, \Diamond_o^{(k,k')}, \Diamond_o^{[k,k']}\}$) is included in the language, then $\Diamond_o$ (resp., $\Diamond_o^{<k}$, $\Diamond_o^{>k}$) can be defined. Similarly, whenever $\Diamond_o^{[k,k']}$ is included, then $\Diamond_o^{=k}$, $\Diamond_o^{\geq k}$, and $\Diamond_o^{(k,k')}$ can be defined.*

Proof. See Table 4.3, right column. □

Theorem 4.5.5. *Let $\mathcal{S}_w = \{\{<\}, \{>\}, \{=\}, \{()\}, \{<, =\}, \{>, ()\}\}$. We have that*

$$w\mathsf{MPNL} = \{\mathsf{MPNL}^S, \mathsf{MPNL}_l^S \mid S \in \mathcal{S}_w\}.$$

[3] In Proposition 4.5.3, we make use of the notion of modal depth of an L formula φ. Let us denote the modal depth of φ by $mdepth(\varphi)$. As usual, $mdepth(\varphi)$ can be inductively defined as follows: (i) $mdepth(p) = 0$, for each $p \in \mathcal{AP}$; (ii) $mdepth(\neg\varphi) = mdepth(\varphi), mdepth(\varphi \vee \psi) = max\{mdepth(\varphi), mdepth(\psi)\}, mdepth(\Diamond\varphi) = mdepth(\varphi) + 1$, for each modality $\Diamond$ of the language.

Proof. As a preliminary step, notice that, by Lemma 4.5.4, it immediately follows $\mathsf{MPNL}^{=} \equiv \mathsf{MPNL}^{<,=}$ and $\mathsf{MPNL}^{()} \equiv \mathsf{MPNL}^{>,()}$. Thus, we can disregard the logics $\mathsf{MPNL}^{<,=}$ and $\mathsf{MPNL}^{>,()}$. Next, we show that both MPNL^{S} and MPNL_l^{S} belong to $w\mathsf{MPNL}$ for each $S \in \mathcal{S}_w$. To this end, we prove that $\mathsf{PNL} \not\preceq \mathsf{MPNL}_l^{S}$, for each $S \in \mathcal{S}_w$. From this, it immediately follows that, for each $S \in \mathcal{S}_w$, $\mathsf{PNL} \not\preceq \mathsf{MPNL}^{S}$ as well. Moreover, by Lemma 4.5.4, we have that $\mathsf{MPNL}_l^{<} \preceq \mathsf{MPNL}_l^{=}$ and $\mathsf{MPNL}_l^{>} \preceq \mathsf{MPNL}_l^{()}$, and thus it suffices to show that $\mathsf{PNL} \not\preceq \mathsf{MPNL}_l^{=}$ and $\mathsf{PNL} \not\preceq \mathsf{MPNL}_l^{()}$.

PNL $\not\preceq$ MPNL$_l^{=}$. It is easy to show that classical, non-metric modal operators of PNL can be expressed using formulae of $\mathsf{MPNL}_l^{=}$ of infinite length. For example, it is possible to express the formula $\Diamond_r p$ of PNL by means the infinite formulae $\Diamond_r^{=0} p \vee \Diamond_r^{=1} p \vee \ldots \Diamond_r^{=i} p \vee \ldots$. Nevertheless, suppose, by contradiction, that there exists a finite formula $\varphi \in \mathsf{MPNL}_l^{=}$ such that $\varphi \equiv \Diamond_r p$. This means that φ contains a finite number of modal operators. Let $t \in \mathbb{N}$ be the largest number such that $\Diamond_r^{=t}$ or $\Diamond_l^{=t}$ occurs in φ, and, for any $t \in \mathbb{N}$, define ${}^t\mathsf{MPNL}_l^{=}$ as the restriction of $\mathsf{MPNL}_l^{=}$ to the set of modalities $\{\Diamond_r^{=k}, \Diamond_l^{=k} \mid 0 \leq k \leq t\}$. Now, let $M = \langle \mathbb{I}(\mathbb{D} = \mathbb{N}), V\rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}' = \mathbb{N}), V'\rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \{[1, t+2]\}$, $V'(p) = \emptyset$, and $Z \subset \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ defined as $Z = \{([i,j], [i',j']) \mid \delta(i,j) \leq t\}$. It is possible to show that Z is a bisimulation for ${}^t\mathsf{MPNL}_l^{=}$. Since $M, [1,1] \Vdash \Diamond_r p$, $M', [1',1'] \not\Vdash \Diamond_r p$, and $[1,1]$ is Z-related with $[1',1']$, we have a contradiction.

PNL $\not\preceq$ MPNL$_l^{()}$. Again, suppose that for some $\varphi \in \mathsf{MPNL}_l^{()}$ it is the case that $\varphi \equiv \Diamond_r p$. Consider $M = \langle \mathbb{I}(\mathbb{D} = \mathbb{N}), V\rangle$, $M' = \langle \mathbb{I}(\mathbb{D}' = \mathbb{N}), V'\rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \{[1,1]\}$, $V'(p) = \emptyset$, and $Z \subset \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ defined as $Z = \{([i,j], [i',j']) \mid i \neq j\}$. As before, Z is a bisimulation for $\mathsf{MPNL}_l^{()}$. Since $M, [0,1] \Vdash \Diamond_r p$, $M', [0',1'] \not\Vdash \Diamond_r p$, and $[0,1]$ is Z-related with $[0',1']$, we have a contradiction.

To complete the proof, we show that no other language belongs to $w\mathsf{MPNL}$, that is, neither MPNL^{S} nor MPNL_l^{S} belongs to $w\mathsf{MPNL}$ for any $S \notin \mathcal{S}_w$. Let $S \subseteq \{\epsilon, <, =, >, \geq, [], ()\}$ such that $S \notin \mathcal{S}_w$. We must show that $\mathsf{PNL} \preceq \mathsf{MPNL}^{S}$ and $\mathsf{PNL} \preceq \mathsf{MPNL}_l^{S}$. Since $\mathsf{MPNL}^{S} \preceq \mathsf{MPNL}_l^{S}$, it suffices to show that $\mathsf{PNL} \preceq \mathsf{MPNL}^{S}$. If $\epsilon \in S$, then clearly $\mathsf{PNL} \preceq \mathsf{MPNL}^{S}$, since $\mathsf{PNL} \equiv \mathsf{MPNL}^{\epsilon}$. If $\geq \in S$ or $[] \in S$, then the result immediately follows from Lemma 4.5.4. If $\{<, >\} \subseteq S$, then the thesis immediately follows by the fact that $\Diamond_o \psi$ is defined by $\Diamond_o^{<1} \psi \vee \Diamond_o^{>0} \psi$ for each $o \in \{r, l\}$. The rest of the cases are consequences of the considered ones and of previous lemmas. ☐

We now establish how the various languages of $w\mathsf{MPNL}$ relate to each other in terms of expressive power.

Theorem 4.5.6. *The relative expressive power of the languages of the class $w\mathsf{MPNL}$ is as depicted in Fig. 4.3, where each arrow means that the language at the top is strictly more expressive than the one at the bottom.*

Proof. By Lemma 4.5.4, we already know that $\mathsf{MPNL}^{<} \preceq \mathsf{MPNL}^{=}$, $\mathsf{MPNL}_l^{<} \preceq \mathsf{MPNL}_l^{=}$, $\mathsf{MPNL}^{>} \preceq \mathsf{MPNL}^{()}$, and that $\mathsf{MPNL}_l^{>} \preceq \mathsf{MPNL}_l^{()}$. To complete the proof,

$$\begin{array}{ccc} \mathsf{MPNL}^{=}_{l} \equiv \mathsf{MPNL}^{<}_{l} & & \mathsf{MPNL}^{()}_{l} \equiv \mathsf{MPNL}^{>}_{l} \\ \uparrow & & \uparrow \\ \mathsf{MPNL}^{=} & & \mathsf{MPNL}^{()} \\ \uparrow & & \uparrow \\ \mathsf{MPNL}^{<} & & \mathsf{MPNL}^{>} \end{array}$$

Figure 4.3: Relative expressive power of metric languages belonging to wMPNL. An arrow going from L to L' denotes that L' is strictly more expressive than L. Logics which are not connected through any path are incomparable

it remains to show that $\mathsf{MPNL}^{=} \not\preceq \mathsf{MPNL}^{<}$, $\mathsf{MPNL}^{=}_{l} \preceq \mathsf{MPNL}^{<}_{l}$, $\mathsf{MPNL}^{()} \not\preceq \mathsf{MPNL}^{>}$, and $\mathsf{MPNL}^{()}_{l} \preceq \mathsf{MPNL}^{>}_{l}$.

MPNL$^{=}$ $\not\preceq$ MPNL$^{<}$. It suffices to show that $\Diamond^{=k}_{r}$ cannot be defined in $\mathsf{MPNL}^{<}$. Suppose the contrary, and let $M = \langle \mathbb{I}(\mathbb{D} = \mathbb{N}), V\rangle$, $M' = \langle \mathbb{I}(\mathbb{D}' = \{0'\}), V'\rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \mathbb{I}(\mathbb{D})$, $V'(p) = \mathbb{I}(\mathbb{D}') = \{[0', 0']\}$, and $Z = \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$. It is possible to show that Z is a bisimulation for $\mathsf{MPNL}^{<}$. Since it holds that $M, [0, 0] \Vdash \Diamond^{=1}_{r} p$, $M', [0', 0'] \not\Vdash \Diamond^{=1}_{r} p$, and $[0, 0]$ is Z-related to $[0', 0']$, we have a contradiction.

MPNL$^{()}$ $\not\preceq$ MPNL$^{>}$. For any $t \in \mathbb{N}$, consider the language ${}^{t}\mathsf{MPNL}^{>}$, that is, as before, the restriction of $\mathsf{MPNL}^{>}$ to the set of modalities $\{\Diamond^{>k}_{r}, \Diamond^{>k}_{l} \mid 0 \leq k \leq t\}$. Let

$\Diamond^{\geq k}_{o}\psi \Leftrightarrow$	$\Diamond^{<1}_{o}\psi \vee \Diamond^{>0}_{o}\psi$	$k = 0$
	$\Diamond^{>k-1}_{o}\psi$	$k > 0$
$\Diamond^{(k,k')}_{o}\psi \Leftrightarrow$	$\Diamond^{=k+1}_{o}\psi \vee \ldots \vee \Diamond^{=k'-1}_{o}\psi \vee \bot$	$k' \neq \infty$
	$\Diamond^{>k}_{o}\psi$	$k' = \infty$
$\Diamond^{[k,k']}_{o}\psi \Leftrightarrow$	$\Diamond^{(k-1,k'+1)}_{o}\psi$	$k > 0, k' \neq \infty$
	$\Diamond^{<k'+1}_{o}\psi$	$k = 0, k' \neq \infty$
	$\Diamond^{(k-1,k')}_{o}\psi$	$k > 0, k' = \infty$
	$\Diamond^{(k,k')}_{o}\psi \vee \Diamond^{<1}_{o}\psi$	$k = 0, k' = \infty$
$\Diamond^{\geq k}_{o}\psi \Leftrightarrow$	$\Diamond_{o}\psi$	$k = 0$
	$\Diamond^{>k-1}_{o}\psi$	$k > 0$
$\Diamond^{<k}_{o}\psi \Leftrightarrow$	$\Diamond^{[0,k-1]}_{o}\psi$	$k > 0$
	$\bot$	$k = 0$
$\Diamond^{>k}_{o}\psi \Leftrightarrow$	$\Diamond^{[k+1,\infty]}_{o}\psi$	
$\Diamond^{[k,k']}_{o}\psi \Leftrightarrow$	$\Diamond_{o}(\ell_{\geq \mathsf{k}} \wedge \ell_{\leq \mathsf{k}'} \wedge \psi)$	$k' \neq \infty$
	$\Diamond_{o}(\ell_{\geq \mathsf{k}} \wedge \psi)$	$k' = \infty$
$\Diamond^{=k}_{o}\psi \Leftrightarrow$	$\Diamond_{o}(\ell_{= \mathsf{k}} \wedge \psi)$	
$\Diamond^{(k,k')}_{o}\psi \Leftrightarrow$	$\Diamond_{o}(\ell_{> \mathsf{k}} \wedge \ell_{< \mathsf{k}'} \wedge \psi)$	$k' \neq \infty$
	$\Diamond_{o}(\ell_{> \mathsf{k}} \wedge \psi)$	$k' = \infty$

Table 4.4: Additional equivalences between metric operators, with $o \in \{r, l\}$

$N \in \mathbb{N}$. Moreover, let $M = \langle \mathbb{I}(\mathbb{D} = \mathbb{N}), V \rangle$, $M' = \langle \mathbb{I}(\mathbb{D}' = \mathbb{N}), V' \rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \{[i,j] \mid \delta(i,j) \text{ is odd and } \delta(i,j) \leq t+1\}$, $V'(p) = \{[i', j'] \mid \delta(i,j) \text{ is odd}, \delta(i,j) \leq t+1, \text{ and } [i',j'] \neq [(a-1)', a']\}$, where $a = (N-1) \cdot (t+1) + 3$, and consider the relation $Z = \{([i,j], [k',l']) \mid \delta(i,j) = \delta(k,l) \leq t+1 \text{ and } [k',l'] \neq [(a-1)', a']\} \cup \{([i,j],[i',k']) \mid \delta(i,j) > t+1 \text{ and } \delta(i,k) > t+1\} \cup \{([a-1,a],[(a-3)', a']), ([a-1,a],[(a-1)',(a+2)'])\} \cup \{([i,j],[(a-1)',a']) \mid \delta(i,j) = 2\}$. It is possible to show that Z represents a winning strategy for Player II with initial configuration $([a,b], [a',b'])$ (for any b) in the N-moves bisimulation game for ${}^t\mathsf{MPNL}^{>}$. However, we have that $M, [a,b] \Vdash \Diamond_l^{(0,2)} p$ and M', $[a',b'] \not\Vdash \Diamond_l^{(0,2)} p$, which means that the formula $\Diamond_l^{(0,2)} p$ cannot be expressed in the language ${}^t\mathsf{MPNL}^{>}$ for any $t, N \in \mathbb{N}$. Thus, we have the result.

$\mathbf{MPNL}_l^{=} \preceq \mathbf{MPNL}_l^{<}$, $\mathbf{MPNL}_l^{()} \preceq \mathbf{MPNL}_l^{>}$. This is immediate by observing that, for each $o \in \{r, l\}$, $\Diamond_o^{=k}\psi$ is defined by $\Diamond_o^{<k+1}(\ell_{=\mathsf{k}} \wedge \psi)$, and that $\Diamond_o^{(k,k')}\psi$ is defined by $\Diamond_o^{>k}(\ell_{<\mathsf{k'}} \wedge \psi)$ (if $k' \neq \infty$) or by $\Diamond_o^{>k}\psi$ (if $k' = \infty$).

From the above results, we have that $\mathsf{MPNL}^{<} \prec \mathsf{MPNL}^{=}$, $\mathsf{MPNL}_l^{<} \equiv \mathsf{MPNL}_l^{=}$, $\mathsf{MPNL}^{>} \prec \mathsf{MPNL}^{()}$, and $\mathsf{MPNL}_l^{>} \equiv \mathsf{MPNL}_l^{()}$. We show that each language in the set $\{ \mathsf{MPNL}^{<}, \mathsf{MPNL}^{=}, \mathsf{MPNL}_l^{=} \}$ is incomparable with any language in the set $\{ \mathsf{MPNL}^{>}, \mathsf{MPNL}^{()}, \mathsf{MPNL}_l^{()} \}$. To this end, it suffices to show that $\mathsf{MPNL}^{<} \not\preceq \mathsf{MPNL}_l^{()}$ and $\mathsf{MPNL}^{>} \not\preceq \mathsf{MPNL}_l^{=}$, which can be done as in Theorem 4.5.5. Finally, we must show that $\mathsf{MPNL}^{=} \prec \mathsf{MPNL}_l^{=}$ and $\mathsf{MPNL}^{()} \prec \mathsf{MPNL}_l^{()}$. It is easy to see that $\mathsf{MPNL}^{=} \preceq \mathsf{MPNL}_l^{=}$ and $\mathsf{MPNL}^{()} \preceq \mathsf{MPNL}_l^{()}$. To show that $\mathsf{MPNL}_l^{=} \not\preceq \mathsf{MPNL}^{=}$, for any $t \in \mathbb{N}$, consider the language ${}^t\mathsf{MPNL}^{=}$, defined as before. Let $N \in \mathbb{N}$. Moreover, let $M = \langle \mathbb{I}(\mathbb{D} = \mathbb{N}), V \rangle$, $M' = \langle \mathbb{I}(\mathbb{D}' = \mathbb{N}), V' \rangle$, $\mathcal{AP} = \emptyset$, $V(p) = V'(p) = \emptyset$, and let Z be the relation $\{([i,j],[i',j']) \mid i,j \in \mathbb{N}\} \cup \{([a,a+1],[a',(a+2)'])\} \cup \{([i,j],[(i+1)',(j+1)']) \mid i,j \in \mathbb{N}\}$, where $a = (N-1) \cdot (t+1)$. It is possible to show that Z represents a winning strategy for Player II with initial configuration $([a,a+1],[a',(a+2)'])$ in the N-moves bisimulation game for ${}^t\mathsf{MPNL}^{=}$. However, $M, [a,a+1] \Vdash \ell_{=1}$ and $M', [a',(a+2)'] \not\Vdash \ell_{=1}$, which means that the formula $\ell_{=1}$ cannot be expressed in the language ${}^t\mathsf{MPNL}^{=}$ for any $t, N \in \mathbb{N}$. Thus, we have the result. By exploiting a very similar argument, it is possible to show that $\mathsf{MPNL}_l^{()} \not\preceq \mathsf{MPNL}^{()}$. □

4.5.2 Expressive power of fragments of MPNL

In this section, we deal with the problem of classifying all the fragments of MPNL with respect to their relative expressive power. Fig. 4.4 shows how the various languages are related to each other.

Lemma 4.5.7. *The following equivalences hold:*

1. $\mathsf{MPNL}^{<,>} \equiv \mathsf{MPNL}^{<,\geq}$;
2. $\mathsf{MPNL}^{<,()} \equiv \mathsf{MPNL}^{=,()} \equiv \mathsf{MPNL}^{=,>} \equiv \mathsf{MPNL}^{=,\geq} \equiv \mathsf{MPNL}^{[]}$;

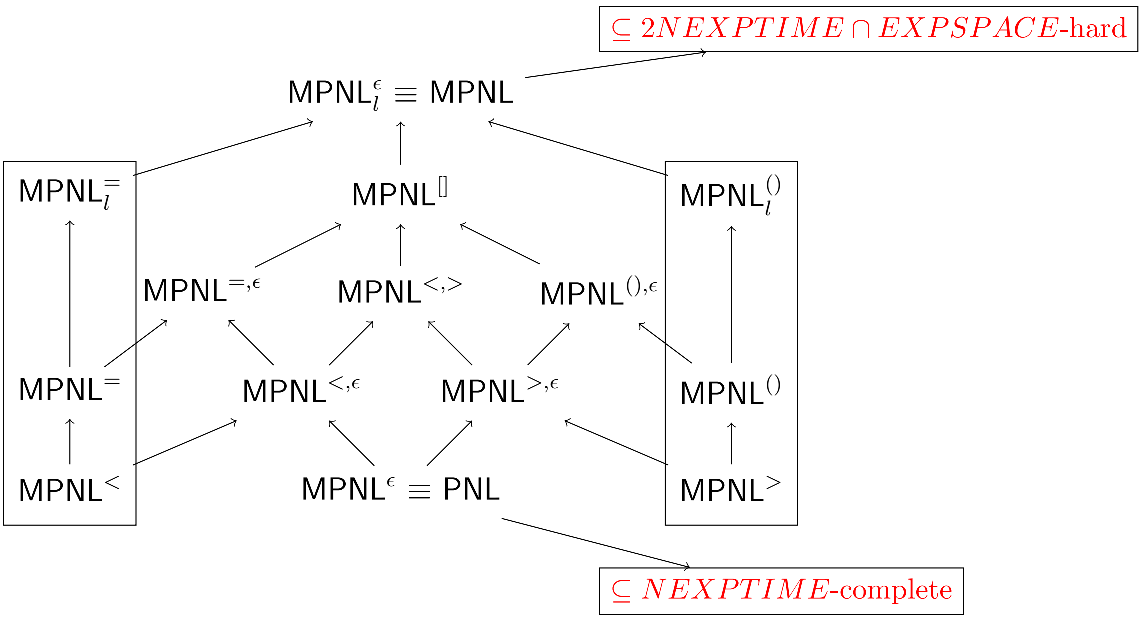


Figure 4.4: Relative expressive power of the fragments of MPNL. Fragments inside the boxes belong to wMPNL (see Fig. 4.3)

3. $\mathsf{MPNL}^{>,\epsilon} \equiv \mathsf{MPNL}^{\geq}$;

4. $\mathsf{MPNL}^{\geq,()} \equiv \mathsf{MPNL}^{(),\epsilon}$.

Proof. It suffices to use Lemma 4.5.4 and the equivalences in Table 4.4. □

Corollary 4.5.8. *If* $S = \{\epsilon, <, =, >, \geq, (), []\}$*, then we have that* $\mathsf{MPNL}^S \equiv \mathsf{MPNL}^{[]}$ *and* $\mathsf{MPNL}_l^S \equiv \mathsf{MPNL}_l^{[]}$.

Theorem 4.5.9. *The relative expressive power of the fragments of* MPNL *is as depicted in Fig. 4.4, where each arrow means that the language at the top is strictly more expressive than the one at the bottom.*

Proof. To prove this result, one can exploit bisimulations (and bisimulation games), as in the previous theorems, plus the equivalences in Table 4.4 and all the above results. For this reason, we only detail the proof of one case, namely, $\mathsf{MPNL}^< \not\preceq \mathsf{MPNL}^{(),\epsilon}$ (the proofs of the other cases are very similar).

For any $t \in \mathbb{N}$, let us define the language ${}^t\mathsf{MPNL}^{(),\epsilon}$ in the same way we did before. Let $N \in \mathbb{N}$. Moreover, let $M = \langle \mathbb{I}(\mathbb{D} = \mathbb{N}), V \rangle$, $M' = \langle \mathbb{I}(\mathbb{D}' = \mathbb{N}), V' \rangle$, $\mathcal{AP} = \{p\}$, $V(p) = \{[i,i],[i,i+1] \mid i \in \mathbb{N}\}, V'(p) = \{[i',i'],[i',(i+1)'] \mid i \in \mathbb{N}\} \setminus \{[a',a']\}$, where $a = (N-1)\cdot(t+1)+2$, and consider the relation $Z = \{([i,j],[k',l']) \mid \delta(i,j) = \delta(k,l)$ and $[k',l'] \neq [a',a']\} \cup \{([a,a],[a',(a+1)']), ([a,a],[(a-1)',a'])\} \cup \{([i, i+2], [a', a']) \mid i \in \mathbb{N}\}$. It is possible to show that Z represents a winning strategy for Player II with initial configuration $([a,b], [a',b'])$ (for any b) in the N-moves bisimulation game for ${}^t\mathsf{MPNL}^{(),\epsilon}$. However, we have that $M, [a, b] \Vdash \Diamond_l^{<1} p$ and M', $[a', b'] \nVdash \Diamond_l^{<1} p$, which means that the formula $\Diamond_l^{<1} p$ cannot be expressed in ${}^t\mathsf{MPNL}^{(),\epsilon}$ for any $t, N \in \mathbb{N}$. Thus, we have the result. $\square$

4.6 Spatial generalization of metric interval logics

The transfer of formalisms, techniques, and results from the temporal context to the spatial one is quite common in computer science and artificial intelligence. However, it (almost) never comes for free: it involves a blow up in complexity, that can possibly yield undecidability. Before concluding the chapter, in this section, we study a spatial generalization of the decidable metric interval temporal logic RPNL+INT [26]. The main goal of spatial formal systems is to capture common-sense knowledge about space and to provide a calculus of spatial information. Information about spatial objects may concern their shape and size, the distance between them, their topological and directional relations. Applications of spatial calculi include, for instance, spatial databases management, geographical information systems, image processing, and autonomous agents. Depending on the considered class of spatial relations, we can distinguish between *topological* and *directional* spatial reasoning. While topological relations between pairs of spatial objects (viewed as sets of points) are preserved under translation, scaling, and rotation, directional relations depend on the relative spatial position of the objects. A comprehensive and sufficiently up-to-date survey, which covers topological, directional, and combined constraint systems and relations, can be found in [1, 41].

