DISSERTATION

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"Arboreal Forcing Notions and Regularity Properties of the Real Line"

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Introduction

The study of the regularity properties of the real line may be approached in several ways. Since the "birth" of Solovay's model, most of the studies in this area have been focused on Lebesgue measurability and Baire property. Furthermore, during the years, the interest of experts in this field has touched many other notions of regularity, like Sacks-, Miller-, Laver- and Silver-measurability, which we will introduce in section 1.1.3.

After Solovay's article [**So70**], one had to wait about fifteen years to see other results on this topic, comparable to those of Solovay, in terms of depth and appeal; in 1984, Shelah introduced a very profound and mysterious tool, called *amalgamation*, to construct Boolean algebras having strong homogeneity, one of the crucial properties of the Levy Collapse used by Solovay. In [**Sh84**], Shelah was able to solve the most intriguing question arisen from Solovay's work, i.e., one could build a model for **ZF**+**DC** plus the statement

$$\mathbf{BP} \equiv \text{“every set of reals has the Baire property”},$$

without using an inaccessible, whereas the analogous statement for Lebesgue measurability (**LM**) could not avoid the use of an inaccessible. Furthermore, in [**Sh85**], Shelah also solved another important problem, which was to separate **LM** from **BP**, i.e., to construct a model

$$\mathbf{N} \models \mathbf{LM} \wedge \neg\mathbf{BP}.$$

(Note that a model for $\mathbf{BP} \wedge \neg\mathbf{LM}$ is an indirect consequence of the result in [**Sh84**]).

The aim of this work is mainly devoted to solve analogous problems for other notions of regularity. Hence, we will analyze statements of the form

$$\Gamma(\mathbb{P}) \equiv^{\text{def}} \text{“every set of reals is } \mathbb{P}\text{-measurable”},$$

where $\mathbb{P}$-measurability will be introduced in definition 18 in a rather general way and it will be exactly our notion of regularity. More precisely, our notion of measurability will be strong enough to capture all of the most popular notions of regularity, such as Baire property, Lebesgue measurability, Sacks-, Silver-, Miller- and Laver-regularity. However, our notion will not consider asymmetric properties, like perfect set property.

In particular, we will try to construct models to separate such statements for different notions of measurability, i.e., to construct models $\mathbf{N}$ such that

$$\mathbf{N} \models \Gamma(\mathbb{P}) \wedge \neg\Gamma(\mathbb{Q}),$$

for different forcings $\mathbb{P}, \mathbb{Q}$.

As we said, we will focus on regularity properties coming from:

Sacks forcing. $\mathbb{S} = \{T \subseteq 2^{<\omega} : T \text{ is a perfect tree}\}$, ordered by $\subseteq$;

Miller forcing. $\mathbb{M} = \{T \subseteq \omega^{<\omega} : T \text{ is a superperfect tree}\}$, ordered by $\subseteq$;

Laver forcing.

$$\mathbb{L} = \{T \subseteq \omega^{<\omega} : T \text{ is a tree } \wedge \forall t \in T (t \trianglerighteq \textsc{Stem}(T) \Rightarrow t \text{ is } \omega\text{-splitting})\}$$

ordered by $\subseteq$;

Silver forcing. $\mathbb{V} = \{f : \mathrm{dom}(f) \to 2 : |\omega \setminus \mathrm{dom}(f)| = \omega\}$, ordered by $f \leq g \Leftrightarrow f \supseteq g$.

For instance, we will show that the implication

every $\mathbf{\Sigma}^1_2$ set is Silver measurable $\Rightarrow$ every $\mathbf{\Sigma}^1_2$ set is Miller measurable,

which is a corollary of proposition 3.7 in [**BLH05**] and theorem 6.1 in [**BL99**], does not extend to the family Γ consisting of all sets of reals. Our work will also concern results of separation regarding the second level of projective hierarchy. At such level many proofs are possible because of Shoenfield's absoluteness theorem; we remark that when one moves to the third level, the situation becomes more difficult; a possible way to preserve some results could be either Jensen's coding or the use of models closed under $\sharp$'s for sets of ordinals; the former has been used by Sy Friedman and David Schrittesser in [**FS10**] to build up a model where all projective sets are Lebesgue measurable, but there exists a $\mathbf{\Delta}^1_3$-set without Baire property. About the latter, Sy Friedman and I currently work on extending some results of separation presented in this thesis to the third level of projective hierarchy as well.

We conclude this introductive section with a schema of the thesis:

(1) Chapter 1 is divided into two different sections: the first one simply consists of a preliminary part concerning the approach to regularity properties of the real line, with some historical remarks; the second one is devoted to introduce some important tools for our work and it is split into two subsections, the first one concerning Shelah's amalgamation and the second one concerning forcings for adding trees of generic reals;

(2) Chapter 2 is itself divided into two main parts. The first one is devoted to results about separation of regularity properties for $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ sets, whereas the second concerns the work for the family Γ of all sets of reals.

(3) Finally, a last brief chapter is devoted to the study of the generalized Cantor space 2^κ, equipped with the topology induced by basic clopen sets $[\eta] = \{z \in 2^\kappa : z \rhd \eta\}$, for every $\eta \in 2^{<\kappa}$. Within this chapter we will analyze the behaviour of the natural generalized regularity properties, showing some known examples which underline the difference with the standard case. The most important result of this chapter will be the definition of a new notion of measure on 2^κ, which will give us a way to measure the Borel sets and to then define a notion of random forcing for 2^κ and the corresponding notion of Lebesgue measurability.

At the end, I would like to acknowledge the indispensable support that I have received from Sy-David Friedman, whose deep and brilliant suggestions have strongly improved the results in this dissertation. The great pleasure of working with him is an experience that I will never forget.

Chapter 1

Basic notions and tools for forcing constructions

This first chapter is divided into two main parts. The first one consists of an introduction to some basic notions and the historical background. The second one is more advanced, and it is dedicated to introduce some important tools, which one will need in the second chapter.

1.1 Basic definitions and preliminary results

1.1.1 Measure and category

As we mentioned in the introduction, the starting point of the study concerning regularity properties is given by the investigation around the Baire property and the Lebesgue measurability. In our work, when we will say "real", we will refer to elements of Baire space ω^ω and Cantor space 2^ω. Since these spaces are "almost homeomorphic", i.e., there is an homeomorphism $h : \omega^\omega \to 2^\omega \setminus C$, where C is countable, our questions about regularity properties are invariant between ω^ω and 2^ω. Hence, we will deal with any question in the space that one considers more suitable for the situation. After this brief remark, one can introduce the notions of measure and category in the Baire space (those for the Cantor space are analogous).

Work into the Baire space ω^ω, consisting of infinite sequences of natural numbers, endowed with the topology generated by basic clopen sets $[s] = \{x \in \omega^\omega : s \lhd x\}$, for all $s \in \omega^{<\omega}$. One can define the family $\textsc{Bor}$ consisting of Borel sets as the smallest family containing all basic clopen sets $[s]$'s and closed under countable union and complementation. Furthermore, one can also define the family $\textsc{Proj}$ of projective sets as follows:

$\boldsymbol{\Sigma}^1_0$ = family of open sets of ω^ω;

$\boldsymbol{\Sigma}^1_1$ = family of projections of closed sets;

$\mathbf{\Pi}^1_1$ = family of complements of $\mathbf{\Sigma}^1_1$ sets;

$\mathbf{\Sigma}^1_{n+1}$ = family of projections of $\mathbf{\Pi}^1_n$;

$\mathbf{\Pi}^1_{n+1}$ = family of complements of $\mathbf{\Sigma}^1_{n+1}$ sets;

$\text{PROJ} = \bigcup_{n\in\omega} \mathbf{\Sigma}^1_n = \bigcup_{n\in\omega} \mathbf{\Pi}^1_n$.

It is well-known that one can also set on the Baire space the standard Lebesgue measure μ such that for every $s \in \omega^{<\omega}$, $\mu([s]) = \Pi_{j<|s|} 2^{-(s(j)+1)}$. For $X \subseteq \omega^\omega$ one says that

X is *nowhere dense* $\Leftrightarrow$ the interior of the closure of X is empty;

X is *meager* $\Leftrightarrow$ X is the union of countably many nowhere dense sets;

X is *null* $\Leftrightarrow$ for every $\epsilon > 0$ there exists an open set O with $\mu(O) < \epsilon$ such that $X \subseteq O$.

One can then define the ideal of meager sets $\mathcal{M}$ and the ideal of null sets $\mathcal{N}$ simply as

$\mathcal{M} = \{X \subseteq \omega^\omega : X \text{ is meager}\}$, and

$\mathcal{N} = \{X \subseteq \omega^\omega : X \text{ is null}\}$

and consequently the notions of regularity associated with category and measure, respectively.

Definition 1. Given $X \subseteq \omega^\omega$. one says that

X has the *Baire property* $\Leftrightarrow \exists B \in \text{BOR}(X \triangle B \in \mathcal{M})$, and

X is *Lebesgue measurable* $\Leftrightarrow \exists B \in \text{BOR}(X \triangle B \in \mathcal{N})$.

It is well-known that every Borel set (and also every $\mathbf{\Sigma}^1_1$ set) is Lebesgue measurable and has the Baire property. Nevertheless, under the axiom of choice **AC**, one can construct "bad" sets, such as the non-principal ultrafilter on ω and Vitali's set, which are examples of sets without Baire property and non-Lebesgue measurable. So the natural question which arises is to understand whether **AC** is really necessary to get non-regular sets, or to realize which family of regular sets of reals is consistent with **AC**.

Both problems were solved by Solovay, and the model named after him, obtained by collapsing an inaccessible cardinal κ to ω_1, has represented (and probably still represents nowadays) the corner-stone and the tip of the iceberg for studying the behaviour of regularity properties. Hence, we have to state this famous result. For a good comprehension we remind the following:

BP $\equiv$ "every set of reals has the Baire property",

$\mathbf{BP}^{\mathbf{L}(\omega^\omega)} \equiv$ "every set of reals in $\mathbf{L}(\omega^\omega)$ has the Baire property",

$\mathbf{LM} \equiv$ "every set of reals is Lebesgue measurable",

$\mathbf{LM}^{\mathbf{L}(\omega^\omega)} \equiv$ "every set of reals in $\mathbf{L}(\omega^\omega)$ is Lebesgue measurable".

Theorem 2 (Solovay,1970). *Let κ be an inaccessible cardinal, i.e., κ is regular and $\forall\alpha < \kappa, 2^\alpha < \kappa$, and let $\boldsymbol{Coll}(\omega, \kappa)$ be the forcing to collapse κ to ω_1, defined as*

$$\begin{aligned}\boldsymbol{Coll}(\omega,\kappa) \quad = \quad & \{p : \mathrm{dom}(p) \subseteq \kappa \times \omega \wedge |p| < \omega \wedge \\ & \wedge \forall(\alpha, n) \in \mathrm{dom}(p)(p(\alpha, n) \in \alpha)\},\end{aligned}$$

ordered by extension. Finally, let G be a $\boldsymbol{Coll}(\omega,\kappa)$-generic filter over $\mathbf{V}$. Then

$$\mathbf{V}[G] \models \boldsymbol{ZFC} + \boldsymbol{BP}^{\mathbf{L}(\omega^\omega)} + \boldsymbol{LM}^{\mathbf{L}(\omega^\omega)}$$

and

$$\mathbf{L}(\omega^\omega)^{\mathbf{V}[G]} \models \boldsymbol{ZF}\text{+}\boldsymbol{DC} + \boldsymbol{BP} + \boldsymbol{LM}.$$

Beyond Solovay's article [**So70**], other presentations of this popular theorem can be found in [**Kan**] (pg. 139) and [**Jech**] (pg. 519). We said that this theorem is the tip of the iceberg of a wide research field as many natural and interesting questions come immediately out.

> **Question (a)**. Is the inaccessible really necessary to get models for **BP** and **LM**?
>
> **Question (b)**. Can one construct a model for **BP** $\wedge\neg$**LM** and, viceversa , for **LM** $\wedge\neg$**BP**?

As we already mentioned in the introduction, these problems were solved by Shelah and we refer the reader to the introduction for the answers.

Crucial notions of forcing, which are used for Lebesgue measurability and Baire property, are random forcing $\mathbb{R}$ and Cohen forcing $\mathbb{C}$, respectively:

> [Random Forcing]. $\mathbb{R} = \{B \subseteq \omega^\omega : B \text{ is closed}, \mu(B) > 0\}$, ordered by inclusion;
>
> [Cohen Forcing]. $\mathbb{C} = \{[s] : s \in \omega^{<\omega}\}$, ordered by inclusion, which is isomorphic to the original Cohen forcing.

One of the most profound connections between these forcings and the related regularity properties is underlined by the following well-known fact.

Lemma 3. *Let $\mathbb{C}(\mathbf{V})$ be the set of Cohen reals over $\mathbf{V}$, and, analogously, let $\mathbb{R}(\mathbf{V})$ be the set of random reals. Then*

$$\mathbb{C}(\mathbf{V}) = \omega^\omega \setminus \bigcup(\mathcal{M} \cap \mathbf{V}), \text{ and } \mathbb{R}(\mathbf{V}) = \omega^\omega \setminus \bigcup(\mathcal{N} \cap \mathbf{V}),$$

where $\bigcup(\mathcal{M} \cap \mathbf{V})$ is the union of all Borel meager sets coded in the ground model $\mathbf{V}$ (and analogously for $\mathcal{N}$). The left-right inclusion (i.e., $\subseteq$) can be proved by an easy density argument, and it is true in a very general setting, whereas the right-left inclusion only holds for ccc forcings.

Beyond Lebesgue measurability and Baire property, Solovay's model satisfies many other regularity properties of the Baire space. For example, set $[s, f] = \{x \in \omega^\omega : x \rhd s \wedge \forall n \in \omega(x(n) \geq f(n))\}$, where $s \in \omega^{<\omega}$, $f \in \omega^\omega$ and $f \rhd s$ (i.e., f extends s); then, consider the topology δ generated by such basic open sets. One can easily remark that δ is finer than the standard topology, and $\textsc{Bor}(\delta) \supseteq \textsc{Bor}$ (where the left member represents the family of Borel sets of ω^ω w.r.t. δ). Furthermore, since the notion of nowhere dense is different, also the notion of Baire property associated with δ (which we call δ-Baire property) is different from the standard Baire property. Nevertheless, with a simple generalization of the proof to get the Baire property of all sets in $\mathbf{L}(\omega^\omega)$ inside Solovay's model, one can also easily get

$$\mathbf{V}[G] \models \text{“every set of reals in } \mathbf{L}(\omega^\omega) \text{ has the } \delta\text{-Baire property”}.$$

Obviously, the forcing associated is the Hechler forcing

$$\mathbb{D} = \{(s, f) : s \in \omega^{<\omega} \wedge f \in \omega^\omega \wedge s \lhd f\},$$

ordered by $(s', f') \leq (s, f) \Leftrightarrow s' \trianglerighteq s \wedge \forall n(f(n) \leq f'(n))$. An analogous of Lemma 3 for the set of Hechler reals $\mathbb{D}(\mathbf{V})$ can be stated also in this case:

$$\mathbb{D}(\mathbf{V}) = \omega^\omega \setminus \bigcup(\mathcal{M}(\delta) \cap \mathbf{V}),$$

where $\mathcal{M}(\delta)$ is the ideal of meager sets w.r.t. δ.

To conclude this paragraph, we remark that a very general result to obtain regularity properties inside Solovay's model is proved by Yurii Khomskii in [**K12**], proposition 2.2.8.

1.1.2 Trees

The notion of tree on ω is very useful to characterize closed sets of ω^ω and furthermore, many interesting forcings has been invented by using trees.

Definition 4. $T \subseteq \omega^{<\omega}$ is an infinite tree if and only if:

(i) if $t \in T$ then $\forall s \trianglelefteq t$, $s \in T$;

(ii) for every $t \in T$ there exists $t' \trianglerighteq t$, $t' \in T$.

Note that (ii) ensures the tree to be well-pruned, i.e., there are no terminal nodes. We also introduce some standard notations.

Definition 5. Given an infinite tree $T \subseteq \omega^{<\omega}$ we define:

- $t = \textsc{Stem}(T)$ iff $\forall t' \in T(t' \trianglelefteq t \vee t' \trianglerighteq t)$;
- $t \in \textsc{Split}(T)$ iff $\exists k_0, k_1 \in \omega(t^\frown k_0 \in T \wedge t^\frown k_1 \in T)$;
- $t \in \textsc{Split}_n(T)$ iff $t \in \textsc{Split}(T) \wedge \exists j_0 < j_1 < \cdots < j_{n-1} \forall i < n(t \restriction j_i \in \textsc{Split}(T))$ and we say that t is an $n+1$-st splitting node of T (for $n = 0$, we set $\textsc{Split}_0(T) = \textsc{Stem}(T)$).
- $n \in \textsc{Succ}(s, T)$ iff $s^\frown n \in T$, (for a fixed $s \in T$);
- $t \in \mathrm{Lv}(n, T)$ iff $|t| = n \wedge t \in T$, where $|t|$ is the length of t;
- $[T] = \{x \in \omega^\omega : \forall n \in \omega(x \restriction n \in T)\}$, ($[T]$ is called the *body* of T).
- $T|t = \{s \in T : s \trianglerighteq t \vee s \trianglelefteq t\}$. Note that we will also use the notation T_t to denote such a subtree;
- $T \restriction n = \{t \in T : |t| \leq n\}$. Note that $T \restriction n$ is necessarily a finite tree.

In some cases we will also deal with finite trees. In this case we will use the following notations:

- $t \in \textsc{Term}(T)$ iff t in a terminal node of T, i.e., there is no $t' \triangleright t$ such that $t' \in T$;
- $\mathrm{ht}(T) = \max\{|t| : t \in \textsc{Term}(T)\}$.

It is straightforward that same definitions can be given for $2^{<\omega}$, in place of $\omega^{<\omega}$.

Fact 6. *For every tree $T \subseteq \omega^{<\omega}$, $[T]$ is closed. Conversely, If $C \subseteq \omega^\omega$ is closed then there exists a tree T such that $[T] = C$.*

The proof immediately follows from the definition.

As we mentioned at the beginning, some specific trees have become very popular, because of their importance in the method of forcing.

Definition 7.

- $T \subseteq 2^{<\omega}$ is a *Sacks tree* (or perfect tree) if and only if for every node $t \in T$, there exists $t' \trianglerighteq t, t' \in T$ such that $t' \in \textsc{Split}(T)$;
- $T \subseteq 2^{<\omega}$ is a *Silver tree* (or uniform tree) iff T is perfect and for every $s, t \in T$, such that $|s| = |t|$, one has $s^\frown 0 \in T \Leftrightarrow t^\frown 0 \in T$ and $s^\frown 1 \in T \Leftrightarrow t^\frown 1 \in T$.
- $T \subseteq \omega^{<\omega}$ is a *Miller tree* (or superperfect tree) if and only if for every $t \in T$ there exists $t' \trianglerighteq t, t' \in T$ such that $t' \in \textsc{Split}(T)$ and $|\textsc{Succ}(t', T)| = \omega$ (we will call such nodes ω-splitting, and we will indicate them with ω-$\textsc{Split}$);
- $T \subseteq \omega^{<\omega}$ is a *Laver tree* if and only if for every $t \in T, t \trianglerighteq \textsc{Stem}(T)$, one has $t \in \omega$-$\textsc{Split}(T)$.

In the introduction we have also presented the related forcing notions $\mathbb{S}, \mathbb{M}$, $\mathbb{V}$ and $\mathbb{L}$, respectively. For Sacks forcing $\mathbb{S}$, a notion of regularity property was introduced by Bernstein and it is known as *Bernstein partition property* (BPP): $X \subseteq 2^\omega$ has the BPP if and only if

$$\forall T \in \mathbb{S} \exists T' \in \mathbb{S}, T' \subseteq T([T'] \subseteq X \vee [T'] \cap X = \emptyset).$$

It is clear that one can analogously define a notion of regularity associated with Miller forcing $\mathbb{M}$, Silver forcing $\mathbb{V}$ and Laver forcing $\mathbb{L}$. To finish this section, we give a proof that the Miller property for every set of reals holds in Solovay's model. Note that such a result is well-known and we decide to present it here only to figure out which properties of Solovay's model are essential for some questions which will come out afterward (see the paragraph in next section concerning Shelah's amalgamation).

Fact 8. *Let* $\mathbf{V}[G]$ *be Solovay's model obtained by collapsing an inaccessible* κ *to* ω_1*. Then*

$$\mathbf{V}[G] \models \text{“every set of reals in } \mathbf{L}(\omega^\omega) \text{ has the Miller property”}.$$

Proof. We remind the following two key lemmata, whose proofs can be found in [**Kan**], proposition 10.21 and lemma 11.12. In both, G is $\mathbf{Coll}(\omega, \kappa)$-generic over $\mathbf{V}$.

Lemma 9. [Factor Lemma] *Let* $x \in \mathrm{On}^\omega \cap \mathbf{V}[G]$*. Then there exists a* $\boldsymbol{Coll}(\omega, \kappa)$*-generic filter* G' *over* $\mathbf{V}[x]$ *such that* $\mathbf{V}[G] = \mathbf{V}[x][G']$.

Lemma 10. *For every formula* ψ *there exists a formula* φ *such that, for every* $x \in \omega^\omega \cap \mathbf{V}[G]$*,*

$$\mathbf{V}[G] \models \psi(x) \Leftrightarrow \mathbf{V}[x] \models \varphi(x).$$

Let $X \subseteq \omega^\omega$ and let ψ and $v \in \mathrm{On}^\omega \cap \mathbf{V}[G]$ such that $X = \{x \in \omega^\omega : \psi(x, v)\}$. By κ-cc, there is $\alpha < \kappa$ such that $v \in \mathbf{V}[G{\restriction}\alpha]$. From now on, $\mathbf{V}[G{\restriction}\alpha]$ will be our new ground model. Consider the formula φ as in the above lemma. Consider the forcing $\mathbb{MT}$ for adding a Miller tree of Miller reals inside any ground model Miller tree, which we will introduce at page 25; one can easily show that such a forcing completely embeds into $\mathbf{Coll}(\omega, \kappa)$. Furthermore, note that $\mathbf{Coll}(\omega, \kappa)/\mathbf{Coll}(\omega, \alpha) = \mathbf{Coll}(\alpha, \kappa) \approx \mathbf{Coll}(\omega, \kappa)$. These two facts together give, in $\mathbf{V}[G{\restriction}\alpha]$, for every $W \in \mathbb{M} \cap \mathbf{V}[G{\restriction}\alpha]$,

$$\Vdash_{\mathbf{Coll}(\alpha,\kappa)} (\exists T \in \mathbb{M})(T \subseteq W)(\forall z \in [T])(z \text{ is Miller over } \mathbf{V}[G{\restriction}\alpha]).$$

Work in $\mathbf{V}[G{\restriction}\alpha]$. Let $P \lessdot \mathbf{Coll}(\alpha, \kappa)$ represents the subforcing equivalent to $\mathbb{M}$. Let $\dot{z}$ be a P-name for a P-generic real. Two cases are possible

$$\exists p \in P(p \Vdash \varphi(\dot{z})) \vee \exists p \in P(p \Vdash \neg\varphi(\dot{z})).$$

W.l.o.g., assume the first holds. For every P-generic filter H such that $p \in H$ one obtains

$$\mathbf{V}[G{\restriction}\alpha][H] \models \varphi(\dot{z}^H).$$

From now on, we will call a P-generic simply Miller real and we will indicate it with z.

Note that the condition $p \in P$ can be seen as a Miller tree and so one can consider in $\mathbf{V}[G]$ a Miller tree T of Miller reals inside p. Hence,

$$\text{for every } z \in [T] \Rightarrow \mathbf{V}[G{\restriction}\alpha][z] \models \varphi(z). \tag{1.1}$$

As a consequence of the two above lemmata one obtains

$$\mathbf{V}[G] \models \forall x \in \omega^\omega (x \in [T] \Rightarrow \psi(x)),$$

and so $\mathbf{V}[G] \models [T] \subseteq X$.

Similarly, the case $p \Vdash \neg\varphi(\dot{z})$ provides a Miller tree T such that $\mathbf{V}[G] \models [T] \cap X = \emptyset$. □

1.1.3 A general approach to regularity properties

In the previous two subsections we have seen several types of regularity properties. In this subsection we will present three different abstract manners to introduce notions of regularity for sets of reals and to then show a more general point of view, whereby we will be able to see under the same light all of the regularity properties previously introduced.

Idealized forcing. This notion is essentially a first generalization of Baire property and Lebesgue measurability.

Definition 11. Let $\mathcal{I}$ be an ideal on ω^ω. For every $X \subseteq \omega^\omega$, one says

$$X \text{ is } \mathcal{I}\textit{-null} \Leftrightarrow X \in \mathcal{I}$$

and

$$X \text{ is } \mathcal{I}\textit{-regular} \Leftrightarrow \exists B \in \textsc{Bor}(X \triangle B \in \mathcal{I}).$$

It is straightforward to note that the $\mathcal{M}$-regularity and $\mathcal{N}$-regularity correspond to the Baire property and to the Lebesgue measurability, respectively. As for the specific cases of Cohen forcing and random forcing, the ideal $\mathcal{I}$ shows a natural way to introduce forcing notions.

Definition 12. Let $\textsc{Bor}^*(\omega^\omega) = \textsc{Bor}(\omega^\omega) \setminus \mathcal{I}$. An *$\mathcal{I}$-forcing* (idealized forcing) is the partial order $\mathbb{P}_\mathcal{I}$ defined as

$$\mathbb{P}_\mathcal{I} = \textsc{Bor}^*(\omega^\omega)/\mathcal{I},$$

ordered by inclusion.

In [**Za00**], Zapletal showed that many well-known forcings are of this form, in particular all of those we are interested in (like Sacks forcing, Miller forcing, Laver forcing and Silver forcing), but the naive notion of $\mathcal{I}$-regularity is the right one only for ccc forcings. As an example, one may consider the case of Sacks forcing; the corresponding ideal $\mathcal{I}_{\mathbb{S}}$ is the family of countable subsets of ω^ω; however, in such a case, being $\mathcal{I}_{\mathbb{S}}$-regular would mean being Borel, which obviously does not correspond to the usual notion of Sacks measurability (i.e. either the set or its complement contains the set of branches through a perfect tree). The more experienced reader could object that we should consider the ideal of $\mathbb{S}$-null sets (see definition 15 later on) instead of the ideal of countable sets; nevertheless, even in that case one would not obtain the appropriate notion of regularity.

Topological forcings and τ-Baire property. Another class of forcings which appears as a generalization of the Cohen forcing can be defined as follows.

Definition 13. A poset $\mathbb{P}$ is called a *topological forcing* whenever one can associate every $p \in \mathbb{P}$ with a basic neighborhood U_p in such a way that the family $\mathcal{U}_{\mathbb{P}} = \{U_p : p \in \mathbb{P}\}$ generates a topology τ on ω^ω, and $\mathcal{U}_{\mathbb{P}}$, ordered by inclusion, is forcing equivalent to $\mathbb{P}$.

The obvious concepts of smallness and regularity related to this class of forcings are those induced by the topology τ. Thus, the small sets are those belonging to the ideal of τ-meager sets $\mathcal{M}(\tau)$ and the regular sets are exactly those having the τ-Baire property. Besides the Cohen forcing $\mathbb{C}$, other forcing notions belonging to this class are the Hechler forcing $\mathbb{D}$, the eventually different forcing $\mathbb{E}$, the Mathias forcing $\mathbb{MA}$. (A detailed study of $\mathbb{D}$ and $\mathbb{E}$ can be found in [**LR95**] and [**L96**], respectively, while for $\mathbb{MA}$ one may see [**Jech**], pg.524-529). As above, this definition of topological posets captures the other forcings as well, such as Sacks, Miller and Laver. Nevertheless, in such cases the notion of regularity property is not the suitable one again.

Arboreal forcings and $\mathbb{P}$-property. Finally, we give a natural generalization of forcings like Sacks, Miller and Laver.

Definition 14. A forcing $\mathbb{P}$ is *arboreal* if every element $T \in \mathbb{P}$ is a perfect tree of $\omega^{<\omega}$ and for every node $t \in T$, one has $T_t \in \mathbb{P}$, where $T_t = \{s \in T : s \trianglerighteq t \lor t \trianglerighteq s\}$, and $\mathbb{P}$ is ordered by inclusion.

One can actually see also the other posets as forcings of this sort.

Cohen forcing. Each basic open set $[s]$ can be seen as the body of the tree $T_s = \{t \in 2^{<\omega} : t \trianglerighteq s\}$. Thus,

$$\mathbb{C} \equiv \{T_s \subseteq 2^{<\omega} : s \in 2^{<\omega}\}.$$

Random forcing. Each closed set can be identified with a tree $T \subseteq 2^{<\omega}$, and therefore

$$\mathbb{R} \equiv \{T \subseteq 2^{<\omega} : \mu([T]) > 0 \wedge \forall t \in T(\mu([T_t]) > 0)\}.$$

Silver forcing. $g \in \mathbb{V}$ is associated with

$$T_g = \{t \in 2^{<\omega} : \forall n \in \mathrm{dom}(t) \cap \mathrm{dom}(g)(t(n) = g(n))\}.$$

About the last one, the idea is to associate a function $g \in \mathbb{V}$ with a tree in such a way that for those $n \notin \mathrm{dom}(g)$ every node of length n is a splitting node, otherwise when $n \in \mathrm{dom}(g)$ for every $s, t \in T_g$ (of length $> n$), $t(n) = s(n) = g(n)$. (Morally speaking, in the second case the tree decides the value of the new real, according to the function g, while in the first it can not decide the value, according to the fact that the function g is not defined).

One can easily note that, if one orders by inclusion each poset just defined, one gets equivalent forms of the usual ones. Note that the list is not complete; in fact we could have added Mathias forcing $\mathbb{MA}$, Hechler forcing $\mathbb{D}$ and eventually different forcing $\mathbb{E}$. Nevertheless, since they will not take part of our study in the next chapter, we have not quoted them in the above list.

As we said above, one may introduce notions of smallness and regularity related to arboreal forcings.

Definition 15. For every $X \subseteq \omega^\omega$,

$$X \text{ is } \mathbb{P}\text{-}null \Leftrightarrow \forall T \in \mathbb{P} \exists S \in \mathbb{P}(S \leq T \wedge [S] \cap X = \emptyset)$$

and

$$X \text{ has the } \mathbb{P}\text{-}property \Leftrightarrow \forall T \in \mathbb{P} \exists S \in \mathbb{P}\big((S \leq T) \wedge ([S] \cap X = \emptyset \vee [S] \subseteq X)\big).$$

The set $\mathcal{J}_\mathbb{P} = \{X \subseteq \omega^\omega : X \text{ is } \mathbb{P}\text{-null}\}$ is an ideal, but in general it is not a σ-ideal (for instance $\mathcal{J}_\mathbb{C}$ is the ideal of nowhere dense sets which is not a σ-ideal). Hence, we will consider its closure under countable unions, i.e.,

$$\mathcal{I}_\mathbb{P} = \{Y \subseteq \omega^\omega : Y \subseteq \bigcup_{n \in \omega} X_n, \text{ for some } X_n \in \mathcal{J}_\mathbb{P}\}.$$

In some cases, the $\mathbb{P}$-property can be verified in an easier way, as the following result shows. Remind that Γ is the family of all sets of reals, and $\Gamma(\mathbb{P})$ is the statement "all sets of reals have the $\mathbb{P}$-property".

Lemma 16. *Let $\mathbb{P} \in \{\mathbb{S}, \mathbb{V}, \mathbb{M}, \mathbb{L}\}$. Then $\Gamma(\mathbb{P})$ is equivalent to require that for every $X \in \Gamma$,*

$$\exists T \in \mathbb{P}([T] \subseteq X \vee [T] \cap X = \emptyset).$$

For a proof one can see [**BL99**], lemma 2.1. Actually, such a result can be proved in a more general setting, replacing the family Γ with any topologically reasonable family Θ, which we will introduce at the beginning of chapter 2.

Remark 17. Unfortunately, even if it has been easy to see that this notion of arboreal forcing captures all of the posets which we are interested in, the related notion of regularity does not. For example, the Baire property does not correspond to the $\mathbb{C}$-property; in fact, in this particular case, the $\mathbb{C}$-property is rather senseless, since not even the set of rationals satisfies the $\mathbb{C}$-property. However, a slight modification of definition 15 will give us the notion of regularity we are interested in.

A unique notion of regularity.

Definition 18. For every $X \subseteq \omega^\omega$,

$$X \text{ is } \mathbb{P}\textit{-measurable} \quad \Leftrightarrow \quad \forall T \in \mathbb{P} \exists S \in \mathbb{P} \\ \big((S \leq T) \wedge ([S] \cap X \in \mathcal{I}_\mathbb{P} \vee [S] \setminus X \in \mathcal{I}_\mathbb{P})\big).$$

(Notation: we will often say Sacks measurable, Miller measurable, Silver measurable and so on).

One may show that, whenever $\mathbb{P}$ allows a fusion argument, $\mathcal{J}_\mathbb{P}$ is actually a σ-ideal and so $\mathcal{J}_\mathbb{P} = \mathcal{I}_\mathbb{P}$. As an easy consequence, one gets that, if $[T] \cap X \in \mathcal{I}_\mathbb{P} = \mathcal{J}_\mathbb{P}$, then there is $T' \leq T$ such that $[T'] \cap X = \emptyset$, simply by definition of $\mathbb{P}$-null. Hence, in such cases, definition 18 is equivalent to definition 15. Furthermore, Ikegami proved in [**Ik10**], that, for any ccc arboreal forcing notion $\mathbb{P}$,

$$\mathcal{I}_\mathbb{P}\text{-regularity} \Leftrightarrow \mathbb{P}\text{-measurability},$$

and so, for all of the forcings of our interest, the notion of $\mathbb{P}$-measurability is exactly the suitable one.

Once again, we remark that a detailed and enlightening exposition to regularity properties, by using idealized forcing in place of arboreal forcings, can be found in [**K12**].

A word about the $\mathbb{P}$-generic filter and the reals added by $\mathbb{P}$. In the previous paragraphs we introduced a general notion of forcing, called arboreal forcing, which will be central in our study throughout the thesis. It is therefore necessary to understand which new objects such forcings add into the model and which properties such objects satisfy. Like Cohen and random forcing, it is clear that one can easily associate a $\mathbb{P}$-generic filter G over $\mathbf{V}$ with a unique generic sequence z_G, i.e.,

$$z_G =^{\mathbf{def}} \bigcup \{\textsc{Stem}(T) : T \in G\} = \bigcap \{[T] : T \in G\}.$$

We will often refer to these sequences by using the word *real* and calling them Silver reals, Sacks reals, Miller reals and so on, according to which forcing we will deal with. The features of such z_G depend on which arboreal forcing $\mathbb{P}$ one considers. In what follows, we will use the following notation:

$$\exists^\infty n \equiv \forall m \exists n \geq m, \text{ and } \forall^\infty n \equiv \exists m \forall n \geq m.$$

We list the main properties with the relative proofs, or sketches of them:

- *Miller reals are unbounded over* $\mathbf{V}$. Remind that $z \in \omega^\omega$ is *unbounded over* $\mathbf{V}$ iff
$$\forall x \in \omega^\omega \cap \mathbf{V} \exists^\infty n(x(n) < z(n)).$$
To show that Miller reals have such a property, one can easily note that, for every $n \in \omega$ and $x \in \omega^\omega \cap \mathbf{V}$,
$$D(n,x) = \{T \in \mathbb{M} : \exists m \geq n(|\textsc{Stem}(T)| = m+1 \wedge \textsc{Stem}(T)(m) > x(m))\}$$
is open dense in $\mathbb{M}$. Hence, given $x \in \omega^\omega \cap \mathbf{V}$, $n \in \omega$ and $T \in G$, since $G \cap D(n,x) \neq \emptyset$, there exists $T' \leq T$, $T' \in G$ such that $T' \in D(n,x)$, and therefore there exists $m \geq n$ for which
$$T' \Vdash z_G(m) = \textsc{Stem}(T')(m) > x(m).$$
An analogous argument works for Cohen reals as well.

- *Laver reals are dominating over* $\mathbf{V}$. Remind that $z \in \omega^\omega$ is *dominating over* $\mathbf{V}$ iff
$$\forall x \in \omega^\omega \cap \mathbf{V} \forall^\infty n(x(n) \leq z(n)).$$
Note that, for every $x \in \omega^\omega \cap \mathbf{V}$, the set
$$D(x) = \{T \in \mathbb{L} : \forall t \trianglerighteq \textsc{Stem}(T)(t(|t|-1) \geq x(|t|-1))\}$$
is open dense in $\mathbb{L}$. For every $x \in \omega^\omega \cap \mathbf{V}$ and $T \in G$, $G \cap D(x) \neq \emptyset$, and therefore there exists $T' \in G$, $T' \leq T$ such that $T' \in D(x)$ and
$$T' \Vdash \forall n \geq |\textsc{Stem}(T')|(z_G(n) \geq x(n)).$$
An analogous argument works for Hechler reals and Mathias reals as well.

Such a list stated some properties satisfied by specific generic reals. However, another interesting point is to find properties which are satisfied by all of the reals added by a specific arboreal forcing $\mathbb{P}$. As before, we state them and we give some sketches of the proofs:

- *Miller forcing and Cohen forcing does not add dominating reals.* The two proofs are different. About Cohen forcing, one can see [**BJ95**], Lemma 3.1.2, page 100. We will give a proof for Miller forcing, by using a fusion argument. Let $\dot{f} \in \omega^\omega \cap \mathbf{V}^{\mathbb{M}}$ and $T \in \mathbb{M}$. We want to build $T' \leq T$ as a limit of a fusion sequence and $z \in \omega^\omega \cap \mathbf{V}$ such that $T' \Vdash \exists^\infty n(\dot{f}(n) < z(n))$. Remind that a fusion sequence $\langle T_n : n \in \omega \rangle$ of Miller trees satisfies $T_{n+1} \leq_n T_n$, where

  $$\begin{aligned} T \leq_n S \Leftrightarrow \quad & T \leq S \wedge \forall j \leq n \forall t \in \textsc{Split}_j(S) \\ & (t \in \textsc{Split}_j(T) \wedge \textsc{Succ}(t,S) = \textsc{Succ}(t,T)). \end{aligned}$$

  (Remind that $\textsc{Split}_j(T)$ is the set of $j+1$-st splitting nodes of T). It is clear that $\bigcap_{n\in\omega} T_n \in \mathbb{M}$.

  We build the fusion sequence by induction; for the construction we need to fix a bijection $\phi : \omega^{<\omega} \leftrightarrow \omega$.

  Start from $T_0 = T$. Then assume T_n already defined. Let $\sigma_n = \{t_w : w \in \omega^n\}$ be the set of $n+1$-st splitting nodes of T_n and $\tau^n_w = \{i_k \in \omega : i_k \in \textsc{Succ}(t_w, T_n)\}$. For every $w \in \omega^n$ and $k \in \omega$, choose $S^k_w \leq T_n | t_w^\frown i_k$ and $a(w,k,n) \in \omega$ such that

  $$S^k_w \Vdash \dot{f}(\phi(w^\frown k)) = a(w,k,n).$$

  Finally, put $T_{n+1} = \bigcup\{S^k_w : w \in \omega^n, k \in \omega\}$. Clearly, $T_{n+1} \leq_n T_n$ and so one can consider $T' = \bigcap_{n\in\omega} T_n$. Furthermore, one defines the suitable $z \in \omega^\omega \cap \mathbf{V}$ as follows:

  for every $j \in \omega$, $z(j) = a(w,k,n)+1$, for w,k,n such that $\phi(w^\frown k) = j$.

  It is left to show that, for every $\mathbb{M}$-generic x over $\mathbf{V}$ belonging to $[T']$,

  $$\mathbf{V}[x] \models \exists^\infty n(\dot{f}(n) < z(n)).$$

  To see that, let Σ be the set of splitting nodes of T' and $\psi : \Sigma \leftrightarrow \omega^{<\omega}$ be the natural isomorphism preserving inclusion.

  Furthermore, for every $t \in \Sigma$, let $e_t : \textsc{Succ}(t, T') \leftrightarrow \omega$ be an enumeration of the successors of t in T'. Hence,

  $$\forall i \in \omega, x{\restriction}i \in \textsc{Split}(T') \Rightarrow \dot{f}^x(m_i) < z(m_i),$$

  where $m_i =^{\mathbf{def}} \phi(\psi(x{\restriction}i)^\frown e_{x{\restriction}i}(x(i)))$. Since there are infinitely many i's for which $x{\restriction}i$ splits, the proof is completed.

- *Sacks forcing, Silver forcing and random forcing are ω^ω-bounding.* Remind that a forcing is ω^ω-bounding if it adds no unbounded reals over the ground model. About random forcing, one can see again [**BJ95**], Lemma 3.1.2 at page 100. The analogous results for Sacks forcing $\mathbb{S}$ and Silver forcing $\mathbb{V}$ are well-known as well, and the proofs use a standard fusion argument.

- *Miller forcing and Laver forcing adds neither Cohen nor random reals.* The method to prove that is completely explained in details in [**BJ95**], and therefore we will only state the leading steps. The key property is called L_f-property for forcings satisfying axiom A. The axiom A will be introduced at page 35; all of the arboreal forcings allowing a fusion argument introduced so far, like $\mathbb{S}$, $\mathbb{V}$, $\mathbb{M}$ and $\mathbb{L}$, satisfy axiom A. In the following definition we define the L_f-property for any forcing P satisfying axiom A, coming from [**BJ95**], definition 7.2.1, page 327.

  Definition 19. Let $f \in \omega^\omega$. One says P has the L_f-property iff for every $p \in P$, $n \in \omega$ and $A \in [\omega]^{<\omega}$ one has:

  if $p \Vdash \dot{a} \in A$, then there exists $q \leq_n p$ and $B \subseteq A$, $|B| \leq f(n)$ such that $q \Vdash \dot{a} \in B$.