Deductive systems for reasoning about topological relations have been proposed in various papers, including Bennett's work [8, 9], later extended by Bennett et al. [10], Nutt's systems for generalized topological relations [89], the modal logic systems for a number of mathematical theories of space described in [2], the logic of connectedness constraints developed by Kontchakov et al. [68], and Lutz and Wolter's modal logic of topological relations [74]. Directional relations have been dealt with following either the algebraic approach or the modal logic one. As for the first one, the most important contributions are those by Güsgen [57] and by Mukerjee and Joe [88], that introduce Rectangle Algebra (RA), later extended by Balbiani et al. in [6, 7]. As for the second one, we mention Venema's Compass Logic [101], whose undecidability has been shown by Marx and Reynolds in [76], Spatial Propositional Neighborhood Logic (SpPNL for short) by Morales et al. [85], that generalizes the logic of temporal neighborhood [53] to the two-dimensional space, and the fragment of SpPNL, called Weak Spatial Propositional Neighborhood Logic (WSpPNL), presented in [29]. As for *quantitative* spatial formalisms, the literature is very scarce. Condotta [43] proposes a generalization of RA including quantitative constraints, and identifies some meaningful tractable fragments of it. Dutta [47] develops an integrated framework for representing spatial constraints between a set of landmarks given imprecise, incomplete, and possibly conflicting quantitative and qualitative information about them, using fuzzy logic. Finally, Sheremet, Tishkovsky, Wolter, and Zakharyaschev [98] devise a logic for reasoning about metric spaces with the induced topologies, which combines the qualitative interior and closure operators with the quantitative operators "somewhere in the sphere of radius r" including or excluding the boundary; similar and related work can be also found in [65, 72].

In the following, we present the Directional Area Calculus (DAC), introduced in [20], that can be viewed as a two-dimensional variant of RPNL+INT [26]. DAC allows one to reason with basic shapes, such as lines, points, and rectangles, directional relations, and (a weak form of) areas. It features two modal operators: *somewhere to the north* and *somewhere to the east*. Moreover, by means of special *atomic propositions* of the form $\ell^{\mathsf{h}}_{=\mathsf{k}}$ (resp., $\ell^{\mathsf{v}}_{=\mathsf{k}}$), it makes it possible to constrain the length of the horizontal (resp., vertical) projections of objects. In the following, we show that, despite its simplicity, DAC allows one to express meaningful spatial properties. As an example, combining horizontal and vertical length constraints, conditions like "*the area of the current object is less than 4 square meters*" can be expressed in DAC. Moreover, we prove that its satisfiability problem is decidable in 2NEXPTIME. Then, we study a proper fragment of DAC, called Weak DAC (WDAC), which is expressive enough to capture meaningful qualitative and quantitative spatial properties. Decidability of WDAC is proved by a decision procedure whose complexity is exponentially lower than that for DAC. Optimality is an open issue for both DAC and WDAC.

4.6.1 Directional Area Calculi (DAC and WDAC)

The languages DAC and WDAC consist of a set of propositional variables $\mathcal{AP}$, the logical connectives $\neg$ and $\vee$, and the modalities $\Diamond_e, \Diamond_n$ (corresponding to the relations somewhere to the east and to the north, respectively), plus an infinite set of special atomic propositions of the form $\ell^{\mathsf{h}}_{=\mathsf{k}}$ and $\ell^{\mathsf{v}}_{=\mathsf{k}}$, with $k \in \mathbb{N}$. Let $p \in \mathcal{AP}$. Well-formed formulae, denoted by $\varphi, \psi, \ldots$, are recursively defined as follows:

$$\varphi ::= \ell^{\mathsf{h}}_{=\mathsf{k}} \mid \ell^{\mathsf{v}}_{=\mathsf{k}} \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond_e\varphi \mid \Diamond_n\varphi.$$

The other logical connectives, as well as the logical constants $\top$ and $\bot$, and the universal modalities $\Box_e$ and $\Box_n$, can be defined in the usual way.

Let $\mathbb{D}_h = \langle D_h, <\rangle$ and $\mathbb{D}_v = \langle D_v, <\rangle$, where D_h (resp., D_v) is (a prefix of) the set of natural numbers $\mathbb{N}$ and $<$ is the usual linear order. Elements of D_h (resp., D_v) will be denoted by $h_a, h_b, \ldots$ (resp., $v_a, v_b, \ldots$). A *spatial frame* is a structure $\mathbb{F} = \mathbb{D}_h \times \mathbb{D}_v$. The set of *objects* (rectangles, lines, and points) is the set $\mathbb{O}(\mathbb{F}) = \{\langle (h_a, v_b), (h_c, v_d)\rangle \mid h_a \leq h_c,\ v_b \leq v_d,\ h_a, h_c \in D_h,\ v_b, v_d \in D_v\}$. The semantics of DAC is given in terms of *spatial models* $M = \langle(\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{V}\rangle$, where $\mathbb{F}$ is a spatial frame, $\mathbb{O}(\mathbb{F})$ is the set of relevant objects, and $\mathcal{V} : \mathbb{O}(\mathbb{F}) \to 2^{\mathcal{AP}}$ is a *spatial valuation function*. The pair $(\mathbb{F}, \mathbb{O}(\mathbb{F}))$ is called *spatial structure*. Given a model M

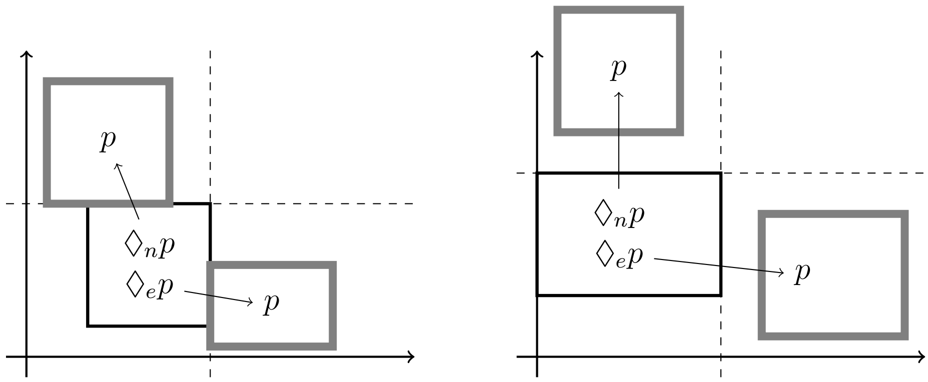


Figure 4.5: Intuitive semantics of DAC (left) and WDAC (right).

and an object $o = \langle (h_a, v_b), (h_c, v_d) \rangle$, the *truth* relation for DAC formulae is defined as follows:

- $M, o \Vdash \ell^{\mathsf{h}}_{=\mathsf{k}}$ (resp., $\ell^{\mathsf{v}}_{=\mathsf{k}}$) iff $(h_c - h_a) = k$ (resp., $(v_d - v_b) = k$);
- $M, o \Vdash p$ iff $p \in \mathcal{V}(o)$, for any $p \in \mathcal{AP}$;
- $M, o \Vdash \neg\varphi$ iff $M, o \nVdash \varphi$;
- $M, o \Vdash \varphi \vee \psi$ iff $M, o \Vdash \varphi$ or $M, o \Vdash \psi$;
- $M, o \Vdash \Diamond_e \varphi$ iff there exist $h_e \in D_h$, with $h_c \leq h_e$, and $v_g, v_i \in D_v$, with $v_g \leq v_i$, such that $M, \langle (h_c, v_g), (h_e, v_i) \rangle \Vdash \varphi$;
- $M, o \Vdash \Diamond_n \varphi$ iff there exist $v_e \in D_v$, with $v_d \leq v_e$, and $h_g, h_i \in D_h$, with $h_g \leq h_i$, such that $M, \langle (h_g, v_d), (h_i, v_e) \rangle \Vdash \varphi$.

The truth relation for WDAC formulae is obtained by replacing the last two rules with the following ones:

- $M, o \Vdash \Diamond_e \varphi$ iff there exist $h_e, h_f \in D_h$, with $h_c \leq h_e \leq h_f$, and $v_g, v_i \in D_v$, with $v_g \leq v_i$, such that $M, \langle (h_e, v_g), (h_f, v_i) \rangle \Vdash \varphi$;
- $M, o \Vdash \Diamond_n \varphi$ iff there exist $v_e, v_f \in D_v$, with $v_d \leq v_e \leq v_f$, and $h_g, h_i \in D_h$, with $h_g \leq h_i$, such that $M, \langle (h_g, v_e), (h_i, v_f) \rangle \Vdash \varphi$.

Length constraints of the form $\ell^{\mathsf{h}}_{>\mathsf{k}}$ or $\ell^{\mathsf{h}}_{<\mathsf{k}}$ can be easily defined in terms of $\ell^{\mathsf{h}}_{=\mathsf{k}}$; the same holds for the vertical ones.

Without loss of generality, we will restrict our attention to the satisfiability problem for DAC and WDAC over an *initial* object $\langle (0,0), (h_0, v_0) \rangle$ (*initial satisfiability*).

4.6.2 Expressive power of DAC

As pointed out in [85], one of the possible measures of the expressive power of a directional-based spatial logic for rectangles is the comparison with Rectangle Algebra (RA) [88]. In RA, one considers a finite set of objects (rectangles) $O_1, \ldots, O_n$, and a set of constraints between pairs of them. Each constraint is a pair of Allen's Interval Algebra relations that capture the relationships between the projections on the x- and the y-axis of the objects. As an example, $O_1(b,d)O_2$ means that $before$ (resp., *during*) is the interval relation between the x-projections (resp., y-projections) of O_1 and O_2. In general, given an *algebraic constraint network*, the main problem is to establish whether or not the network is consistent, that is, if all constraints can be jointly satisfied. In [85], it has been shown that SpPNL is powerful enough to express and to check the consistency of an RA-constraint network. In [29], the authors show that the same can be done with its decidable fragment WSpPNL. Here, we consider the problem of checking the consistency of an *augmented interval and rectangle network* [43], which can be viewed as the metric version of the consistency problem for an RA-constraint network. An augmented network is basically an RA-constraint network enriched with a set of point-based constraints of the forms $O_i^{X^+} - O_j^{X^-} = k$ or $O_i^{Y^+} - O_j^{Y^-} = k$. Such point-based constraints allow one to relate the endpoints of the various objects; thus, for example, one can force the object O_1 and the object O_2 to be 3 units far from each other along the x-axis, with O_2 after O_1, by means of $O_2^{X^-} - O_1^{X^+} = 3$. Moreover, an augmented network makes it possible to constrain the horizontal and/or the vertical lengths of an object by imposing suitable constraints on the distance between its endpoints.

Augmented networks can be embedded in DAC as follows. As a preliminary step, we introduce the *universal operator* $\Box_u$ and *nominals*. The universal operator forces a formula φ to be true everywhere in a model M; nominals are propositional letters which are true only over the current (spatial) object. It can be easily shown that both the universal operator and nominals can be defined in DAC. The former is defined as follows:

$$\Box_u \varphi \equiv \Box_e \Box_n \varphi.$$

As for the latter ones, in order to express the property "*p is true over the current object and false everywhere else*", we use the following formula, that exploits the fresh propositional letters p^e and p^n:

$$\begin{gathered} p \wedge \Box_u(p \rightarrow \Diamond_e p^e \wedge \Diamond_n p^n) \wedge \\ \Box_u(\Diamond_e p \rightarrow \Box_e(\ell^{\mathsf{h}}_{>0} \rightarrow \Box_e \neg p)) \wedge \\ \Box_u(\Diamond_n p \rightarrow \Box_n(\ell^{\mathsf{v}}_{>0} \rightarrow \Box_n \neg p)) \wedge \\ \Box_u(\Diamond_e p^e \rightarrow \Box_e(\ell^{\mathsf{h}}_{>0} \rightarrow \Box_e \neg p^e)) \wedge \\ \Box_u(\Diamond_n p^n \rightarrow \Box_n(\ell^{\mathsf{v}}_{>0} \rightarrow \Box_n \neg p^n)) \end{gathered}$$

Notice that degenerate spatial objects (lines and points) play an essential role in the definitions of the universal operator and nominals.

The encoding is defined as follows. For every object O_i in the network, we introduce a distinct propositional variable p_{O_i} and we force it to be a nominal. Metric constraints are expressed by means of the metric component of DAC. As an example, the constraint $O_2^{X^-} - O_1^{X^+} = 3$ can be encoded by the formula:

$$\Box_u(p_{O_1} \rightarrow \Diamond_e(\ell^{\mathsf{h}}_{=3} \wedge \Diamond_e p_{O_2})).$$

One can prove that the conjunction of the resulting DAC formulae is satisfiable if and only if the network is consistent.

Finally, DAC allows one to express natural spatial statements. Let $\mathsf{area}_{=\mathsf{k}}$ be a shorthand for $(\ell^{\mathsf{h}}_{=1} \wedge \ell^{\mathsf{v}}_{=\mathsf{k}}) \vee (\ell^{\mathsf{h}}_{=2} \wedge \ell^{\mathsf{v}}_{=\frac{\mathsf{k}}{2}}) \vee \ldots$, where all and only admissible combinations of horizontal and vertical constraints occur (in a similar way, one can define $\mathsf{area}_{>\mathsf{k}}$ and $\mathsf{area}_{<\mathsf{k}}$). The condition "*The area of the current object is less then 4 square meters*" can be expressed by means of the formula $\mathsf{area}_{<4}$. Similarly, the condition "*If the area of the current object is greater than 6 square meters, then there exists a line of length 12 meters to the north of it with the property q, and a point with the property p to the east of it*" is captured by the formula:

$$\mathsf{area}_{>6} \rightarrow \Diamond_n(\ell^{\mathsf{v}}_{=0} \wedge \ell^{\mathsf{h}}_{=12} \wedge q) \wedge \Diamond_e(\ell^{\mathsf{h}}_{=0} \wedge \ell^{\mathsf{v}}_{=0} \wedge p).$$

4.6.3 DAC: decidability and complexity

4.6.3.1 Basic notions

Let φ be a DAC formula to be checked for satisfiability and let $\mathcal{AP}$ be the set of its propositional variables. We define the notions of *closure, spatial request, atom*, and *fulfilling labeled spatial structure* as follows.

Definition 4.6.1. The *closure* $\mathrm{CL}(\varphi)$ of φ is the set of all sub-formulae of φ and of their negations (we identify $\neg\neg\psi$ with ψ). Let $\bigcirc_e \in \{\Diamond_e, \Box_e, \neg\Diamond_e, \neg\Box_e\}$ (resp., $\bigcirc_n \in \{\Diamond_n, \Box_n, \neg\Diamond_n, \neg\Box_n\}$). The set of *horizontal* (resp., *vertical*) *spatial requests* of φ is the set $\mathrm{HF}(\varphi)$ (resp., $\mathrm{VF}(\varphi)$) of all horizontal (resp., vertical) spatial formulae in $\mathrm{CL}(\varphi)$, that is, $\mathrm{HF}(\varphi) = \{\bigcirc_e\psi \mid \bigcirc_e\psi \in \mathrm{CL}(\varphi)\}$ (resp., $\mathrm{VF}(\varphi) = \{\bigcirc_n\psi \mid \bigcirc_n\psi \in \mathrm{CL}(\varphi)\}$).

Definition 4.6.2. A φ-*atom* is a set $A \subseteq \mathrm{CL}(\varphi)$ such that i) for every $\psi \in \mathrm{CL}(\varphi)$, $\psi \in A$ iff $\neg\psi \notin A$, and ii) for every $\psi_1 \vee \psi_2 \in \mathrm{CL}(\varphi)$, $\psi_1 \vee \psi_2 \in A$ iff $\psi_1 \in A$ or $\psi_2 \in A$.

We denote the set of all φ-atoms by A_φ. Let $|\varphi|$ (the size of φ) be the number of symbols of φ. By induction on the structure of φ, one can easily prove that $|\mathrm{CL}(\varphi)|$ is linear and $|A_\varphi|$ is exponential in $|\varphi|$. Atoms are connected by the binary relation R^h_φ (resp., R^v_φ) over $A_\varphi \times A_\varphi$ such that, for every pair of atoms $(A, A') \in A_\varphi \times A_\varphi$, $A\ R^h_\varphi\ A'$ (resp., $A\ R^v_\varphi\ A'$) if and only if, for every $\Box_e\psi \in \mathrm{CL}(\varphi)$ (resp., $\Box_n\psi \in \mathrm{CL}(\varphi)$), if $\Box_e\psi \in A$ (resp., $\Box_n\psi \in A$), then $\psi \in A'$.

We now introduce a suitable labeling of spatial structures based on φ-atoms.

Definition 4.6.3. A *φ-labeled spatial structure* (LSS for short) is a pair $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$, where $(\mathbb{F}, \mathbb{O}(\mathbb{F}))$ is a spatial structure and $\mathcal{L} : \mathbb{O}(\mathbb{F}) \rightarrow A_\varphi$ is a *labeling function* such that, for every pair of objects $\langle(h_a, v_b), (h_c, v_d)\rangle$ and $\langle(h_c, v_e), (h_f, v_g)\rangle$, $\mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)\ R^h_\varphi\ \mathcal{L}(\langle(h_c, v_e), (h_f, v_g)\rangle)$, and for every pair of objects $\langle(h_a, v_b), (h_c, v_d)\rangle$ and $\langle(h_e, v_d), (h_f, v_g)\rangle$, $\mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)\ R^v_\varphi\ \mathcal{L}(\langle(h_e, v_d),(h_f,v_g)\rangle)$.

An LSS $\mathbf{L}$ is said to be:

- *horizontally* (resp., *vertically*) *fulfilling* if for every formula of the type $\Diamond_e\psi$ (resp., $\Diamond_n\psi$) in CL(φ) and every object $\langle(h_a, v_b),(h_c, v_d)\rangle$, if $\Diamond_e\psi \in \mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)$ (resp., $\Diamond_n\psi \in \mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)$), then there exists an object $\langle(h_c,v_e),(h_f,v_g)\rangle$ (resp., $\langle(h_e,v_d),(h_f,v_g)\rangle$) such that ψ belongs to $\mathcal{L}(\langle(h_c, v_e),(h_f,v_g)\rangle)$ (resp., $\mathcal{L}(\langle(h_e,v_d),(h_f,v_g)\rangle)$);

- *length fulfilling* if for every length constraint $\ell^{\mathsf{h}}_{=\mathsf{k}}$ (resp., $\ell^{\mathsf{v}}_{=\mathsf{k}}$) in CL($\varphi$) and every object $\langle(h_a, v_b), (h_c, v_d)\rangle$, $\ell^{\mathsf{h}}_{=\mathsf{k}}$ (resp., $\ell^{\mathsf{v}}_{=\mathsf{k}}$) belongs to $\mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)$ iff $(h_c - h_a) = k$ (resp., $(v_d - v_b) = k$);

- *fulfilling* if and only if it is horizontally, vertically, and length fulfilling.

It is quite straightforward to prove that a formula φ is satisfiable if and only if there exists a fulfilling LSS such that φ belongs to the labeling of some initial object $\langle(0, 0), (h_0, v_0)\rangle$. This allows us to reduce the satisfiability problem for φ to the problem of finding a fulfilling LSS with an initial object labeled by φ. From now on, we say that a fulfilling LSS $\mathbf{L}$ *satisfies* φ if and only if $\varphi \in \mathcal{L}(\langle(0, 0), (h_0, v_0)\rangle)$ for some $h_0, v_0 \geq 0$.

4.6.3.2 The Elimination Lemma

Since fulfilling LSSs satisfying φ may be arbitrarily large or even infinite, we must find a way to finitely establish their existence. In the following, we will show how the techniques developed in [26] and [15] for the metric temporal logics RPNL+INT and MPNL, respectively, can be exploited to prove the decidability of DAC. We first give a bound on the size of finite fulfilling LSSs and then we show that, in the infinite case, we can safely restrict ourselves to infinite fulfilling LSSs with a finite bounded representation. To prove these results, we take advantage of the following two fundamental properties of LSSs: i) the labelings of all objects that share the rightmost horizontal (resp., topmost vertical) coordinate must agree on horizontal (resp., vertical) spatial formulae, that is, for every $\psi \in HF(\varphi)$ (resp., $\psi \in VF(\varphi)$), $\psi \in \mathcal{L}(\langle(h_a, v_b),(h_c, v_d)\rangle)$ if and only if $\psi \in \mathcal{L}(\langle(h_e,v_f),(h_c,v_g)\rangle)$ (resp., $\psi \in \mathcal{L}(\langle(h_a, v_b), (h_c, v_d)\rangle)$ if and only if $\psi \in \mathcal{L}(\langle(h_e,v_f),(h_g,v_d)\rangle)$); ii) $\frac{|\mathrm{HF}(\varphi)|}{2}$ different objects of the type $\langle(h_c, v_e),(h_f, v_g)\rangle$ are sufficient to fulfill existential horizontal spatial formulae belonging to the labeling of an object $\langle(h_a, v_b),(h_c, v_d)\rangle$ (and symmetrically for the vertical axis).

Definition 4.6.4. Given an LSS $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ and $h_c \in D_h$ (resp., $v_d \in D_v$), we denote by $\mathrm{REQ}_h(h_c)$ (resp., $\mathrm{REQ}_v(v_d)$) the set of all and only the horizontal (resp., vertical) requests belonging to the labelings of objects of the type $\langle(h_a, v_b),(h_c, v_d)\rangle$. The set $\mathrm{REQ}_h(\varphi)$ (resp., $\mathrm{REQ}_v(\varphi)$) is the set of all possible sets of horizontal (resp., vertical) requests for the formula φ.

In order to bound the size of finite LSSs that we must take into consideration when checking the satisfiability of a given formula φ, we determine the maximum number of times any set in $\mathrm{REQ}_h(\varphi)$ (resp., $\mathrm{REQ}_v(\varphi)$) may appear in a given LSS.

Definition 4.6.5. Given an LSS $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$, we say that a *horizontal* (resp., *vertical*) *k-sequence in* $\mathbf{L}$ is a sequence of k consecutive points in D_h (resp., D_v). Given a horizontal sequence σ in $\mathbf{L}$, its *sequence of horizontal requests* $\mathrm{REQ}_h(\sigma)$ is defined as the sequence of horizontal requests at the points in σ, and similarly for the vertical component. We say that $h \in D_h$ *starts a horizontal k-sequence* σ if the horizontal requests at $h, \ldots, h+k-1$ define an occurrence of $\mathrm{REQ}_h(\sigma)$, and similarly for the vertical component.

Hereafter, let $m_h = \frac{|\mathrm{HF}(\varphi)|}{2}$, $m_v = \frac{|\mathrm{VF}(\varphi)|}{2}$, and $m = max\{m_h, m_v\}$, and let $k = max\{k', 1\}$, where either $\ell^{\mathsf{h}}_{=\mathsf{k}'}$ or $\ell^{\mathsf{v}}_{=\mathsf{k}'}$ occurs in φ and for all $\ell^{\mathsf{h}}_{=\mathsf{k}''}$ and $\ell^{\mathsf{v}}_{=\mathsf{k}''}$ occurring in φ, $k' \geq k''$.

Definition 4.6.6. Given an LSS $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$, any sequence of horizontal requests $\mathrm{REQ}_h(\sigma)$ is said to be *abundant* in $\mathbf{L}$ if and only if it has at least $k \cdot (m^2 + m) \cdot |\,\mathrm{REQ}_h(\varphi)|^2 + (m^2 + 3 \cdot m) \cdot |\,\mathrm{REQ}_h(\varphi)| + 1$ distinct occurrences in D_h. The case of an abundant sequence of vertical requests is defined similarly.

The above definition shows a quadratic increase in complexity from RPNL+INT. In the temporal case, indeed, a number of occurrences linear in m and $\mathrm{REQ}(\varphi)$ suffices to declare a sequence of requests as abundant. For any given horizontal k-sequence σ in $\mathbf{L}$, we will denote by h^σ_q the first point of the q-th occurrence of σ. Hereafter, whenever σ will be evident from the context, we will write h_q for h^σ_q. The next Lemma is analogous to Lemma 5.12 in [26]. However, in the spatial setting, to be able to reduce the size of the model we must also guarantee the existence of a certain number of occurrences of the sequence *before* a given point h_q.