  (We explicitely introduce such a property because it will be necessary in next chapter, in the paragraph concerning the separation of $\mathbf{\Sigma}^1_2(\mathbb{V})$ and $\mathbf{\Delta}^1_2(\mathbb{C})$.)

  Several results shows that both $\mathbb{L}$ and $\mathbb{M}$ have the L_f-property, for the appropriate $f \in \omega^\omega$ (see [**BJ95**], theorem 7.3.29, page 353 and theorem 7.3.45, page 360) and that L_f-property implies neither Cohen reals nor random reals are added (see [**BJ95**], lemma 7.2.2 and lemma 7.2.3, page 328).

  We also remark that the same is true for Sacks forcing $\mathbb{S}$ and Silver forcing $\mathbb{V}$ as well. The fact that they do not add Cohen reals was already known, since they are ω^ω-bounding. Furthermore, it is not hard to see that they both have the L_f-property and therefore they do not even add random reals.

1.2 Tools for forcing constructions

Two important tools in our work will be the amalgamation of Boolean algebras, introduced by Shelah in [**Sh84**], and some notions of forcing to add trees of generic reals. The amalgamation will be introduced in the first part, where we will also explain why it is so important for our purpose, while the second part will be dedicated to introduce some forcing notions to add trees whose branches are generic reals. In the end of the section, we will also give an easy but enlightening application of such tools. In the part dedicated to Shelah's amalgamation, we will deal with Boolean algebras instead of forcing notions, simply because the argument can be more easily handled.

1.2.1 Homogeneous algebras and Amalgamation

The study of models for Lebesgue measurability and Baire property, presented by Solovay in [**So70**], shows that the main property that a Boolean algebra $\mathbf{B}$ should have to be useful in Solovay-like proofs, is the, so called, *reflection property.*

Definition 20. A Boolean algebra $\mathbf{B}$ has the *reflection property* if and only if for any formula φ with parameters in $\mathbf{V}$ and for any $\mathbf{B}$-name for a real $\dot{x}$, one has $||\varphi(\dot{x})||_{\mathbf{B}} \in \mathbf{B}_{\dot{x}}$, where $\mathbf{B}_{\dot{x}}$ is the Boolean algebra generated by $\dot{x}$, i.e., $\mathbf{B}_{\dot{x}}$ is generated by $\{||s \lhd \dot{x}||_{\mathbf{B}} : s \in \omega^{<\omega}\}$.

The meaning of the definition is that, to evaluate $\varphi(\dot{x})$ in $\mathbf{V}^{\mathbf{B}}$, it suffices to know its value in a certain partial extension obtained from a subalgebra of $\mathbf{B}$, namely $\mathbf{B}_{\dot{x}}$. It is not hard to show that a particular family of Boolean algebras, satisfying the reflection property, is the class of strongly homogeneous algebras, which is part of the next definition.

Definition 21. A Boolean algebra $\mathbf{B}$ is *strongly homogeneous* if and only if for every pair of σ-generated complete subalgebras $\mathbf{B}_0, \mathbf{B}_1 \lessdot \mathbf{B}$ and every $\phi^* : \mathbf{B}_0 \to \mathbf{B}_1$ isomorphism, one can extend such ϕ^* to an automorphism $\phi : \mathbf{B} \to \mathbf{B}$.

Lemma 9.8.3 in [**BJ95**] shows that

$$\text{if } \mathbf{B} \text{ is strongly homogeneous, then } \mathbf{B} \text{ has the reflection property.} \tag{1.2}$$

Hence, we need a method to construct strongly homogeneous algebras. One may note that the strong homogeneity is a strengthening of the notion of *weak homogeneity*, saying that given a Boolean algebra $\mathbf{B}$, for any $a, b \in \mathbf{B}$ there exists an automorphism $\phi : \mathbf{B} \to \mathbf{B}$ such that $\phi(a)$ is compatible with b. To get weakly homogeneous algebras, a rather simple argument works, as the following result shows.

Fact 22. *Let* $\mathbf{B}$ *be a Boolean algebra and* $\mathbf{B}(\omega) = \prod_{n\in\omega} \mathbf{B}$ *with finite support. Then* $\mathbf{B}(\omega)$ *is weakly homogeneous.*

Proof. Let $p, q \in \mathbf{B}(\omega)$ and let $I(p), I(q)$ be the supports of p and q, respectively. Define $f : \omega \to \omega$ such that

1. $\forall n \in I(p) \cap I(q)(p_n \perp q_n \Rightarrow f(n) \notin I(p) \cap I(q) \wedge f(f(n)) = n)$;
2. otherwise $f(n) = n$.

Then, define the function

$$\phi : \mathbf{B}(\omega) \to \mathbf{B}(\omega), \text{ such that } \phi(p)_j = p_{f(j)}.$$

(Intuitively, $\phi(p)$ is just p endowed with the "new" support). Note that, for every $n \in I(\phi(p)) \cap I(q)$, $p_n \parallel q_n$, and so $\phi(p) \parallel q$. It is left to show that ϕ is an automorphism. To check that ϕ is onto, pick arbitrarily $q \in \mathbf{B}$, $q = \langle q_1, q_2, \dots, q_n, 1, 1, \dots \rangle$. Then, if we pick $q'_j = q_{f^{-1}(j)}$, one gets $\phi(q')_j = q'_{f(j)} = q_j$, which gives $\phi(q') = q$. The proof that ϕ is order preserving is immediate. $\square$

Unfortunately, this construction is not sufficient to get strong homogeneity too. Anyway, the idea behind the proof of fact 22 should not be entirely thrown away. In fact, for any pair $\mathbf{B}_0, \mathbf{B}_1$ of isomorphic σ-generated complete subalgebras of $\mathbf{B}$, Shelah's amalgamation carefully constructs a subalgebra of $\mathbf{B}(\omega)$. Furthermore, since we have to be able to extend any isomorphism between each of these pairs, one has to iterate this process ω_1-many times.

We will not go into details, but we would like to give an idea about such a construction. Our presentation will not be exhaustive, and we will only give basic definitions and a list of the main properties, which we will use later on. For a complete and detailed exposition of Shelah's amalgamation one may see [**JR93**], and we will often refer to that paper for the proofs.

Definition 23. Let $\mathbf{B}$ be a complete Boolean algebra and $\mathbf{B}_0 \lessdot \mathbf{B}$. One defines the *projection* $\pi : \mathbf{B} \to \mathbf{B}_0$ as the surjective map such that, for every $b \in \mathbf{B}$, $\pi(b) = \prod\{b \leq b_0 : b_0 \in \mathbf{B}_0\}$.

Definition 24. Let $\mathbf{B}$ be a complete algebra and $\mathbf{B}_1, \mathbf{B}_2$ two isomorphic complete subalgebras of $\mathbf{B}$ and ϕ_0 the isomorphism between them. One defines the *amalgamation of* $\mathbf{B}$ *over* ϕ_0, say $\mathbf{Am}(\mathbf{B}, \phi_0)$, as follows: first, let

$$\mathbf{B} \times_{\phi_0} \mathbf{B} =^{\mathbf{def}} \{(b', b'') \in \mathbf{B} \times \mathbf{B} : \phi_0(\pi_1(b')) \cdot \pi_2(b'') \neq \mathbf{0}\},$$

where $\pi_j : \mathbf{B} \to \mathbf{B}_j$ is the projection, for $j = 1, 2$, and consider on such $\mathbf{B} \times_{\phi_0} \mathbf{B}$ simply the product order. Then set $\mathbf{Am}(\mathbf{B}, \phi_0) =^{\mathbf{def}} B(\mathbf{B} \times_{\phi_0} \mathbf{B})$, i.e., the complete Boolean algebra generated by $\mathbf{B} \times_{\phi_0} \mathbf{B}$.

One can easily see that $e_j : \mathbf{B} \to \mathbf{Am}(\mathbf{B}, \phi_0)$ such that

$$e_1(b) = (\mathbf{1}, b) \text{ and } e_2(b) = (b, \mathbf{1})$$

are both complete embeddings (for a proof, see [**JR93**], lemma 3.1). Further, for any $b_1 \in \mathbf{B}_1$, one can show that

$$(\mathbf{1}, b_1) \text{ is equivalent to } (\phi_0(b_1), \mathbf{1}). \tag{1.3}$$

In fact, given $(b', b'') \in \mathbf{Am}(\mathbf{B}, \phi_0)$, one has

$$\phi_0(\pi_1(b_1 \cdot b')) \cdot \pi_2(b'') = \phi_0(b_1) \cdot \phi_0(\pi_1(b')) \cdot \pi_2(b'') = \phi_0(\pi_1(b')) \cdot \pi_2(\phi_0(b_1) \cdot b'')$$

and so, if either $(\mathbf{1}, b_1) \leq (b', b'')$ or $(\phi_0(b_1), \mathbf{1}) \leq (b', b'')$, then $(b_1 \cdot b', \phi_0(b_1) \cdot b'') \in \mathbf{B} \times_{\phi_0} \mathbf{B}$.

Moreover, if one considers $\phi_1 : e_1[\mathbf{B}] \to e_2[\mathbf{B}]$ such that, for every $b \in \mathbf{B}$, $\phi_1(b, \mathbf{1}) = (\mathbf{1}, b)$, one obtains nothing more than an isomorphism between two copies of $\mathbf{B}$ into $\mathbf{Am}(\mathbf{B}, \phi_0)$, which may be seen as an extension of ϕ_0 (since for every $b_1 \in \mathbf{B}_1$, by (1.3) above, $e_1(b_1) = (\mathbf{1}, b_1) = (\phi_0(b_1), \mathbf{1}) = e_2(\phi_0(b_1))$ and so $\phi_1 \circ e_2 = e_1$).

Hence, if one considers $e_1[\mathbf{B}], e_2[\mathbf{B}]$ as two isomorphic complete subalgebras of $\mathbf{Am}(\mathbf{B}, \phi_0)$, one can repeat the same procedure to construct

$$2\text{-}\mathbf{Am}(\mathbf{B}, \phi_0) =^{\mathbf{def}} \mathbf{Am}(\mathbf{Am}(\mathbf{B}, \phi_0), \phi_1)$$

and ϕ_2 the isomorphism between two copies of $\mathbf{Am}(\mathbf{B}, \phi_0)$ extending ϕ_1.

It is clear that one can continue such a construction, in order to define, for every $n \in \omega$,

$$n+1\text{-}\mathbf{Am}(\mathbf{B}, \phi_0) =^{\mathbf{def}} \mathbf{Am}(n\text{-}\mathbf{Am}(\mathbf{B}, \phi_0), \phi_n)$$

and ϕ_{n+1} the isomorphism between two copies of n-$\mathbf{Am}(\mathbf{B}, \phi_0)$ extending ϕ_n.

Finally, putting

(i) ω-$\mathbf{Am}(\mathbf{B}, \phi_0)$ = direct limit of n-$\mathbf{Am}(\mathbf{B}, \phi_0)$'s, and

(ii) $\phi_\omega = \lim_{n\in\omega} \phi_n$,

one obtains $\mathbf{B}_0, \mathbf{B}_1 \lessdot \omega$-$\mathbf{Am}(\mathbf{B}, \phi)$ and ϕ_ω automorphism of ω-$\mathbf{Am}(\mathbf{B}, \phi)$ extending ϕ_0.

Obviously, that is not sufficient to get a strongly homogeneous algebra, since the construction only works for two subalgebras, fixed at the beginning. The following crucial result completes the construction. In next theorem, and for the rest of the paper, limit cases will be direct limits.

Theorem 25. (Shelah, 1984). *Let $\langle \mathbf{B}_\alpha : \alpha < \omega_1 \rangle$ be a sequence of Boolean algebras of size $\leq\omega_1$, such that $\mathbf{B}_\alpha \lessdot \mathbf{B}_\beta$, whenever $\alpha < \beta$, and let $\mathbf{B}_{\omega_1} = \lim_{\alpha<\omega_1} \mathbf{B}_\alpha$. Furthermore, by using a book-keeping argument, we require also that whenever $\mathbf{B}_{\alpha_0} \lessdot \mathbf{B}' \lessdot \mathbf{B}_{\omega_1}$ and $\mathbf{B}_{\alpha_0} \lessdot \mathbf{B}'' \lessdot \mathbf{B}_{\omega_1}$ are such that*

(i) $\mathbf{V}^{\mathbf{B}_{\alpha_0}} \models$ "$(\mathbf{B}' : \mathbf{B}_{\alpha_0})$ and $(\mathbf{B}'' : \mathbf{B}_{\alpha_0})$ are σ-generated algebras",

(ii) $\phi_0 : \mathbf{B}' \to \mathbf{B}''$ is an isomorphism s.t. $\phi_0 \restriction \mathbf{B}_{\alpha_0} = \mathrm{Id}_{\mathbf{B}_{\alpha_0}}$,

then one can find a sequence of functions in order to extend the isomorphism ϕ_0 to an automorphism $\Phi : \mathbf{B}_{\omega_1} \to \mathbf{B}_{\omega_1}$, i.e., $\exists\langle \alpha_\eta : \eta < \omega_1 \rangle$ increasing, cofinal in ω_1, and $\exists\langle \phi_\eta : \eta < \omega_1 \rangle$ such that $\mathrm{dom}(\phi_\eta) \supseteq \mathbf{B}_{\alpha_\eta}$ and

$$(\mathbf{B}_{\alpha_{1+\eta}+1}) = \omega\text{-}\mathbf{Am}(\mathbf{B}_{\alpha_{1+\eta}}, \phi_\eta),$$

$\phi_\lambda = \lim_{\eta<\lambda} \ \phi_\eta$, whenever λ is a limit ordinal, and

$$\Phi = \lim_{\eta<\omega_1} \phi_\eta.$$

Then the Boolean algebra $\mathbf{B}_{\omega_1}$ is strongly homogeneous.

Hence, we have a general method to build strongly homogeneous algebras. At this point a natural question arises: where did Solovay's inaccessible end up?

In fact, our construction only uses a direct limit of length ω_1 and, at first sight, it does not seem to need an inaccessible. However, the point is that the amalgamation does not preserve ccc, which is helpful to preserve ω_1 and to therefore absorb the real parameter (used for a set in $\mathbf{L}(\omega^\omega)$) into the ground model. To get a particular regularity property is necessary to add into the construction a specific forcing notion adding a "suitable" set of generic reals and, in many cases, such a forcing notion affects ccc. For example, in the case of Lebesgue measurability and Baire property, these forcing notions are the Amoeba forcing for measure $\mathbb{A}$ and for category $\mathbb{UM}$, respectively, defined as follows:

1. $\mathbb{A} = \{T \subseteq 2^{<\omega} : T \text{ is a perfect tree} \wedge \mu([T]) > 1/2\}$, with $T' \leq T$ iff $T' \subseteq T$;
2. $\mathbb{UM} = \{(T, \mathfrak{T}) : \mathfrak{T} \text{ is a nowhere dense tree} \wedge \exists n \in \omega(T = \mathfrak{T}{\restriction}n)\}$, with $(T', \mathfrak{T}') \leq (T, \mathfrak{T}) \Leftrightarrow \mathfrak{T}' \supseteq \mathfrak{T} \wedge T' \sqsupseteq^{\text{end}} T$, where $\sqsupseteq^{\text{end}}$ means that $T' \supseteq T$ and $T'{\restriction}\text{ht}(T) = T$.

$\mathbb{A}$ makes the set of random reals a measure one set, whereas $\mathbb{UM}$ makes the set of Cohen reals a comeager set. In other cases, the right choice is to involve in the construction a forcing adding a particular tree of generic reals (as we will see at the end of this section). Anyway, except for some particular cases, like when $\mathbb{UM}$ is used to get the Baire property (see [**Sh84**]), one has to lengthen the construction to an inaccessible κ in order to get κ-cc, which allows us to absorb the real parameter into the ground model. For a more detailed explanation we refer the reader to section 2.2.4 at the end of next chapter.

Definition 26. [**Full amalgamation model**]. If one considers the same construction introduced in theorem 25, but of length κ inaccessible, instead of ω_1, and moreover, one requires for every $\alpha < \kappa$, $|\mathbf{B}_\alpha| < \kappa$ to hold, then one obtains a κ-cc algebra. Furthermore, one can add into such a construction any forcing of size $< \kappa$, without losing κ-cc. If G is $\mathbf{B}_\kappa$-generic over $\mathbf{V}$ we will call $\mathbf{V}[G]$ the *full amalgamation model.* (One may see that this construction provides a complete Boolean algebra which is forcing-equivalent to the Levy collapse).

Remark 27. Sometimes it will be enough (and necessary) to amalgamate not over any pair of isomorphic σ-generated subalgebras, but only over any $\mathbf{B}_0, \mathbf{B}_1$ isomorphic to $B(\mathbb{P})$, for a certain forcing $\mathbb{P}$, where $B(\mathbb{P})$ is the Boolean algebra generated by $\mathbb{P}$. This can be simply done by replacing, in theorem 25, condition (i) with

$$\mathbf{V}^{\mathbf{B}_{\alpha_0}} \models \text{“}(\mathbf{B}' : \mathbf{B}_{\alpha_0}) \text{ and } (\mathbf{B}'' : \mathbf{B}_{\alpha_0}) \text{ are isomorphic to } B(\mathbb{P})\text{”}.$$

In this case, we will say that the Boolean algebra obtained is *strongly* $\mathbb{P}$-*homogeneous.*

1.2.2 Trees of generic reals

This section is devoted to introduce several ways to add trees of generic reals. The importance of this work has two aspects: firstly, such constructions are intrinsically interesting and some of these need new forcing notions; secondly (and probably mainly), they play a crucial role in questions concerning regularity properties, as the following simple and well-known example shows.

Example 28. Consider the forcing consisting of finite trees $T \subset 2^{<\omega}$ such that all $t \in \textsc{Term}(T)$ have the same length, ordered by end-extension $\sqsupseteq^{\text{end}}$. It is well-known, and not hard to verify, that such a forcing adds a perfect tree of Cohen reals, more precisely a perfect tree whose branches are Cohen reals. Moreover, since this forcing is countable, it is actually the Cohen forcing $\mathbb{C}$. Hence,

$$\mathbb{C} \Vdash \text{“}\exists \dot{T} \text{ perfect tree of Cohen reals over } \mathbf{V}\text{”}. \tag{1.4}$$

Now, we are going to use (1.4) to prove

$$\mathbf{V}[G] \models \text{“every } \mathbf{\Delta}^1_2\text{-set of reals is Sacks measurable”},$$

where G is a $\mathbb{C}_{\omega_1}$-generic filter over $\mathbf{V}$.

Let $X = \{x \in 2^\omega : \varphi_0(v_0, x)\} = \{x \in 2^\omega : \neg\varphi_1(v_1, x)\}$ be a set of reals in $\mathbf{V}[G]$, where φ_0, φ_1 are Π^1_2-formula and v_0, v_1 real-parameters. Such reals can be absorbed into $\mathbf{V}[G{\restriction}\alpha]$, for some $\alpha < \omega_1$, and the tail $\mathbb{C}_{\omega_1}/G{\restriction}\alpha$ is still forcing equivalent to $\mathbb{C}_{\omega_1}$. Let c be the Cohen real added by $G(\alpha)$; one has two possible cases:

$$\mathbf{V}[G{\restriction}\alpha][c] \models \varphi_0(v_0, c) \text{ or } \mathbf{V}[G{\restriction}\alpha][c] \models \varphi_1(v_1, c).$$

Assume the first case holds (the argument is analogous in the other case). Argue in $\mathbf{V}[G{\restriction}\alpha]$. Since $\varphi_0(v_0, c)$ holds, there exists $t \in \mathbb{C}$ such that $t \Vdash \varphi_0(v_0, \dot{c})$. In $\mathbf{V}[G{\restriction}\alpha][c]$, by shrinking and translating the perfect tree in (1.4), one can get a perfect tree T of Cohen reals over $\mathbf{V}[G{\restriction}\alpha]$ contained into $[t]$. Hence, we get

$$\mathbf{V}[G{\restriction}\alpha][c] \models \forall x \in 2^\omega (x \in [T] \Rightarrow \varphi_0(v_0, x))$$

and, by absoluteness of Π^1_2-formulas, also

$$\mathbf{V}[G] \models \forall x \in 2^\omega (x \in [T] \Rightarrow \varphi_0(v_0, x)).$$

Therefore, $\mathbf{V}[G] \models [T] \subseteq X$.

It is clear that, from the second case, one can deduce that there exists a tree T such that $\mathbf{V}[G] \models [T] \cap X = \emptyset$, which completes the proof.

Remark 29. Given a Cohen real c, the quotient $\mathbb{C}_{\omega_1}/c \approx \mathbb{C}_{\omega_1}$. As a consequence, we get $\mathbb{C}_{\omega_1}$ is strongly Cohen homogeneous, and so one can actually get the Sacks measurability for every projective set of reals, simply by using the same argument above and the property that if $p \Vdash \varphi(v, \dot{c})$, for some $p \in \mathbb{C}_{\omega_1}$, then there exists $t \in \mathbb{C}$ such that $t \Vdash \varphi^*(v, \dot{c})$, where φ^* is a translation of φ to a statement about the single Cohen extension.