Lemma 4.6.7. *Let $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ be an LSS and σ be a horizontal k-sequence in $\mathbf{L}$ such that $\mathrm{REQ}_h(\sigma)$ is abundant in $\mathbf{L}$. Then, there exists an index q such that:*

1. *for every pair $(\mathrm{REQ}_h(h), \mathrm{REQ}_h(h'))$ such that $h \in D^-_h = \{\overline{h} \mid h_q \leq \overline{h} < h_{q+1}\}$ and $h' - h \leq k$, there exist at least $m^2 + m$ distinct pairs of points h'', h''' in $D_h \setminus D^-_h$ such that $h''' - h'' = h' - h$ and $(\mathrm{REQ}_h(h''), \mathrm{REQ}_h(h''')) = (\mathrm{REQ}_h(h), \mathrm{REQ}_h(h'))$;*

2. *for every element $\mathcal{R} \in \{REQ_h(h) \mid h \in D^-_h\}$, $\mathcal{R}$ occurs at least $m^2 + m$ times before h_q and at least $2 \cdot m$ times after $h_{q+1} + k$.*

Proof. (sketch) By Definition 4.6.6, there exist at least $k \cdot (m^2+m) \cdot |\operatorname{REQ}_h(\varphi)|^2 + (m^2+3\cdot m)\cdot|\operatorname{REQ}_h(\varphi)|+1$ points $h_i \in D_h$ such that h_i is the first element of a distinct occurrence of σ. For every index i, if there exists a pair $(\operatorname{REQ}_h(h), \operatorname{REQ}_h(h'))$, with $h_i \leq h < h_{i+1}$ and $h' - h \leq k$, such that there exist no m^2+m distinct pairs of points h'', h''' in $D_h \setminus \{h \mid h_i \leq h < h_{i+1}\}$ with $h''' - h'' = h' - h$ and $(\operatorname{REQ}_h(h''), \operatorname{REQ}_h(h''')) = (\operatorname{REQ}_h(h), \operatorname{REQ}_h(h'))$, then q cannot be equal to i. By an easy combinatorial argument, we can prove that there exist at most $k \cdot (m^2 + m) \cdot |\operatorname{REQ}_h(\varphi)|^2$ such indexes, where $|\operatorname{REQ}_h(\varphi)|^2$ is the number of possible pairs $(\operatorname{REQ}_h(h), \operatorname{REQ}_h(h'))$, k is the number of possible values for $h'-h$, and, for any pair $(\operatorname{REQ}_h(h), \operatorname{REQ}_h(h'))$ and any distance $h'-h$, m^2+m is the greatest number of occurrences of a pair $(\operatorname{REQ}_h(h), \operatorname{REQ}_h(h'))$ that may lead to a violation of condition 1. Since σ is abundant in $\mathbf{L}$, we can conclude that there exist at least $(m^2+3\cdot m)\cdot |\operatorname{REQ}_h(\varphi)|+1$ indexes in D_h that satisfy condition 1. Let us now restrict our attention to these indexes. In the worst case, for at most $(m^2+m)\cdot|\operatorname{REQ}_h(\varphi)|$ indexes i it may happen that there exist no m^2+m occurrences of $\mathcal{R}$ before h_i for some $\mathcal{R} \in \{REQ_h(h) \mid h_i \leq h < h_{i+1}\}$. Hence, there exist at least $2\cdot m\cdot |\operatorname{REQ}_h(\varphi)|+1$ indexes that satisfy both the above conditions. By applying the same argument, we can conclude that for at most $2\cdot m \cdot |\operatorname{REQ}_h(\varphi)|$ indexes i it may happen that there exist no $2\cdot m$ occurrences of $\mathcal{R}$ after $h_{i+1}+k$ for some $\mathcal{R} \in \{REQ_h(h) \mid h_i \leq h < h_{i+1}\}$. This allows us to conclude that there exists at least one index i that satisfies the conditions of the lemma. □

Lemma 4.6.8. *(Horizontal Elimination Lemma) Let* $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ *be a fulfilling* LSS *that satisfies* φ*. Suppose that there exists an abundant* k*-sequence of horizontal requests* $\operatorname{REQ}_h(\sigma)$ *and let* D_h^- *be the set whose existence is guaranteed by Lemma 4.6.7. Then, there exists a fulfilling* LSS $\overline{\mathbf{L}} = ((\overline{\mathbb{F}}, \mathbb{O}(\overline{\mathbb{F}})), \overline{\mathcal{L}})$ *that satisfies* φ*, with* $\overline{D}_h = D_h \setminus D_h^-$ *and* $\overline{D}_v = D_v$*.*

Proof. Let us fix a fulfilling LSS $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ satisfying φ at some $\langle(0,0),(h_0,v_0)\rangle$ and an abundant k-sequence of horizontal requests $\operatorname{REQ}_h(\sigma)$. Moreover, let D_h^- be the set whose existence is guaranteed by Lemma 4.6.7 and $D'_h = D_h \setminus D_h^-$. Finally, let $\overline{\mathbf{L}} = ((\mathbb{F}', \mathbb{O}(\mathbb{F}')), \mathcal{L}')$ be the restriction of $\mathbf{L}$ to D'_h. $\overline{\mathbf{L}}$ is still an LSS, but not necessarily a fulfilling one. To recover the property of fulfillment, we must suitably redefine the evaluation of objects.

Fixing lengths. First, we must revise the labeling of those objects whose horizontal length was greater than k before the elimination of points in D_h^- and has become less than or equal to k in D'_h. In general, the horizontal length of every object $\langle(h_a,v_b),(h_c,v_d)\rangle$, with $h_a < h_q$ and $h_c \geq h_{q+1}$, becomes $h_{q+1} - h_q$ units shorter. This is critical for those objects whose horizontal length in D_h was less than or equal to $k + (h_{q+1} - h_q)$. To cope with these cases, for every $h < h_q$, $v_a, v_b \in D_v$, and $0 \leq r < k$, we put $\mathcal{L}'(\langle(h,v_a),(h_{q+1}+r,v_b)\rangle) = \mathcal{L}(\langle(h,v_a),(h_q+r,v_b)\rangle)$. (Notice that, in D'_h, h_{q+1} turns out to be the immediate successor of $h_q - 1$.)

Fixing defects. Once the above relabeling has been accomplished, we may still have four types of defects (some of them have been introduced by the elimination, others by the length-fixing process itself):

1. there is a formula $\Diamond_e\psi \in \text{REQ}_h(h_a)$, for some $h_a \in D'_h$, that is not fulfilled anymore. For this to be the case, it must be that some object $\langle(h_a,v_b), (h,v_c)\rangle$ either has been eliminated because $h \in D^-_h$ or its labeling has been changed by the previous step. In both cases, the critical objects are those such that $(h - h_a) > k$ in the original model. Since there are at least $2 \cdot m$ points $h_1, \ldots, h_{2 \cdot m}$ after $h_{q+1} + k$ with the same set of requests of h, for at least one of them, say h_i, either the label of the object $\langle(h_a,v_b),(h_i, v_c)\rangle$ satisfies neither vertical requests from $\text{REQ}_v(v_b)$ nor horizontal requests from $\text{REQ}_h(h_a)$, or it satisfies only requests that are satisfied elsewhere. So, we put $\mathcal{L}'(\langle(h_a,v_b),(h_i, v_c)\rangle) = \mathcal{L}(\langle(h_a,v_b),(h,v_c)\rangle)$, thus fixing the defect;

2. there is a formula $\Diamond_n\psi \in \text{REQ}_v(v_a)$, for some $v_a \in D_v$, that is not fulfilled anymore. As in the previous case, this may happen either because of the elimination of some object $\langle(h_b,v_a),(h, v_c)\rangle$, where $h \in D^-_h$ and $h_b \in D'_h$, or because of the relabeling of some object $\langle(h_b,v_a),(h,v_c)\rangle$, where $h, h_b \in D'_h$. Again, for this to be the case, it must be that $(h - h_b) > k$. To fix this defect, we proceed exactly as in the previous case;

3. there is a formula $\Diamond_n\psi \in \text{REQ}_v(v_a)$, for some $v_a \in D_v$, that is not fulfilled anymore because of the elimination of some object $\langle(h,v_a), (h_b,v_c)\rangle$, where $h, h_b \in D^-_h$ and $h_b - h \le k$. By Lemma 4.6.7, there are at least $m^2 + m$ distinct pairs $(h_1, h'_1), \ldots, (h_{m^2+m}, h'_{m^2+m})$ such that for all i $(1 \le i \le m^2 + m)$, $h_i, h'_i \in D_h \setminus D^-_h$, $h'_i - h_i = h_b - h$, and $(\text{REQ}_h(h_i), \text{REQ}_h(h'_i)) = (\text{REQ}_h(h), \text{REQ}_h(h_b))$. Let $\{\Diamond_e\tau_1, \ldots, \Diamond_e\tau_q\} \subseteq \text{REQ}_h(h)$, with $q \le m$, be the set of horizontal requests at h. We look for an index i such that we can force h_i to satisfy ψ, as well as all horizontal requests τ_j $(1 \le j \le q)$, exactly (that is, at the same vertical coordinates) as h did, that is, $\psi \in \mathcal{L}'(\langle(h_i, v_a), (h'_i, v_c)\rangle)$ and, for every j $(1 \le j \le q)$, $\tau_j \in \mathcal{L}'(\langle(h_i, v_{\tau_j}), (h'_{\tau_j}, v'_{\tau_j})\rangle)$ if and only if $\tau_j \in \mathcal{L}(\langle(h, v_{\tau_j}), (h_{\tau_j}, v'_{\tau_j})\rangle)$, with $h'_{\tau_j} - h_i = h_{\tau_j} - h$. In order to accomplish such a relabeling process, we must be careful not to introduce defects. The operation is safe with respect to horizontal defects, since, by construction, $REQ_h(h_i) = REQ_h(h)$. As for possible vertical defects, the replacement of object labels may cause vertical requests fulfilled by overwritten labels to become unfulfilled, thus introducing vertical defects. However, thanks to the presence of sufficiently many points h_i with the same set of horizontal requests as h (candidates for the relabeling process), we are guaranteed of the existence of an index i such that the objects whose labels we overwrite either do not satisfy any vertical requests or satisfy only vertical requests that are also satisfied by other objects (other candidates for the relabeling process).

4. there is a formula $\Diamond_n\psi \in \text{REQ}_v(v_a)$, for some $v_a \in D_v$, that is not fulfilled anymore because of the elimination of some object $\langle(h,v_a),(h_b,v_c)\rangle)$, where

$h \in D_h^-$, and $(h_b - h) > k$. To fix this defect, we proceed exactly as in case 3, using one of the $m^2 + m$ "copies" of h before h_q as left horizontal coordinate.

In this way, we can fix all defects. At the end of the process, $\overline{\mathbf{L}}$ is a fulfilling LSS, as claimed. □

Similarly, we have:

Lemma 4.6.9. *(Vertical Elimination Lemma) Let* $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ *be a fulfilling* LSS *that satisfies* φ*. Suppose that there exists an abundant* k*-sequence of vertical requests* $\mathrm{REQ}_v(\sigma)$ *and let* D_v^- *be the set whose existence is guaranteed by the (vertical version of) Lemma 4.6.7. Then, there exists a fulfilling* LSS $\overline{\mathbf{L}} = ((\overline{\mathbb{F}}, \mathbb{O}(\overline{\mathbb{F}})), \overline{\mathcal{L}})$ *that satisfies* φ*, with* $\overline{D}_v = D_v \setminus D_v^-$ *and* $\overline{D}_h = D_h$*.*

Lemma 4.6.8 and 4.6.9 are the spatial counterpart of the Elimination Lemma for RPNL+INT [26]. However, while in the temporal case only defects of type 1 may occur, the interaction between the two spatial operators of DAC introduces other types of defect.

4.6.3.3 DAC satisfiability

Thanks to the horizontal and vertical elimination lemmas above, the following theorem holds.

Theorem 4.6.10 (Small Model Theorem)**.** *If* φ *is any finitely satisfiable formula of* DAC*, then it is satisfiable in a finite* LSS $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ *such that* $|D_h| \leq (k \cdot (m^2 + m) \cdot |\mathrm{REQ}_h(\varphi)|^2 + (m^2 + 3 \cdot m) \cdot |\mathrm{REQ}_h(\varphi)|) \cdot |\mathrm{REQ}_h(\varphi)|^k + k - 1$*, and* $|D_v| \leq (k \cdot (m^2 + m) \cdot |\mathrm{REQ}_v(\varphi)|^2 + (m^2 + 3 \cdot m) \cdot |\mathrm{REQ}_v(\varphi)|) \cdot |\mathrm{REQ}_v(\varphi)|^k + k - 1$*.*

Corollary 4.6.11. *Finite satisfiability for* DAC *is decidable.*

Infinite structures can be dealt with in a similar way. As a preliminary step, we distinguish among three types of infinite LSSs, depending on whether only one domain is infinite (and which one) or both. For each of these types, an appropriate representation can be obtained as follows.

Definition 4.6.12. Any LSS $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ is *horizontally ultimately periodic*, with prefix Pre_H, period $Per_H \geq 0$ and threshold k, if and only if:

1. for every h, h', with $h' \geq Pre_H$ and $(h' - h) > k$, and every v, v', $\mathcal{L}(\langle (h, v), (h', v') \rangle) = \mathcal{L}(\langle (h, v), (h' + Per_H, v') \rangle)$;
2. for every object $\langle (h,v), (h', v') \rangle$, with $h \geq Pre_H$, $\mathcal{L}(\langle (h, v), (h', v') \rangle) = \mathcal{L}(\langle (h + Per_H, v), (h' + Per_H, v') \rangle)$.

The notion of *vertically ultimately periodic* LSS can be defined in a similar way. Finally, a LSS is simply *ultimately periodic* if it is (i) both horizontally and vertically ultimately periodic, or (ii) horizontally ultimately periodic and vertically finite, or (iii) horizontally finite and vertically ultimately periodic.

It is immediate to see that every ultimately periodic LSS is finitely presentable.

Lemma 4.6.13. *Let* $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ *be an horizontally infinite, vertically finite* LSS *that satisfies* φ*. Then, there exists an ultimately periodic* LSS $\overline{\mathbf{L}}$ *that satisfies* φ*.*

An analogous of Lemma 4.6.13 can be stated for the vertical component, and, thus, any infinite LSS can be transformed into a ultimately periodic one.

Theorem 4.6.14 (Periodic Small Model Theorem)**.** *Let* $\mathbf{L} = ((\mathbb{F}, \mathbb{O}(\mathbb{F})), \mathcal{L})$ *be an* LSS *that satisfies* φ*. Then, there exists an ultimately periodic* LSS $\overline{\mathbf{L}}$ *such that (i)* $\overline{\mathbf{L}}$ *satisfies* φ *and (ii) the length of the horizontal prefix and the horizontal period are bounded by* $(k \cdot (m^2+m) \cdot |\,\mathrm{REQ}_h(\varphi)|^2 + (m^2+3\cdot m) \cdot |\,\mathrm{REQ}_h(\varphi)|) \cdot |\,\mathrm{REQ}_h(\varphi)|^k + k - 1$ *(similarly for the vertical component).*

Once again, the spatial features of DAC causes a quadratic increase in the size of (prefixes and periods of) the model with respect to the metric temporal logic RPNL+INT [26].

Corollary 4.6.15. *The satisfiability problem for* DAC *is decidable.*

4.6.3.4 Complexity Issues

In [29], it has been shown that the satisfiability problem for the non-metric version of DAC is NEXPTIME-complete. Hence, DAC is at least NEXPTIME-hard. To correctly state the complexity of the satisfiability problem for DAC, we have to consider three different cases, depending on the representation of length constraints. As a direct consequence of the theorems given in previous sections, a nondeterministic decision procedure that guesses an ultimately periodic model satisfying the formula φ can be easily built. Such a procedure works in $\mathrm{NTIME}(2^{|\varphi|\cdot k})$, and its exact complexity class depends on how the metric constants are encoded.

Theorem 4.6.16. *The satisfiability for* DAC *is:*

1. *NEXPTIME-complete, if* k *is a constant;*
2. *NEXPTIME-complete, if* k *is represented in unary;*
3. *between EXPSPACE and 2NEXPTIME, if* k *is represented in binary.*

NEXPTIME inclusion (cases 1 and 2) can be proved by simply observing that $O(2^{|\varphi|\cdot k}) = O(2^{|\varphi|})$ if k is constant or represented in unary (with respect to the length of the formula); NEXPTIME-hardness is a consequence of NEXPTIME-hardness for SpPNL [29]. In these cases, there is not a complexity increase with respect to the temporal counterpart RPNL+INT, which is NEXPTIME-hard as well [26]. On the contrary, when k is represented in binary (case 3), RPNL+INT is EXPSPACE-complete, and thus DAC is at least EXPSPACE-hard. However, since $k = O(2^{|\varphi|})$, the non-deterministic procedure runs in time $O(2^{2^{|\varphi|}})$, giving us a 2NEXPTIME upper bound on the complexity. We do not know yet which is the exact complexity class for DAC in this case, and whether the switch from temporal logic to its spatial counterpart causes an increase in complexity or not.

4.6.4 Weak Directional Area Calculus (WDAC)

In this section, we discuss expressive power, decidability, and complexity of WDAC, and we briefly compare it with full DAC.

First of all, formulae of WDAC can be translated into DAC formulae by replacing any sub-formula of the form $\Diamond_e\psi$ (resp., $\Diamond_n\psi$) by $\Diamond_e\Diamond_e\psi$ (resp., $\Diamond_n\Diamond_n\psi$). By a *bisimulation* argument, we can prove that the converse does not hold. We will show that, for every $k \geq 0$, there exist two models M_1^k and M_2^k that are bisimilar with respect to WDAC formulae with maximum metric constant k, but can be easily distinguished by a DAC formula. Let $k \geq 0$ and $\mathcal{AP} = \{p\}$. The two spatial models $M_1 = \langle(\mathbb{F}_1, \mathbb{O}(\mathbb{F}_1)), \mathcal{V}_1\rangle$ and $M_2 = \langle(\mathbb{F}_2, \mathbb{O}(\mathbb{F}_2)), \mathcal{V}_2\rangle$ are defined as follows.

- $\mathbb{F}_1 = \mathbb{F}_2 = \mathbb{N} \times \mathbb{N}$
- $\mathcal{V}_1(\langle(1, v_a), (k+4, v_b)\rangle) = \mathcal{V}_1(\langle(3, v_a), (k+4, v_b)\rangle) = \{p\}$, for all $v_a, v_b \in \mathbb{N}$;
- $\mathcal{V}_2(\langle(3, v_a), (k+4, v_b)\rangle) = \{p\}$, for all $v_a, v_b \in \mathbb{N}$;
- p is false everywhere else.

The following relation $Z^k \subseteq \mathbb{O}(\mathbb{F}_1) \times \mathbb{O}(\mathbb{F}_2)$ is a WDAC-bisimulation between M_1^k and M_2^k:

- $(\langle(h_a, v_b), (h_c, v_d)\rangle, \langle(h_a, v_b), (h_c, v_d)\rangle) \in Z^k$ for all $(h_a, h_c) \neq (1, k+4)$;
- $(\langle(1, v_b), (k+4, v_d)\rangle, \langle(3, v_b), (k+4, v_d)\rangle) \in Z^k$;
- $(\langle(2, v_b), (k+4, v_d)\rangle, \langle(1, v_b), (k+4, v_d)\rangle) \in Z^k$.

Since the DAC formula $\Diamond_e p$ is true over the object $\langle(0,0),(1,1)\rangle$ in M_1^k, but it is false in M_2^k for every value of k, and since bisimilar models must satisfy the same set of WDAC formulae, $\Diamond_e p$ cannot be translated to any WDAC formula.

Theorem 4.6.17. WDAC *is strictly less expressive than* DAC.

Despite being strictly less expressive than DAC, WDAC is powerful enough to express the augmented interval and rectangle network consistency problem discussed in Section 4.6.2, at the price of a more complex encoding.

Decidability of WDAC trivially follows from the decidability of DAC. However, its weaker semantics allows us to lower the complexity bound. The modal operators are *transitive* in WDAC: if a formula $\Box_e\psi$ holds over an object, then it holds over

PNL^{π}	NEXPTIME-complete	$\mathsf{FO}^2[=,<]$ [25]	NEXPTIME-complete [90]
MPNL	2NEXPTIME, EXPSPACE-hard	$\mathsf{FO}^2_r[\mathbb{N},=,<,s]$	2NEXPTIME, NEXPTIME-hard
MPNL^+	undecidable	$\mathsf{FO}^2[\mathbb{N},=,<,s]$	undecidable

Table 4.5: Complexity and expressive completeness results

any object to the east of it (and symmetrically for $\Box_n\psi$), while in full DAC this is not necessarily the case. This implies that if a formula $\Box_e\psi \in \mathrm{REQ}_h(h_a)$ (resp., $\Box_n\psi \in \mathrm{REQ}_v(v_a)$) for some $h_a \in D_h$ (resp., $v_a \in D_v$), then $\Box_e\psi \in \mathrm{REQ}_h(h_b)$ for every $h_b > h_a$ (resp., $\Box_n\psi \in \mathrm{REQ}_v(v_b)$ for every $v_b > v_a$). By exploiting this property, we can provide a bound on the size of LSS satisfying a WDAC formula that is exponentially smaller than the one given for DAC in Theorem 4.6.14.

Theorem 4.6.18 (Weak Periodic Small Model Theorem)**.** *Let φ be a satisfiable* WDAC *formula. Then, there exists a ultimately periodic fulfilling* LSS *satisfying φ with horizontal and vertical prefix bounded by $(2 \cdot m + 1) \cdot (k + 1) + 1$, horizontal and vertical period bounded by $2 \cdot m \cdot (k + 1)$, and threshold k.*

As a direct consequence of Theorem 4.6.18, a nondeterministic decision procedure that guesses an ultimately periodic model satisfying the formula φ can be easily built. Such a procedure works in NTIME($k \cdot |\varphi|$), and its exact complexity class depends on how the metric constants are encoded.

Theorem 4.6.19. *The satisfiability for* WDAC *is:*

- *NP-complete, if k is a constant;*
- *NP-complete, if k is represented in unary;*
- *in NEXPTIME, if k is represented in binary.*

NP-completeness of the problem when k is constant or in unary encoding follows from the NP-completeness of SAT. We do not know yet if WDAC with binary encoding is NEXPTIME-hard or not.

4.7 Concluding remarks

In this chapter, we have proposed and studied metric extensions of Propositional Neighborhood Logics (PNL) over the interval structure of natural numbers $\mathbb{N}$. We have demonstrated that these are expressive and natural languages to reason about that structure by proving the complexity and expressive completeness results summarized in Table 4.5. First, we have considered a very expressive language in this class, called MPNL, and shown the decidability of its satisfiability problem. Then, we have identified an appropriate fragment of $\mathsf{FO}^2[\mathbb{N},=,<,s]$ (the two-variable fragment of first-order logic with equality, order, successor, and any family of binary relations, interpreted on the structure of natural numbers), denoted by $\mathsf{FO}^2_r[\mathbb{N}, =, <, s]$, and we have proved that MPNL is expressively complete for such a fragment. Decidability of $\mathsf{FO}^2_r[\mathbb{N}, =, <, s]$ immediately follows. Then, we have shown how to extend MPNL in order to obtain an interval logic that is expressively complete for full $\mathsf{FO}^2[\mathbb{N}, =, <, s]$, which we have proved to be undecidable. Finally, we have discussed the variety of fragments of MPNL and studied their expressiveness.

The results obtained here are amenable to some fairly straightforward generalizations, e.g., from $\mathbb{N}$ to $\mathbb{Z}$. A more challenging generalization involves the set $\mathbb{Q}$ and presents the basic problem of defining dense metric constraints. Nevertheless, the positive results about decidability and tableau over $\mathbb{Q}$ for classical (non-metric) PNL are encouraging. An important open problem is to find the exact complexity of the satisfiability problem for MPNL, when constraints are represented in binary, as well as the identification of the fragments of MPNL where the complexity jumps occur. Another interesting open problem is to determine more precisely the (un)decidability border in the family of metric propositional neighborhood logics by identifying maximal decidable extensions of MPNL.

From a more practical point of view, we plan to implement the decision procedure for MPNL presented in this paper, and to study the application of the logic in the modeling and verification of reactive systems.

The last contribution of this chapter is the introduction and analysis of a spatial generalization of the (decidable) future fragment RPNL+INT of MPNL. The proposed formalism is a new modal logic of directional relations, called DAC, that pairs qualitative and quantitative spatial reasoning about points, lines, and rectangles over natural number frames. We proved that the satisfiability problem for DAC is decidable. Moreover, we showed that, when a binary encoding of length constraints is provided, it lies in between EXPSPACE and 2NEXPTIME. The exact complexity class is an open problem. Then, we analyzed the satisfiability problem for a proper expressive fragment of DAC, called WDAC, and we proved that it belongs to NEXPTIME. As in the case of DAC, the exact complexity class, when a binary encoding of length constraints is provided, is an open problem.

5

Undecidable extensions of (metric) PNL

In this chapter, in quest of more and more expressive decidable interval logics, we explore two different, but related, ways of extending the expressive power of PNL and its metric extension MPNL. Our results will be given with respect to interval structures based on the ordering over the natural numbers $\langle \mathbb{N}, < \rangle$. As a matter of fact, most of them also hold with respect to interval structure over $\mathbb{Z}$, as well as over various other linear orders.

First, we consider the possibility of adding some features from hybrid logics to (metric) PNL [45]. Since the *difference* modality is already definable in PNL [53], nominals can be simulated there, and thus their addition is unproblematic with regards to decidability. On the contrary, it is not difficult to show that the addition of binders over state variables immediately leads to undecidability. However, the most natural, and useful, binders for (metric) PNL are not those on state variables ranging over intervals, but those on positive integer variables ranging over *lengths of intervals*, that make it possible to store the length of the current interval and to further refer to it. As we will see, if we add length binders, no metric constraints are needed to cross the decidability/undecidability border. Therefore, the first result of the chapter will be a proof of the undecidability of the extension of PNL, interpreted over natural numbers, with length binders (PNL+LB for short). Notice that PNL+LB is expressively not comparable with MPNL, as MPNL does not feature length binders, but it allows one to constrain the length of the current interval to be equal to (resp., less than, greater than) a certain positive integer k.

Second, we consider a first-order extension of PNL, called PNL+FO, which is obtained by possibly replacing propositional variables by first-order formulae [46]. Unlike some point-based temporal logics, which have been successfully extended with first-order constructs preserving decidability [64], a very limited first-order extension of PNL suffices to get undecidability. In fact, we will show that a single modal operator is enough for undecidability, that is, we replace PNL by its future fragment Right PNL (RPNL for short) [31] and we prove that its first-order extension, called RPNL+FO, is undecidable. Moreover, it is possible to show that the same undecidability result holds for other classes of linear orders, such as dense orders and finite orders.

5.1 Related work

In this section, we briefly survey more or less directly related work about hybrid and first-order extensions of propositional (metric) temporal logics.

The length variables and binders we deal with in this chapter bear natural resemblance with the interval length variables used in Duration Calculus (DC/ITL) [35, 60], an extension of Moszkowski's Interval Temporal Logic (ITL) [87], developed by Chaochen, Hoare, and Ravn [37]. The original version of ITL involves only one, binary modal operator C, called *chop*, where $\varphi C \psi$ states that the current interval $[a, b]$ can be split (chopped) into two consecutive intervals $[a, c]$ and $[c, b]$ such that $[a, c]$ satisfies φ and $[c, b]$ satisfies ψ. DC/ITL is a real-time extension of ITL that adds *state expressions* to the language of ITL to make it possible to model the states of the system; moreover, it allows one to associate a *duration* with state expressions, in order to constrain the length of the time period during which the system remains in the given state. In [34], the authors have proposed a version of DC based on Neighborhood Logic (NL), denoted DC/NL, which features the two interval neighborhood modalities $\Diamond_r$ and $\Diamond_l$ of NL. It is possible to show that DC/NL subsumes the original DC/ITL. The satisfiability/validity problem for ITL, and thus those for DC/ITL and DC/NL, turns out to be undecidable over all relevant classes of linear orders.

A lot of work has been done in the search for decidable variants and fragments of DC/ITL (and of ITL). As an example, an interval-based version of DC/ITL, called *Interval Duration Logic* (IDL), has been developed by Pandya in [93]. In its full generality, such a logic is undecidable. However, it admits meaningful fragments, such as LIDL−, which can be proved to be decidable by exploiting an automata theoretic argument [93]. The problem of checking IDL formulae for validity has been further investigated in [33]. In such a work, Chakravorty and Pandya provide a syntactic characterization of the proper subset of IDL formulae that satisfy the property of *strong closure under inverse digitalization*, and they show that the problem of checking the validity of formulae belonging to such a subset can be reduced to that for *Discrete Time Duration Calculus*, a discrete-time logic whose validity problem has been shown to be decidable following an automaton-based approach in [92] (a complexity improvement to such a decidability result has been given in [70]). In [50], Fränzle and Hansen prove the decidability of a quite expressive fragment of DC/ITL, properly extending the work on linear duration invariants by Chaochen et al. [38]. Other fragments of DC/ITL have been studied in [61]. In particular, the Restricted Duration Calculus, abbreviated RDC_1, allows one to constrain the length of the current interval to be equal to a given constant value. RDC_1 is decidable over discrete linear orders and undecidable over dense ones. Richer fragments, such as, for instance, RDC_3, that allows one to quantify over the variable denoting the length of the current interval, turn out to be undecidable over both discrete and dense linear orders.

Since the original formulation of ITL by Moszkowski [87], an alternative path to decidability has been the enforcement of *locality*: all atomic propositions are evaluated point-wise, meaning that their truth over an interval is defined as truth at its initial point. The assumption of locality has been exploited in DC/ITL as well to recover decidability. In [12], Bolander et al. describe a hybrid extension of local DC/ITL, introducing interval binders (not length ones) and nominals, that allow one to refer to specific intervals, and prove its decidability over natural numbers. This does not come as a surprise, as the locality assumption essentially boils down the logic to a point-based one, and it eventually reduces its satisfiability problem to the one for monadic second-order logic (over the same linear order).

A limited amount of work has been devoted to the model checking problem for duration calculi. In [14], Bouajjani at al. address the problem of specifying and verifying hybrid systems in the framework of duration calculi, exploiting techniques borrowed from hybrid automata. Model checking algorithm for duration calculi have been developed in [49, 51, 77].

First-order point-based temporal logics have been systematically studied by Hodkinson et al. in [64]. They show that (un)decidability of such logics depends on both the classical (first-order) and temporal components of the language. In particular, they prove that the two-variable fragment of first-order Linear Temporal Logic (LTL), with Since and Until, interpreted over $\mathbb{N}$ and $\mathbb{Z}$, is undecidable. The same results hold for LTL with Next and Future modalities only. Then, they show that decidability can be recovered by restricting the first-order component to a decidable fragment of first-order logic and the temporal component to monodic formulae, that is, formulae whose sub-formulae with a temporal operator as their outermost operator have at most one free variable. In particular, they prove that the two-variable fragment of monodic first-order LTL (without equality and function symbols, and with constant first-order domains) is decidable over various linear time structures, including $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ (the latter holds for finite first-order domains only). In the following, we will show that there is not a counterpart of these decidability results in the setting of first-order extensions of PNL: the first-order one-variable extension of PNL, with a finite first-order domain, interpreted over $\mathbb{N}$, is already undecidable.

5.2 Hybrid and First-Order extensions of (metric) PNL

In this section, we introduce two meaningful extensions of (metric) PNL, namely, hybrid and first-order extensions. In the first part, we focus our attention on hybrid extensions. We start by showing that some typical components of hybrid logics, such as nominals, can actually be defined in PNL. Then, we briefly discuss the effects of the addition of binders to (metric) PNL. In the second part, we introduce first-order extensions of PNL. Both extensions will be investigated in detail in the following sections.

5.2.1 (Metric) Hybrid extensions

Despite its simplicity, PNL makes it possible to define significant hybrid features such as nominals. Let $[G]$ be the *universal modality*, that is, given a formula φ, $[G]\varphi$ is true over an interval if and only if φ is true over all intervals. The universal modality can be defined in all variants of PNL. For instance, if the non-strict semantics is assumed, it can be defined as follows: $[G]\varphi \equiv \Box_l\Box_l\Box_r\Box_r\varphi$. The same holds for the *difference modality* $[\neq]$. In the strict semantics, as shown in [53], it can be defined as follows:

$$[\neq]\varphi \equiv \Box_l\Box_l\Box_r\varphi \wedge \Box_l\Box_r\Box_r\varphi \wedge \Box_r\Box_l\Box_l\varphi \wedge \Box_r\Box_r\Box_l\varphi.$$

Such a formula can be easily modified to define the modality $[\neq]$ when non-strict semantics is assumed (the revised formula makes an essential use of the modal constant π for point-intervals). Thus, nominals over intervals can be simulated in PNL, and therefore this (basic) hybrid extension of PNL remains decidable over a large family of linear orders, including $\mathbb{N}$. However, it is quite easy to see that the addition of stronger hybrid features, such as binders or quantifiers over intervals, immediately leads to undecidability, even under very week assumptions about the class of linear orders.

In this chapter, we focus our attention on the addition of length binders to MPNL. Besides binders on state variables, ranging over intervals, one may introduce binders on integer variables, ranging over *interval lengths*. In its “classical” version, MPNL features metric constraints expressed by constants. As an example, $\Diamond_r(\ell_{=5} \wedge p \rightarrow \Diamond_l\Diamond_r q)$ is a well-formed MPNL formula, while $\Diamond_r(\ell_{=x} \wedge p)$, for some variable x, is not. This means that, despite the fact that MPNL can be considered a quite expressive interval logic (as witnessed by a number of meaningful examples in [18]), there are simple and natural properties that it cannot express, such as, for instance, *the right neighbor interval, whose length is equal to the length of the current interval, satisfies the property q*. To deal with properties like this one, we extend the language of MPNL with a sort of hybrid machinery making it possible to store the length of the current interval and to use it further in formulae.

Let us denote by MPNL+LB the hybrid extension of MPNL with length binders, which is defined as follows. First, we introduce a binder $\downarrow$, called *length binder*, a countable set of *length variables* $DVar = \{x, y, \ldots\}$, where $DVar \cap \mathcal{AP} = \emptyset$, and a set of *hybrid metric constraints* of the form $\ell_{\mathcal{C}x}$, for each $\mathcal{C} \in \{<, \leq, =, \geq, >\}$ and $x \in DVar$. Semantics of MPNL+LB formulae is defined as usual, pairing the classical valuation function for propositional letters with a *length assignment* $g : DVar \rightarrow \mathbb{N}$. An MPNL+LB model over $\mathbb{N}$ is a quadruple $M = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), V, g\rangle$, where $\mathbb{I}(\mathbb{N})$ is the interval structure on $\mathbb{N}$, $V : \mathcal{AP} \rightarrow 2^{\mathbb{I}(\mathbb{N})}$ is the valuation function for propositional letters, and g is the length assignment. For any pair of length assignments g, g' and any variable x, we write $g' \sim_x g$ to mean that g' possibly differs from g on the value of x only. Formally, formulae are defined by the following grammar:

$$\varphi ::= p \mid \ell_{\mathcal{C}\mathsf{k}} \mid \ell_{\mathcal{C}\mathsf{x}} \mid \neg\varphi \mid \psi \vee \varphi \mid \Diamond_r\varphi \mid \Diamond_l\varphi \mid \downarrow_x \varphi,$$

where $k \in \mathbb{N}$ and $x \in DVar$.

Let $M = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), V, g\rangle$. The semantic rules for MPNL+LB consist of those for MPNL plus the following clauses:

- $M, [a, b] \Vdash \ell_{\mathcal{C}\mathsf{x}}$ iff $\delta(a, b)\mathcal{C}g(x)$;
- $M, [a, b] \Vdash\downarrow_x \varphi$ iff $M', [a, b] \Vdash \varphi$ for $M' = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), V, g'\rangle$, where g' is a length assignment such that $g' \sim_x g$ and $g'(x) = \delta(a, b)$.

It is worth pointing out that a universal analogue of the hybrid operator @, with the following semantics:

- $M, [a, b] \Vdash @_x\varphi$ iff for any interval $[c, d]$ such that $\delta(c, d) = g(x)$ it is the case that $M, [c, d] \Vdash \varphi$.

can be easily defined in MPNL+LB as follows: $@_x\varphi := [G](\ell_{=\mathsf{x}} \to \varphi)$. The same holds for the existential analogue of @.

We show now that undecidability of almost all extensions of MPNL with length binders can be easily proved by a reduction from the satisfiability problem for undecidable fragments of HS, the only difficult case being that of the extension of MPNL with equality constraints over length variables ($\ell_{=\mathsf{x}}$, with $x \in DVar$), which will be dealt with in Section 5.3. As a matter of fact, we prove a stronger result showing that the fragment PNL+LB of MPNL+LB, devoid of atomic propositions for length constraints over constants ($\ell_{\mathcal{C}\mathsf{k}}$, with $k \in \mathbb{N}$), is already undecidable.

Unlike what happens with atomic propositions for length constraints over constants, that is, $\ell_{=\mathsf{k}}$, $\ell_{>\mathsf{k}}$, $\ell_{\geq\mathsf{k}}$, $\ell_{<\mathsf{k}}$, and $\ell_{\leq\mathsf{k}}$, with $k \in \mathbb{N}$, which are known to be definable in terms of each other, no general interdefinability rules are known for constraints over length variables. As an example, it can be easily shown that $\ell_{\geq\mathsf{x}}$ is equivalent to $\neg\ell_{<\mathsf{x}}$, but we are not aware of any way of expressing $\ell_{\leq\mathsf{x}}$ or $\ell_{<\mathsf{x}}$ in terms of $\ell_{=\mathsf{x}}$. The undecidability of full PNL+LB immediately follows from that of HS as HS operators $\langle B\rangle$, $\langle E\rangle$, $\langle \overline{B}\rangle$, and $\langle \overline{E}\rangle$, which suffice to define all other HS operators when non-strict semantics is assumed, can be easily defined in it:

$$\begin{array}{rcl}\langle B\rangle p & \equiv & \downarrow_x \Diamond_l\Diamond_r(p \wedge \ell_{<\mathsf{x}}),\\ \langle E\rangle p & \equiv & \downarrow_x \Diamond_r\Diamond_l(p \wedge \ell_{<\mathsf{x}}).\\ \langle \overline{B}\rangle p & \equiv & \downarrow_x \Diamond_l\Diamond_r(p \wedge \ell_{>\mathsf{x}}),\\ \langle \overline{E}\rangle p & \equiv & \downarrow_x \Diamond_r\Diamond_l(p \wedge \ell_{>\mathsf{x}}).\end{array}$$

Theorem 5.2.1. *The satisfiability problem for full* PNL+LB*, and thus that for full* MPNL+LB*, interpreted over* $\mathbb{N}$*, is undecidable.*

As shown in [16], the HS fragment featuring the pair of modalities $\langle B \rangle$ and $\langle E \rangle$ (resp., $\langle B \rangle$ and $\langle \overline{E} \rangle$, $\langle \overline{B} \rangle$ and $\langle E \rangle$, $\langle \overline{B} \rangle$ and $\langle \overline{E} \rangle$) only, interpreted over $\mathbb{N}$, is undecidable. Hence, the fragments of PNL+LB featuring only one length variable and only one (type of) constraint among $\{<, \leq, >, \geq\}$ are already undecidable.

5.2.2 First-Order extensions

Let us consider now a completely different extension of PNL, interpreted over natural numbers, which is obtained by lifting it to the first-order setting (we call the resulting logic PNL+FO). It essentially consists of the replacement of proposition letters by *predicate symbols* $P, Q, \ldots$ of fixed arity (proposition letters can be recovered as 0-ary predicate symbols), and the addition of a set of *individual variables* $x, y, \ldots$, a set of *individual constants* $c_1, c_2, \ldots$, that is, functions of arity 0 (for the sake of simplicity, we exclude function symbols of arity greater than 0), and the *universal (first-order) quantifier* $\forall$. *Terms* $\tau_1, \tau_2, \ldots$ are either individual variables or individual constants. As usual, the existential (first-order) quantifier can be defined in terms of the universal one as follows: $\exists x \varphi(x) \equiv \neg \forall x \neg \varphi(x)$, where x is an individual variable. A *first-order interval model* is a tuple $M = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), \mathfrak{D}, \mathcal{I} \rangle$, where $\langle \mathbb{N}, \mathbb{I}(\mathbb{N}) \rangle$ is an interval structure, $\mathfrak{D}$ is the *first-order domain* of M, and $\mathcal{I}$ is the *interpretation function*, mapping every interval of $\mathbb{I}(\mathbb{D})$ into a *first-order structure*:

$$\mathcal{I}([a, b]) = \langle \mathfrak{D}, P^{\mathcal{I}([a,b])}, Q^{\mathcal{I}([a,b])}, \ldots \rangle.$$

For every interval $[a, b]$ and predicate symbol P, $P^{\mathcal{I}([a,b])}$ is a relation on $\mathfrak{D}$ with the same arity as P (for proposition letters, it is simply *true* or *false*).

An *assignment* λ is a function that maps terms into elements of $\mathfrak{D}$. We assume constants to be *rigid*, that is, we assume that each constant refers to the same element of $\mathfrak{D}$ regardless of which is the current interval.

The set of semantic clauses for PNL+FO is obtained from that for PNL by adding the assignment as an additional parameter, by replacing the clause for proposition letters by a clause for predicates, and by introducing a clause for the universal quantifier:

- $M, [a, b], \lambda \Vdash P(\tau_1, \ldots, \tau_n)$ iff $P^{\mathcal{I}([a,b])}(\lambda(\tau_1), \ldots, \lambda(\tau_n))$, for each predicate symbol P;

- $M, [a, b], \lambda \Vdash \forall x \psi$ iff $M, [a, b], \lambda' \Vdash \psi$ for any assignment λ' that differs from λ at most for the value of x.

PNL+FO can thus be viewed as a *limited* first-order generalization of PNL: it allows one to move along the time domain by applying the modal operators and to formulate specific statements about what is true over a given interval by using first-order constructs.

We conclude the section by recalling some important contributions that address somehow related topics. In the first-order setting, there exist at least two important decidability results to mention: (i) the decidability of the two-variable fragment of first-order logic [13], and (ii) the decidability of the two-variable fragment of first-order logic interpreted over ordered domains, in particular, over the class of all linear orders and over $\mathbb{N}$ [90]. In the framework of temporal logics, as we have already pointed out, it has been shown that extending LTL (with Since and Until, but the result also applies to the fragment with Future and Next only) with a first-order machinery with two distinct variables yields undecidability [64]. To recover decidability, one must restrict the language by allowing one variable only. We will show that in the interval setting the situation is way worse. the addition of very elementary first-order ingredients to PNL suffices to cross the border of undecidability.

5.3 Undecidability of PNL+LB

As we have already shown in Section 5.2.1, the fragments of PNL+LB with only one length variable and only one type of constraint from the set $\{<, \leq, >, \geq\}$ are undecidable. In this section, we provide a reduction from the Finite Tiling Problem to the satisfiability problem for the fragment of PNL+LB with only one length variable and length constraints of the form $\ell_{=\mathsf{x}}$ only, thus proving its undecidability. In addition to those given in Section 5.2.1, this result allows one to conclude that extending PNL with length binders always leads to undecidability, even when only one length variable is used and regardless to the kind of length constraints allowed.

As a matter of fact, we will show that the use of π in the undecidability proof is unessential: we first give an encoding of Finite Tiling Problem that makes use of π, and then we show how to revise it to do without π.

The u-chain is defined by the following set of formulae:

$\mathsf{u} \wedge \mathsf{Start} \wedge \Box_r \neg \mathsf{Start} \wedge \Diamond_r \Diamond_r (\mathsf{Stop} \wedge \mathsf{u})$	(5.1)	*starts the* u*-chain*
$[G](\mathsf{u} \vee \mathsf{Start} \vee \mathsf{Stop} \rightarrow \ell_{=\mathsf{x}})$	(5.2)	Start, u, *and* Stop *are equal*
$[G](\Diamond_r \mathsf{Start} \rightarrow \Box_r(\neg\pi \rightarrow \Box_r \neg \mathsf{Start}))$	(5.3)	Start *is unique*
$[G](\Diamond_r \mathsf{Stop} \rightarrow \Box_r(\neg\pi \rightarrow \Box_r \neg \mathsf{Stop}))$	(5.4)	Stop *is unique*
$[G](\mathsf{u} \wedge \neg \mathsf{Stop} \rightarrow \Diamond_r \mathsf{u})$	(5.5)	u*-chain to the right*
$[G]((\mathsf{Start} \rightarrow \Box_l \Box_l \neg \mathsf{u}) \wedge (\mathsf{Stop} \rightarrow \Box_r \Box_r \neg \mathsf{u}))$	(5.6)	*no* u *out of the chain*
$(5.1) \wedge \ldots \wedge (5.6)$	(5.7)	

Lemma 5.3.1. *Let* $M = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), V, g \rangle$ *be a model for* PNL+LB *such that*

$$M, [a, b] \Vdash \downarrow_x (5.7).$$

Then, there exists a finite sequence of points $b_0 < b_1 < \ldots < b_k$*, with* $k > 0$*, such that* $b_0 = a$*,* $b_1 = b$*, and:*

1. *All intervals* $[b_i, b_{i+1}]$*, for* $0 \leq i \leq k-1$*, have the same length* $b - a > 0$.

2. $M, [b_i, b_{i+1}] \Vdash \mathsf{u}$ *for each* $0 \leq i \leq k-1$.

3. $M, [c, d] \Vdash \mathsf{u}$ *holds for no other interval* $[c, d]$.

Proof. First of all, by (5.1), the interval $[a, b](= [b_0, b_1])$ satisfies Start and it is not a point-interval as it satisfies $\Box_r\neg\mathsf{Start}$ as well. Moreover, by (5.2), all u-, Start-, and Stop-intervals have the same length (equal to $b - a$). Hence, two different Start-intervals (resp., Stop-intervals, u-intervals) cannot start at the same point. Then, from (5.3) (resp., (5.4)), it follows that the interval satisfying Start (resp., Stop) is unique. Next, by (5.5), the interval $[b_0, b_1]$ starts a finite chain of u-intervals $[b_i, b_{i+1}]$, with $0 \leq i \leq k-1$. The finiteness follows from the fact that, by (5.1), some future u-interval satisfies Stop and, by (5.6), there are no u-intervals starting to the right of it. Moreover, the (unique) Stop-interval must belong to the u-chain, otherwise, by (5.5), the u-chain would go beyond the Stop-interval, that is, there would be a u-interval following the Stop-interval, thus contradicting (5.6). Hence, the (unique) Stop-interval must be the last u-interval of the u-chain, that is, the interval $[b_{k-1}, b_k]$. To conclude the proof, suppose, by contradiction, that there exists a u-interval $[c, d]$ not belonging to the u-chain. By (5.6), it can be neither to the left of the Start-interval $[b_0, b_1]$ nor to the right of the Stop-interval $[b_{k-1}, b_k]$. Thus, it must start another chain of u-intervals, all of the same length $b-a$ (by (5.2)), whose u-intervals overlap those of the first u-chain. However, the unique interval satisfying Stop cannot belong to this second u-chain, and thus it will be crossed by it, leading to a contradiction with (5.6). ☐

We now define the Id-chain with the following formulae:

$[G]((\mathsf{u} \leftrightarrow \mathsf{tile} \vee *) \wedge (* \rightarrow \neg\mathsf{tile}))$	(5.8)	u *is either* tile *or* $*$
$[G]((\Diamond_r\mathsf{Start} \leftrightarrow \Diamond_r\mathsf{IdStart}) \wedge (\Diamond_l\mathsf{Stop} \leftrightarrow \Diamond_l\mathsf{IdStop}))$	(5.9)	*first and last* Id
$[G](\mathsf{IdStart} \vee \mathsf{IdStop} \rightarrow \mathsf{Id}) \wedge (\mathsf{IdStart} \rightarrow \neg\mathsf{IdStop})$	(5.10)	IdStart, IdStop *def*
$[G](\mathsf{Id} \rightarrow \ell_{=\mathsf{x}} \wedge \Diamond_r\Diamond_l\mathsf{tile})$	(5.11)	Id *s same length*
$[G](\Diamond_r\mathsf{Id} \leftrightarrow \Diamond_r *)$	(5.12)	Id *s start with* *
$[G]((\mathsf{Id} \wedge \neg\mathsf{IdStop} \rightarrow \Diamond_r\mathsf{Id}) \wedge (\mathsf{Id} \wedge \neg\mathsf{IdStart} \rightarrow \Diamond_l\mathsf{Id}))$	(5.13)	Id*-chain*
(5.8) $\wedge \ldots \wedge$ (5.13) $\wedge\ \mathsf{IdStart} \wedge \Box_r\neg\mathsf{IdStart}$	(5.14)	

Formula (5.14) guarantees that the interval satisfying IdStart, and hence any Id-interval, is not a point-interval.

Lemma 5.3.2. *Let* $M = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), V, g\rangle$ *be a model for* PNL+LB *such that*

$$M, [a, b] \Vdash \downarrow_x (5.7) \wedge \Diamond_l\Diamond_r \downarrow_x (5.14).$$

Then, there exist two positive integers h, v *and a finite sequence of points* $a = b_1^0 < b_1^1 < \ldots < b_1^h = b_2^0 < \ldots < b_2^h = b_3^0 < \ldots < b_{v-1}^h = b_v^0 < \ldots < b_v^h$ *such that for each* $1 \leq j \leq v$*, we have:*

1. $M, [b_j^0, b_j^1] \Vdash *$.
2. $M, [b_j^i, b_j^{i+1}] \Vdash \mathsf{tile}$ *for each* $0 < i < h$.
3. $M, [b_j^0, b_j^h] \Vdash \mathsf{Id}$.

Moreover, no other interval satisfies $*$, tile, *or* Id.

Proof. First of all, by Lemma 5.3.1, there is a finite sequence of points $a = b_0 < b_1 < \ldots < b_k$, which defines a finite chain of u-intervals. By (5.8), each of these u-intervals is either a $*$-interval or a tile-interval and no other interval is a $*$-interval or a tile-interval. Furthermore, by (5.12), every $*$-interval starts an Id-interval and every Id-interval starts with a $*$-interval, and, by (5.11), every Id-interval ends with a tile-interval. Thus, every Id-interval spans several u-intervals. Therefore, there must be finitely many Id-intervals. Let v be the number of Id-intervals. The first u-interval $[b_0, b_1]$, which is also the unique Start-interval, starts an Id-interval, say it $[b_0, b_h]$, for some $h < k$, that satisfies IdStart, by (5.9). The unique Stop-interval, which is the last u-interval, ends the last Id-interval, also labeled by IdStop, again by (5.9). Since all u-intervals have the same length $b_1 - b_0 (= b - a)$ and all Id-intervals have the same length $b_h - b_0$, every Id-interval spans exactly h u-intervals. Hence, the sequence $b_0 < b_1 < \ldots < b_k$ can be written as $b_1^0 < b_1^1 < \ldots < b_1^h = b_2^0 < \ldots < b_2^h = b_3^0 < \ldots < b_{v-1}^h = b_v^0 < \ldots < b_v^h$, as required. Now, points 1, 2, and 3 of the lemma immediately follow. To complete the proof, we only need to show that no other interval satisfies Id, $*$, or tile. Let us focus on the propositional letter Id. By (5.13), we have that every Id-interval, different from the IdStop-interval, starts a chain of Id-intervals. By (5.11) and (5.8), such a chain must terminate. By (5.13), it must end with the unique IdStop-interval, which, in its turn, ends with the unique Stop-interval. Similarly, by (5.9), the first Id-interval starts with the first u-interval, which is the unique Start-interval. Thus, no other Id-interval exists in M, but those of the type $[b_j^0, b_j^h]$. The fact that there are no other $*$- and tile-intervals immediately follows. □

The above lemma guarantees the existence of a unique Id-chain. Now, we want to force the propositional letter up_rel to correctly encode the relation that connects pairs of tiles of the rectangle that are vertically adjacent. We do it with the following set of formulae.