Hence, adding a particular tree of generic reals, can be helpful for our topic concerning regularity properties. Furthermore, as we said above, the problem of adding trees of generic reals is of intrinsic interest and some natural questions arise, like adding a perfect tree of random reals. At first sight, the more natural manner to do that would seem the use of random forcing $\mathbb{R}$; however, the following result of Bartoszynski and Judah shows that it is not the right way.

Theorem 30. *Let r be a random real over $\mathbf{V}$. Then*

$$\mathbf{V}[r] \models \text{“}\mathbb{R}(\mathbf{V}) \textit{ does not contain a perfect set”}.$$

For a proof one may see ([**JR93**], theorem 3.2.17, page 114).

The rest of this section is dedicated to several examples and constructions of forcing notions to figure such questions out.

Adding a perfect tree of random reals. Consider the forcing $\mathbb{RT}$ consisting of pairs $(T, \mathfrak{T})$ such that:

(i) $\mathfrak{T}$ is a perfect tree such that for every $t \in \mathfrak{T}$, $\mu([\mathfrak{T}_t]) > 0$;

(ii) $T = \mathfrak{T}{\restriction}n$, for some $n \in \omega$,

ordered by

$$(T', \mathfrak{T}') \leq (T, \mathfrak{T}) \Leftrightarrow \mathfrak{T}' \subseteq \mathfrak{T} \wedge T' \sqsupseteq^{\text{end}} T$$

It is clear that, for every null set $N \in \mathbf{V}$, the set

$$D_N = \{(T, \mathfrak{T}) \in \mathbb{RT} : [\mathfrak{T}] \cap N = \emptyset\}$$

is open dense in $\mathbb{RT}$. To see that, one can simply note that for $t \in \textsc{Term}(T)$, $\mathfrak{T}_t$ is a perfect tree of positive measure which can be shrunk to a perfect tree $\mathfrak{T}'_t$, still of positive measure, such that $[\mathfrak{T}'_t] \cap N = \emptyset$; therefore, if one sets $\mathfrak{T}' = \bigcup\{\mathfrak{T}'_t : t \in \textsc{Term}(T)\}$ one precisely obtains a stronger condition $(T, \mathfrak{T}') \in D_N$.

Hence, the branches of the generic $T_G = \bigcup\{T : (T, \mathfrak{T}) \in G\}$ are random reals. Moreover, it is clear that T_G is itself a perfect tree, and so

$$\Vdash_{\mathbb{RT}} \text{“}[\dot{T}_G] \text{ is a perfect set of random reals over } \mathbf{V}\text{”}.$$

Note that, in (i) above, if one drops the condition $\mu([\mathfrak{T}]) > 0$, one gets a forcing, say $\mathbb{ST}$, adding a perfect tree of Sacks reals.

Adding a Miller tree of Miller reals. First of all, note that, by using a similar argument of example 28, it is easy to add a Miller tree of Cohen reals. We want to show that one can also define a forcing notion for adding a Miller tree of Miller reals. Note the analogy with $\mathbb{RT}$ above.

Definition 31. We use the following notation:

$$\mathfrak{T}[n] =^{\mathbf{def}} \{t \in \mathfrak{T} : \exists s \in \textsc{Split}_n(\mathfrak{T})(t \trianglelefteq s)\}.$$

Consider the following forcing notion:

$$\mathbb{MT} = \{(T, \mathfrak{T}) : \mathfrak{T} \text{ is a Miller tree} \wedge \exists n \in \omega(T = \mathfrak{T}[n])\}$$

ordered by

$$(T', \mathfrak{T}') \leq (T, \mathfrak{T}) \Leftrightarrow \mathfrak{T}' \subseteq \mathfrak{T} \wedge T' \sqsupseteq^{\text{end}} T$$

Fact 32. *Let G be $\mathbb{MT}$-generic over $\mathbf{V}$. Then*

$$\mathbf{V}[G] \models \text{“}T_G \textit{ is a Miller tree of Miller reals”},$$

where, as usual, $T_G = \bigcup\{T : (T, \mathfrak{T}) \in G\}$.

Proof. What we have to do is to make sure that any branch through T_G is in each ground model open dense subset $D \subseteq \mathbb{M}$. Therefore, fix such D arbitrarily and let

$$E_D = \{(T, \mathfrak{T}) \in \mathbb{MT} : \forall t \in T(\mathfrak{T}_t \in D)\},$$

Pick any $(T, \mathfrak{T}) \in \mathbb{MT}$. We are done when we find $(T', \mathfrak{T}') \leq (T, \mathfrak{T})$ such that $(T', \mathfrak{T}') \in E_D$. Let $\Lambda = \{\mathfrak{T}_t : t \in \textsc{Term}(T)\}$; obviously, any $\mathfrak{T}_t \in \Lambda$ can be shrunk to a tree $\mathfrak{T}'_t \in D$, simply by density of $D \subseteq \mathbb{M}$. Furthermore, one can extend any terminal node $t \in T$ to $t' \in T'$ such that t' is ω-splitting (this is to make sure that T_G will be a Miller tree). It is therefore clear that, if we put $\mathfrak{T}'$ to be the union of such $\mathfrak{T}'_t$'s, then $(T', \mathfrak{T}') \leq (T, \mathfrak{T})$ and $(T', \mathfrak{T}') \in E_D$. □

Remark 33. Unfortunately, this construction cannot be generalized for any arboreal forcing $\mathbb{P}$; in fact, one can easily notice that such a method does not work for Laver trees and Silver trees; about the latter, while the union of Miller (Sacks) trees is again a Miller (Sacks) tree, the same is not true for Silver trees, since we lose the uniformity; however, a slight refinement of the proof above is sufficient in this case, as the following paragraph shows. On the contrary, for Laver trees the problem seems to be more complicate to solve, since, when one shrinks the second coordinate, in order to make it belonging to the dense subset D, one may lose the possibility to have infinitely many immediate successors.

Adding a Silver tree of Silver reals. Consider the forcing $\mathbb{VT}$ defined like $\mathbb{ST}$ at page 23, but with Silver trees in place of Sacks trees. As above, we want to show that

$$\mathbf{V}[G] \models \text{“}T_G \text{ is a Silver tree of Silver reals”}.$$

Fix an open dense $D \subseteq \mathbb{V}$. By remark 33, one has to find a finer method to make sure that the union of Silver trees $\mathfrak{T}_t$'s is a Silver tree. First of all, let $t_0, t_1, \dots, t_k$ be an enumeration of all terminal nodes in T. Before starting the construction, we need the following notation: for any tree T and $t \in 2^{<\omega}$ such that $|t| \leq |\textsc{Stem}(T)|$, let

$$T \oplus t = \{t' \in 2^{<\omega} : \forall n < |t|(t'(n) = t(n)) \wedge \exists t'' \in T \forall n \geq |t|(t''(n) = t'(n))\}.$$

(Intuitively, $T \oplus t$ is the translation of T above t). Consider the following construction:

- firstly, let $\mathfrak{T}^0_{t_0} \subseteq \mathfrak{T}_{t_0}$ be in D and let $\mathfrak{T}^0_{t_1} = \mathfrak{T}^0_{t_0} \oplus t_1$;
- then, let $\mathfrak{T}^1_{t_1} \subseteq \mathfrak{T}^0_{t_1}$ be in D and let $\mathfrak{T}^1_{t_2} = \mathfrak{T}^1_{t_1} \oplus t_2$; note that $\mathfrak{T}^1_{t_1} \oplus t_0 \subseteq \mathfrak{T}^0_{t_0}$ and so $\mathfrak{T}^1_{t_1} \oplus t_0 \in D$;
- continue this construction for every $j \leq k$;
- finally, let $\mathfrak{T}'_{t_j} = \mathfrak{T}^k_{t_k} \oplus t_j$, for every $j \leq k$.

It follows from the construction that $\mathfrak{T}' = \bigcup\{\mathfrak{T}'_{t_j} : j \leq k\}$ is a Silver tree. The rest of the argument works as in the proof of fact 32.

Remark 34. It is noteworthy that such forcings $\mathbb{RT}, \mathbb{ST}, \mathbb{VT}$ and $\mathbb{MT}$ are rather different from their counterparts $\mathbb{R}, \mathbb{S}, \mathbb{V}$ and $\mathbb{M}$. A difference between $\mathbb{R}$ and $\mathbb{RT}$ has been already underlined, since the latter adds a perfect set of random reals, whereas the former does not. About forcing $\mathbb{VT}$, we will see at page 35 that, rather surprisingly, it adds a dominating real (and a similar proof could be given for $\mathbb{ST}$ and $\mathbb{MT}$ as well). If one goes into the construction of such a dominating real, one can note that it does not work for forcing $\mathbb{RT}$; In fact, one can prove that the latter does not add dominating reals (as a corollary of theorem 3.2.23, lemma 6.5.10 and theorem 6.5.11 of [**BJ95**]). This last fact also implies that $\mathbb{RT}$ is not forcing equivalent to the Amoeba for measure $\mathbb{A}$, since the latter adds Hechler reals. As a further information we therefore obtain that the perfect set of random reals added by $\mathbb{RT}$ is measure zero; in fact, since such a perfect tree of random reals exists inside any positive set of the ground model, if it were positive, one would obtain by translation that it would have measure one, which is impossible since the latter would imply adding dominating reals.

Remark 35. Note that, any forcing $\mathbb{PT}$ just introduced, adds a tree in $\mathbb{P}$ of $\mathbb{P}$-generic reals *below any ground model condition in* $\mathbb{P}$. This simply follows noting that the forcing $\mathbb{PT}_W = \{(T, \mathfrak{T}) \in \mathbb{PT} : \mathfrak{T} \subseteq W\}$ is isomorphic to $\mathbb{PT}$, for any $W \in \mathbb{P}$. As an immediate consequence, one gets

$$\Vdash_{\mathbb{PT}} (\forall W \in \mathbb{P} \cap \mathbf{V})(\exists T \in \mathbb{P})(T \subseteq W)(\forall z \in [T])(z \text{ is } \mathbb{P}\text{-generic over } \mathbf{V}),$$

We conclude this section with a standard application of the tools introduced so far.

Example 36. Consider a Boolean algebra $\mathbf{B}_\kappa$ obtained as in definition 26, but only amalgamating over Silver forcing $\mathbb{V}$, and adding also cofinally often the forcing $\mathbb{VT}$ into the construction, i.e., for cofinally many α's, let $\mathbf{B}_{\alpha+1} = \mathbf{B}_\alpha * \dot{\mathbb{VT}}$. We want to show that

$$\mathbf{B}_\kappa \Vdash \text{"every set of reals in } \mathbf{L}(\omega^\omega) \text{ is } \mathbb{V}\text{-measurable"}.$$

Proof. Fix arbitrarily $X \subseteq 2^\omega$ and let φ and $v \in \mathrm{On}^\omega$ such that $X = \{x \in 2^\omega : \varphi(x, v)\}$. Let $\alpha < \kappa$ be such that $v \in \mathbf{V}[G{\restriction}\alpha + 1]$ and $\mathbf{B}_{\alpha+1} = \mathbf{B}_\alpha * \dot{\mathbb{VT}}$. To lighten the notation, let $G_{\alpha+1} = G{\restriction}\alpha + 1$.

By construction, we know

$$\mathbf{V}[G_{\alpha+1}] \models \text{"}\mathbf{B}_\kappa / G_{\alpha+1} \text{ is strongly } \mathbb{V}\text{-homogeneous"}.$$

Let $\mathbf{N} = \mathbf{V}[G_{\alpha+1}]$ and $\mathbf{B}^* = \mathbf{B}_\kappa / G_{\alpha+1}$.

Let H be the tail of the generic filter G, i.e., H is $\mathbf{B}^*$-generic over $\mathbf{N}$ and $\mathbf{N}[H] = \mathbf{V}[G]$. Hence, for every $W \in \mathbf{N} \cap \mathbb{V}$,

$$\mathbf{N}[H] \models (\exists T \in \mathbb{V})(T \subseteq W)(\forall z \in [T])(z \text{ is } \mathbb{V}\text{-generic over } \mathbf{N}). \tag{1.5}$$

The following result is easy to check.

Fact 37. *Let $\dot{x}$ be a $\mathbf{B}$-name such that $\Vdash_{\mathbf{B}}$ "$\dot{x}$ is a $\mathbb{V}$-generic real over $\mathbf{N}$", and assume for every Borel set $B \notin \mathcal{I}_{\mathbb{V}}$, $\|\dot{x} \in B\|_{\mathbf{B}} \neq \mathbf{0}$. Then there exists an isomorphism*

$$f : B(\mathbb{V}) \to \mathbf{B}_{\dot{x}}, \text{ such that } \Vdash_{\mathbf{B}} f(\dot{v}) = \dot{x},$$

where $\dot{v}$ is the canonical $\mathbb{V}$-name for the $\mathbb{V}$-generic real.

Let $\dot{x}$ be a name for a Silver real and assume $\|\varphi(\dot{x})\|_{\mathbf{B}^*} \neq \mathbf{0}$. Since we have in the construction iteration of $\mathbb{VT}$, one obviously obtains $B(\mathbb{V}) \lessdot \mathbf{B}^*$. Hence, because of 37, together with strong $\mathbb{V}$-homogeneity, one can consider $A \in B(\mathbb{V})$ such that $A = \|\varphi(\dot{v})\|_{B(\mathbb{V})} \neq \emptyset$. Then, pick $T \in \mathbb{V}$ as in (1.5) such that $[T] \subseteq A$. The next observation follows from Solovay's lemma, stated for $\mathbb{V}$-generic (Silver) reals in place of random reals.

Remark 38. Suppose $\mathbf{N}[H] \models$ "z is a Silver real over $\mathbf{N}$". Then

$$\mathbf{N}[H] \models \text{"}z \in A \Leftrightarrow \varphi(z)\text{"}.$$

Thus, since every $z \in [T]$ is $\mathbb{V}$-generic over $\mathbf{N}$ and, by construction, $z \in A$, then for every $z \in [T]$, $\mathbf{N}[H] \models \varphi(z)$. Hence, one obtains

$$\mathbf{V}[G] \models [T] \subseteq X.$$

It is left to show the case $\|\varphi(\dot{x})\|_{\mathbf{B}^\star} = \mathbf{0}$. In this case, $\|\neg\varphi(\dot{x})\|_{\mathbf{B}^\star} \neq \mathbf{0}$ and then, arguing in the same way, one gets a Silver tree T such that for every $z \in [T]$, $\mathbf{N}[H] \models \neg\varphi(z)$, and therefore

$$\mathbf{V}[G] \models [T] \cap X = \emptyset.$$

□

Chapter 2

Separation of regularity properties

As we said in the introduction, this chapter is dedicated to prove results of separation between regularity properties, or in other words to construct models

$$\mathbf{M} \models \Gamma(\mathbb{P}) \wedge \neg\Gamma(\mathbb{Q}),$$

for different arboreal forcings $\mathbb{P}$ and $\mathbb{Q}$. Note that, the statements $\Gamma(\mathbb{P})$ defined in the introduction may be seen as particular cases of a more general type of statements, i.e.,

$$\Theta(\mathbb{P}) \equiv^{\text{def}} \text{“every set of reals in } \Theta \text{ is } \mathbb{P}\text{-measurable”}.$$

In this fashion, the family Γ considered in the introduction exactly consists of all sets of reals. Obviously, such statements are well defined for every family of sets of reals Θ; however, some nice results can be proved when this family Θ satisfies certain particular properties, that are to be closed under continuous preimages and to be closed under intersections with closed sets; such families are called *topologically reasonable families.* A detailed study of this topic may be found in [**BL99**]. Let us note that the family Γ obviously satisfies such properties, and moreover also the families consisting of projective sets, of $\mathbf{\Delta}^1_2$-sets, $\mathbf{\Sigma}^1_2$-sets and so on are topologically reasonable.

Our work is essentially devoted to the separation of statements of the form $\Gamma(\mathbb{P})$; furthermore, another interesting point is the study of statements of the form $\mathbf{\Delta}^1_2(\mathbb{P})$ and $\mathbf{\Sigma}^1_2(\mathbb{P})$. About this second point, most of the results concerning these two families are corollaries of some interesting characterizations due to Solovay, Shelah, Brendle and Löwe; other results are a little more complicate and they will require some sophisticated argument, like that one we will present at page 35 to answer question 3 of [**Ha03**]. The chapter is therefore divided into two main sections for dealing with those two different subjects. We remark that for $\mathbf{\Sigma}^1_1$ this sort of issues do not occur, since one can rather easily show in **ZFC** that any $\mathbf{\Sigma}^1_1$-set is $\mathbb{P}$-measurable, for all $\mathbb{P}$'s of our interest (for a general proof one can see [**K12**], proposition 2.2.3).

2.1 Regularity properties for $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ sets

We divide this section in several parts, each of ones concerns a particular parallel between two regularity properties. The following diagram shows the known implicatons.

$$\begin{array}{ccccc}
\mathbf{\Delta}^1_2(\mathbb{R}) & \longrightarrow & \longrightarrow & \mathbf{\Delta}^1_2(\mathbb{V}) \longrightarrow & \mathbf{\Sigma}^1_2(\mathbb{S}) \\
\uparrow & & & \uparrow & \uparrow \\
 & & \mathbf{\Delta}^1_2(\mathbb{C}) \longrightarrow & \mathbf{\Sigma}^1_2(\mathbb{V}) \longrightarrow & \mathbf{\Sigma}^1_2(\mathbb{M}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{\Sigma}^1_2(\mathbb{R}) & \longrightarrow & \mathbf{\Sigma}^1_2(\mathbb{C}) \longrightarrow & \longrightarrow & \mathbf{\Sigma}^1_2(\mathbb{L})
\end{array}$$

We will call it the **RP-diagram**. The reader may note the lack of the statements $\mathbf{\Delta}^1_2(\mathbb{S})$, $\mathbf{\Delta}^1_2(\mathbb{M})$ and $\mathbf{\Delta}^1_2(\mathbb{L})$. The reason is that such assertions are equivalent to their corresponding counterparts for $\mathbf{\Sigma}^1_2$, as we will see in the next paragraphs.

Almost all of the implications are known results proved by Shelah, Solovay, Brendle, Löwe and Halbeisen. We only have to prove the following.

Fact 39. $\mathbf{\Delta}^1_2(\mathbb{R}) \Rightarrow \mathbf{\Delta}^1_2(\mathbb{V})$. *Actually, for every perfect tree T such that, for every $t \in T$, $\mu([T_t]) > 0$, there exists $T' \subseteq T$ such that $T' \in \mathbb{V}$.*

Proof. Let $T \subseteq 2^{<\omega}$ be a perfect tree of positive measure, with the further condition that for every $t \in T$, also $\mu([T_t]) > 0$. The proof is essentially a consequence of the well-known density lemma for measure. In particular, this lemma implies that, given any positive measure tree $[T]$, one can find $x \in [T]$ such that

$$\lim_{n<\omega} \frac{\mu([T_{x\restriction n}])}{\mu([x\restriction n])} = 1,$$

which means that for every $\varepsilon > 0$ there exists $n \in \omega$ such that

$$\mu([T_{x\restriction n}]) > (1-\varepsilon)\mu([x\restriction n]) \tag{2.1}$$

Such x is called *density branch*. The construction of the Silver tree T' is done by induction.

STEP 0. Pick $x \in [T]$ density branch. Apply (2.1) for $\varepsilon = \frac{1}{2}$; to lighten the notation we put $t = x\restriction n$. Furthermore, (2.1) also implies that t is a splitting node of T. Hence, we have $\mu([T_t]) > \frac{1}{2}\mu([t])$. Put

$T_0 = T_{t^\frown 0}$, $T_1 = T_{t^\frown 1}$ and $T_1^* = T_1 \oplus t^\frown 0$, where we remind the latter is the translation of T_1 over $t^\frown 0$.

We claim that $[T_1^*] \cap [T_0] \neq \emptyset$. To reach a contradiction, assume not. Then, on the one hand, by $[T_1^*] \cup [T_0] \subseteq [t^\frown 0]$ and (2.1), it follows

$$\mu([T_1^*]) + \mu([T_0]) = \mu([T_1^*] \cup [T_0]) \leq \mu([t^\frown 0]) = \frac{1}{2}\mu([t]).$$

On the other hand, since $\mu([T_1^*]) = \mu([T_1])$, it follows

$$\mu([T_1^*]) + \mu([T_0]) = \mu([T_1] \cup [T_0]) = \mu([T_t]) > \frac{1}{2}\mu([t]),$$

and so one obtains a contradiction.

Furthermore, we know that the intersection is not only non-empty, but is closed and with positive measure; put $S_0 = T_1^* \cap T_0$. We also remark that for every $t \in S_0$, $\mu([S_0]) > 0$. Therefore, by using density lemma again, one can pick $x_0 \in [S_0]$ density branch, such that $x_0 \oplus t^\frown 1 \in [S_0 \oplus t^\frown 1]$ is a density branch as well. Hence, we have found a way to lengthen the splitting node t with two uniform density branches in $[T]$.