$[G](\mathsf{up\_rel} \rightarrow \ell_{=\mathsf{x}} \wedge \Diamond_l \Diamond_r \mathsf{tile})$	(5.15)	up_rel *and* Id *are equally long*
$[G](\mathsf{tile} \rightarrow (\Diamond_r \Diamond_r \mathsf{IdStop} \leftrightarrow \Diamond_l \Diamond_r \mathsf{up\_rel}))$	(5.16)	tile *starts* up_rel
(5.15) $\wedge$ (5.16)	(5.17)	

Lemma 5.3.3. *Let* $M = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), V, g \rangle$ *be a model for* PNL+LB *such that*

$$M, [a, b] \Vdash \downarrow_x (5.7) \wedge \Diamond_l \Diamond_r \downarrow_x ((5.14) \wedge (5.17))$$

and let $a = b_1^0 < b_1^1 < \ldots < b_1^h = b_2^0 < \ldots < b_2^h = b_3^0 < \ldots < b_v^0 < \ldots < b_v^h$ be the sequence of points whose existence is guaranteed by Lemma 5.3.2. Then, for each $1 \leq j < v$ and $1 \leq i < h$, the interval $[b_j^i, b_{j+1}^i]$ satisfies up_rel*, and no other interval satisfies* up_rel*.*

Proof. By (5.15), we have that up_rel-intervals have the same length of ld-intervals ((5.14) and (5.17) are in the scope of the same length binder $\downarrow_x$). By (5.16), each tile-interval, but the ones belonging to the last ld-interval, starts a up_rel-interval. Finally, by (5.15), each up_rel-interval is started by a tile-interval. Given that the length of all u-intervals is the same and every ld-interval spans the same number of u-intervals, the claim follows immediately from Lemma 5.3.2. □

Finally, we can force all color-matching conditions to be fulfilled by means of the following set of formulae, where $\mathcal{T}_r$ (resp., $\mathcal{T}_l$, $\mathcal{T}_u$, $\mathcal{T}_d$) is the subset of $\mathcal{T}$ in which all tiles have the right (resp., left, up, down) side colored with \$.

$$[G](((\mathsf{tile} \wedge \Diamond_r *) \vee (\mathsf{tile} \wedge \mathsf{Stop})) \leftrightarrow \mathsf{Rtile}) \quad (5.18) \quad \textit{right side tiles}$$

$$[G](\mathsf{tile} \wedge \Diamond_l * \leftrightarrow \mathsf{Ltile}) \quad (5.19) \quad \textit{left side tiles}$$

$$[G]((\mathsf{tile} \leftrightarrow \bigvee_{t_q \in \mathcal{T}} \mathtt{t_q}) \wedge \bigwedge_{t_q, t_u \in \mathcal{T}, t_q \neq t_u} \neg(\mathtt{t_q} \wedge \mathtt{t_u})) \quad (5.20) \quad \textit{tiles are tiles}$$

$$[G](\mathsf{tile} \wedge \Diamond_r \mathsf{tile} \rightarrow \bigvee_{right(t_q)=left(t_u)} (\mathtt{t_q} \wedge \Diamond_r \mathtt{t_u})) \quad (5.21) \quad \textit{right-left constraint}$$

$$[G](\mathsf{up\_rel} \rightarrow \bigvee_{up(t_q)=down(t_u)} (\Diamond_l \Diamond_r \mathtt{t_q} \wedge \Diamond_r \mathtt{t_u})) \quad (5.22) \quad \textit{up-down constraint}$$

$$[G](\Diamond_l \mathsf{ldStart} \rightarrow \Box_l \Box_l (\mathsf{tile} \rightarrow \bigvee_{t_q \in \mathcal{T}_d} \mathtt{t_q})) \quad (5.23) \quad \textit{down side constraint}$$

$$[G](\Diamond_r \mathsf{ldStop} \rightarrow \Box_r \Box_r (\mathsf{tile} \rightarrow \bigvee_{t_q \in \mathcal{T}_u} \mathtt{t_q})) \quad (5.24) \quad \textit{up side constraint}$$

$$[G](\mathsf{Ltile} \rightarrow \bigvee_{t_q \in \mathcal{T}_l} \mathtt{t_q}) \quad (5.25) \quad \textit{left side constraint}$$

$$[G](\mathsf{Rtile} \rightarrow \bigvee_{t_q \in \mathcal{T}_r} \mathtt{t_q}) \quad (5.26) \quad \textit{right side constraint}$$

$$(5.18) \wedge \ldots \wedge (5.26) \quad (5.27)$$

Theorem 5.3.4. *Given any finite set of tile types $\mathcal{T}$ and a side color \$, the formula*

$$\Phi :=\downarrow_x (5.7) \wedge \Diamond_l \Diamond_r \downarrow_x ((5.14) \wedge (5.17) \wedge (5.27))$$

is satisfiable in $\mathbb{N}$ if and only if $\mathcal{T}$ can tile some finite rectangle $\mathcal{R}$ with side color \$.

Proof. (Only if:): Suppose that $M, [a, b] \Vdash \Phi$. Then, by Lemma 5.3.2, there is a sequence of points $a = b_1^0 < b_1^1 < \ldots < b_1^h = b_2^0 < \ldots < b_2^h = b_3^0 < \ldots < b_v^0 < \ldots <$

$b_v^h = b_k$. We put $X = h - 1$ and $Y = v$. We have that $M, [b_r^s, b_r^{s+1}] \Vdash \mathsf{tile}$ if and only if $s > 0$, and thus, by (5.20), $M, [b_r^s, b_r^{s+1}] \Vdash \mathtt{t_q}$ for a unique $\mathtt{t_q}$. Now, for all s, r, where $1 \leq s \leq X$, $1 \leq r \leq Y$, define $f(s, r) = t_q$ if and only if $M, [b_r^s, b_r^{s+1}] \Vdash \mathtt{t_q}$. From Lemma 5.3.1, Lemma 5.3.2, and Lemma 5.3.3, it follows that the function $f : \mathcal{R} \to \mathcal{T}$ defines a correct tiling of $\mathcal{R} = \{(x, y) \mid 1 \leq x \leq X \text{ and } 1 \leq y \leq Y\}$.

(If:) Let $f : \mathcal{R} \to \mathcal{T}$ define a correct tiling of the rectangle $\mathcal{R} = \{(x, y) \mid 1 \leq x \leq X \text{ and } 1 \leq y \leq Y\}$ for some X, Y, and a given border color \$. We will show that there exists a model M and an interval $[a, b]$ such that $M, [a, b] \Vdash \Phi$. Let $n = (X+1) \cdot Y$, we define a model $M = \langle \mathbb{N}, \mathbb{I}(\mathbb{N}), V, g \rangle$ such that $M, [0, 1] \Vdash \Phi$. Since the only length variable occurring in Φ is x and it has no free occurrences there, any valuation of x would be as good as any other, so we put $g(x) = 1$. The valuation function is defined as follows.

$$\begin{array}{rcl} V(\mathsf{u}) & = & \{[i, i+1] \mid 0 \leq i < n\}; \\ V(\mathsf{Start}) & = & \{[0, 1]\}; \\ V(\mathsf{Stop}) & = & \{[n-1, n]\}. \end{array}$$

This guarantees that $\downarrow_x$ (5.7) is satisfied. Now, in order to satisfy the remaining part of Φ on $[0, 1]$, it suffices to show that the formula $\downarrow_x ((5.14) \wedge (5.17) \wedge (5.27))$ can be satisfied on the interval $[0, X+1]$, i.e., $(5.14) \wedge (5.17) \wedge (5.27)$ can be satisfied on $[0, X+1]$ by a valuation assigning value $X+1$ to the length variable x. In the following, we define the valuation for the remaining propositional letters:

$$\begin{array}{rcl} V(\mathsf{Id}) & = & \{[i \cdot (X+1), (i+1) \cdot (X+1)] \mid 0 \leq i < Y\}; \\ V(*) & = & \{[i \cdot (X+1), i \cdot (X+1) + 1] \mid 0 \leq i < Y\}; \\ V(\mathsf{tile}) & = & V(\mathsf{u}) \setminus V(*); \\ V(\mathsf{IdStart}) & = & \{[0, X+1]\}; \\ V(\mathsf{IdStop}) & = & \{[(X+1) \cdot (Y-1), (X+1) \cdot Y]\}; \\ V(\mathsf{up\_rel}) & = & \{[i, j] \mid \delta(i, j) = X+1, [i, j] \notin V(\mathsf{Id}), 0 \leq i, j < n\}; \\ V(\mathsf{Ltile}) & = & \{[i \cdot (X+1) + 1, i \cdot (X+1) + 2] \mid 0 \leq i < Y\}; \\ V(\mathsf{Rtile}) & = & \{[i \cdot (X+1) - 1, i \cdot (X+1)] \mid 0 < i \leq Y\}. \end{array}$$

Finally, we evaluate the propositional letters of the set $\mathtt{T} = \{\mathtt{t_1}, \ldots \mathtt{t_k}\}$, also called *tile-variables*, as follows. For each $\mathtt{t} \in \mathtt{T}$:

$$V(\mathtt{t}) := \{[i + (j-1) \cdot (X+1), i + (j-1) \cdot (X+1) + 1)] \mid f(i, j) = t\}.$$

It is straightforward to check that $M, [0, 1] \Vdash \Phi$, hence the claim.

□

From Theorem 5.3.4, the undecidability of the fragment of PNL+LB with only one length variable and length constraints of the form $\ell_{=\mathsf{x}}$ only immediately follows.

We conclude the section by showing that the removal of the modal constant π does not suffice to recover decidability.

First of all, we observe that the modal constant π is used in formulae (5.3) and (5.4) only, to force the uniqueness of the u-intervals satisfying Start and Stop, and consequently the uniqueness of the u-chain. We prove that the uniqueness of Stop can also be forced by the following formulae that make no use of π:

$$[G](\mathsf{Stop} \rightarrow \mathsf{u} \wedge \Box_r \Box_r \neg \mathsf{Stop}) \tag{5.28}$$

$$[G](\mathsf{Stop} \rightarrow \Diamond_l (\mathsf{u} \wedge \Box_l \Box_r (\Diamond_r \mathsf{Stop} \leftrightarrow \mathsf{u}))). \tag{5.29}$$

By contradiction, let us assume that there exist two distinct Stop-intervals, say, $[a, b]$ and $[c, d]$. Since Stop-intervals have the same length, it must hold that $a \neq c$. Without loss of generality, we assume $a < c$, and thus $b < d$. Two cases are possible. If $b \leq c$, then formula (5.28), over $[a, b]$, is false. Otherwise, if $c < b$, then $[a, b]$ overlaps $[c, d]$ $(a < c < b < d)$, and formula (5.29), over $[c, d]$, is false. Indeed, consider the u-interval immediately to the left of the Stop-interval $[c, d]$, say it $[c', c]$. Since u-intervals have the same length of Stop-intervals, $c' < a$. Moreover, by (5.29), $[c', c]$ must satisfy $\Box_l \Box_r (\Diamond_r \mathsf{Stop} \leftrightarrow \mathsf{u})$. In particular, $[c', a]$ must satisfy $\Diamond_r \mathsf{Stop} \leftrightarrow \mathsf{u}$. However, $[c', a]$ satisfies $\Diamond_r \mathsf{Stop}$, but it is not a u-interval as it is shorter than $[c', c]$, which is a u-interval (contradiction). In a completely symmetric way, it possible to guarantee the uniqueness of Start-intervals.

Theorem 5.3.5. *The satisfiability problem for the fragment of* PNL+LB *devoid of the modal constant π and with one length variable only is undecidable (over natural numbers).*

5.3.1 Undecidability in the strict semantics

Another minor modification of some of the formulae above can reduce the Finite Tiling Problem to the satisfiability problem of the logic PNL+LB interpreted over $\mathbb{N}$ with strict semantics, thus excluding point intervals. Essentially the only necessary changes in the formulae used in the encoding of the tiling problem are to replace formulae of the type $\Box_r \Box_r \psi$ with $\Box_r \psi \wedge \Box_r \Box_r \psi$, likewise $\Box_l \Box_l \psi$ with $\Box_l \psi \wedge \Box_l \Box_l \psi$, and, respectively, $\Diamond_r \Diamond_r \psi$ with $\Diamond_r \psi \wedge \Diamond_r \Diamond_r \psi$, likewise $\Diamond_l \Diamond_l \psi$ with $\Diamond_l \psi \wedge \Diamond_l \Diamond_l \psi$. The rest is essentially the same, save for the fact that the complications coming from the point intervals will now disappear, as in the strict semantics it is directly possible to define the nominals.

5.4 Undecidability of (R)PNL+FO

As it becomes clear from what we said at the beginning, there are a number of possible parameters to be set for PNL+FO. Beside the usual possible choices for the temporal domain, that is, discrete, dense, finite, bounded, unbounded, and so on, we can vary on the first-order component by assuming that the first-order domain is finite, infinite, constant, variable, expanding, or assuming other specific properties for it (linearity, discreteness, denseness, and so on), and also by limiting the number of distinct variables in formulae. Since we are interested in tight undecidability results, in contrast with decidability results for first-order point-based temporal logic, we focus our attention on very restrictive assumptions.

Actually we extend the one-modality fragment of PNL, namely RPNL, with a very small first-order language, with countable and constant first-order domain $\mathfrak{D}$, only one variable, no first-order constants and no free variables in formulae. The encoding is given over a finite linear order. Nevertheless, the results presented in this section hold even over the linear order based on $\mathbb{N}$. For sake of readability, in the following construction we omit the variable assignment λ.

The proof hinges on the fact that introducing first-order constructs makes it possible to express properties of the following type: "*if an interval satisfies φ, then all its beginning intervals (resp., ending intervals, strict sub-intervals) do not satisfy ψ*", where the strict sub-intervals of an interval $[a,b]$ are all intervals $[c,d]$ such that $a < c < d < b$. In order to express such properties, we force a predicate of the type $P(x)$ in such a way that if $P(x)$ is true, for some x, over an interval $[a,b]$, then it can be possibly true (for the same x) only over intervals that start from a and it must be false over all intervals starting from some different point $c \neq a$. For example, given an interval $[a,b]$ that satisfies $P^{\mathcal{I}([a,b])}(C)$, for some constant C, we force $\neg P^{\mathcal{I}([c,d])}(C)$ to hold over each interval $[c,d]$, with $c \neq a$. To this end, we exploit the following formula:

$$\Box_r\Box_r(\exists x \Diamond_r P(x) \wedge \forall x(\Diamond_r P(x) \rightarrow \Box_r(\neg\pi \rightarrow \Box_r \neg P(x)))). \qquad (5.30)$$

Lemma 5.4.1. *Let M be a model for* RPNL+FO *and $[a,b]$ any interval on it. If*

$$M,[a,b] \Vdash (5.30),$$

then, for each $c \in D$ such that $c \geq b$ there exists a point $d \geq c$ such that $P^{\mathcal{I}([c,d])}(C)$ holds for some $C \in \mathfrak{D}$, and for any $e, f > c$, $\neg P^{\mathcal{I}([e,f])}(C)$ holds.

Proof. Suppose that $M,[a,b] \Vdash (5.30)$. Now, consider any point $c \geq b$ and any interval $[c,d]$; by the semantics of $\Box_r\Box_r$, the formula $\exists x \Diamond_r P(x) \wedge \forall x(\Diamond_r P(x) \rightarrow \Box_r(\neg\pi \rightarrow \Box_r \neg P(x)))$ is true on some interval ending at c. Therefore, there must be some element of $\mathfrak{D}$, let us call it C, such that $P(C)$ is true on $[c,d]$. Since also the formula $\forall x(\Diamond_r P(x) \rightarrow \Box_r(\neg\pi \rightarrow \Box_r \neg P(x)))$ is true on some interval ending at c, and since $P(C)$ is true on $[c,d]$, then it must hold for every $[e,f]$ with $c < e$ that $P(C)$ is false on $[e,f]$. □

With the help of the above formula, we can express properties about beginning intervals, ending intervals, and strict sub-intervals of any given interval. For example, it is easy to see that the following formulae correctly define the operator $[B^{\varphi}_{\psi}]$ (resp., $[E^{\varphi}_{\psi}]$, $[D^{\varphi}_{\psi}]$), expressing the property: "*if an interval satisfies the property φ, then each beginning interval (resp., ending interval, strict sub-interval) satisfies the property ψ*", thus 'simulating' the modal operator $[B]$ (resp., $[E]$, $[D]$) of the logic

HS, corresponding to the (universal version of) Allen's relation *begins* (resp., *ends*, *during*):

$$[B^{\varphi}_{\psi}] \equiv \Box_r\Box_r\forall x(\Diamond_r(\varphi \wedge \Diamond_r P(x)) \rightarrow \Box_r(\Diamond_r(\neg\pi \wedge \Diamond_r P(x)) \rightarrow \psi))$$
$$[E^{\varphi}_{\psi}] \equiv \Box_r\Box_r\forall x(\Diamond_r(\varphi \wedge \Diamond_r P(x)) \rightarrow \Box_r(\neg\pi \rightarrow \Box_r(\Diamond_r P(x) \rightarrow \psi)))$$
$$[D^{\varphi}_{\psi}] \equiv \Box_r\Box_r\forall x(\Diamond_r(\varphi \wedge \Diamond_r P(x)) \rightarrow \Box_r(\neg\pi \rightarrow \Box_r(\Diamond_r(\neg\pi \wedge \Diamond_r P(x)) \rightarrow \psi)))$$

Notice that we are not able to properly define the HS operators $[B]$, $[E]$, and $[D]$, since we cannot capture beginning, ending, and during intervals of the current one.

To define the u-chain we use the following formulae:

$$\neg\mathsf{u} \wedge \Diamond_r(\neg\pi \wedge \mathsf{u}) \quad (5.31) \quad \textit{starts the u-chain}$$
$$\Box_r\Box_r(\mathsf{u} \rightarrow (\neg\pi \wedge (\Diamond_r\mathsf{u} \vee \Box_r\pi))) \quad (5.32) \quad \textit{completes the u-chain}$$
$$[B^{\mathsf{u}}_{\neg\mathsf{u}}] \wedge [B^{\mathsf{u}}_{\neg\pi\rightarrow\neg\Diamond_r\mathsf{u}}] \quad (5.33) \quad \textit{makes the u-chain unique}$$
$$(5.30) \wedge (5.31) \wedge (5.32) \wedge (5.33) \quad (5.34)$$

Lemma 5.4.2. *Let* $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathfrak{D}, \mathcal{I}\rangle$ *be a model for* RPNL+FO *based on a finite linearly ordered temporal domain such that*

$$M, [a, b] \Vdash (5.34).$$

Then, there exists a finite sequence of points $b = b_0 < b_1 < \ldots < b_n$*, with* $n > 0$*, such that:*

1. $M, [b_l, b_{l+1}] \Vdash \mathsf{u}$ *for each* $0 \leq l \leq n-1$*;*
2. $M, [c, d] \Vdash \mathsf{u}$ *holds for no other interval* $[c, d]$*, unless* $c < b$*.*

Proof. If $M, [a, b] \Vdash (5.34)$, then, by (5.31), for some $c > b$ the interval $[b, c]$ is a u-interval. By (5.32), $b(= b_0)$ starts a finite chain of u-intervals $[b_l, b_{l+1}]$, with $l \geq 0$. The satisfiability of (5.32) over finite temporal domains follows from the fact that the last point of the temporal domain satisfies $\Box_r\pi$. Now suppose, by contradiction, that for some interval $[c, d]$, it is the case that $[c, d]$ is a u-interval but $[c, d] \neq [b_l, b_{l+1}]$ for any $l \geq 0$. Then, either $c = b_l$ for some l, contradicting the first conjunct of (5.33), or $b_l < c < b_{l+1}$, contradicting the second conjunct of (5.33). ☐

We now define the Id-chain with the following formulae:

$$\neg\mathsf{Id} \wedge \Diamond_r\mathsf{Id} \wedge \Box_r\Box_r((\Diamond_r\mathsf{Id} \rightarrow \Diamond_r\mathsf{u})\wedge$$
$$(\mathsf{Id} \rightarrow \neg\pi \wedge \neg\mathsf{u} \wedge (\Diamond_r\mathsf{Id} \vee \Box_r\pi))) \quad (5.35) \quad \textit{constructs the Id-chain}$$
$$[B^{\mathsf{Id}}_{\neg\mathsf{Id}}] \wedge [B^{\mathsf{Id}}_{\neg\pi\rightarrow\neg\Diamond_r\mathsf{Id}}] \quad (5.36) \quad \textit{makes the u-chain unique}$$
$$(5.35) \wedge (5.36) \quad (5.37)$$

Lemma 5.4.3. *Let $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathfrak{D}, \mathcal{I} \rangle$ be a model for* RPNL+FO *based on a finite linearly ordered temporal domain, such that*

$$M, [a, b] \Vdash (5.34) \wedge (5.37).$$

Then, there exist a positive integer v, a finite sequence of positive integers $m_1, \ldots, m_v$, and a finite sequence of points $b_0^1 < b_1^1 < \ldots < b_{m_1}^1 = b_0^2 < \ldots < b_{m_2}^2 = \ldots = b_0^{v-1} < \ldots < b_{m_{v-1}}^{v-1} = b_0^v < \ldots < b_{m_v}^v$ such that, for each $1 \leq s \leq v$, we have $M, [b_0^s, b_{m_s}^s] \Vdash$ Id, and no other interval $[c, d]$ satisfies Id*, unless $c < b$.*

Proof. First of all, by Lemma 5.4.2, there is a finite sequence of points $b_0 < b_1 < \ldots < b_n$, with $n > 0$, which defines a finite chain of u-intervals. By (5.35), b_0 starts a Id-interval, which must end at some $b_l > b_1$. By (5.35) again, each Id-interval that does not end at the last point of the linear order is followed by another Id-interval, and each Id-interval must end at some b_l. Thus, every Id-interval spans several u-intervals, and there are finitely many Id-intervals. Let their number be v. Hence, the sequence $b_0 < b_1 < \ldots < b_n$ can be written as $b_0^1 < b_1^1 < \ldots < b_{m_1}^1 = b_0^2 < \ldots < b_{m_2}^2 = \ldots = b_0^{v-1} < \ldots < b_{m_{v-1}}^{v-1} = b_0^v < \ldots < b_{m_v}^v$, as required. We want to show that there are no other Id-interval beside those of the type $[b_0^s, b_{m_s}^s]$. This can be shown exactly as in Lemma 5.4.2, by using (5.36). □

The above lemma guarantees the existence of an Id-chain. Now, we want to force the propositional letter up_rel to correctly encode the relation that connects pairs of tiles of the rectangle that are vertically adjacent. Formally, we define two u-intervals $[b_l, b_{l+1}]$ and $[b_{l'}, b_{l'+1}]$ to be *above-connected* if and only if $[b_{l+1}, b_{l'}]$ is an up_rel-interval. At the same time, we want to make sure that each Id-interval spans the same number of tile-intervals. Intuitively, these two properties can be guaranteed by assuring that each u-interval of an Id-interval is connected with exactly one u-interval of the next Id-interval and with exactly one Id-interval of the previous level. To this end, firstly we suitably label u-intervals belonging to the last Id-interval with the propositional letter Final. Then, we constraint each u-interval not belonging to the last Id-interval to be connected to at least one u-interval in the future (formula (5.39)) and at least one interval in the past (formula (5.45)) by means of an up_rel-interval. In order to guarantee the correct correspondence between u-intervals of consecutive Id-intervals and to guarantee that each u-interval is connected with at most one u-interval in the future and at most one u-interval in the past, we force the condition that no up_rel-interval is a beginning interval (resp., ending, strict sub-interval) of any other up_rel-interval. Finally, in order to guarantee that up_rel-intervals connect u-intervals belonging to consecutive Id-intervals, we have to make sure that no Id-interval is a beginning interval (resp., ending interval, strict sub-interval, strict super-interval) of an up_rel-interval.

$\Box_r\Box_r(\mathsf{u} \wedge \Box_r\Box_r\neg\mathsf{Id} \leftrightarrow \mathsf{Final})$	(5.38)	*sets* Final *at the end*
$\Box_r\Box_r(\mathsf{u} \rightarrow (\neg\mathsf{Final} \leftrightarrow \Diamond_r\mathsf{up\_rel}))$	(5.39)	*starts the* up_rel*-chain*
$\Box_r\Box_r(\mathsf{up\_rel} \rightarrow \neg\mathsf{Id} \wedge \neg\pi \wedge \neg\mathsf{u} \wedge \Diamond_r\mathsf{u})$	(5.40)	up_rel *spans various* us
$\neg\mathsf{up\_rel} \wedge \neg\Diamond_r\mathsf{up\_rel} \wedge \Box_r\Box_r(\Diamond_r\mathsf{up\_rel} \rightarrow \Diamond_r\mathsf{u})$	(5.41)	up_rel *starts with a* u
$[B^{\mathsf{up\_rel}}_{\neg\mathsf{up\_rel}}] \wedge [E^{\mathsf{up\_rel}}_{\neg\mathsf{up\_rel}}] \wedge [D^{\mathsf{up\_rel}}_{\neg\mathsf{up\_rel}}]$	(5.42)	up_rel*s are unique*
$[B^{\mathsf{up\_rel}}_{\neg\mathsf{Id}}] \wedge [E^{\mathsf{up\_rel}}_{\neg\mathsf{Id}}] \wedge [D^{\mathsf{up\_rel}}_{\neg\mathsf{Id}}]$	(5.43)	*the* up_rel*-chain...*
$[D^{\mathsf{Id}}_{\neg\mathsf{up\_rel}}]$	(5.44)	*...overlaps the* Id*-chain*
$\forall x(\Diamond_r(\mathsf{Id} \wedge \Diamond_r(\Diamond_r\mathsf{u} \wedge \Diamond_r P(x))) \rightarrow$ $\Diamond_r\Diamond_r(\mathsf{up\_rel} \wedge \Diamond_r P(x)))$	(5.45)	*no* tile *is skipped*
(5.38) $\wedge \ldots \wedge$ (5.45)	(5.46)	

Lemma 5.4.4. *Let $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathfrak{D}, \mathcal{I}\rangle$ be a model for* RPNL+FO *based on a finite linearly ordered temporal domain such that*

$$M, [a, b] \Vdash (5.34) \wedge (5.37) \wedge (5.46)$$

and let $b^1_0 < b^1_1 < \ldots < b^1_{m_1} = b^2_0 < \ldots < b^2_{m_2} = \ldots = b^{v-1}_0 < \ldots < b^{v-1}_{m_{v-1}} = b^v_0 < \ldots < b^v_{m_v}$ be the sequence of point guaranteed by Lemma 5.4.3. Then, we have that, for each $0 < s < v$ and each $0 \le l < m_s$, $M, [b^s_{l+1}, b^{s+1}_l] \Vdash$ up_rel*, and no other interval $[c, d]$ satisfies* up_rel*, unless $c < b$. Moreover, we have that for each $1 \le s < s' \le v$, $m_s = m_{s'}$.*

Proof. Consider any u-interval $[b^s_l, b^s_{l+1}]$ not belonging to the last Id-interval. Formula (5.39) guarantees that b^s_{l+1} starts an up_rel-interval, which cannot be point-interval and must end at some point of the type $b^{s'}_{l'} > b^s_{l+2}$. Now, observe that $b^{s'}_{l'} \ge b^{s+1}_0$, otherwise we would have a contradiction with (5.44). Similarly, we have that $b^{s'}_{l'} < b^{s+1}_{m_{s+1}}$, in order to avoid a contradiction with (5.43). Suppose by contradiction that $[b^s_0, b^s_1]$ is above-connected with $[b^{s+1}_l, b^{s+1}_{l+1}]$, with $l > 0$, for some s. By (5.45), there must be an up_rel-interval ending in b^{s+1}_0 and starting from a point $b^s_{l'}$, with $l' > 0$. It must also be $l' > 1$, otherwise there would be two different up_rel-intervals starting at the same point b^s_1, contradicting the first conjunct of (5.42). So, it must be that the up_rel-interval $[b^s_{l'}, b^{s+1}_0]$ is a strict sub-interval of the up_rel-interval $[b^s_1, b^{s+1}_l]$, contradicting the third conjunct of (5.42). By applying a similar argument, and assuming that, up to a given l, $[b^s_l, b^s_{l+1}]$ is above-connected to $[b^{s+1}_l, b^{s+1}_{l+1}]$, it is easy to show also that $[b^s_{l+1}, b^s_{l+2}]$ (if there exists such an interval) is above-connected to $[b^{s+1}_{l+1}, b^{s+2}_{l+2}]$. From (5.42) it follows that each u-interval can be connected with at most one u-interval in the future and at most one in the past, so we can conclude that for each $0 \le s < s' \le v$, $m_s = m_{s'}$. ∎

Finally, we can force all tile-matching conditions to be respected, by using the following formulae, where $\mathcal{T}_r$ (resp., $\mathcal{T}_l$, $\mathcal{T}_u$, $\mathcal{T}_d$) is the subset of $\mathcal{T}$ containing all tiles having the right (resp., left, up, down) side colored with \$.

$$\Box_r\Box_r(\mathsf{u} \rightarrow \bigvee_{t_q\in\mathcal{T}} \mathtt{t_q} \wedge \bigwedge_{t_q\neq t_{q'}} \neg(\mathtt{t_q} \wedge \mathtt{t_{q'}})) \quad (5.47) \quad \textit{puts the tiles}$$

$$\Box_r\Box_r(\bigvee_{t_q\in\mathcal{T}} \mathtt{t_q} \rightarrow \mathsf{u}) \quad (5.48) \quad \textit{tiles are only } \mathsf{us}$$

$$\Box_r\Box_r(\mathsf{u} \wedge \neg\Diamond_r\mathsf{Id} \wedge \neg\Box_r\pi \rightarrow \bigvee_{right(t_q)=left(t_{q'})}(\mathtt{t_q} \wedge \Diamond_r\mathtt{t_{q'}})) \quad (5.49) \quad \textit{right-left constraint}$$

$$\Box_r\Box_r(\mathsf{u} \wedge \Diamond_r\mathsf{up\_rel} \rightarrow \bigvee_{up(t_q)=down(t_{q'})}(\mathtt{t_q} \wedge \Diamond_r(\mathsf{up\_rel} \wedge \Diamond_r\mathtt{t_{q'}}))) \quad (5.50) \quad \textit{up-down constraint}$$

$$\Box_r\Box_r(\Diamond_r\mathsf{Id} \rightarrow (\Diamond_r \bigvee_{t_q\in\mathcal{T}_l} \mathtt{t_q}) \wedge (\mathsf{u} \rightarrow \bigvee_{t_q\in\mathcal{T}_r} \mathtt{t_q})) \quad (5.51) \quad \textit{left and right side}$$

$$\forall x(\Diamond_r(\mathsf{Id} \wedge \Diamond_r P(x)) \rightarrow \Box_r\Box_r(\mathsf{u} \wedge \Diamond_r\Diamond_r P(x) \rightarrow \bigvee_{t_q\in\mathcal{T}_d} \mathtt{t_q})) \quad (5.52) \quad \textit{down side}$$

$$\Box_r\Box_r(\mathsf{u} \wedge \mathsf{Final} \rightarrow \bigvee_{t_q\in\mathcal{T}_u} \mathtt{t_q}) \quad (5.53) \quad \textit{up side}$$

$$(5.47) \wedge \ldots \wedge (5.53) \quad (5.54)$$

Theorem 5.4.5. *Given any finite set of tiles $\mathcal{T}$ and a side color \$, the formula*

$$\Phi_{\mathcal{T}} := (5.34) \wedge (5.37) \wedge (5.46) \wedge (5.54)$$

is satisfiable in a finite linearly ordered temporal domain if and only if $\mathcal{T}$ can tile a finite rectangle $\mathcal{R} = \{(x, y) \mid 1 \leq x \leq X$ and $1 \leq y \leq Y\}$, for some $X, Y \in \mathbb{N}$, with side color \$.

Proof. (Only if:): Suppose that $M, [a, b] \Vdash \Phi_{\mathcal{T}}$. Then, by Lemma 5.4.3, there is a sequence of points $b_0 = b_0^1 < b_1^1 < \ldots < b_{m_1}^1 = b_0^2 < \ldots < b_{m_2}^2 = \ldots = b_0^{v-1} < \ldots < b_{m_{v-1}}^{v-1} = b_0^v < \ldots < b_{m_v}^v = b_n$, and by Lemma 5.4.4, for each $1 \leq s < s' \leq v$, $m_s = m_{s'}$. We put $X = m_s$ and $Y = v$. For all l, s, where $0 \leq l \leq X-1$, $1 \leq s \leq Y$, define $f(l+1, s) = t_q$ if and only if $M, [b_l^s, b_{l+1}^s] \Vdash \mathtt{t_q}$. From Lemma 5.4.2, 5.4.3, and 5.4.4 it follows that the function $f : \mathcal{R} \rightarrow \mathcal{T}$ defines a correct tiling of $\mathcal{R}$.

(If:) Let $f : \mathcal{R} \rightarrow \mathcal{T}$ be a correct tiling function of the rectangle $\mathcal{R}$ with border color \$. For convenience, we will identify the tile-variables $\mathtt{t_1}, \mathtt{t_2}, \ldots \in \mathtt{T}$ with their corresponding tiles $t_1, t_2, \ldots \in \mathcal{T}$. We will show that there exist a model M and an interval $[a, b]$ such that $M, [a, b] \Vdash \Phi_{\mathcal{T}}$. Let $D = \mathfrak{D} = \mathbb{N}\mid_{X\cdot Y+1}$, and let M the model for RPNL+FO built over these two domains. We want to define an interpretation $\mathcal{I}$ in such a way that $M, [0, 1] \Vdash \Phi_{\mathcal{T}}$. Then, we put

$$\mathsf{u}^{\mathcal{I}([i,i+1])} \quad \forall i.0 < i < X \cdot Y,$$

to guarantee that (5.34) is satisfied. Now, in order to satisfy the remaining part of $\Phi_{\mathcal{T}}$ on $[0,1]$, it suffices to define the valuation for the remaining propositional letters and the predicate symbol P:

$$\begin{array}{rl} P^{\mathcal{I}([i,j])}(i) & \forall i,j>0 \\ \mathsf{Id}^{\mathcal{I}([i\cdot X+1,(i+1)\cdot X+1])} & \forall i.0\leq i\leq Y-1 \\ \mathsf{up\_rel}^{\mathcal{I}([i,i+X-1])} & \forall i.2\leq i\leq X\cdot(Y-1)+1 \\ \mathsf{Final}^{\mathcal{I}([i,i+1])} & \forall i.X\cdot(Y-1)+1\leq i\leq X\cdot Y \end{array}$$

Finally, we evaluate the tile-variables as follows. For each $\mathtt{t}\in\mathtt{T}$:

$$\mathtt{t_q}^{\mathcal{I}([i,i+1])}\Leftrightarrow f(l,s)=t_q \qquad \forall i(i=X\cdot(s-1)+l).$$

□

The following theorem states the undecidability of the satisfiability problem of the extension of RPNL with First-Order machinery (RPNL+FO), even if only one first-order variable is included into the language.

Theorem 5.4.6. *The satisfiability problem of* RPNL+FO *with only one first-order variable, interpreted over (finite prefixes of) natural numbers, is undecidable.*

5.5 Conclusions

In this chapter, we have investigated further extensions of (metric versions of) PNL with classical machinery, namely, hybrid and first-order constructs. In these cases, even very weak extensions immediately yield undecidability. In particular, on the one hand, we have proved that extending (metric versions of) PNL with variables and binders over interval lengths leads to undecidability even in very restricted fragments. On the other hand, we have shown that another classical extension for temporal logics, obtained by generalizing propositional letters into first-order formulae, oversteps the barrier of decidability, even in a very restricted case such as that of monadic first-order formulae with finite domains. At a first glance, this result may appear discouraging, concerning our aim of finding more expressive, yet decidable, interval temporal logics. Nevertheless, it should be pointed out that, while for the logic PNL+LB the use of both modal operators of PNL is essential in the undecidability proof, for RPNL+FO the modal constant π plays an important role in the reduction (in this case only one modality of PNL is actually necessary). Thus, in both cases there still are sub-fragments to be explored. Moreover, while a sound and complete axiomatic system for PNL and its metric extension MPNL over natural numbers seems to be very difficult to devise, the extensions that we have defined here, thanks to their enhanced expressive power, might be actually easier to axiomatize. Finally, it might be worth considering the satisfiability problem of (R)PNL+FO restricted with some (natural) syntactic rules that constrain the relationship between the modal and the first-order components.

Conclusions

This dissertation has addressed expressiveness, decidability, and undecidability issues for the most studied interval temporal logic, namely, Halpern and Shoham's Modal Logic of Time Intervals (HS).

Its main contributions are described in Chapters 2-5. In Chapter 2, we have provided a complete classification, with respect to their expressiveness, of all the fragments of HS, when strict semantics is adopted and no assumptions on the class of linear orderings are made. To this end, we have massively exploited the notion of bisimulation between interval models in order to determine a sound, complete, and optimal set of inter-definability equations among all modal operators of HS. Such a classification has a number of important applications. As an example, it allows one to properly identify the (small) set of HS fragments for which the decidability of the satisfiability problem is still an open problem. It should be emphasized that the set of inter-definability equations we have provided (and the resulting classification) generally does not apply if the non-strict semantics is considered, or if some restrictions, such as denseness or discreteness, are imposed on the class of linear orders. In that regard, we have shown some examples of additional inter-definability equations holding when either denseness or discreteness is assumed or when non-strict semantics is adopted. The complete classification of the expressiveness of HS fragments with respect to the non-strict semantics, as well as over specific classes of linear orders, e.g., the class of all discrete/dense linear orderings, is still missing.

In Chapter 3, we have systematically investigated the decidable/undecidable status of the satisfiability problem for a number of previously unclassified HS fragments, showing that, once more, undecidability is the rule and decidability the exception. In particular, we have provided a number of undecidability results that actually subsume most of the previous ones (except for the undecidability result for the fragments D and $\overline{\mathsf{D}}$, given in [78]). Such results are based on reductions from (suitable versions of) the tiling problem. Pairing the results given here with existing ones, the long-standing goal of obtaining a complete classification of all HS fragments with respect to their satisfiability problem is now almost achieved. Different natural classes of linear orderings have been considered. In particular, when no assumptions are made on the class of linear orderings (the class of all linear orderings), it has turned out that the only one-modality and two-modalities fragments that remain unclassified are D, $\overline{\mathsf{D}}$, $\mathsf{D\overline{D}}$, AE, $\mathsf{A\overline{E}}$, $\mathsf{\overline{A}B}$, $\overline{\mathsf{AB}}$, LD, $\mathsf{L\overline{D}}$, $\mathsf{\overline{L}D}$, $\overline{\mathsf{LD}}$, BD, $\mathsf{B\overline{D}}$, $\mathsf{\overline{B}D}$, $\overline{\mathsf{BD}}$, ED, $\mathsf{E\overline{D}}$, $\mathsf{\overline{E}D}$, $\overline{\mathsf{ED}}$ (19 out of 76). As for fragments with more than two modalities, the open cases are represented by fragments that neither contain one of the undecidable fragments listed above nor are contained in one of the two decidable fragments $\mathsf{AB\overline{B}\overline{L}}$ and $\mathsf{\overline{A}E\overline{B}L}$. Among such open problems, the most interesting and challenging ones are surely those about the fragments D and $\mathsf{D}\overline{\mathsf{D}}$. The interest for these fragments has several reasons. First of all, they feature very natural relations, which, apparently, do not present strong conceptual difficulties (models for these two fragments are quite simple to figure out). In [73], Lodaya conjectured decidability of D and undecidability of $\mathsf{D}\overline{\mathsf{D}}$. After more than 10 years and several attempts, such problems, in their full generality, are still open, even if both the fragments have been classified when either discreteness or denseness is assumed. In particular, they represent two of the few cases in which the status of a fragments depends on the class of linear orderings on which it is interpreted (both of them are decidable over dense linear orderings and undecidable over discrete linear ones). Finally, it is worth to point out that D is the only one-modality fragment that is still unclassified, and proving its undecidability would mean to solve almost all the open cases.

In Chapter 4, we have presented and studied metric extensions of Propositional Neighborhood Logic over the interval structure of natural numbers $\mathbb{N}$. We have demonstrated that these are expressive and natural languages by proving the complexity and several expressive completeness results. First, we have considered a very expressive language of this class, MPNL, and shown that its satisfiability problem is decidable in 2NEXPTIME. Then, we have proved that such a language is expressively complete with respect to a well-defined sub-fragment of the two-variable fragment $\mathsf{FO}^2[\mathbb{N}, =, <, s]$ of first-order logic for linear orders with successor function, interpreted over natural numbers. We have shown that MPNL can be extended in a natural way to cover full $\mathsf{FO}^2[\mathbb{N}, =, <, s]$, but, unexpectedly, the latter (and hence the former) turns out to be undecidable. Finally, we have discussed the variety of fragments of MPNL and studied their relative expressive power. The results obtained here are amenable to some fairly straightforward generalizations, e.g., from $\mathbb{N}$ to $\mathbb{Z}$, as well as, with respect to different distance functions. The most important open problem is to find the exact complexity of the satisfiability problem for MPNL (the problem is EXPSPACE-hard), when constraints are represented in binary.

In Chapter 5, we have investigated further extensions of (metric versions of) PNL with classical machinery, namely, hybrid and first-order constructs. In these cases, even very weak extensions immediately yield undecidability. In particular, on the one hand, we have proved that extending (metric versions of) PNL with variables and binders over interval lengths leads to undecidability even in very restricted fragments. While these results are somewhat disappointing, they suggest that strong restrictions must be imposed on the application of length binders in order to retain decidability. The question whether an essential gain of expressiveness can be obtained by adding some hybrid machinery to interval logic, while preserving decidability, is still open. On the other hand, we have shown that yet another classical extension for temporal logics, obtained by replacing propositional letters by first-order formulae, oversteps the barrier of decidability, even in a very restrictive case such as that of monadic first-order formulae with finite domains. At a first glance, this result may appear discouraging, concerning our aim of finding decidable first-order interval temporal logics. Nevertheless, it should be pointed out that the modal constant π plays an important role in the reduction. Thus, it could be worth considering the satisfiability problem of the language devoid of such an operator, as well as the satisfiability problem of some version of the language restricted by means of some natural syntactic rules constraining the relationship between the modal and the first-order components.

Before concluding this dissertation, it is worth to point out that another interesting problem to be addressed in the area of interval temporal logics concerns model checking. Even if this topic has been extensively studied and successfully applied to real-world domains in the context of point-based logics, it is still quite unexplored in the interval setting. The major difficulty concerns the problem of finding a convenient way to (finitely) represent the models to be checked.

A

Classification of HS fragments with respect to the satisfiability problem: the state of the art

In this Chapter, we provide the state of the art of the classification of HS fragments with respect to the decidability status of the satisfiability problem. We outline the picture with respect to the strict semantics and to the class of all linear orders, as, under these assumptions, we are able to identify and to count all the expressively different HS fragments (see Chapter 2).

Additionally, to the web page http://itl.dimi.uniud.it/content/logic-hs, it is possible to run a collection of web tools, allowing one to verify the status (decidable/undecidable/unknown) of any specific fragment with respect to the satisfiability problem, over various classes of linear orders (all, dense, discrete, and finite) and considering both strict and non-strict semantics, as weel as to compare relative expressive power of any pair of HS fragments. The web page will be kept up to date about all new results concerning both satisfiability and axpressiveness.

A.1 Classification in the strict semantics, over the class of all linear orderings

Number of syntactically different fragments:	**4096**	
Number of expressively different fragments (taking into account the inter-definability equations of Chapter 2):	**1347**	
Decidable fragments:	**44**	(3.27 %)
Undecidable fragments:	**1203**	(89.31 %)
Unknown fragments:	**100**	(7.42 %)

A.1.1 Decidable fragments

A	B	AB	E	L	LB	LE	$\overline{A}$	A$\overline{A}$	E$\overline{A}$	L$\overline{A}$
LE$\overline{A}$	$\overline{B}$	A$\overline{B}$	B$\overline{B}$	AB$\overline{B}$	L$\overline{B}$	LB$\overline{B}$	$\overline{E}$	E$\overline{E}$	L$\overline{E}$	LE$\overline{E}$
$\overline{AE}$	E$\overline{AE}$	L$\overline{AE}$	LE$\overline{AE}$	$\overline{L}$	A$\overline{L}$	B$\overline{L}$	AB$\overline{L}$	E$\overline{L}$	L$\overline{L}$	LB$\overline{L}$
LE$\overline{L}$	$\overline{LB}$	A$\overline{LB}$	B$\overline{LB}$	AB$\overline{LB}$	L$\overline{LB}$	LB$\overline{LB}$	$\overline{LE}$	E$\overline{LE}$	L$\overline{LE}$	LE$\overline{LE}$

A.1.2 Undecidable fragments

AD	ABD	BE	ABE	AED	LBE
O	AO	BO	ABO	DO	ADO
BDO	ABDO	EO	AEO	BEO	ABEO
EDO	AEDO	LO	LBO	LDO	LBDO
LEO	LBEO	LEDO	D$\overline{A}$	AD$\overline{A}$	BD$\overline{A}$
ABD$\overline{A}$	BE$\overline{A}$	ABE$\overline{A}$	ED$\overline{A}$	AED$\overline{A}$	LD$\overline{A}$
LBD$\overline{A}$	LBE$\overline{A}$	LED$\overline{A}$	O$\overline{A}$	AO$\overline{A}$	BO$\overline{A}$
ABO$\overline{A}$	DO$\overline{A}$	ADO$\overline{A}$	BDO$\overline{A}$	ABDO$\overline{A}$	EO$\overline{A}$
AEO$\overline{A}$	BEO$\overline{A}$	ABEO$\overline{A}$	EDO$\overline{A}$	AEDO$\overline{A}$	LO$\overline{A}$
LBO$\overline{A}$	LDO$\overline{A}$	LBDO$\overline{A}$	LEO$\overline{A}$	LBEO$\overline{A}$	LEDO$\overline{A}$
AD$\overline{B}$	ABD$\overline{B}$	E$\overline{B}$	AE$\overline{B}$	BE$\overline{B}$	ABE$\overline{B}$
ED$\overline{B}$	AED$\overline{B}$	O$\overline{B}$	AO$\overline{B}$	BO$\overline{B}$	ABO$\overline{B}$
DO$\overline{B}$	ADO$\overline{B}$	BDO$\overline{B}$	ABDO$\overline{B}$	LO$\overline{B}$	LBO$\overline{B}$
LDO$\overline{B}$	LBDO$\overline{B}$	AB$\overline{AB}$	D$\overline{AB}$	AD$\overline{AB}$	BD$\overline{AB}$
ABD$\overline{AB}$	E$\overline{AB}$	AE$\overline{AB}$	BE$\overline{AB}$	ABE$\overline{AB}$	ED$\overline{AB}$
AED$\overline{AB}$	LD$\overline{AB}$	LBD$\overline{AB}$	O$\overline{AB}$	AO$\overline{AB}$	BO$\overline{AB}$
ABO$\overline{AB}$	DO$\overline{AB}$	ADO$\overline{AB}$	BDO$\overline{AB}$	ABDO$\overline{AB}$	LO$\overline{AB}$
LBO$\overline{AB}$	LDO$\overline{AB}$	LBDO$\overline{AB}$	A$\overline{D}$	AB$\overline{D}$	AD$\overline{D}$
ABD$\overline{D}$	AE$\overline{D}$	BE$\overline{D}$	ABE$\overline{D}$	AED$\overline{D}$	LBE$\overline{D}$
O$\overline{D}$	AO$\overline{D}$	BO$\overline{D}$	ABO$\overline{D}$	DO$\overline{D}$	ADO$\overline{D}$
BDO$\overline{D}$	ABDO$\overline{D}$	EO$\overline{D}$	AEO$\overline{D}$	BEO$\overline{D}$	ABEO$\overline{D}$
EDO$\overline{D}$	AEDO$\overline{D}$	LO$\overline{D}$	LBO$\overline{D}$	LDO$\overline{D}$	LBDO$\overline{D}$
LEO$\overline{D}$	LBEO$\overline{D}$	LEDO$\overline{D}$	$\overline{AD}$	A$\overline{AD}$	B$\overline{AD}$
AB$\overline{AD}$	D$\overline{AD}$	AD$\overline{AD}$	BD$\overline{AD}$	ABD$\overline{AD}$	E$\overline{AD}$
AE$\overline{AD}$	BE$\overline{AD}$	ABE$\overline{AD}$	ED$\overline{AD}$	AED$\overline{AD}$	L$\overline{AD}$
LB$\overline{AD}$	LD$\overline{AD}$	LBD$\overline{AD}$	LE$\overline{AD}$	LBE$\overline{AD}$	LED$\overline{AD}$
O$\overline{AD}$	AO$\overline{AD}$	BO$\overline{AD}$	ABO$\overline{AD}$	DO$\overline{AD}$	ADO$\overline{AD}$
BDO$\overline{AD}$	ABDO$\overline{AD}$	EO$\overline{AD}$	AEO$\overline{AD}$	BEO$\overline{AD}$	ABEO$\overline{AD}$
EDO$\overline{AD}$	AEDO$\overline{AD}$	LO$\overline{AD}$	LBO$\overline{AD}$	LDO$\overline{AD}$	LBDO$\overline{AD}$
LEO$\overline{AD}$	LBEO$\overline{AD}$	LEDO$\overline{AD}$	A$\overline{BD}$	AB$\overline{BD}$	AD$\overline{BD}$
ABD$\overline{BD}$	E$\overline{BD}$	AE$\overline{BD}$	BE$\overline{BD}$	ABE$\overline{BD}$	ED$\overline{BD}$
AED$\overline{BD}$	O$\overline{BD}$	AO$\overline{BD}$	BO$\overline{BD}$	ABO$\overline{BD}$	DO$\overline{BD}$
ADO$\overline{BD}$	BDO$\overline{BD}$	ABDO$\overline{BD}$	LO$\overline{BD}$	LBO$\overline{BD}$	LDO$\overline{BD}$