STEP $j+1$. Assume $S_j \subseteq T$ and x_j already defined. Applying the same argument of previous step, there exists $n \in \omega$, such that, for every $\sigma \in 2^j$ and $i = 0,1$, one has $t_{\sigma^\frown i} =^{\mathbf{def}} (x_j \oplus t_\sigma^\frown i){\restriction}n$ is a splitting node and $\mu([S_j \oplus t_\sigma^\frown i] \cap [t_{\sigma^\frown i}]) > \frac{1}{2}\mu([t_{\sigma^\frown i}])$. (Note that we consider $t_\emptyset$ to be t of STEP 0). Furthermore, we also get S_{j+1} and x_{j+1} such that:

- $S_{j+1} \subseteq S_j$ such that $\forall t \in S_{j+1}$, $\mu([S_{j+1}]) > 0$;
- $x_{j+1} \in [S_{j+1}]$ is a density branch;
- for every $\sigma \in 2^{j+1}$, for $i = 0,1$, one has $(x_{j+1} \oplus t_\sigma^\frown i) \in [S_{j+1} \oplus t_\sigma^\frown i]$.

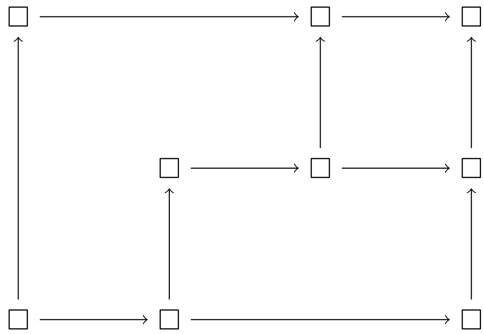
One can therefore uniformly extend all splitting nodes t_σ, for $\sigma \in 2^{j+1}$.

It is clear that such a recursive construction satisfies our requirements. More precisely,

$$T' =^{\mathbf{def}} \{t_\sigma \in 2^{<\omega} : t_\sigma \text{ as defined above}, \sigma \in 2^{<\omega}\}$$

is a Silver tree inside T. □

Going back to the RP-diagram, note that one may assign a white square □ to mean that the corresponding statement in the diagram holds, and a black square ■ to mean that the corresponding statement does not hold. For instance, if one considers an ω_1-iteration of Amoeba for measure $\mathbb{A}$, one obtains a model satisfying $\mathbf{\Sigma}^1_2(\mathbb{R})$, and hence such a model satisfies the following RP-diagram:



In this manner, like for Cichom's diagram for cardinal invariants, one can ask whether one can construct a model for each allowed combination of white and black squares. In the concluding part of this section we will summarize all of the results which we are going to present in the next paragraphs in terms of combinations of the RP-diagram.

Baire property vs Lebesgue measurability. We start from the following two theorems, due to Solovay and Shelah, which are part of the folklore of our subject. Remind that $\mathbb{P}(\mathbf{V})$ is the set of $\mathbb{P}$-generic reals over $\mathbf{V}$.

Theorem 40 (Shelah,1978)**.**

(i) $\mathbf{\Delta}^1_2(\mathbb{R}) \Leftrightarrow \forall x \in \omega^\omega(\mathbb{R}(\mathbf{L}[x]) \neq \emptyset)$*;*

(ii) $\mathbf{\Delta}^1_2(\mathbb{C}) \Leftrightarrow \forall x \in \omega^\omega(\mathbb{C}(\mathbf{L}[x]) \neq \emptyset)$*.*

Theorem 41 (Solovay,1970)**.**

(i) $\mathbf{\Sigma}^1_2(\mathbb{R}) \Leftrightarrow \forall x \in \omega^\omega(\mathbb{R}(\mathbf{L}[x])$ *has measure one);*

(ii) $\mathbf{\Sigma}^1_2(\mathbb{C}) \Leftrightarrow \forall x \in \omega^\omega(\mathbb{C}(\mathbf{L}[x])$ *is comeager).*

Proofs of such results may be found in [**BJ95**], sections 9.2 and 9.3. As an immediate corollary we also get

$$\mathbf{L}^{\mathbb{R}_{\omega_1}} \models \mathbf{\Delta}^1_2(\mathbb{R}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{C}), \text{ and}$$

$$\mathbf{L}^{\mathbb{C}_{\omega_1}} \models \mathbf{\Delta}^1_2(\mathbb{C}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{R}),$$

since random forcing $\mathbb{R}$ does not add Cohen reals, and viceversa, Cohen forcing $\mathbb{C}$ does not add random reals. Note that, rather surprisingly, the same separation cannot be done for $\mathbf{\Sigma}^1_2$. In fact, on the one hand

$$\mathbf{L}^{\mathbb{UM}_{\omega_1}} \models \mathbf{\Sigma}^1_2(\mathbb{C}) \wedge \neg\mathbf{\Sigma}^1_2(\mathbb{R}),$$

where $\mathbb{UM}$ is the Amoeba for category, the forcing to make the union of the ground model meager sets a meager set; on the other hand, we have $\mathbf{\Sigma}^1_2(\mathbb{R}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{C})$ (this result is due to Raisonnier; for a proof see also [**BJ95**], theorem 9.3.4). The symptoms of that may be already noted because the Amoeba for measure $\mathbb{A}$, which is the natural forcing to make the union of ground model null sets a null set, also makes the union of the ground model meager sets a meager set, and so

$$\mathbf{L}^{\mathbb{A}_{\omega_1}} \models \mathbf{\Sigma}^1_2(\mathbb{R}) \wedge \mathbf{\Sigma}^1_2(\mathbb{C}).$$

This implication does not extend to the family Γ of all sets of reals, as we mentioned in the introduction. Furthermore, in [**FS10**], Sy Friedman and David Schrittesser showed an even deeper result, constructing a model

$$\mathbf{M} \models \textsc{Proj}(\mathbb{R}) \wedge \neg\mathbf{\Delta}^1_3(\mathbb{C}).$$

Note that such a result is optimal, since $\mathbf{M} \models \mathbf{\Sigma}^1_2(\mathbb{C})$ by Raisonnier's theorem stated above. This construction is the core of David Schrittesser's PhD thesis.

Baire property vs Miller measurability. In this case we remark two interesting results proved in [**BL99**], corollary 3.5 and theorem 6.1.

Remark 42. For every set of reals X, one has

$$X \text{ has the Baire property} \Rightarrow X \text{ is Miller measurable.}$$

Theorem 43. $\mathbf{\Delta}^1_2(\mathbb{M}) \Leftrightarrow \mathbf{\Sigma}^1_2(\mathbb{M}) \Leftrightarrow \forall x \in \omega^\omega(\omega^\omega \cap \mathbf{L}[x]$ *is not dominating*).

Hence, to get a model for $\mathbf{\Sigma}^1_2(\mathbb{M}) \wedge \neg\mathbf{\Sigma}^1_2(\mathbb{C})$ is sufficient to consider an ω_1-iteration of Cohen, i.e.

$$\mathbf{L}^{\mathbb{C}_{\omega_1}} \models \mathbf{\Sigma}^1_2(\mathbb{M}) \wedge \neg\mathbf{\Sigma}^1_2(\mathbb{C}).$$

Note that the failure of $\mathbf{\Sigma}^1_2(\mathbb{C})$ in such a model in due to theorem 44.

However, such a model also satisfies $\mathbf{\Delta}^1_2(\mathbb{C})$. To obtain a model for $\mathbf{\Sigma}^1_2(\mathbb{M}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{C})$, we need to iterate a forcing adding unbounded reals, without adding Cohen reals; for such a proof we refer the reader to the paragraph on Laver measurability vs Baire property.

Lebesgue measurability vs Miller measurability. The situation for Lebesgue measurability is slightly different, since we do not have an analogous of theorem 42 also for Lebesgue measurability. In fact, since random forcing does not add unbounded reals, we get

$$\mathbf{L}^{\mathbb{R}_{\omega_1}} \models \mathbf{\Delta}^1_2(\mathbb{R}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{M}).$$

On the contrary, the situation for $\mathbf{\Sigma}^1_2$ is different; in fact, because of theorem 42 and Raisonnier's result stated above, we get $\mathbf{\Sigma}^1_2(\mathbb{R}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{M})$. So for $\mathbf{\Sigma}^1_2$ we get again the result that we already had for the Baire property. In section 2.2.1, we will see that such an implication does not extend to the family Γ.

Baire property vs Laver measurability. First of all, note that in this case we do not have an analogous of theorem 42. However, the following results proved by Löwe and Brendle in [**BL99**], theorem 4.1 and theorem 5.8, allows us to make some interesting observations. In the following theorem, remind that $\mathbb{D}(\mathbf{L}[x])$ denotes the set of Hechler reals over $\mathbf{L}[x]$.

Theorem 44 (Brendle-Löwe,1999)**.**

(i) $\mathbf{\Sigma}^1_2(\mathbb{C}) \Leftrightarrow \forall x \in \omega^\omega(\mathbb{D}(\mathbf{L}[x]) \neq \emptyset)$*;*

(ii) $\mathbf{\Delta}^1_2(\mathbb{L}) \Leftrightarrow \mathbf{\Sigma}^1_2(\mathbb{L}) \Leftrightarrow \forall x \in \omega^\omega(\omega^\omega \cap \mathbf{L}[x]$ *is bounded*$)$*.*

As an immediate corollary we get $\mathbf{\Sigma}^1_2(\mathbb{C}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{L})$. The interesting observation is that such an implication does not reverse. Actually, we can prove something even stronger, constructing a model

$$\mathbf{M} \models \mathbf{\Sigma}^1_2(\mathbb{L}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{C}).$$

The method to do that is simply an ω_1-iteration of $\mathbb{L}$, with countable support, as the following fact shows.

Fact 45. *Let G be an $\mathbb{L}_{\omega_1}$-generic over $\mathbf{L}$, where $\mathbb{L}_{\omega_1}$ is an iteration of Laver forcing $\mathbb{L}$ of length ω_1, with countable support. Then*

$$\mathbf{L}[G] \models \mathbf{\Sigma}^1_2(\mathbb{L}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{C}).$$

Proof. The core of the proof his the fact that a countable support iteration of Laver forcing $\mathbb{L}$ satisfies Laver property ([**BJ95**], theorem 6.3.34) and the latter in turn implies that no Cohen reals are added ([**BJ95**], Lemma 7.3.33). The rest of the argument is standard. Since ω_1 is preserved by properness, any real parameter r can be absorbed at some stage of the iteration $\alpha < \omega_1$, i.e., $r \in \mathbf{L}[G{\restriction}\alpha]$. Since $\mathbb{L}$ adds dominating reals, we therefore have $\omega^\omega \cap \mathbf{L}[r]$ is bounded, for any real r, and hence, by theorem 44-(i), we get $\mathbf{L}[G] \models \mathbf{\Sigma}^1_2(\mathbb{L})$. On the contrary, since $\mathbb{L}_{\omega_1}$ does not add Cohen reals, we obtain, by theorem 40, $\mathbf{L}[G] \models \neg\mathbf{\Delta}^1_2(\mathbb{C})$. □

It is straightforward that such a result also implies that one can separate $\mathbf{\Sigma}^1_2(\mathbb{M})$ and $\mathbf{\Delta}^1_2(\mathbb{C})$, simply since the former is implied by $\mathbf{\Sigma}^1_2(\mathbb{L})$.

Furthermore, we may remark that in the implication $\mathbf{\Sigma}^1_2(\mathbb{C}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{L})$, there is no hope to weaken the left side, since

$$\mathbf{L}^{\mathbb{C}_{\omega_1}} \models \mathbf{\Delta}^1_2(\mathbb{C}) \wedge \neg\mathbf{\Sigma}^1_2(\mathbb{L}).$$

Laver measurability vs Miller measurability. There is nothing interesting to say in this case, since, on the one hand we obviously have that every Laver measurable set of reals is Miller measurable as well, and on the other hand, a simple ω_1-iteration of Cohen forcing $\mathbb{C}_{\omega_1}$ gives

$$\mathbf{L}^{\mathbb{C}_{\omega_1}} \models \mathbf{\Sigma}^1_2(\mathbb{M}) \wedge \neg\mathbf{\Sigma}^1_2(\mathbb{L}).$$

Lebesgue measurability vs Laver measurability. First of all, one may note that, again by Lemma 7.3.33 in [**BJ95**], Laver forcing $\mathbb{L}$ does not add random reals and so, a similar argument used in the proof of Fact 45 shows that

$$\mathbf{L}^{\mathbb{L}_{\omega_1}} \models \mathbf{\Sigma}^1_2(\mathbb{L}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{R}).$$

For the converse, one can again note that $\mathbf{\Sigma}^1_2(\mathbb{R}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{L})$, because of Raisonnier's result and Theorem 44. Moreover, this implication is optimal, since a simple ω_1-iteration of random forcing gives

$$\mathbf{L}^{\mathbb{R}_{\omega_1}} \models \mathbf{\Delta}^1_2(\mathbb{R}) \wedge \neg\mathbf{\Sigma}^1_2(\mathbb{L}).$$

Sacks measurability vs all the others. Sacks measurability is the weakest among all regularity properties that we are considering. In fact, it is clear that it is implied by each of the other ones, i.e.,

$$X \text{ is } \mathbb{P}\text{-measurable} \Rightarrow X \text{ is Sacks measurable},$$

where $\mathbb{P}$ is any of the arboreal forcing in the RP-diagram. The question which could arise is whether is the case of

$$\mathbf{\Delta}^1_2(\mathbb{S}) \Rightarrow \mathbf{\Delta}^1_2(\mathbb{P}) \text{ or } \mathbf{\Sigma}^1_2(\mathbb{S}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{P}).$$

Once more, an enlightening characterization of the statements $\mathbf{\Delta}^1_2(\mathbb{S})$ and $\mathbf{\Sigma}^1_2(\mathbb{S})$, due to Brendle and Löwe, is presented in [**BL99**], theorem 7.1, and we state here for completeness.

Theorem 46. $\mathbf{\Delta}^1_2(\mathbb{S}) \Leftrightarrow \mathbf{\Sigma}^1_2(\mathbb{S}) \Leftrightarrow \forall x \in \omega^\omega(\omega^\omega \cap \mathbf{L}[x] \neq \omega^\omega)$.

Hence, if we consider an ω_1-iteration of Sacks forcing $\mathbb{S}_{\omega_1}$, with countable support, we immediately get

$$\mathbf{L}^{\mathbb{S}_{\omega_1}} \models \mathbf{\Sigma}^1_2(\mathbb{S}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{C}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{M}),$$

since Sacks forcing $\mathbb{S}$ does not add unbounded reals. Moreover, since one may also prove that Sacks forcing does not add random reals, we get

$$\mathbf{L}^{\mathbb{S}_{\omega_1}} \models \mathbf{\Sigma}^1_2(\mathbb{S}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{R}).$$

Silver measurability vs Laver measurability and Lebesgue measurability. A result due to Halbeisen (see [**Ha03**], page 176) shows that Cohen forcing $\mathbb{C}$ adds a Silver tree of Cohen reals. Hence, by a standard argument, an ω_1-iteration of $\mathbb{C}$ with finite support gives us a model for $\mathbf{\Sigma}^1_2(\mathbb{V}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{L}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{R})$. Conversely, since $\mathbb{R}$ does not add unbounded reals, one has $\mathbf{L}^{\mathbb{R}_{\omega_1}} \models \mathbf{\Delta}^1_2(\mathbb{R}) \wedge \neg\mathbf{\Sigma}^1_2(\mathbb{V})$, and since $\mathbb{V}$ satisfies the Laver property, one has $\mathbf{L}^{\mathbb{V}_{\omega_1}} \models \mathbf{\Delta}^1_2(\mathbb{V}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{R})$. On the contrary, the implication $\mathbf{\Sigma}^1_2(\mathbb{L}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{V})$ is still unsolved.

Silver measurability vs Baire property. In the previous paragraph we have seen that to separate Silver measurability and Laver measurability a simple iteration of Cohen forcing is sufficient. Furthermore, that model also satisfies $\mathbf{\Sigma}^1_2(\mathbb{V}) \wedge \neg\mathbf{\Sigma}^1_2(\mathbb{C})$. Nevertheless, the method adds Cohen reals, and so such a model satisfies $\mathbf{\Delta}^1_2(\mathbb{C})$. Hence, the natural question turning out is how to separate $\mathbf{\Sigma}^1_2(\mathbb{V})$ and $\mathbf{\Delta}^1_2(\mathbb{C})$. This question was asked by Halbeisen in [**Ha03**], as question 3 in the last page of the paper. Our idea to answer this question is to find a forcing notion whereby one can add a Silver tree of generic reals, without adding Cohen reals. We want to prove that the right choice is the forcing $\mathbb{VT}$ introduced in the previous chapter at page 25. In that paragraph we have already shown that such forcing $\mathbb{VT}$ adds a Silver tree of Silver reals. Hence, only two things are left:

1. to verify that $\mathbb{VT}$ is proper;
2. to verify that $\mathbb{VT}$ does not add Cohen reals.

Obviously $\mathbb{V} \lessdot \mathbb{VT}$. The following observation points out that they are not equivalent.

Remark 47. Consider the following definition:

- for every tree $\mathfrak{T}$, let $\mathtt{SL}_{\mathfrak{T}}(n) = |t|$, where $t \in \mathfrak{T}$ is an $n+1$-st splitting node;
- for every $T \subseteq \omega^{<\omega}$ finite, let $\mathsf{ns}(T)$ = number of splitting levels of T;
- let $\delta_T(n) = \mathtt{SL}_T(n+1) - \mathtt{SL}_T(n)$ and set

$$\Delta_T = \langle \delta_T(0), \delta_T(1), \dots, \delta_T(\mathsf{ns}(T)-1) \rangle.$$

Finally, if G is $\mathbb{VT}$-generic over $\mathbf{V}$, let $h_G = \bigcup\{\Delta_T : (T, \mathfrak{T}) \in G\}$.

Claim: $\Vdash_{\mathbb{VT}}$ " $\dot{h}_G$ is dominating over $\mathbf{V}$".

To see that, fix an increasing $x \in \omega^\omega \cap \mathbf{V}$ and $(T, \mathfrak{T}) \in \mathbb{VT}$, arbitrarily. Pick $\mathfrak{T}' \subseteq \mathfrak{T}$, $\mathfrak{T}' \restriction \mathrm{ht}(T) = T$ such that for every $n \geq \mathsf{ns}(T)$, $\mathtt{SL}_{\mathfrak{T}'}(n+1) - \mathtt{SL}_{\mathfrak{T}'}(n) > x(n)$.

To prove the properness, we will actually show that $\mathbb{VT}$ satisfies *Axiom A*, which is defined as follows.

Definition 48. A forcing P satisfies *Axiom A* if and only if there exists a sequence $\{\leq_n : n \in \omega\}$ of orderings of P such that:

1. for every $p, p' \in P$, for every $n \in \omega$, $p' \leq_{n+1} p$ implies both $p' \leq_n p$ and $p' \leq p$;
2. for every sequence $\langle p_n : n \in \omega \rangle$ of conditions in P such that for every $n \in \omega, p_{n+1} \leq_n p_n$, there exists $q \in P$ such that for every $n \in \omega$, $q \leq_n p_n$;

3. for every antichain $A \subseteq P$, $p \in P$, $n \in \omega$, there exists $q \leq_n p$ such that $\{p' \in A : p' \text{ is compatible with } q\}$ is countable.

We define the sequence of orderings on $\mathbb{VT}$ as follows:

$$(T', \mathfrak{T}') \leq_n (T, \mathfrak{T}) \quad \Leftrightarrow \quad (T', \mathfrak{T}') \leq (T, \mathfrak{T}) \\ T' = T \wedge \forall k \leq n(\mathtt{SL}_{\mathfrak{T}'}(k) = \mathtt{SL}_{\mathfrak{T}}(k)).$$

Clearly, conditions 1 and 2 of the above definition are satisfied. To obtain condition 3, the following observation is crucial.

Remark 49. Let $D \subseteq \mathbb{VT}$ be open dense and fix $(T, \mathfrak{T}) \in \mathbb{VT}$ arbitrarily. Let $T^0 = \mathfrak{T}\restriction(\mathtt{SL}_{\mathfrak{T}}(h_0) + 1)$, where the h_0-th splitting nodes are the first splitting nodes occurring above T. Furthermore, let $\{T^0_j : j < 3\}$ be an enumeration of all uniform finite trees such that $T \subseteq T^0_j \subseteq T^0$, $\text{ht}(T^0_j) = \text{ht}(T^0)$ and $T^0_j\restriction\text{ht}(T) = T$

Notation: given $\mathfrak{T}$ infinite tree and T finite tree, put

$$\mathfrak{T} \otimes T = \{t \in 2^{<\omega} : \exists t' \in \mathfrak{T} \exists t'' \in \textsc{Term}(T) \quad , \quad \forall n < |t''|(t(n) = t''(n)) \\ \wedge \forall n \geq |t''|(t(n) = t'(n))\}.$$

(Intuitively, $\mathfrak{T} \otimes T$ is the translation of $\mathfrak{T}$ over T).

Starting from such T^0, one develops the following construction along $i \geq h_0$, and $j < 3^{i-h_0+1}$.