LBDO$\overline{\text{BD}}$	$\overline{\text{ABD}}$	A$\overline{\text{ABD}}$	B$\overline{\text{ABD}}$	AB$\overline{\text{ABD}}$	D$\overline{\text{ABD}}$
AD$\overline{\text{ABD}}$	BD$\overline{\text{ABD}}$	ABD$\overline{\text{ABD}}$	E$\overline{\text{ABD}}$	AE$\overline{\text{ABD}}$	BE$\overline{\text{ABD}}$
ABE$\overline{\text{ABD}}$	ED$\overline{\text{ABD}}$	AED$\overline{\text{ABD}}$	L$\overline{\text{ABD}}$	LB$\overline{\text{ABD}}$	LD$\overline{\text{ABD}}$
LBD$\overline{\text{ABD}}$	O$\overline{\text{ABD}}$	AO$\overline{\text{ABD}}$	BO$\overline{\text{ABD}}$	ABO$\overline{\text{ABD}}$	DO$\overline{\text{ABD}}$
ADO$\overline{\text{ABD}}$	BDO$\overline{\text{ABD}}$	ABDO$\overline{\text{ABD}}$	LO$\overline{\text{ABD}}$	LBO$\overline{\text{ABD}}$	LDO$\overline{\text{ABD}}$
LBDO$\overline{\text{ABD}}$	B$\overline{\text{E}}$	AB$\overline{\text{E}}$	AD$\overline{\text{E}}$	BD$\overline{\text{E}}$	ABD$\overline{\text{E}}$
BE$\overline{\text{E}}$	ABE$\overline{\text{E}}$	AED$\overline{\text{E}}$	LB$\overline{\text{E}}$	LBD$\overline{\text{E}}$	LBE$\overline{\text{E}}$
O$\overline{\text{E}}$	AO$\overline{\text{E}}$	BO$\overline{\text{E}}$	ABO$\overline{\text{E}}$	DO$\overline{\text{E}}$	ADO$\overline{\text{E}}$
BDO$\overline{\text{E}}$	ABDO$\overline{\text{E}}$	EO$\overline{\text{E}}$	AEO$\overline{\text{E}}$	BEO$\overline{\text{E}}$	ABEO$\overline{\text{E}}$
EDO$\overline{\text{E}}$	AEDO$\overline{\text{E}}$	LO$\overline{\text{E}}$	LBO$\overline{\text{E}}$	LDO$\overline{\text{E}}$	LBDO$\overline{\text{E}}$
LEO$\overline{\text{E}}$	LBEO$\overline{\text{E}}$	LEDO$\overline{\text{E}}$	B$\overline{\text{AE}}$	AB$\overline{\text{AE}}$	D$\overline{\text{AE}}$
AD$\overline{\text{AE}}$	BD$\overline{\text{AE}}$	ABD$\overline{\text{AE}}$	AE$\overline{\text{AE}}$	BE$\overline{\text{AE}}$	ABE$\overline{\text{AE}}$
ED$\overline{\text{AE}}$	AED$\overline{\text{AE}}$	LB$\overline{\text{AE}}$	LD$\overline{\text{AE}}$	LBD$\overline{\text{AE}}$	LBE$\overline{\text{AE}}$
LED$\overline{\text{AE}}$	O$\overline{\text{AE}}$	AO$\overline{\text{AE}}$	BO$\overline{\text{AE}}$	ABO$\overline{\text{AE}}$	DO$\overline{\text{AE}}$
ADO$\overline{\text{AE}}$	BDO$\overline{\text{AE}}$	ABDO$\overline{\text{AE}}$	EO$\overline{\text{AE}}$	AEO$\overline{\text{AE}}$	BEO$\overline{\text{AE}}$
ABEO$\overline{\text{AE}}$	EDO$\overline{\text{AE}}$	AEDO$\overline{\text{AE}}$	LO$\overline{\text{AE}}$	LBO$\overline{\text{AE}}$	LDO$\overline{\text{AE}}$
LBDO$\overline{\text{AE}}$	LEO$\overline{\text{AE}}$	LBEO$\overline{\text{AE}}$	LEDO$\overline{\text{AE}}$	$\overline{\text{BE}}$	A$\overline{\text{BE}}$
B$\overline{\text{BE}}$	AB$\overline{\text{BE}}$	D$\overline{\text{BE}}$	AD$\overline{\text{BE}}$	BD$\overline{\text{BE}}$	ABD$\overline{\text{BE}}$
E$\overline{\text{BE}}$	AE$\overline{\text{BE}}$	BE$\overline{\text{BE}}$	ABE$\overline{\text{BE}}$	ED$\overline{\text{BE}}$	AED$\overline{\text{BE}}$
L$\overline{\text{BE}}$	LB$\overline{\text{BE}}$	LD$\overline{\text{BE}}$	LBD$\overline{\text{BE}}$	O$\overline{\text{BE}}$	AO$\overline{\text{BE}}$
BO$\overline{\text{BE}}$	ABO$\overline{\text{BE}}$	DO$\overline{\text{BE}}$	ADO$\overline{\text{BE}}$	BDO$\overline{\text{BE}}$	ABDO$\overline{\text{BE}}$
LO$\overline{\text{BE}}$	LBO$\overline{\text{BE}}$	LDO$\overline{\text{BE}}$	LBDO$\overline{\text{BE}}$	$\overline{\text{ABE}}$	A$\overline{\text{ABE}}$
B$\overline{\text{ABE}}$	AB$\overline{\text{ABE}}$	D$\overline{\text{ABE}}$	AD$\overline{\text{ABE}}$	BD$\overline{\text{ABE}}$	ABD$\overline{\text{ABE}}$
E$\overline{\text{ABE}}$	AE$\overline{\text{ABE}}$	BE$\overline{\text{ABE}}$	ABE$\overline{\text{ABE}}$	ED$\overline{\text{ABE}}$	AED$\overline{\text{ABE}}$
L$\overline{\text{ABE}}$	LB$\overline{\text{ABE}}$	LD$\overline{\text{ABE}}$	LBD$\overline{\text{ABE}}$	O$\overline{\text{ABE}}$	AO$\overline{\text{ABE}}$
BO$\overline{\text{ABE}}$	ABO$\overline{\text{ABE}}$	DO$\overline{\text{ABE}}$	ADO$\overline{\text{ABE}}$	BDO$\overline{\text{ABE}}$	ABDO$\overline{\text{ABE}}$
LO$\overline{\text{ABE}}$	LBO$\overline{\text{ABE}}$	LDO$\overline{\text{ABE}}$	LBDO$\overline{\text{ABE}}$	A$\overline{\text{ED}}$	B$\overline{\text{ED}}$
AB$\overline{\text{ED}}$	AD$\overline{\text{ED}}$	BD$\overline{\text{ED}}$	ABD$\overline{\text{ED}}$	AE$\overline{\text{ED}}$	BE$\overline{\text{ED}}$
ABE$\overline{\text{ED}}$	AED$\overline{\text{ED}}$	LB$\overline{\text{ED}}$	LBD$\overline{\text{ED}}$	LBE$\overline{\text{ED}}$	O$\overline{\text{ED}}$
AO$\overline{\text{ED}}$	BO$\overline{\text{ED}}$	ABO$\overline{\text{ED}}$	DO$\overline{\text{ED}}$	ADO$\overline{\text{ED}}$	BDO$\overline{\text{ED}}$
ABDO$\overline{\text{ED}}$	EO$\overline{\text{ED}}$	AEO$\overline{\text{ED}}$	BEO$\overline{\text{ED}}$	ABEO$\overline{\text{ED}}$	EDO$\overline{\text{ED}}$
AEDO$\overline{\text{ED}}$	LO$\overline{\text{ED}}$	LBO$\overline{\text{ED}}$	LDO$\overline{\text{ED}}$	LBDO$\overline{\text{ED}}$	LEO$\overline{\text{ED}}$
LBEO$\overline{\text{ED}}$	LEDO$\overline{\text{ED}}$	$\overline{\text{AED}}$	A$\overline{\text{AED}}$	B$\overline{\text{AED}}$	AB$\overline{\text{AED}}$
D$\overline{\text{AED}}$	AD$\overline{\text{AED}}$	BD$\overline{\text{AED}}$	ABD$\overline{\text{AED}}$	E$\overline{\text{AED}}$	AE$\overline{\text{AED}}$
BE$\overline{\text{AED}}$	ABE$\overline{\text{AED}}$	ED$\overline{\text{AED}}$	AED$\overline{\text{AED}}$	L$\overline{\text{AED}}$	LB$\overline{\text{AED}}$
LD$\overline{\text{AED}}$	LBD$\overline{\text{AED}}$	LE$\overline{\text{AED}}$	LBE$\overline{\text{AED}}$	LED$\overline{\text{AED}}$	O$\overline{\text{AED}}$
AO$\overline{\text{AED}}$	BO$\overline{\text{AED}}$	ABO$\overline{\text{AED}}$	DO$\overline{\text{AED}}$	ADO$\overline{\text{AED}}$	BDO$\overline{\text{AED}}$
ABDO$\overline{\text{AED}}$	EO$\overline{\text{AED}}$	AEO$\overline{\text{AED}}$	BEO$\overline{\text{AED}}$	ABEO$\overline{\text{AED}}$	EDO$\overline{\text{AED}}$
AEDO$\overline{\text{AED}}$	LO$\overline{\text{AED}}$	LBO$\overline{\text{AED}}$	LDO$\overline{\text{AED}}$	LBDO$\overline{\text{AED}}$	LEO$\overline{\text{AED}}$
LBEO$\overline{\text{AED}}$	LEDO$\overline{\text{AED}}$	AD$\overline{\text{L}}$	ABD$\overline{\text{L}}$	BE$\overline{\text{L}}$	ABE$\overline{\text{L}}$
AED$\overline{\text{L}}$	LBE$\overline{\text{L}}$	O$\overline{\text{L}}$	AO$\overline{\text{L}}$	BO$\overline{\text{L}}$	ABO$\overline{\text{L}}$

DO$\overline{\mathrm{L}}$	ADO$\overline{\mathrm{L}}$	BDO$\overline{\mathrm{L}}$	ABDO$\overline{\mathrm{L}}$	EO$\overline{\mathrm{L}}$	AEO$\overline{\mathrm{L}}$
BEO$\overline{\mathrm{L}}$	ABEO$\overline{\mathrm{L}}$	EDO$\overline{\mathrm{L}}$	AEDO$\overline{\mathrm{L}}$	LO$\overline{\mathrm{L}}$	LBO$\overline{\mathrm{L}}$
LDO$\overline{\mathrm{L}}$	LBDO$\overline{\mathrm{L}}$	LEO$\overline{\mathrm{L}}$	LBEO$\overline{\mathrm{L}}$	LEDO$\overline{\mathrm{L}}$	AD$\overline{\mathrm{LB}}$
ABD$\overline{\mathrm{LB}}$	E$\overline{\mathrm{LB}}$	AE$\overline{\mathrm{LB}}$	BE$\overline{\mathrm{LB}}$	ABE$\overline{\mathrm{LB}}$	ED$\overline{\mathrm{LB}}$
AED$\overline{\mathrm{LB}}$	O$\overline{\mathrm{LB}}$	AO$\overline{\mathrm{LB}}$	BO$\overline{\mathrm{LB}}$	ABO$\overline{\mathrm{LB}}$	DO$\overline{\mathrm{LB}}$
ADO$\overline{\mathrm{LB}}$	BDO$\overline{\mathrm{LB}}$	ABDO$\overline{\mathrm{LB}}$	LO$\overline{\mathrm{LB}}$	LBO$\overline{\mathrm{LB}}$	LDO$\overline{\mathrm{LB}}$
LBDO$\overline{\mathrm{LB}}$	A$\overline{\mathrm{LD}}$	AB$\overline{\mathrm{LD}}$	AD$\overline{\mathrm{LD}}$	ABD$\overline{\mathrm{LD}}$	AE$\overline{\mathrm{LD}}$
BE$\overline{\mathrm{LD}}$	ABE$\overline{\mathrm{LD}}$	AED$\overline{\mathrm{LD}}$	LBE$\overline{\mathrm{LD}}$	O$\overline{\mathrm{LD}}$	AO$\overline{\mathrm{LD}}$
BO$\overline{\mathrm{LD}}$	ABO$\overline{\mathrm{LD}}$	DO$\overline{\mathrm{LD}}$	ADO$\overline{\mathrm{LD}}$	BDO$\overline{\mathrm{LD}}$	ABDO$\overline{\mathrm{LD}}$
EO$\overline{\mathrm{LD}}$	AEO$\overline{\mathrm{LD}}$	BEO$\overline{\mathrm{LD}}$	ABEO$\overline{\mathrm{LD}}$	EDO$\overline{\mathrm{LD}}$	AEDO$\overline{\mathrm{LD}}$
LO$\overline{\mathrm{LD}}$	LBO$\overline{\mathrm{LD}}$	LDO$\overline{\mathrm{LD}}$	LBDO$\overline{\mathrm{LD}}$	LEO$\overline{\mathrm{LD}}$	LBEO$\overline{\mathrm{LD}}$
LEDO$\overline{\mathrm{LD}}$	A$\overline{\mathrm{LBD}}$	AB$\overline{\mathrm{LBD}}$	AD$\overline{\mathrm{LBD}}$	ABD$\overline{\mathrm{LBD}}$	E$\overline{\mathrm{LBD}}$
AE$\overline{\mathrm{LBD}}$	BE$\overline{\mathrm{LBD}}$	ABE$\overline{\mathrm{LBD}}$	ED$\overline{\mathrm{LBD}}$	AED$\overline{\mathrm{LBD}}$	O$\overline{\mathrm{LBD}}$
AO$\overline{\mathrm{LBD}}$	BO$\overline{\mathrm{LBD}}$	ABO$\overline{\mathrm{LBD}}$	DO$\overline{\mathrm{LBD}}$	ADO$\overline{\mathrm{LBD}}$	BDO$\overline{\mathrm{LBD}}$
ABDO$\overline{\mathrm{LBD}}$	LO$\overline{\mathrm{LBD}}$	LBO$\overline{\mathrm{LBD}}$	LDO$\overline{\mathrm{LBD}}$	LBDO$\overline{\mathrm{LBD}}$	AD$\overline{\mathrm{LE}}$
AED$\overline{\mathrm{LE}}$	O$\overline{\mathrm{LE}}$	AO$\overline{\mathrm{LE}}$	DO$\overline{\mathrm{LE}}$	ADO$\overline{\mathrm{LE}}$	EO$\overline{\mathrm{LE}}$
AEO$\overline{\mathrm{LE}}$	EDO$\overline{\mathrm{LE}}$	AEDO$\overline{\mathrm{LE}}$	LO$\overline{\mathrm{LE}}$	LDO$\overline{\mathrm{LE}}$	LEO$\overline{\mathrm{LE}}$
LEDO$\overline{\mathrm{LE}}$	$\overline{\mathrm{LBE}}$	A$\overline{\mathrm{LBE}}$	D$\overline{\mathrm{LBE}}$	AD$\overline{\mathrm{LBE}}$	E$\overline{\mathrm{LBE}}$
AE$\overline{\mathrm{LBE}}$	ED$\overline{\mathrm{LBE}}$	AED$\overline{\mathrm{LBE}}$	L$\overline{\mathrm{LBE}}$	LD$\overline{\mathrm{LBE}}$	O$\overline{\mathrm{LBE}}$
AO$\overline{\mathrm{LBE}}$	DO$\overline{\mathrm{LBE}}$	ADO$\overline{\mathrm{LBE}}$	LO$\overline{\mathrm{LBE}}$	LDO$\overline{\mathrm{LBE}}$	A$\overline{\mathrm{LED}}$
AD$\overline{\mathrm{LED}}$	AE$\overline{\mathrm{LED}}$	AED$\overline{\mathrm{LED}}$	O$\overline{\mathrm{LED}}$	AO$\overline{\mathrm{LED}}$	DO$\overline{\mathrm{LED}}$
ADO$\overline{\mathrm{LED}}$	EO$\overline{\mathrm{LED}}$	AEO$\overline{\mathrm{LED}}$	EDO$\overline{\mathrm{LED}}$	AEDO$\overline{\mathrm{LED}}$	LO$\overline{\mathrm{LED}}$
LDO$\overline{\mathrm{LED}}$	LEO$\overline{\mathrm{LED}}$	LEDO$\overline{\mathrm{LED}}$	$\overline{\mathrm{O}}$	A$\overline{\mathrm{O}}$	B$\overline{\mathrm{O}}$
AB$\overline{\mathrm{O}}$	D$\overline{\mathrm{O}}$	AD$\overline{\mathrm{O}}$	BD$\overline{\mathrm{O}}$	ABD$\overline{\mathrm{O}}$	E$\overline{\mathrm{O}}$
AE$\overline{\mathrm{O}}$	BE$\overline{\mathrm{O}}$	ABE$\overline{\mathrm{O}}$	ED$\overline{\mathrm{O}}$	AED$\overline{\mathrm{O}}$	L$\overline{\mathrm{O}}$
LB$\overline{\mathrm{O}}$	LD$\overline{\mathrm{O}}$	LBD$\overline{\mathrm{O}}$	LE$\overline{\mathrm{O}}$	LBE$\overline{\mathrm{O}}$	LED$\overline{\mathrm{O}}$
O$\overline{\mathrm{O}}$	AO$\overline{\mathrm{O}}$	BO$\overline{\mathrm{O}}$	ABO$\overline{\mathrm{O}}$	DO$\overline{\mathrm{O}}$	ADO$\overline{\mathrm{O}}$
BDO$\overline{\mathrm{O}}$	ABDO$\overline{\mathrm{O}}$	EO$\overline{\mathrm{O}}$	AEO$\overline{\mathrm{O}}$	BEO$\overline{\mathrm{O}}$	ABEO$\overline{\mathrm{O}}$
EDO$\overline{\mathrm{O}}$	AEDO$\overline{\mathrm{O}}$	LO$\overline{\mathrm{O}}$	LBO$\overline{\mathrm{O}}$	LDO$\overline{\mathrm{O}}$	LBDO$\overline{\mathrm{O}}$
LEO$\overline{\mathrm{O}}$	LBEO$\overline{\mathrm{O}}$	LEDO$\overline{\mathrm{O}}$	$\overline{\mathrm{AO}}$	A$\overline{\mathrm{AO}}$	B$\overline{\mathrm{AO}}$
AB$\overline{\mathrm{AO}}$	D$\overline{\mathrm{AO}}$	AD$\overline{\mathrm{AO}}$	BD$\overline{\mathrm{AO}}$	ABD$\overline{\mathrm{AO}}$	E$\overline{\mathrm{AO}}$
AE$\overline{\mathrm{AO}}$	BE$\overline{\mathrm{AO}}$	ABE$\overline{\mathrm{AO}}$	ED$\overline{\mathrm{AO}}$	AED$\overline{\mathrm{AO}}$	L$\overline{\mathrm{AO}}$
LB$\overline{\mathrm{AO}}$	LD$\overline{\mathrm{AO}}$	LBD$\overline{\mathrm{AO}}$	LE$\overline{\mathrm{AO}}$	LBE$\overline{\mathrm{AO}}$	LED$\overline{\mathrm{AO}}$
O$\overline{\mathrm{AO}}$	AO$\overline{\mathrm{AO}}$	BO$\overline{\mathrm{AO}}$	ABO$\overline{\mathrm{AO}}$	DO$\overline{\mathrm{AO}}$	ADO$\overline{\mathrm{AO}}$
BDO$\overline{\mathrm{AO}}$	ABDO$\overline{\mathrm{AO}}$	EO$\overline{\mathrm{AO}}$	AEO$\overline{\mathrm{AO}}$	BEO$\overline{\mathrm{AO}}$	ABEO$\overline{\mathrm{AO}}$
EDO$\overline{\mathrm{AO}}$	AEDO$\overline{\mathrm{AO}}$	LO$\overline{\mathrm{AO}}$	LBO$\overline{\mathrm{AO}}$	LDO$\overline{\mathrm{AO}}$	LBDO$\overline{\mathrm{AO}}$
LEO$\overline{\mathrm{AO}}$	LBEO$\overline{\mathrm{AO}}$	LEDO$\overline{\mathrm{AO}}$	$\overline{\mathrm{BO}}$	A$\overline{\mathrm{BO}}$	B$\overline{\mathrm{BO}}$
AB$\overline{\mathrm{BO}}$	D$\overline{\mathrm{BO}}$	AD$\overline{\mathrm{BO}}$	BD$\overline{\mathrm{BO}}$	ABD$\overline{\mathrm{BO}}$	E$\overline{\mathrm{BO}}$
AE$\overline{\mathrm{BO}}$	BE$\overline{\mathrm{BO}}$	ABE$\overline{\mathrm{BO}}$	ED$\overline{\mathrm{BO}}$	AED$\overline{\mathrm{BO}}$	L$\overline{\mathrm{BO}}$
LB$\overline{\mathrm{BO}}$	LD$\overline{\mathrm{BO}}$	LBD$\overline{\mathrm{BO}}$	O$\overline{\mathrm{BO}}$	AO$\overline{\mathrm{BO}}$	BO$\overline{\mathrm{BO}}$
ABO$\overline{\mathrm{BO}}$	DO$\overline{\mathrm{BO}}$	ADO$\overline{\mathrm{BO}}$	BDO$\overline{\mathrm{BO}}$	ABDO$\overline{\mathrm{BO}}$	LO$\overline{\mathrm{BO}}$

$LBO\overline{BO}$	$LDO\overline{BO}$	$LBDO\overline{BO}$	$\overline{ABO}$	$A\overline{ABO}$	$B\overline{ABO}$
$AB\overline{ABO}$	$D\overline{ABO}$	$AD\overline{ABO}$	$BD\overline{ABO}$	$ABD\overline{ABO}$	$E\overline{ABO}$
$AE\overline{ABO}$	$BE\overline{ABO}$	$ABE\overline{ABO}$	$ED\overline{ABO}$	$AED\overline{ABO}$	$L\overline{ABO}$
$LB\overline{ABO}$	$LD\overline{ABO}$	$LBD\overline{ABO}$	$O\overline{ABO}$	$AO\overline{ABO}$	$BO\overline{ABO}$
$ABO\overline{ABO}$	$DO\overline{ABO}$	$ADO\overline{ABO}$	$BDO\overline{ABO}$	$ABDO\overline{ABO}$	$LO\overline{ABO}$
$LBO\overline{ABO}$	$LDO\overline{ABO}$	$LBDO\overline{ABO}$	$\overline{DO}$	$A\overline{DO}$	$B\overline{DO}$
$AB\overline{DO}$	$D\overline{DO}$	$AD\overline{DO}$	$BD\overline{DO}$	$ABD\overline{DO}$	$E\overline{DO}$
$AE\overline{DO}$	$BE\overline{DO}$	$ABE\overline{DO}$	$ED\overline{DO}$	$AED\overline{DO}$	$L\overline{DO}$
$LB\overline{DO}$	$LD\overline{DO}$	$LBD\overline{DO}$	$LE\overline{DO}$	$LBE\overline{DO}$	$LED\overline{DO}$
$O\overline{DO}$	$AO\overline{DO}$	$BO\overline{DO}$	$ABO\overline{DO}$	$DO\overline{DO}$	$ADO\overline{DO}$
$BDO\overline{DO}$	$ABDO\overline{DO}$	$EO\overline{DO}$	$AEO\overline{DO}$	$BEO\overline{DO}$	$ABEO\overline{DO}$
$EDO\overline{DO}$	$AEDO\overline{DO}$	$LO\overline{DO}$	$LBO\overline{DO}$	$LDO\overline{DO}$	$LBDO\overline{DO}$
$LEO\overline{DO}$	$LBEO\overline{DO}$	$LEDO\overline{DO}$	$\overline{ADO}$	$A\overline{ADO}$	$B\overline{ADO}$
$AB\overline{ADO}$	$D\overline{ADO}$	$AD\overline{ADO}$	$BD\overline{ADO}$	$ABD\overline{ADO}$	$E\overline{ADO}$
$AE\overline{ADO}$	$BE\overline{ADO}$	$ABE\overline{ADO}$	$ED\overline{ADO}$	$AED\overline{ADO}$	$L\overline{ADO}$
$LB\overline{ADO}$	$LD\overline{ADO}$	$LBD\overline{ADO}$	$LE\overline{ADO}$	$LBE\overline{ADO}$	$LED\overline{ADO}$
$O\overline{ADO}$	$AO\overline{ADO}$	$BO\overline{ADO}$	$ABO\overline{ADO}$	$DO\overline{ADO}$	$ADO\overline{ADO}$
$BDO\overline{ADO}$	$ABDO\overline{ADO}$	$EO\overline{ADO}$	$AEO\overline{ADO}$	$BEO\overline{ADO}$	$ABEO\overline{ADO}$
$EDO\overline{ADO}$	$AEDO\overline{ADO}$	$LO\overline{ADO}$	$LBO\overline{ADO}$	$LDO\overline{ADO}$	$LBDO\overline{ADO}$
$LEO\overline{ADO}$	$LBEO\overline{ADO}$	$LEDO\overline{ADO}$	$\overline{BDO}$	$A\overline{BDO}$	$B\overline{BDO}$
$AB\overline{BDO}$	$D\overline{BDO}$	$AD\overline{BDO}$	$BD\overline{BDO}$	$ABD\overline{BDO}$	$E\overline{BDO}$
$AE\overline{BDO}$	$BE\overline{BDO}$	$ABE\overline{BDO}$	$ED\overline{BDO}$	$AED\overline{BDO}$	$L\overline{BDO}$
$LB\overline{BDO}$	$LD\overline{BDO}$	$LBD\overline{BDO}$	$O\overline{BDO}$	$AO\overline{BDO}$	$BO\overline{BDO}$
$ABO\overline{BDO}$	$DO\overline{BDO}$	$ADO\overline{BDO}$	$BDO\overline{BDO}$	$ABDO\overline{BDO}$	$LO\overline{BDO}$
$LBO\overline{BDO}$	$LDO\overline{BDO}$	$LBDO\overline{BDO}$	$\overline{ABDO}$	$A\overline{ABDO}$	$B\overline{ABDO}$
$AB\overline{ABDO}$	$D\overline{ABDO}$	$AD\overline{ABDO}$	$BD\overline{ABDO}$	$ABD\overline{ABDO}$	$E\overline{ABDO}$
$AE\overline{ABDO}$	$BE\overline{ABDO}$	$ABE\overline{ABDO}$	$ED\overline{ABDO}$	$AED\overline{ABDO}$	$L\overline{ABDO}$
$LB\overline{ABDO}$	$LD\overline{ABDO}$	$LBD\overline{ABDO}$	$O\overline{ABDO}$	$AO\overline{ABDO}$	$BO\overline{ABDO}$
$ABO\overline{ABDO}$	$DO\overline{ABDO}$	$ADO\overline{ABDO}$	$BDO\overline{ABDO}$	$ABDO\overline{ABDO}$	$LO\overline{ABDO}$
$LBO\overline{ABDO}$	$LDO\overline{ABDO}$	$LBDO\overline{ABDO}$	$\overline{EO}$	$A\overline{EO}$	$D\overline{EO}$
$AD\overline{EO}$	$E\overline{EO}$	$AE\overline{EO}$	$ED\overline{EO}$	$AED\overline{EO}$	$L\overline{EO}$
$LD\overline{EO}$	$LE\overline{EO}$	$LED\overline{EO}$	$O\overline{EO}$	$AO\overline{EO}$	$DO\overline{EO}$
$ADO\overline{EO}$	$EO\overline{EO}$	$AEO\overline{EO}$	$EDO\overline{EO}$	$AEDO\overline{EO}$	$LO\overline{EO}$
$LDO\overline{EO}$	$LEO\overline{EO}$	$LEDO\overline{EO}$	$\overline{AEO}$	$A\overline{AEO}$	$D\overline{AEO}$
$AD\overline{AEO}$	$E\overline{AEO}$	$AE\overline{AEO}$	$ED\overline{AEO}$	$AED\overline{AEO}$	$L\overline{AEO}$
$LD\overline{AEO}$	$LE\overline{AEO}$	$LED\overline{AEO}$	$O\overline{AEO}$	$AO\overline{AEO}$	$DO\overline{AEO}$
$ADO\overline{AEO}$	$EO\overline{AEO}$	$AEO\overline{AEO}$	$EDO\overline{AEO}$	$AEDO\overline{AEO}$	$LO\overline{AEO}$
$LDO\overline{AEO}$	$LEO\overline{AEO}$	$LEDO\overline{AEO}$	$\overline{BEO}$	$A\overline{BEO}$	$D\overline{BEO}$
$AD\overline{BEO}$	$E\overline{BEO}$	$AE\overline{BEO}$	$ED\overline{BEO}$	$AED\overline{BEO}$	$L\overline{BEO}$
$LD\overline{BEO}$	$O\overline{BEO}$	$AO\overline{BEO}$	$DO\overline{BEO}$	$ADO\overline{BEO}$	$LO\overline{BEO}$
$LDO\overline{BEO}$	$\overline{ABEO}$	$A\overline{ABEO}$	$D\overline{ABEO}$	$AD\overline{ABEO}$	$E\overline{ABEO}$

$\mathrm{AE}\overline{\mathrm{ABEO}}$	$\mathrm{ED}\overline{\mathrm{ABEO}}$	$\mathrm{AED}\overline{\mathrm{ABEO}}$	$\mathrm{L}\overline{\mathrm{ABEO}}$	$\mathrm{LD}\overline{\mathrm{ABEO}}$	$\mathrm{O}\overline{\mathrm{ABEO}}$
$\mathrm{AO}\overline{\mathrm{ABEO}}$	$\mathrm{DO}\overline{\mathrm{ABEO}}$	$\mathrm{ADO}\overline{\mathrm{ABEO}}$	$\mathrm{LO}\overline{\mathrm{ABEO}}$	$\mathrm{LDO}\overline{\mathrm{ABEO}}$	$\overline{\mathrm{EDO}}$
$\mathrm{A}\overline{\mathrm{EDO}}$	$\mathrm{D}\overline{\mathrm{EDO}}$	$\mathrm{AD}\overline{\mathrm{EDO}}$	$\mathrm{E}\overline{\mathrm{EDO}}$	$\mathrm{AE}\overline{\mathrm{EDO}}$	$\mathrm{ED}\overline{\mathrm{EDO}}$
$\mathrm{AED}\overline{\mathrm{EDO}}$	$\mathrm{L}\overline{\mathrm{EDO}}$	$\mathrm{LD}\overline{\mathrm{EDO}}$	$\mathrm{LE}\overline{\mathrm{EDO}}$	$\mathrm{LED}\overline{\mathrm{EDO}}$	$\mathrm{O}\overline{\mathrm{EDO}}$
$\mathrm{AO}\overline{\mathrm{EDO}}$	$\mathrm{DO}\overline{\mathrm{EDO}}$	$\mathrm{ADO}\overline{\mathrm{EDO}}$	$\mathrm{EO}\overline{\mathrm{EDO}}$	$\mathrm{AEO}\overline{\mathrm{EDO}}$	$\mathrm{EDO}\overline{\mathrm{EDO}}$
$\mathrm{AEDO}\overline{\mathrm{EDO}}$	$\mathrm{LO}\overline{\mathrm{EDO}}$	$\mathrm{LDO}\overline{\mathrm{EDO}}$	$\mathrm{LEO}\overline{\mathrm{EDO}}$	$\mathrm{LEDO}\overline{\mathrm{EDO}}$	$\overline{\mathrm{AEDO}}$
$\mathrm{A}\overline{\mathrm{AEDO}}$	$\mathrm{D}\overline{\mathrm{AEDO}}$	$\mathrm{AD}\overline{\mathrm{AEDO}}$	$\mathrm{E}\overline{\mathrm{AEDO}}$	$\mathrm{AE}\overline{\mathrm{AEDO}}$	$\mathrm{ED}\overline{\mathrm{AEDO}}$
$\mathrm{AED}\overline{\mathrm{AEDO}}$	$\mathrm{L}\overline{\mathrm{AEDO}}$	$\mathrm{LD}\overline{\mathrm{AEDO}}$	$\mathrm{LE}\overline{\mathrm{AEDO}}$	$\mathrm{LED}\overline{\mathrm{AEDO}}$	$\mathrm{O}\overline{\mathrm{AEDO}}$
$\mathrm{AO}\overline{\mathrm{AEDO}}$	$\mathrm{DO}\overline{\mathrm{AEDO}}$	$\mathrm{ADO}\overline{\mathrm{AEDO}}$	$\mathrm{EO}\overline{\mathrm{AEDO}}$	$\mathrm{AEO}\overline{\mathrm{AEDO}}$	$\mathrm{EDO}\overline{\mathrm{AEDO}}$
$\mathrm{AEDO}\overline{\mathrm{AEDO}}$	$\mathrm{LO}\overline{\mathrm{AEDO}}$	$\mathrm{LDO}\overline{\mathrm{AEDO}}$	$\mathrm{LEO}\overline{\mathrm{AEDO}}$	$\mathrm{LEDO}\overline{\mathrm{AEDO}}$	$\overline{\mathrm{LO}}$
$\mathrm{A}\overline{\mathrm{LO}}$	$\mathrm{B}\overline{\mathrm{LO}}$	$\mathrm{AB}\overline{\mathrm{LO}}$	$\mathrm{D}\overline{\mathrm{LO}}$	$\mathrm{AD}\overline{\mathrm{LO}}$	$\mathrm{BD}\overline{\mathrm{LO}}$
$\mathrm{ABD}\overline{\mathrm{LO}}$	$\mathrm{E}\overline{\mathrm{LO}}$	$\mathrm{AE}\overline{\mathrm{LO}}$	$\mathrm{BE}\overline{\mathrm{LO}}$	$\mathrm{ABE}\overline{\mathrm{LO}}$	$\mathrm{ED}\overline{\mathrm{LO}}$
$\mathrm{AED}\overline{\mathrm{LO}}$	$\mathrm{L}\overline{\mathrm{LO}}$	$\mathrm{LB}\overline{\mathrm{LO}}$	$\mathrm{LD}\overline{\mathrm{LO}}$	$\mathrm{LBD}\overline{\mathrm{LO}}$	$\mathrm{LE}\overline{\mathrm{LO}}$
$\mathrm{LBE}\overline{\mathrm{LO}}$	$\mathrm{LED}\overline{\mathrm{LO}}$	$\mathrm{O}\overline{\mathrm{LO}}$	$\mathrm{AO}\overline{\mathrm{LO}}$	$\mathrm{BO}\overline{\mathrm{LO}}$	$\mathrm{ABO}\overline{\mathrm{LO}}$
$\mathrm{DO}\overline{\mathrm{LO}}$	$\mathrm{ADO}\overline{\mathrm{LO}}$	$\mathrm{BDO}\overline{\mathrm{LO}}$	$\mathrm{ABDO}\overline{\mathrm{LO}}$	$\mathrm{EO}\overline{\mathrm{LO}}$	$\mathrm{AEO}\overline{\mathrm{LO}}$
$\mathrm{BEO}\overline{\mathrm{LO}}$	$\mathrm{ABEO}\overline{\mathrm{LO}}$	$\mathrm{EDO}\overline{\mathrm{LO}}$	$\mathrm{AEDO}\overline{\mathrm{LO}}$	$\mathrm{LO}\overline{\mathrm{LO}}$	$\mathrm{LBO}\overline{\mathrm{LO}}$
$\mathrm{LDO}\overline{\mathrm{LO}}$	$\mathrm{LBDO}\overline{\mathrm{LO}}$	$\mathrm{LEO}\overline{\mathrm{LO}}$	$\mathrm{LBEO}\overline{\mathrm{LO}}$	$\mathrm{LEDO}\overline{\mathrm{LO}}$	$\overline{\mathrm{LBO}}$
$\mathrm{A}\overline{\mathrm{LBO}}$	$\mathrm{B}\overline{\mathrm{LBO}}$	$\mathrm{AB}\overline{\mathrm{LBO}}$	$\mathrm{D}\overline{\mathrm{LBO}}$	$\mathrm{AD}\overline{\mathrm{LBO}}$	$\mathrm{BD}\overline{\mathrm{LBO}}$
$\mathrm{ABD}\overline{\mathrm{LBO}}$	$\mathrm{E}\overline{\mathrm{LBO}}$	$\mathrm{AE}\overline{\mathrm{LBO}}$	$\mathrm{BE}\overline{\mathrm{LBO}}$	$\mathrm{ABE}\overline{\mathrm{LBO}}$	$\mathrm{ED}\overline{\mathrm{LBO}}$
$\mathrm{AED}\overline{\mathrm{LBO}}$	$\mathrm{L}\overline{\mathrm{LBO}}$	$\mathrm{LB}\overline{\mathrm{LBO}}$	$\mathrm{LD}\overline{\mathrm{LBO}}$	$\mathrm{LBD}\overline{\mathrm{LBO}}$	$\mathrm{O}\overline{\mathrm{LBO}}$
$\mathrm{AO}\overline{\mathrm{LBO}}$	$\mathrm{BO}\overline{\mathrm{LBO}}$	$\mathrm{ABO}\overline{\mathrm{LBO}}$	$\mathrm{DO}\overline{\mathrm{LBO}}$	$\mathrm{ADO}\overline{\mathrm{LBO}}$	$\mathrm{BDO}\overline{\mathrm{LBO}}$
$\mathrm{ABDO}\overline{\mathrm{LBO}}$	$\mathrm{LO}\overline{\mathrm{LBO}}$	$\mathrm{LBO}\overline{\mathrm{LBO}}$	$\mathrm{LDO}\overline{\mathrm{LBO}}$	$\mathrm{LBDO}\overline{\mathrm{LBO}}$	$\overline{\mathrm{LDO}}$
$\mathrm{A}\overline{\mathrm{LDO}}$	$\mathrm{B}\overline{\mathrm{LDO}}$	$\mathrm{AB}\overline{\mathrm{LDO}}$	$\mathrm{D}\overline{\mathrm{LDO}}$	$\mathrm{AD}\overline{\mathrm{LDO}}$	$\mathrm{BD}\overline{\mathrm{LDO}}$
$\mathrm{ABD}\overline{\mathrm{LDO}}$	$\mathrm{E}\overline{\mathrm{LDO}}$	$\mathrm{AE}\overline{\mathrm{LDO}}$	$\mathrm{BE}\overline{\mathrm{LDO}}$	$\mathrm{ABE}\overline{\mathrm{LDO}}$	$\mathrm{ED}\overline{\mathrm{LDO}}$
$\mathrm{AED}\overline{\mathrm{LDO}}$	$\mathrm{L}\overline{\mathrm{LDO}}$	$\mathrm{LB}\overline{\mathrm{LDO}}$	$\mathrm{LD}\overline{\mathrm{LDO}}$	$\mathrm{LBD}\overline{\mathrm{LDO}}$	$\mathrm{LE}\overline{\mathrm{LDO}}$
$\mathrm{LBE}\overline{\mathrm{LDO}}$	$\mathrm{LED}\overline{\mathrm{LDO}}$	$\mathrm{O}\overline{\mathrm{LDO}}$	$\mathrm{AO}\overline{\mathrm{LDO}}$	$\mathrm{BO}\overline{\mathrm{LDO}}$	$\mathrm{ABO}\overline{\mathrm{LDO}}$
$\mathrm{DO}\overline{\mathrm{LDO}}$	$\mathrm{ADO}\overline{\mathrm{LDO}}$	$\mathrm{BDO}\overline{\mathrm{LDO}}$	$\mathrm{ABDO}\overline{\mathrm{LDO}}$	$\mathrm{EO}\overline{\mathrm{LDO}}$	$\mathrm{AEO}\overline{\mathrm{LDO}}$
$\mathrm{BEO}\overline{\mathrm{LDO}}$	$\mathrm{ABEO}\overline{\mathrm{LDO}}$	$\mathrm{EDO}\overline{\mathrm{LDO}}$	$\mathrm{AEDO}\overline{\mathrm{LDO}}$	$\mathrm{LO}\overline{\mathrm{LDO}}$	$\mathrm{LBO}\overline{\mathrm{LDO}}$
$\mathrm{LDO}\overline{\mathrm{LDO}}$	$\mathrm{LBDO}\overline{\mathrm{LDO}}$	$\mathrm{LEO}\overline{\mathrm{LDO}}$	$\mathrm{LBEO}\overline{\mathrm{LDO}}$	$\mathrm{LEDO}\overline{\mathrm{LDO}}$	$\overline{\mathrm{LBDO}}$
$\mathrm{A}\overline{\mathrm{LBDO}}$	$\mathrm{B}\overline{\mathrm{LBDO}}$	$\mathrm{AB}\overline{\mathrm{LBDO}}$	$\mathrm{D}\overline{\mathrm{LBDO}}$	$\mathrm{AD}\overline{\mathrm{LBDO}}$	$\mathrm{BD}\overline{\mathrm{LBDO}}$
$\mathrm{ABD}\overline{\mathrm{LBDO}}$	$\mathrm{E}\overline{\mathrm{LBDO}}$	$\mathrm{AE}\overline{\mathrm{LBDO}}$	$\mathrm{BE}\overline{\mathrm{LBDO}}$	$\mathrm{ABE}\overline{\mathrm{LBDO}}$	$\mathrm{ED}\overline{\mathrm{LBDO}}$
$\mathrm{AED}\overline{\mathrm{LBDO}}$	$\mathrm{L}\overline{\mathrm{LBDO}}$	$\mathrm{LB}\overline{\mathrm{LBDO}}$	$\mathrm{LD}\overline{\mathrm{LBDO}}$	$\mathrm{LBD}\overline{\mathrm{LBDO}}$	$\mathrm{O}\overline{\mathrm{LBDO}}$
$\mathrm{AO}\overline{\mathrm{LBDO}}$	$\mathrm{BO}\overline{\mathrm{LBDO}}$	$\mathrm{ABO}\overline{\mathrm{LBDO}}$	$\mathrm{DO}\overline{\mathrm{LBDO}}$	$\mathrm{ADO}\overline{\mathrm{LBDO}}$	$\mathrm{BDO}\overline{\mathrm{LBDO}}$
$\mathrm{ABDO}\overline{\mathrm{LBDO}}$	$\mathrm{LO}\overline{\mathrm{LBDO}}$	$\mathrm{LBO}\overline{\mathrm{LBDO}}$	$\mathrm{LDO}\overline{\mathrm{LBDO}}$	$\mathrm{LBDO}\overline{\mathrm{LBDO}}$	$\overline{\mathrm{LEO}}$
$\mathrm{A}\overline{\mathrm{LEO}}$	$\mathrm{D}\overline{\mathrm{LEO}}$	$\mathrm{AD}\overline{\mathrm{LEO}}$	$\mathrm{E}\overline{\mathrm{LEO}}$	$\mathrm{AE}\overline{\mathrm{LEO}}$	$\mathrm{ED}\overline{\mathrm{LEO}}$
$\mathrm{AED}\overline{\mathrm{LEO}}$	$\mathrm{L}\overline{\mathrm{LEO}}$	$\mathrm{LD}\overline{\mathrm{LEO}}$	$\mathrm{LE}\overline{\mathrm{LEO}}$	$\mathrm{LED}\overline{\mathrm{LEO}}$	$\mathrm{O}\overline{\mathrm{LEO}}$
$\mathrm{AO}\overline{\mathrm{LEO}}$	$\mathrm{DO}\overline{\mathrm{LEO}}$	$\mathrm{ADO}\overline{\mathrm{LEO}}$	$\mathrm{EO}\overline{\mathrm{LEO}}$	$\mathrm{AEO}\overline{\mathrm{LEO}}$	$\mathrm{EDO}\overline{\mathrm{LEO}}$
$\mathrm{AEDO}\overline{\mathrm{LEO}}$	$\mathrm{LO}\overline{\mathrm{LEO}}$	$\mathrm{LDO}\overline{\mathrm{LEO}}$	$\mathrm{LEO}\overline{\mathrm{LEO}}$	$\mathrm{LEDO}\overline{\mathrm{LEO}}$	$\overline{\mathrm{LBEO}}$
$\mathrm{A}\overline{\mathrm{LBEO}}$	$\mathrm{D}\overline{\mathrm{LBEO}}$	$\mathrm{AD}\overline{\mathrm{LBEO}}$	$\mathrm{E}\overline{\mathrm{LBEO}}$	$\mathrm{AE}\overline{\mathrm{LBEO}}$	$\mathrm{ED}\overline{\mathrm{LBEO}}$
$\mathrm{AED}\overline{\mathrm{LBEO}}$	$\mathrm{L}\overline{\mathrm{LBEO}}$	$\mathrm{LD}\overline{\mathrm{LBEO}}$	$\mathrm{O}\overline{\mathrm{LBEO}}$	$\mathrm{AO}\overline{\mathrm{LBEO}}$	$\mathrm{DO}\overline{\mathrm{LBEO}}$
$\mathrm{ADO}\overline{\mathrm{LBEO}}$	$\mathrm{LO}\overline{\mathrm{LBEO}}$	$\mathrm{LDO}\overline{\mathrm{LBEO}}$	$\overline{\mathrm{LEDO}}$	$\mathrm{A}\overline{\mathrm{LEDO}}$	$\mathrm{D}\overline{\mathrm{LEDO}}$

$\text{AD}\overline{\text{LEDO}}$	$\text{E}\overline{\text{LEDO}}$	$\text{AE}\overline{\text{LEDO}}$	$\text{ED}\overline{\text{LEDO}}$	$\text{AED}\overline{\text{LEDO}}$	$\text{L}\overline{\text{LEDO}}$
$\text{LD}\overline{\text{LEDO}}$	$\text{LE}\overline{\text{LEDO}}$	$\text{LED}\overline{\text{LEDO}}$	$\text{O}\overline{\text{LEDO}}$	$\text{AO}\overline{\text{LEDO}}$	$\text{DO}\overline{\text{LEDO}}$
$\text{ADO}\overline{\text{LEDO}}$	$\text{EO}\overline{\text{LEDO}}$	$\text{AEO}\overline{\text{LEDO}}$	$\text{EDO}\overline{\text{LEDO}}$	$\text{AEDO}\overline{\text{LEDO}}$	$\text{LO}\overline{\text{LEDO}}$
$\text{LDO}\overline{\text{LEDO}}$	$\text{LEO}\overline{\text{LEDO}}$	$\text{LEDO}\overline{\text{LEDO}}$			

A.1.3 Unknown fragments

D	BD	AE	ED	LD	LBD	LED	$\text{B}\overline{\text{A}}$
$\text{AB}\overline{\text{A}}$	$\text{AE}\overline{\text{A}}$	$\text{LB}\overline{\text{A}}$	$\text{D}\overline{\text{B}}$	$\text{BD}\overline{\text{B}}$	$\text{LD}\overline{\text{B}}$	$\text{LBD}\overline{\text{B}}$	$\overline{\text{AB}}$
$\text{A}\overline{\text{AB}}$	$\text{B}\overline{\text{AB}}$	$\text{L}\overline{\text{AB}}$	$\text{LB}\overline{\text{AB}}$	$\overline{\text{D}}$	$\text{B}\overline{\text{D}}$	$\text{D}\overline{\text{D}}$	$\text{BD}\overline{\text{D}}$
$\text{E}\overline{\text{D}}$	$\text{ED}\overline{\text{D}}$	$\text{L}\overline{\text{D}}$	$\text{LB}\overline{\text{D}}$	$\text{LD}\overline{\text{D}}$	$\text{LBD}\overline{\text{D}}$	$\text{LE}\overline{\text{D}}$	$\text{LED}\overline{\text{D}}$
$\overline{\text{BD}}$	$\text{B}\overline{\text{BD}}$	$\text{D}\overline{\text{BD}}$	$\text{BD}\overline{\text{BD}}$	$\text{L}\overline{\text{BD}}$	$\text{LB}\overline{\text{BD}}$	$\text{LD}\overline{\text{BD}}$	$\text{LBD}\overline{\text{BD}}$
$\text{A}\overline{\text{E}}$	$\text{D}\overline{\text{E}}$	$\text{AE}\overline{\text{E}}$	$\text{ED}\overline{\text{E}}$	$\text{LD}\overline{\text{E}}$	$\text{LED}\overline{\text{E}}$	$\text{A}\overline{\text{AE}}$	$\overline{\text{ED}}$
$\text{D}\overline{\text{ED}}$	$\text{E}\overline{\text{ED}}$	$\text{ED}\overline{\text{ED}}$	$\text{L}\overline{\text{ED}}$	$\text{LD}\overline{\text{ED}}$	$\text{LE}\overline{\text{ED}}$	$\text{LED}\overline{\text{ED}}$	$\text{D}\overline{\text{L}}$
$\text{BD}\overline{\text{L}}$	$\text{AE}\overline{\text{L}}$	$\text{ED}\overline{\text{L}}$	$\text{LD}\overline{\text{L}}$	$\text{LBD}\overline{\text{L}}$	$\text{LED}\overline{\text{L}}$	$\text{D}\overline{\text{LB}}$	$\text{BD}\overline{\text{LB}}$
$\text{LD}\overline{\text{LB}}$	$\text{LBD}\overline{\text{LB}}$	$\overline{\text{LD}}$	$\text{B}\overline{\text{LD}}$	$\text{D}\overline{\text{LD}}$	$\text{BD}\overline{\text{LD}}$	$\text{E}\overline{\text{LD}}$	$\text{ED}\overline{\text{LD}}$
$\text{L}\overline{\text{LD}}$	$\text{LB}\overline{\text{LD}}$	$\text{LD}\overline{\text{LD}}$	$\text{LBD}\overline{\text{LD}}$	$\text{LE}\overline{\text{LD}}$	$\text{LED}\overline{\text{LD}}$	$\overline{\text{LBD}}$	$\text{B}\overline{\text{LBD}}$
$\text{D}\overline{\text{LBD}}$	$\text{BD}\overline{\text{LBD}}$	$\text{L}\overline{\text{LBD}}$	$\text{LB}\overline{\text{LBD}}$	$\text{LD}\overline{\text{LBD}}$	$\text{LBD}\overline{\text{LBD}}$	$\text{A}\overline{\text{LE}}$	$\text{D}\overline{\text{LE}}$
$\text{AE}\overline{\text{LE}}$	$\text{ED}\overline{\text{LE}}$	$\text{LD}\overline{\text{LE}}$	$\text{LED}\overline{\text{LE}}$	$\overline{\text{LED}}$	$\text{D}\overline{\text{LED}}$	$\text{E}\overline{\text{LED}}$	$\text{ED}\overline{\text{LED}}$
$\text{L}\overline{\text{LED}}$	$\text{LD}\overline{\text{LED}}$	$\text{LE}\overline{\text{LED}}$	$\text{LED}\overline{\text{LED}}$				

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