- Start from $i = h_0$:
  - Substep $j = 0$: if there exists $\mathfrak{S} \subseteq \mathfrak{T}$ such that $(T^0_0, \mathfrak{S}) \in D$, then put $\mathfrak{T}^0_0 = \mathfrak{S}$; otherwise put $\mathfrak{T}^0_0 = \mathfrak{T}$;
  - Substep $j+1$: if there exists $\mathfrak{S} \subseteq \mathfrak{T}^0_j \otimes T^0_{j+1}$ such that $(T^0_{j+1}, \mathfrak{S}) \in D$, then put $\mathfrak{T}^0_{j+1} = \mathfrak{S}$; otherwise let $\mathfrak{T}^0_{j+1} = \mathfrak{T}^0_j$;
  - when the operation is done for every $j < 3$, put $\mathfrak{T}^0_* = \mathfrak{T}^0_2 \otimes T^0$ and $T^1 = \mathfrak{T}^0_*\restriction(\mathtt{SL}_{\mathfrak{T}^0_*}(h_0+1)+1)$; furthermore, let $\{T^1_j : j < 3^2\}$ be the enumeration of all the uniform finite trees such that $T^1_j \subseteq T^1$, $\text{ht}(T^1_j) = \text{ht}(T^1)$ and $T^1_j\restriction\text{ht}(T) = T$;
- Step $i = h_0 + k + 1$:
  - Substep $j = 0$: if there exists $\mathfrak{S} \subseteq \mathfrak{T}^k_*$ such that $(T^{k+1}_0, \mathfrak{S}) \in D$, then put $\mathfrak{T}^{k+1}_0 = \mathfrak{S}$; otherwise let $\mathfrak{T}^{k+1}_0 = \mathfrak{T}^k_*$;
  - Substep $j + 1$: if there exists $\mathfrak{S} \subseteq \mathfrak{T}^{k+1}_j \otimes T^{k+1}_{j+1}$ such that $(T^{k+1}_{j+1}, \mathfrak{S}) \in D$, then put $\mathfrak{T}^{k+1}_{j+1} = \mathfrak{S}$; otherwise let $\mathfrak{T}^{k+1}_{j+1} = \mathfrak{T}^{k+1}_j$;
  - when the operation is done for every $j < 3^{k+2}$, put $\mathfrak{T}^{k+1}_* = \mathfrak{T}^{k+1}_{3^{k+2}-1} \otimes T^{k+1}$ and $T^{k+2} = \mathfrak{T}^{k+1}_*\restriction(\mathtt{SL}_{\mathfrak{T}^{k+1}_*}(i+1)+1)$; furthermore,

let $\{T^{k+2}_j : j < 3^{k+3}\}$ be the enumeration of all the uniform finite trees such that $T^{k+2}_j \subseteq T^{k+2}$, $\mathrm{ht}(T^{k+2}_j) = \mathrm{ht}(T^{k+2})$ and $T^{k+2}_j \restriction \mathrm{ht}(T) = T$.

Once that such a construction is finished, one obtains a sequence $\langle \mathfrak{T}^k_* : k \in \omega \rangle$ such that $\mathfrak{T}^{k+1}_* \leq_{h_0+k} \mathfrak{T}^k_*$. Hence, the tree $\mathfrak{T}^*$ obtained by fusion, i.e., $\mathfrak{T}^* = \bigcap_{k\in\omega} \mathfrak{T}^k_*$, is a Silver tree, and so the pair $(T, \mathfrak{T}^*)$ belongs to $\mathbb{VT}$ and $(T, \mathfrak{T}^*) \leq_{h_0} (T, \mathfrak{T})$.

Let $\mathfrak{T}^* {\downarrow} S = \{t \in \mathfrak{T}^* : \exists s \in S(t \trianglerighteq s \vee t \trianglelefteq s)\}$. By construction, one gets

$$\forall (S, \mathfrak{S}) \leq (T, \mathfrak{T}^*),\ \text{if } (S, \mathfrak{S}) \in D \text{ then } (S, \mathfrak{T}^* {\downarrow} S) \in D. \tag{2.2}$$

Condition (2.2) is the core of the next lemma.

Lemma 50. *Let $A \subset P$ be a maximal antichain and $(T, \mathfrak{T}) \in \mathbb{VT}$. Then there exists $\mathfrak{T}^* \subseteq \mathfrak{T}$ such that $(T, \mathfrak{T}^*)$ only has countably many compatible elements in A.*

Proof. Fix a condition $(T, \mathfrak{T}) \in \mathbb{VT}$. Let D_A be the open dense subset associated with A, i.e., $D_A = \{p \in \mathbb{VT} : \exists q \in A(p \leq q)\}$. Let $\mathfrak{T}^*$ be as in remark 49. To reach a contradiction, assume there are uncountably many elements in A compatible with $(T, \mathfrak{T}^*)$, i.e., there is a sequence $\langle (T_\alpha, \mathfrak{T}_\alpha) : \alpha < \omega_1 \rangle$ of distinct elements of A and there are $(S_\alpha, \mathfrak{S}_\alpha)$'s such that, for every $\alpha < \omega_1$,

$$(S_\alpha, \mathfrak{S}_\alpha) \leq (T_\alpha, \mathfrak{T}_\alpha), (T, \mathfrak{T}^*).$$

Note that $(S_\alpha, \mathfrak{S}_\alpha) \in D_A$. Thus, by remark 49, one obtains $(S_\alpha, \mathfrak{T}^* {\downarrow} S_\alpha) \in D_A$, and therefore

$$(S_\alpha, \mathfrak{T}^* {\downarrow} S_\alpha) \leq (T_\alpha, \mathfrak{T}_\alpha), (T, \mathfrak{T}^*).$$

Note that there are only countably many different $(S_\alpha, \mathfrak{T}^* {\downarrow} S_\alpha)$'s and therefore there exist $\alpha_0, \alpha_1 < \omega_1$ such that $(S_{\alpha_0}, \mathfrak{T}^* {\downarrow} S_{\alpha_0}) = (S_{\alpha_1}, \mathfrak{T}^* {\downarrow} S_{\alpha_1})$, and this contradicts $(T_{\alpha_0}, \mathfrak{T}_{\alpha_0}) \perp (T_{\alpha_1}, \mathfrak{T}_{\alpha_1})$. □

Note also that in remark 49, for any $n \in \omega$, one could repeat the construction starting from $h_0 = n$, in order to get $(T, \mathfrak{T}^*) \leq_n (T, \mathfrak{T})$. Thus, in lemma 50 one can actually pick $(T, \mathfrak{T}^*) \leq_n (T, \mathfrak{T})$, and therefore one obtains condition 3 of definition 48.

Futhermore, one can prove that $\mathbb{VT}$ satisfies the L_f-property (see definition 19), which is the content of the next result.

Lemma 51. *Let $A \in [\omega]^{<\omega}$ and $\dot{a}$ a $\mathbb{VT}$-name for an element of A. Then for any condition $(T, \mathfrak{T}) \in \mathbb{VT}$, for every $n \in \omega$, there exists $(T, \mathfrak{T}^*) \leq_n (T, \mathfrak{T})$ and $B \subseteq A$, $B \subseteq 3^n$ such that*

$$(T, \mathfrak{T}^*) \Vdash \dot{a} \in B.$$

Proof. We will need the following result.

Lemma 52. (Pure Decision of $\mathbb{VT}$). *Let* $(T,\mathfrak{T}) \in \mathbb{VT}$ *and* $\varphi_0,\dots,\varphi_k$ *be a finite list of statements such that* $(T,\mathfrak{T}) \Vdash \bigvee_{i\le k}\varphi_i$*. Then there exists* $\mathfrak{T}^* \subseteq \mathfrak{T}$ *and* $i \le k$ *such that*

$$(T,\mathfrak{T}^*) \Vdash \varphi_i.$$

Proof of Lemma 52. Note that if, for some $i \le k$, there exists $(T',\mathfrak{T}') \le (T,\mathfrak{T})$ such that every $t' \in T' \setminus T$ is not splitting, and $(T',\mathfrak{T}') \Vdash \varphi_i$, then $(T,\mathfrak{T}') \Vdash \varphi_i$ as well. Hence, w.l.o.g., one can assume that does not happen.

To reach a contradiction, assume there exists a minimal $(T^0,\mathfrak{T}^0) \le (T,\mathfrak{T})$ such that $(T^0,\mathfrak{T}^0) \Vdash \varphi_i$, for some $i \le k$, where minimal means that there is no $T' \subset T^0$ (and $\mathfrak{T}' \in \mathbb{V}$) such that $(T',\mathfrak{T}') \Vdash \varphi_i$.

Let $\{T^0_j : j \le k_0\}$ be the list of all finite trees such that $T \subseteq T^0_j \subseteq T^0$ and $T^0_j{\restriction}\mathrm{ht}(T) = T$, for every $j \le k_0$.

> STEP 0: pick $(S^1_0,\mathfrak{T}^1_0) \le (T^0_0,\mathfrak{T}^0)$ such that $(S^1_0,\mathfrak{T}^1_0) \Vdash \varphi_i$ and put $\mathfrak{S}^0_1 = \mathfrak{T}^1_0 \otimes T^0_1$;
>
> STEP J: pick $(S^1_j,\mathfrak{T}^1_j) \le (T^0_j,\mathfrak{S}^0_j)$ such that $(S^1_j,\mathfrak{T}^1_j) \Vdash \varphi_i$ and put $\mathfrak{S}^0_{j+1} = \mathfrak{T}^1_j \otimes T^0_{j+1}$;
>
> Finally, once the procedure has been done for every $j \le k_0$, put $\mathfrak{T}^* = \bigcup_{j\le k_0}\mathfrak{T}^1_{k_0} \otimes T^0_j$.

By construction $\mathfrak{T}^*$ is perfect and uniform. Hence $(T,\mathfrak{T}^*)$ is well-defined, that means $(T,\mathfrak{T}^*) \in \mathbb{VT}$.

Furthermore, for every $(S,\mathfrak{S}) \le (T,\mathfrak{T}^*)$, either $S \le T^0$ (and then $(S,\mathfrak{T}^*) \Vdash \varphi_i$ simply because $(S,\mathfrak{T}^*) \le (T^0,\mathfrak{T}^0)$), or $S = T^0_j$, for the appropriate $j \le k_0$, and therefore there is $S^1_j \le S$ such that

$$(S^1_j,\mathfrak{S}{\downarrow}S^1_j) \le (S,\mathfrak{S}) \text{ and } (S^1_j,\mathfrak{T}^*{\downarrow}S^1_j) \Vdash \varphi_i.$$

By density, that means $(T,\mathfrak{T}^*) \Vdash \varphi_i$, which contradicts the minimality of T_0. □

We now proceed with the proof of lemma 51. Fix $n \in \omega$ arbitrarily and consider $\{T_j : j \le J_n\}$ the list of all finite trees such that $T_j \supseteq T$, $T_j{\restriction}\mathrm{ht}(T) = T$ and $\mathrm{ht}(T_j) = h_n$, where h_n is the level of the n-th splitting nodes. (The development of the proof will show that the case $h_n \le \mathrm{ht}(T)$ is trivial, and so one can assume $h_n > \mathrm{ht}(T)$). Note also that $J_n \le 3^n$.

Let $A = \{a_i : i \le u\}$, for some $u \in \omega$. Note that, for every $j \le J_n$,

$$(T_j,\mathfrak{T}) \Vdash \bigvee_{i\le u} \dot{a} = a_i.$$

As usual, one proceeds by steps:

- start from T_0. Let $\mathfrak{T}_0 = \mathfrak{T}{\downarrow}T_0$ and, by applying lemma 52, choose $\mathfrak{S}_0 \leq \mathfrak{T}_0$ such that $(T_0, \mathfrak{S}_0) \Vdash \dot{a} = a_{i_0}$, for some $i_0 \leq u$.
- Let $\mathfrak{T}_{j+1} = \mathfrak{S}_j \otimes T_{j+1}$ and, by applying lemma 52, choose $\mathfrak{S}_{j+1} \leq \mathfrak{T}_{j+1}$ such that $(T_{j+1}, \mathfrak{S}_{j+1}) \Vdash \dot{a} = a_{i_{j+1}}$, for some $i_{j+1} \leq u$.
- Finally, once the construction has been done for every $j \leq J_n$, set

$$\mathfrak{T}^* = \mathfrak{T}_{J_n} \otimes (\mathfrak{T}{\restriction}h_n) \text{ and } B = \{a_{i_j} : j \leq J_n\}.$$

Hence, by construction, $(T, \mathfrak{T}^*) \leq (T, \mathfrak{T})$ and $(T, \mathfrak{T}^*) \Vdash \dot{a} \in B$, where $|B| \leq J_n \leq 3^n$.

□

We are now able to prove the main result of this section.

Theorem 53. *Let $\mathbb{VT}_{\omega_1}$ be an iteration of length ω_1 with countable support of $\mathbb{VT}$ and let G be a $\mathbb{VT}_{\omega_1}$-generic over $\mathbf{L}$. Then*

$$\mathbf{L}[G] \models \boldsymbol{\Sigma}^1_2(\mathbb{V}) \wedge \neg\boldsymbol{\Delta}^1_2(\mathbb{C}).$$

Proof. Let $X = \{x \in 2^\omega : \varphi(r, x)\} \in \mathbf{L}[G]$, where φ is a Σ^1_2-formula and r is the real parameter. As usual, one can find $\alpha < \omega_1$ such that $r \in \mathbf{L}[G{\restriction}\alpha]$. Consider the formula $\varphi(\dot{v})$, where $\dot{v}$ is the canonical name for the $\mathbb{V}$-generic real. Furthermore, since X is $\boldsymbol{\Sigma}^1_2$, one can find ω_1-many Borel sets B_γ coded in $\mathbf{L}[G{\restriction}\alpha]$ such that $X = \cup_{\gamma<\omega_1} B_\gamma$. Two cases are therefore possible. The first one is that for every $\gamma \in \omega_1$, $B_\gamma \in \mathcal{I}_\mathbb{V}$. If that happens, then simply consider the generic Silver tree T_α of $\mathbb{V}$-generic reals added at stage α. Since every $\mathbb{V}$-generic real avoids all the Borel sets in $\mathcal{I}_\mathbb{V} \cap \mathbf{L}[G{\restriction}\alpha]$, one therefore gets that $[T_\alpha] \cap X = \emptyset$. Hence, one obtains

$$\mathbf{L}[G{\restriction}\alpha][G(\alpha)] \models \forall x \in [T_\alpha](\neg\varphi(x)),$$

which is a Π^1_2-formula, and so also

$$\mathbf{L}[G] \models [T_\alpha] \cap X = \emptyset.$$

In the second case there exists $\gamma \in \omega_1$ such that $B_\gamma \notin \mathcal{I}_\mathbb{V}$. Hence, since every Borel set is $\mathbb{V}$-measurable, there is $T \in \mathbb{V}$, $[T] \subseteq B_\gamma$. Hence, one gets

$$\mathbf{L}[G{\restriction}\alpha][G(\alpha)] \models \forall x \in [T](x \in B_\gamma),$$

which is a Π^1_1-formula, and so, again by absoluteness,

$$\mathbf{L}[G] \models [T] \subseteq B_\gamma \subseteq X.$$

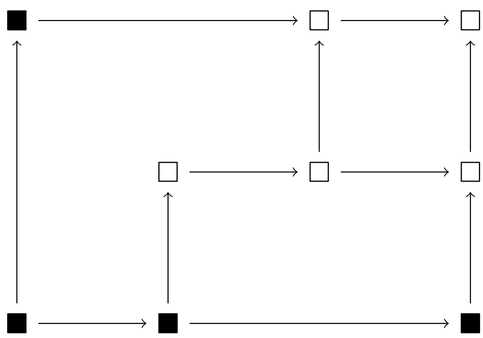
□

Silver measurability vs Miller measurability. The parallel between these two regularity properties is interesting. In fact, even if they seem rather different, Brendle, Löwe and Halbeisen showed that $\mathbf{\Sigma}^1_2(\mathbb{V}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{M})$ (see [**BLH05**], proposition 3.7). At the same time, it is clear that one cannot replace the left side with $\mathbf{\Delta}^1_2(\mathbb{V})$, since the ω^ω-boundedness of $\mathbb{V}$ implies that $\mathbb{V}_{\omega_1}$ forces $\mathbf{\Delta}^1_2(\mathbb{V}) \wedge \neg\mathbf{\Sigma}^1_2(\mathbb{M})$. On the contrary, it is known that $\mathbb{M}_{\omega_1}$ does not add splitting reals, and therefore it provides a model for $\mathbf{\Sigma}^1_2(\mathbb{M}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{V})$ (because of proposition 2.4 in [**BLH05**], saying that $\mathbf{\Delta}^1_2(\mathbb{V})$ implies for all $x \in \omega$, there exists a splitting reals over $\mathbf{L}[x]$).

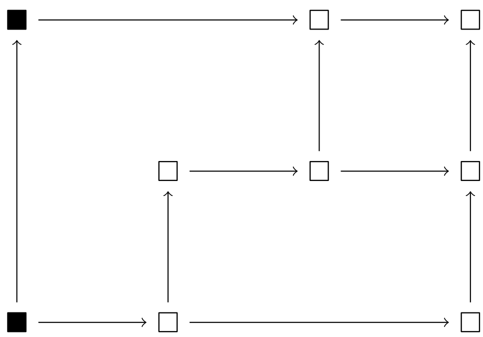
2.1.1 Concluding remarks.

We summarize all of the results of the previous paragraphs in terms of the RP-diagram.

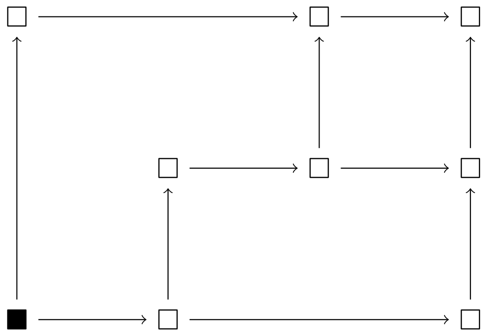
- $\mathbb{C}_{\omega_1}$ forces



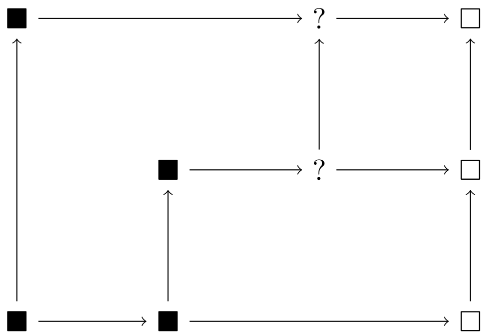
- $\mathbb{UM}_{\omega_1}$ forces



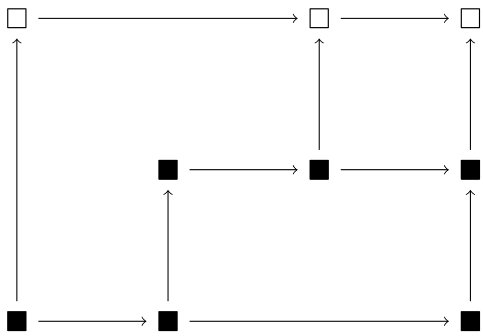
In the latter, to get $\mathbf{\Delta}^1_2(\mathbb{R})$ without $\mathbf{\Sigma}^1_2(\mathbb{R})$ is sufficient to consider a mixed ω_1-iteration of $\mathbb{UM}$ and $\mathbb{R}$, say $(\mathbb{UM} * \mathbb{R})_{\omega_1}$, which therefore provides a model for



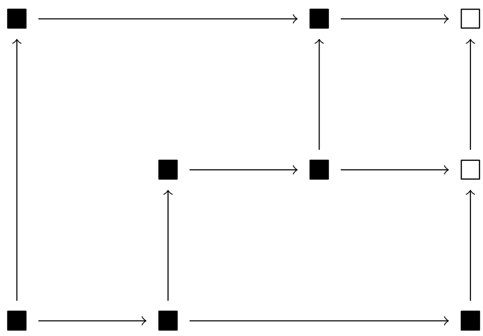
- $\mathbb{L}_{\omega_1}$ forces



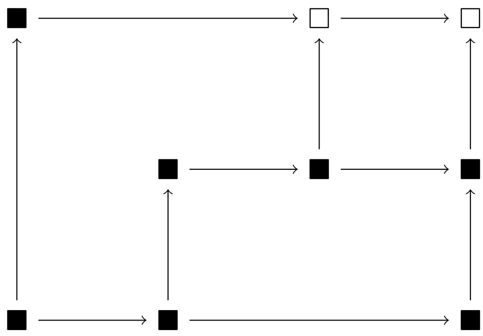
- $\mathbb{R}_{\omega_1}$ forces



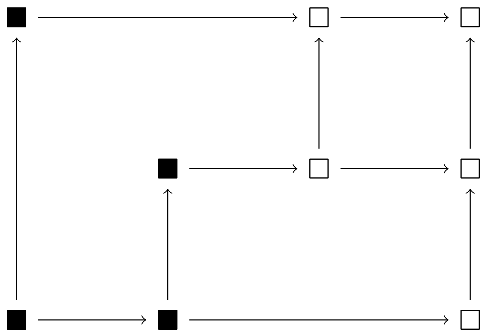
- $\mathbb{M}_{\omega_1}$ forces



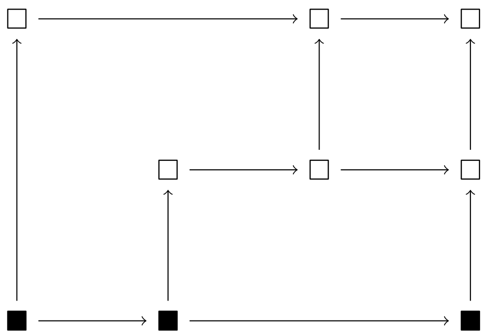
- $\mathbb{V}_{\omega_1}$ forces



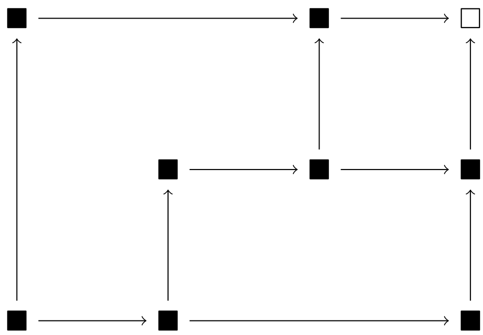
- $\mathbb{VT}_{\omega_1}$ forces



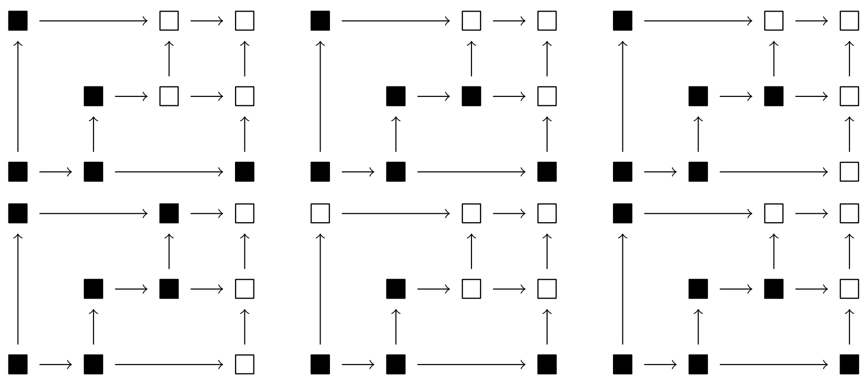
- $(\mathbb{C} * \mathbb{R})_{\omega_1}$ forces



- $\mathbb{S}_{\omega_1}$ forces



Remark 54. From the question marks left in the previous diagrams, we realize that the problem to get a complete work for all possible combinations is still open. In particular the following diagrams are still without models:



2.2 Regularity properties for Γ

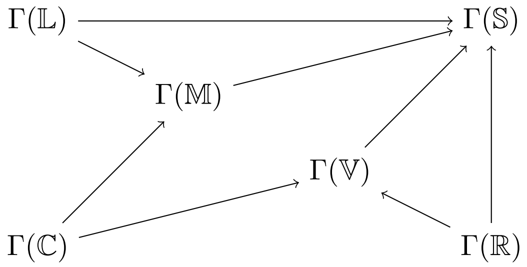
In this section we will go into the second topic of our work, proving some results of separation between regularity properties for the family Γ. Remind that separation means that we will show that, in some cases, statements of the form

$$\Gamma(\mathbb{P}) \equiv^{\text{def}} \text{“every set of reals is } \mathbb{P}\text{-measurable”}$$

are independent one from the other, which means that given a pair of different arboreal forcings $\mathbb{P}, \mathbb{Q}$ among those considered in this thesis, we will be able to construct a model $\mathbf{N}$ such that

$$\mathbf{N} \models \Gamma(\mathbb{P}) \wedge \neg\Gamma(\mathbb{Q}).$$

Before starting our work, we give a survey on what we already know. The following diagram, which will be called the **Γ-RP-diagram**, summarize the known implications existing between such statements.



Some first comments on the Γ-RP-diagram Some implications are trivial, such as $\Gamma(\mathbb{L}) \Rightarrow \Gamma(\mathbb{M})$, $\Gamma(\mathbb{M}) \Rightarrow \Gamma(\mathbb{S})$ and $\Gamma(\mathbb{V}) \Rightarrow \Gamma(\mathbb{S})$. Furthermore, to get $\Gamma(\mathbb{C}) \Rightarrow \Gamma(\mathbb{M})$ one only has to note that any comeager set contains the body of a Miller tree (see remark 42), while $\Gamma(\mathbb{R}) \Rightarrow \Gamma(\mathbb{V})$ follows from fact 39, which we proved at page 29. Finally, $\Gamma(\mathbb{C}) \Rightarrow \Gamma(\mathbb{V})$ can be obtained by the following remark.

Fact 55. *Any comeager set contains the body of a Silver tree.*

Proof. Let $Y \supseteq \bigcap_{n\in\omega} D_n$, where all D_n's are open dense. Consider the following recursive construction:

- let $t_{\langle 0\rangle} \in 2^{<\omega}$ such that $[t_{\langle 0\rangle}] \subseteq D_0$;
- Assume for every $r \in 2^n$, t_r already defined, in order to have $[t_r] \subseteq D_n$. Let $\{t_j : j < 2^{n+1}\}$ be an enumaration of $\{t_r^\frown i : r \in 2^n, i = 0, 1\}$. Then consider the following construction along $j < 2^{n+1}$:

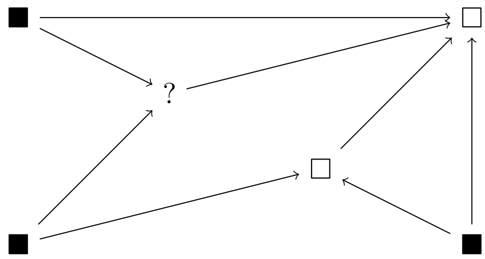
  for $j = 0$, pick $s^0 \trianglerighteq t_0$ such that $[s^0] \subseteq D_{n+1}$;

  for $j + 1$, pick $s^{j+1} \trianglerighteq s^j \oplus t_{j+1}$ such that $[s^{j+1}] \subseteq D_{n+1}$.

  Then put $t_{r^\frown i} = s^{2^{n+1}-1} \oplus t_j$, where $t_j = t_r^\frown i$.

Finally, set $T = \bigcup\{t_r : r \in 2^{<\omega}, t_r \text{ as defined in the construction}\}$. It is clear that T is a Silver tree such that for every $z \in [T]$, $z \in \bigcap_{n\in\omega} D_n$. □

For separating $\Gamma(\mathbb{S})$ and $\Gamma(\mathbb{V})$ from the others, one can simply recall that an ω_1-iteration of $\mathbb{C}$ with finite support forces every set of reals in $\mathbf{L}(\omega^\omega)$ to be $\mathbb{V}$-measurable (see [**Ha03**]), and so, going into $\mathbf{L}(\omega^\omega)$ of such an extension, we get $\Gamma(\mathbb{S})$ and $\Gamma(\mathbb{V})$ without $\Gamma(\mathbb{L}), \Gamma(\mathbb{C})$ and $\Gamma(\mathbb{R})$ (by several results stated in the previous section about characterization of the statements $\mathbf{\Sigma}^1_2(\mathbb{P})$ and $\mathbf{\Delta}^1_2(\mathbb{P})$), while we do not know the behavior of $\Gamma(\mathbb{M})$ in that model. The corresponding Γ-**RP-diagram** is the following:



The coming section is devoted to obtain a model to separate $\Gamma(\mathbb{V})$ from $\Gamma(\mathbb{M})$.

2.2.1 Silver measurability without Miller measurability.

This section is devoted to show how we may separate Silver measurability from Miller measurability. As we mentioned in the introduction, this work is mainly inspired by the fact that $\mathbf{\Sigma}^1_2(\mathbb{V}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{M})$. This implication can be obtained by proposition 3.7 in [**BLH05**], stating that

$$\mathbf{\Sigma}^1_2(\mathbb{V}) \Rightarrow \forall x \in \omega^\omega (\omega^\omega \cap \mathbf{L}[x] \text{ is not dominating}),$$

and theorem 43. Hence, a natural question is whether such an implication extends to the family Γ. The task of this section is precisely to give a negative answer. In fact, we will construct a model

$$\mathbf{N} \models \Gamma(\mathbb{V}) \wedge \neg\Gamma(\mathbb{M}).$$

Note that we will not separate the projective $\mathbb{V}$-measurability from the projective $\mathbb{M}$-measurability. In fact, our method will be to construct a set Y non-$\mathbb{M}$-measurable (and not projective), and to then amalgamate over Silver forcing $\mathbb{V}$, with respect to such Y. The construction will give us a Boolean algebra $\mathbf{B}_\kappa$ forcing Silver measurability of every set of reals in $\mathbf{L}(\omega^\omega, Y)$, which means that we will build up a model

$$\mathbf{V}[G] \models \text{"every set of reals in } \mathbf{L}(\omega^\omega, Y) \text{ is } \mathbb{V}\text{-measurable and } Y \text{ is not } \mathbb{M}\text{-measurable"},$$

where G is $\mathbf{B}_\kappa$-generic over $\mathbf{V}$. We will therefore get the desired model picking $\mathbf{L}(\omega^\omega, Y)^{\mathbf{V}[G]}$.

In this case, one has to make sure that, not only all sets in $\mathbf{L}(\omega^\omega)$ are regular, but all sets in $\mathbf{L}(\omega^\omega, Y)$ are such. In this spirit, one introduces the following notion.

Definition 56. Let $\mathbf{B}$ be a Boolean algebra and $\dot{Y}$ a $\mathbf{B}$-name. One says that $\mathbf{B}$ is $(\mathbb{V}, \dot{Y})$-homogeneous if and only if any isomorphism ϕ_0 between two complete subalgebras $\mathbf{B}_1, \mathbf{B}_2$ of $\mathbf{B}$, such that, for $j = 1, 2$, $\mathbf{B}_j$ is generated by $\mathbf{A}_j \cup a_j$, for some $\mathbf{A}_j \cong B(\mathbb{V})$ and $a_j \in \mathbf{B}$, there exists $\phi : \mathbf{B} \to \mathbf{B}$ automorphism extending ϕ_0 such that $\Vdash_{\mathbf{B}} \phi(\dot{Y}) = \dot{Y}$. (Intuitively, we want a $\mathbf{B}$-name fixed by any automorphism constructed by the amalgamation).

Furthermore, since we will use Silver tree of Silver reals to get Silver measurability, one only has to amalgamate over Silver forcing. So, as usual, one starts from a ground model $\mathbf{V}$ containing an inaccessible cardinal κ. Define a Boolean algebra $\mathbf{B}_\kappa$ as a direct limit of κ-many Boolean algebras $\mathbf{B}_\alpha$'s of size $< \kappa$, such that for every $\alpha < \gamma < \kappa$, $\mathbf{B}_\alpha \lessdot \mathbf{B}_\gamma$, and one simultaneously constructs a set $\dot{Y}$ of $\mathbf{B}_\kappa$-names of reals. Such a set is constructed step by step, that means, for every $\alpha < \kappa$, one defines $\dot{Y}_\alpha$ in a specific way and one finally puts $\dot{Y} = \bigcup_{\alpha<\kappa} \dot{Y}_\alpha$. The two sequences $\langle \mathbf{B}_\alpha : \alpha < \kappa \rangle$ and $\langle \dot{Y}_\alpha : \alpha < \kappa \rangle$ are defined as follows:

- Firstly, to ensure the $(\mathbb{V}, \dot{Y})$-homogeneity, we use a standard bookkeeping argument as follows: whenever $\mathbf{B}_\alpha \lessdot \mathbf{B}' \lessdot \mathbf{B}_\kappa$ and $\mathbf{B}_\alpha \lessdot \mathbf{B}'' \lessdot \mathbf{B}_\kappa$ are such that $\mathbf{B}_\alpha$ forces $(\mathbf{B}' : \mathbf{B}_\alpha)$ and $(\mathbf{B}'' : \mathbf{B}_\alpha)$ to be as in definition 56 and $\phi : \mathbf{B}' \to \mathbf{B}''$ an isomorphism s.t. $\phi_0 \restriction \mathbf{B}_\alpha = \mathrm{Id}_{\mathbf{B}_\alpha}$, then there exists a sequence of functions in order to extend the isomorphism ϕ_0 to an automorphism $\phi : \mathbf{B}_\kappa \to \mathbf{B}_\kappa$, i.e., $\exists \langle \alpha_\eta : \eta < \kappa \rangle$ increasing, cofinal in κ, and $\exists \langle \phi_\eta : \eta < \kappa \rangle$ such that $\mathrm{dom}(\phi_\eta) \supseteq \mathbf{B}_{\alpha_\eta}$ and

$$(\mathbf{B}_{\alpha_{1+\eta}+1}) = \omega\text{-}\mathbf{Am}(\mathbf{B}_{\alpha_{1+\eta}}, \phi_\eta),$$

  Moreover, since one needs to close the set of names under each of such automorphisms ϕ_η, one puts

$$\dot{Y}_{\alpha_{1+\eta}+1} = \{\phi^j_{\eta+1}(\dot{y}) : \dot{y} \in \dot{Y}_{\alpha_{1+\eta}}, j \in \mathbb{Z}\}.$$

- Secondly, to ensure the Silver measurability of every set of reals in $\mathbf{L}(\omega^\omega, Y)$ and that Y will not be Miller measurable, one has to add the following operations into the construction of $\mathbf{B}_\kappa$:

  1. iteration with $\mathbb{VT}$ cofinally often, and so, for cofinally many α's,

$$\mathbf{B}_{\alpha+1} = \mathbf{B}_\alpha * \dot{\mathbb{VT}}.$$

     In this case, put $\dot{Y}_{\alpha+1} = \dot{Y}_\alpha$.
  2. for cofinally many α's, $\mathbf{B}_{\alpha+1} = \mathbf{B}_\alpha * \dot{\mathbb{M}}$ and $\dot{Y}_{\alpha+1} = \dot{Y}_\alpha$;

3. for cofinally many α's, $\mathbf{B}_{\alpha+1} = \mathbf{B}_\alpha * \dot{\mathbb{M}}$ and

$$\dot{Y}_{\alpha+1} = \dot{Y}_\alpha \cup \{\dot{y}_T : T \in \mathbb{M}\},$$

where $\dot{y}_T$ is a name for an $\mathbb{M}$-generic real over $\mathbf{V}^{\mathbf{B}_\alpha}$ belonging to $[T]$, for every $T \in \mathbf{V}^{\mathbf{B}_\alpha}$.

- Finally, for any limit ordinal λ, $\dot{Y}_\lambda = \bigcup_{\alpha<\lambda} \dot{Y}_\alpha$ and $\mathbf{B}_\lambda = \lim_{\alpha<\lambda} \mathbf{B}_\alpha$.

The proof of the main theorem splits into the following two lemmata.

Lemma 57. *Let G be $\mathbf{B}_\kappa$-generic over $\mathbf{V}$. Then*

$$\mathbf{V}[G] \models \text{“every set of reals in } \mathbf{L}(\omega^\omega, Y) \text{ is Silver measurable”}.$$

Proof. The proof of this lemma is basically the same presented in example 36. Let us remind the main steps. Fix arbitrarily $X \subseteq 2^\omega$ and Φ and r such that $X = \{x \in 2^\omega : \Phi(x, r)\}$. Let $\alpha < \kappa$ be such that $r \in \mathbf{V}[G{\restriction}\alpha + 1]$ and $\mathbf{B}_{\alpha+1} = \mathbf{B}_\alpha * \dot{\mathbb{VT}}$. Note that, by construction,

$$\mathbf{V}[G{\restriction}\alpha + 1] \models \text{“}\mathbf{B}_\kappa/G{\restriction}\alpha + 1 \text{ is } (\mathbb{V}, \dot{Y})\text{-homogeneous”}.$$

The next step is to show the reflection property for Φ over Silver reals, which is the content of the next observation.

Remark 58. Let $\mathbf{A}$ be $(\mathbb{V}, \dot{Y})$-homogeneous algebra and $\Phi(x, y)$ be a formula with only parameters in the ground model and Y as parameter, then $||\Phi(\dot{Y}, \dot{v})||_\mathbf{A} \in \mathbf{A}_{\dot{v}}$, where $\dot{v}$ is a name for a Silver real.

The proof is pretty standard and we give a sketch of it for completeness. To reach a contradiction, assume $||\Phi(\dot{Y}, \dot{v})||_\mathbf{A} \notin \mathbf{A}_{\dot{v}}$. Let $\mathbf{A}'$ be the complete Boolean algebra generated by $\mathbf{A}_{\dot{v}} \cup ||\Phi(\dot{Y}, \dot{v})||_\mathbf{A}$. It is well-known that there exists $\rho : \mathbf{A}' \to \mathbf{A}'$ automorphism such that $\rho(||\Phi(\dot{Y}, \dot{v})||_\mathbf{A}) \neq ||\Phi(\dot{Y}, \dot{v})||_\mathbf{A}$ and ρ is the identity over $\mathbf{A}_{\dot{v}}$. By $(\mathbb{V}, \dot{Y})$-homogeneity, there exists $\phi : \mathbf{A} \to \mathbf{A}$ automorphism extending ρ such that $\Vdash_\mathbf{A} \phi(\dot{Y}) = \dot{Y}$. Hence, the following equalities yields a contradiction:

$$\rho(||\Phi(\dot{Y}, \dot{v})||_\mathbf{A}) = \phi(||\Phi(\dot{Y}, \dot{v})||_\mathbf{A}) = ||\Phi(\phi(\dot{Y}), \phi(\dot{v}))||_\mathbf{A} = ||\Phi(\dot{Y}, \dot{v})||_\mathbf{A}.$$

From now on the proof exactly continues as in example 36. □

Lemma 59. *Let G be a $\mathbf{B}_\kappa$-generic filter over $\mathbf{V}$. Then*

$$\mathbf{V}[G] \models \text{“}Y \text{ is not Miller measurable”}.$$

Proof. In $\mathbf{V}[G]$, we want to show that there is a tree $T \in \mathbb{M}$ such that for every tree $S \in \mathbb{M}, S \leq T$, both

$$Y \cap [S] \neq \emptyset \text{ and } [S] \nsubseteq Y.$$

Fix $S \in \mathbb{M}$. Let $\dot{S}$ be a $\mathbf{B}_\kappa$-name. By construction, there is $\alpha < \kappa$ such that $\dot{S}$ is a $\mathbf{B}_\alpha$-name for S, $\mathbf{B}_{\alpha+1} = \mathbf{B}_\alpha * \dot{\mathbb{M}}$ and $\dot{Y}_{\alpha+1} = \dot{Y}_\alpha \cup \{\dot{y}_T : T \in \mathbb{M}\}$. Consider $\dot{y}_S$ name for a Miller real over $\mathbf{V}[G{\restriction}\alpha+1]$ such that $\mathbf{V}[G] \models \dot{y}_S^G \in [S]$. Thus,

$$V[G] \models \dot{y}_S^G \in Y \cap [S].$$

On the other hand, there is also $\gamma < \kappa$, such that $\dot{S}$ is a $\mathbf{B}_\gamma$-name for S, $\mathbf{B}_{\gamma+1} = \mathbf{B}_\gamma * \dot{\mathbb{M}}$ and $\dot{Y}_{\gamma+1} = \dot{Y}_\gamma$. Let $\dot{g}$ be a name for a Miller real over $\mathbf{V}[G{\restriction}\gamma+1]$ such that $\mathbf{V}[G] \models \dot{g}^G \in [S]$. Obviously, $\mathbf{V}[G] \models \dot{g}^G \notin Y_\gamma$, since it occurs at stage $\gamma + 1$, and thus,

$$\mathbf{V}[G] \models \dot{g}^G \in [S] \setminus \dot{Y}^G_{\gamma+1},$$

since $\dot{Y}_{\gamma+1} = \dot{Y}_\gamma$. It is left to show that $\mathbf{V}[G] \models \dot{g}^G \notin Y \setminus \dot{Y}^G_{\gamma+1}$. This follows from the following two general results.

Fact 60. *Let $\dot{x}$ be a $\mathbf{B}$-name for an element of ω^ω such that*

$$\Vdash_{\mathbf{B}} \text{“}\dot{x} \text{ is unbounded over both } \mathbf{V}^{\mathbf{B}_1} \text{ and } \mathbf{V}^{\mathbf{B}_2}\text{”},$$

where $\mathbf{B}_1, \mathbf{B}_2 \lessdot \mathbf{B}$ and $\phi_0 : \mathbf{B}_1 \to \mathbf{B}_2$ isomorphism. Then, for every $n \in \omega$,

$$\Vdash_{\omega\text{-}\mathbf{Am}(\mathbf{B},\phi_0)} \text{“}\phi^n_\omega(\dot{x}) \text{ and } \phi^{-n}_\omega(\dot{x}) \text{ are both unbounded over } \mathbf{V}^{\mathbf{B}}\text{”}.$$

where ϕ_ω is the automorphism of ω-$\mathbf{Am}(\mathbf{B}, \phi_0)$ extending ϕ_0.

For a proof one may see lemma 3.4 in [**JR93**].

Fact 61. *Let $\mathbf{B}_0 \lessdot \mathbf{B}' \lessdot \mathbf{B}$ and $\mathbf{B}_0 \lessdot \mathbf{B}'' \lessdot \mathbf{B}$ such that*

$$\Vdash_{\mathbf{B}_0} \text{“}(\mathbf{B} : \mathbf{B}') \text{ and } (\mathbf{B} : \mathbf{B}'') \text{ are isomorphic to } \mathbf{Q}\text{”}$$

where $\mathbf{Q}$ does not add unbounded reals. Assume $\phi_0 : \mathbf{B}' \to \mathbf{B}''$ isomorphism such that $\phi_0{\restriction}\mathbf{B}_0 = \mathrm{Id}_{\mathbf{B}_0}$. Then for every $\dot{x}$ $\mathbf{B}$-name for an element of ω^ω such that $\Vdash_{\mathbf{B}}$ “$\dot{x}$ is unbounded over $\mathbf{V}^{\mathbf{B}_0}$, one has, for every $n \in \omega$,

$$\Vdash_{\omega\text{-}\mathbf{Am}(\mathbf{B},\phi_0)} \text{“}\phi^n_\omega(\dot{x}) \text{ and } \phi^{-n}_\omega(\dot{x}) \text{ are both unbounded over } \mathbf{V}^{\mathbf{B}}\text{”}.$$

The proof of that is a simple corollary of fact 60 and the assumption that $\mathbf{Q}$ is ω^ω-bounding.

Hence, since Silver forcing is ω^ω-bounding, the results apply to our case. Thus, we have found a $\mathbf{B}_\kappa$-name $\dot{g}$ for a real such that

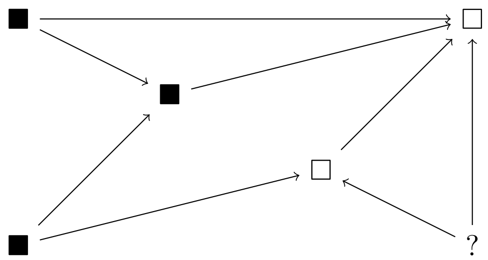
$$\mathbf{V}[G] \models \dot{g}^G \in [S] \setminus Y,$$

which completes the proof. □

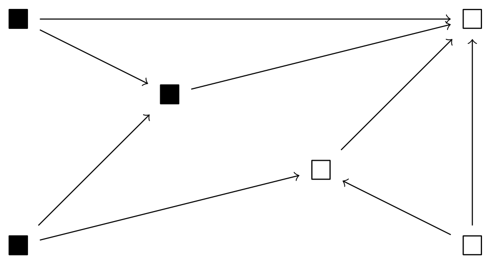
Hence, if one considers the inner model $\mathbf{L}(\omega^\omega, Y)$ of $\mathbf{V}[G]$, one obtains

$$\mathbf{L}(\omega^\omega, Y)^{\mathbf{V}[G]} \models \Gamma(\mathbb{V}) \wedge \neg\Gamma(\mathbb{M}),$$

which corresponds to the following Γ-RP-diagram:



Furthermore, it is straightforward to note that if, in the previous construction, one simultaneously amalgamates over random forcing and iterates Amoeba for measure cofinally often, one obtains a model satisfying $\Gamma(\mathbb{R})$ as well, without affecting the rest of the proof, since random forcing is ω^ω-bounding. Therefore, as an immediate corollary, the following diagram can be obtained as well:



An interesting question could be how one can modify the construction of this section in order to get $\neg\Gamma(\mathbb{R})$. The natural way of doing that would be to add random reals in Y cofinally often, but we do not have an immediate proof to show that this is sufficient.

2.2.2 Miller measurability without Baire property

In [**DT98**], Di Prisco and Todorcevic introduced a way to show that, starting from a choiceless Solovay's model $\mathbf{N}$ (i.e., the $\mathbf{L}(\omega^\omega)$ of a model obtained by collapsing an inaccessible to ω_1), and adding a generic ultrafilter U, one obtains

$$\mathbf{N}[U] \models \text{"every set of reals has the perfect set property and there exists a set without Baire property".}$$

The idea was essentially to use some nice properties of Mathias forcing $\mathbb{MA}$. Such a forcing can be defined in several ways; following the spirit of the rest of our work, we give a definition in terms of trees in $\omega^{<\uparrow\omega}$, i.e., the set of finite increasing sequences of natural numbers.

$$T_{s,A} = \{t \in \omega^{\uparrow<\omega} : t \trianglerighteq s \wedge \textsc{Succ}(t) \subseteq A \setminus \max(t) + 1\},$$

ordered by inclusion. Let us now briefly recall what the model $\mathbf{N}[U]$ is and why it is so interesting. We start from a choiceless Solovay's model $\mathbf{N}$ and we add a generic ultrafilter U, by using forcing $\mathbb{W}$, consisting of infinite subsets of ω modulo the ideal of finite sets, ordered by almost inclusion $\subseteq^*$, i.e.,

$$\mathbb{W} = [\omega]^\omega/\mathsf{Fin}, \text{ and } a \leq b \Leftrightarrow a \subseteq^* b \Leftrightarrow b \setminus a \in \mathsf{Fin}.$$

Since $\mathbb{W}$ is σ-closed, it easily follows that such a forcing does not add reals and so the generic ultrafilter U added by $\mathbb{W}$ is still an ultrafilter in the extension.

The importance of $\mathbf{N}[U]$ is that it may be seen as a model "between" two Solovay's models. Let us explain this fact. It is well-known that, if $\mathbf{N}$ and $\mathbf{N}^*$ are two Solovay's models over $\mathbf{V}$ such that $\omega^\omega \cap \mathbf{N} \subseteq \omega^\omega \cap \mathbf{N}^*$, then there exists an elementary embedding

$$j : \mathbf{N} \to \mathbf{N}^*, \text{ such that } \forall \alpha \in \mathrm{On} \forall r \in \omega^\omega (j(\alpha) = \alpha \wedge j(r) = r).$$

The following two results are well-known as well.

Fact 62. *Let $\mathbb{MA}_U$ be the U-Mathias forcing, consisting of those elements in $\mathbb{MA}$ with second coordinate in U. Then $\mathbb{MA}$ is forcing equivalent to $\mathbb{W} * \mathbb{MA}_U$.*

Fact 63. *If $\mathbf{N}$ is a Solovay's model over $\mathbf{V}$ and G is an $\mathbb{MA}$-generic filter over $\mathbf{N}$, then $\mathbf{N}[G]$ is a Solovay's model over $\mathbf{V}$. (In other words, if one adds a Mathias real into a Solovay's model, one obtains a Solovay's model again.)*

For a proof, see [**DT98**], proposition 2.4.

We have therefore obtained

$$\mathbf{N} \subseteq \mathbf{N}[U] \subseteq \mathbf{N}[U][G_U] = \mathbf{N}[G],$$

where G_U is an $\mathbb{MA}_U$-generic filter over $\mathbf{N}[U]$, while G is an $\mathbb{MA}$-generic filter over $\mathbf{N}$. (We will indicate $\mathbf{N}[G]$ with $\mathbf{N}[m]$, where m is the Mathias real related to G.)

One can now prove the main result of this section.

Theorem 64. *Let $\mathbf{N}$ be a Solovay's model over $\mathbf{V}$ and let U be a generic untrafilter added by $\mathbb{W}$. Then*

$$\mathbf{N}[U] \models \Gamma(\mathbb{M}) \wedge \neg\Gamma(\mathbb{C}) \wedge \neg\Gamma(\mathbb{R}).$$

If one looks at the proof of Di prisco and Todorcevic, one can easily realize that such a method does not work when one directly deals with symmetric properties, like Miller measurability. In fact, the proof does not work for Baire property and Lebesgue measurability, which we know to fail in $\mathbf{N}[U]$. Hence we need a trick to prove Miller measurability inside $\mathbf{N}[U]$, but viewing it under a different light.

Definition 65. We say that a set $X \subseteq \omega^\omega$ is K_σ-regular if and only if either X in bounded or there exists $T \in \mathbb{M}$ such that $[T] \subseteq X$.

It is straightforward that if X is K_σ-regular, then it is $\mathbb{M}$-measurable as well, since the complement of a bounded set contains the branches through a Miller tree.

proof of theorem 64. First of all, one has to show that K_σ-regularity holds in Solovay's model.

Lemma 66. *Let $\mathbf{V}[G]$ be a Solovay's model obtained by collapsing κ inaccessible to ω_1. Then*

$$\mathbf{V}[G] \models \text{“}\textit{every set of reals in } \mathbf{L}(\omega^\omega) \textit{ is } K_\sigma\textit{-regular”}.$$

Proof. Let $X = \{x \in \omega^\omega : \psi(x)\}$ be an unbounded set of reals (it is known that one can consider any parameter inside the ground model). Pick $x \in X$ such that x is unbounded over $\mathbf{V}$. Consider the formula φ as in lemma 10. Furthermore, by lemma 9, there exists a forcing $P \lessdot \mathbf{Coll}$ of countable size such that $x \in \mathbf{V}[H]$, where H is P-generic over $\mathbf{V}$. Hence, there exist $p \in P$ and $\dot{x}$ P-name for x such that

$$p \Vdash \text{“}\varphi(\dot{x}) \wedge \dot{x} \text{ is unbounded over } \mathbf{V}\text{”}.$$

Moreover, let $\{D_n : n \in \omega\}$ be a countable enumeration of all open dense subsets of P. Consider the following recursive construction:

$j = 0$: pick $p_\emptyset \leq p$ such that $p_\emptyset \in D_0$ and let $\sigma_\emptyset$ be the initial segment of $\dot{x}$ decided by $p_\emptyset$;

$j = 1$: for every $n \in \omega$, pick $p_{\langle n \rangle} \leq p_\emptyset$ such that $p_{\langle n \rangle} \in D_1$ and $p_{\langle n \rangle} \Vdash \dot{x}(k_1) > n$, where $\dot{x}(k_1)$ is not already decided by $p_\emptyset$ (note that can be done, by virtue of the unboundedness of $\dot{x}$). Lastly, let $\sigma_{\langle n \rangle}$ be the initial segment of $\dot{x}$ decided by $p_{\langle n \rangle}$.

$j+1$: for $t \in \omega^j$ and $n \in \omega$, pick $p_{t^\frown\langle n \rangle} \leq p_t$ such that $p_{t^\frown\langle n \rangle} \in D_{j+1}$ and $p_{t^\frown\langle n \rangle} \Vdash \dot{x}(k^t_{j+1}) > n$, where $\dot{x}(k^t_{j+1})$ is not already decided by p_t. Lastly, let $\sigma_{t^\frown\langle n \rangle}$ be the initial segment of $\dot{x}$ decided by $p_{t^\frown\langle n \rangle}$.

Finally, set $\Sigma = \{\sigma_t : t \in \omega^{<\omega}, \sigma_t \text{ defined as above}\}$. By construction, Σ is a tree and for every $z \in [\Sigma]$, $\dot{x}^H = z$, for some P-generic filter H containing p. Furthermore, by construction, one can consider a Miller tree $T \subseteq \Sigma$. Hence, we have found a Miller tree T such that, for every $z \in [T]$, $\mathbf{V}[z] \models \varphi(z)$, that means

$$\mathbf{V}[G] \models \exists T \text{ Miller tree } ([T] \subseteq X).$$

□

Now, fix any unbounded set of reals $X \in \mathbf{N}[U]$. In $\mathbf{N}$, let $\dot{X}$ be a $\mathbb{W}$-name for X. Let m be an $\mathbb{MA}$-generic real over $\mathbf{N}$. Recall there exists an elementary embedding

$$j : \mathbf{N} \to \mathbf{N}[m],$$

which fixes ordinals and reals. Define the set

$$X' = \{x \in \omega^\omega \cap \mathbf{N}[m] : m \Vdash x \in j(\dot{X})\},$$

Note that $X' \in \mathbf{N}[m]$, by definition. Therefore, since $\mathbf{N}[m]$ is a Solovay's model, two cases are possible:

Case 1. $\mathbf{N}[m] \models \exists T \in \mathbb{M}([T] \subseteq X')$. Remind that $m \subseteq^* u$, for every $u \in U$. Then, fixed $u \in U$ arbitrarily, we have

$$\mathbf{N}[m] \models \exists a \subseteq^* u \exists T \in \mathbb{M}\big[a \Vdash [\dot{T}] \subseteq j(\dot{X})\big],$$

and hence, by elementarity,

$$\mathbf{N} \models \exists a \subseteq^* u \exists T \in \mathbb{M}\big[a \Vdash [\dot{T}] \subseteq \dot{X}\big],$$

and so also $\mathbf{N}[U] \models \exists T \in \mathbb{M}([T] \subseteq X')$.

Case 2. $\mathbf{N}[m] \models$ "X' is bounded", that means

$$\mathbf{N}[m] \models \exists a \subseteq^* u \exists f \in \omega^\omega \big[a \Vdash \forall x \in j(\dot{X}) \forall^\infty n \in \omega (x(n) \leq \dot{f}(n))\big],$$

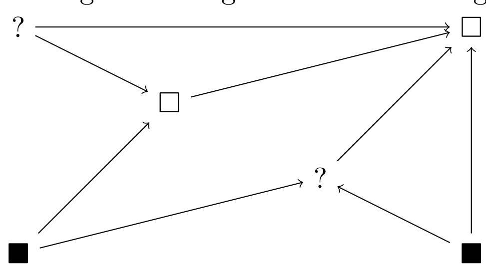
and again, by elementarity,

$$\mathbf{N} \models \exists a \subseteq^* u \exists f \in \omega^\omega \big[a \Vdash \forall x \in \dot{X} \forall^\infty n \in \omega (x(n) \leq \dot{f}(n))\big],$$

that means $\mathbf{N}[U] \models$ "X is bounded", contradicting the assumption on X.

□

The corresponding Γ-RP-diagram is the following:



2.2.3 A brief digression: from Sacks to Miller

In section 2.2.1, we saw that Sacks measurability and Miller measurability can be separated. However, there is a little space between Sacks and Miller, in the sense of the following definition.

Definition 67. (a) A tree $T \subseteq \omega^{<\omega}$ is an *n-perfect* tree iff

$$\forall t \exists t' (t' \trianglerighteq t \wedge |\textsc{Succ}(t')| \geq n).$$

We call the forcing consisting of such trees the n-Sacks forcing $\mathbb{S}_n$, ordered in the usual way.

(b) A set of reals X is $\mathbb{S}_n$*-measurable* iff $\exists T \in \mathbb{S}_n$ such that

$$\text{either } [T] \subseteq X \text{ or } [T] \cap X = \emptyset.$$

Hence, it could be interesting to analyze such properties. Later on, we will see how this work could suggest a possible way to separate Miller measurability from Laver measurability.

Let us now go a little into the study of these "new" regularity properties turning out between Sacks and Miller. The fact that $\Gamma(\mathbb{S}_n)$ holds in a $\mathbb{C}_{\omega_1}$-extension is clear, and the proof is exactly the same used for the Sacks measurability. What we actually want to do is to prove a stronger property, which is somehow very close to the Miller measurability.

Definition 68.

(a) A sequence of trees $\langle T_n : n \in \omega \rangle$ is *good* iff $\forall n \in \omega (T_n \in \mathbb{S}_n \wedge T_n \subseteq T_{n+1})$.

(b) A set of reals X is $\mathbb{S}^\omega$*-measurable* iff there exists a good sequence $\langle T_n : n \in \omega \rangle$ such that

$$\text{either } \forall n([T_n] \subseteq X) \text{ or } \forall n([T_n] \cap X = \emptyset).$$

Fact 69. *There exists a forcing $\mathbb{P}$ equivalent to the Cohen forcing $\mathbb{C}$ such that*

$$\mathbb{P} \Vdash \text{“}\exists \langle T_n : n \in \omega \rangle \text{ good sequence } \wedge \forall n([T_n] \subseteq \mathbb{C}(\mathbf{V}))\text{”},$$

where $\mathbb{C}(\mathbf{V})$ is the set of Cohen reals over $\mathbf{V}$.

Proof. Let us define the forcing $\mathbb{P}$ as a finite support ω-iteration as follows:

Step 0. First of all, consider

$$\mathbb{P}_0 = \text{the forcing for adding a perfect tree of Cohen reals},$$

and let T_0 be the $\mathbb{P}_0$-generic perfect tree of Cohen reals over $\mathbf{V}$;

STEP 1. In $\mathbf{V}[T_0] = \mathbf{V}_1$, let

$$\mathbb{P}_1 = \{T \subseteq \omega^{<\omega} : T \text{ is finite } \wedge T_0 \sqsubseteq T \wedge T \in \mathbb{S}_3\},$$

where $T_0 \sqsubseteq T$ means that $T_0 \restriction \mathrm{ht}(T) \subseteq T$. As above, let T_1 be the $\mathbb{P}_1$-generic over $\mathbf{V}_1$;

STEP $n+1$. In $\mathbf{V}_n[T_n] = \mathbf{V}_{n+1}$, let

$$\mathbb{P}_{n+1} = \{T \subseteq \omega^{<\omega} : T \text{ is finite } \wedge T_n \sqsubseteq T \wedge T \in \mathbb{S}_{n+3}\},$$

and let T_{n+1} be the $\mathbb{P}_{n+1}$-generic over $\mathbf{V}_{n+1}$.

STEP ω. Finally, let $\mathbb{P} = \lim_{n\in\omega} \mathbb{P}_n$.

($*$) In each of these cases the order is always the end-extension.

First of all, note that each $\mathbb{P}_n$ is countable and so it is equivalent to the Cohen forcing $\mathbb{C}$, and since $\mathbb{P}$ is the direct limit of such $\mathbb{P}_n$'s, then $\mathbb{P} \approx \mathbb{C}$ as well.

Fix arbitrarily $n \in \omega$. It is straightforward to show that T_n is an n-perfect tree, since, for each $k \in \omega$, $D_k = \{T \in \mathbb{P}_n : \mathrm{ht}(T) > k\}$ is dense in $\mathbb{P}_n$. It is left to show that each branch in $[T_n]$ is a Cohen real over $\mathbf{V}$. To see that, one has to prove, for every nowhere dense tree $S \in \mathbf{V}$,

$$D_S = \{T' \in \mathbb{P}_n : \forall t \in T'(t \in \mathrm{TERM}(T') \Rightarrow t \notin S)\}$$

is dense in $\mathbb{P}_n$. To this aim, fix a nowhere dense tree $S \in \mathbf{V}$ and $T \in \mathbb{P}_n$. The goal is to find $T' \leq T$, $T' \in D_S$. Obviously, every terminal node $t \in T$ can be extended to a node $e(t) \notin S$. Therefore,

$$T' = T \cup \{t' \trianglelefteq e(t) : t \in \mathrm{TERM}(T)\}$$

is the tree we wanted. □

Thus, one has obtained that, if c is a Cohen real over $\mathbf{V}$, then

$$\mathbf{V}[c] \models \text{“ } \exists \langle T_n : n \in \omega \rangle \text{ good sequence } \wedge \forall n([T_n] \subseteq \mathbb{C}(\mathbf{V})) \text{ ”}.$$

Theorem 70. *Let G be a $\mathbb{C}_{\omega_1}$-generic over $\mathbf{V}$.*

$$\mathbf{V}[G] \models \text{“every set of reals in } \mathbf{L}(\omega^\omega) \text{ is } \mathbb{S}^\omega\text{-measurable”}.$$

Proof. The proof is similar to that of example 28, by using the strong Cohen-homogeneity of $\mathbb{C}_{\omega_1}$ to replace the argument using the absoluteness of Σ^1_2-formulae. □

Another question which we would like to deal with is how one can modify such an argument to get some results about Miller measurability. In fact, the reader may notice that if one considers the forcing $\mathbb{P}$ introduced above, but with full support, one obtains a forcing notion adding a Miller tree of Cohen reals, and this forcing is exactly an ω_1-iteration of $\mathbb{C}$, with countable support, say $\mathbb{C}^{\omega}_{\omega_1}$. Since $\mathbb{C}^{\omega}_{\omega_1}$ is proper, the usual argument to absorb the real parameter in $\mathbf{V}[G{\restriction}\alpha]$, for some $\alpha < \omega_1$, is valid. The point is that is not clear whether $\mathbb{C}^{\omega}_{\omega_1}$ is strongly Cohen-homogeneous. Thus the following remains open:

Conjecture. Let G be $\mathbb{C}^{\omega}_{\omega_1}$-generic over $\mathbf{V}$. Then

$$\mathbf{V}[G] \models \text{"every set of reals in } \mathbf{L}(\omega^{\omega}) \text{ is } \mathbb{M}\text{-measurable"}.$$

Remark 71. In case the above conjecture were true one could obtain a nice result about our work concerning separation of regularity properties. In fact, it is well-known that, $\mathbb{C}^{\omega}_{\omega_1}$ does not add dominating reals over $\mathbf{V}$. Therefore, by theorem 4.1 and theorem 5.8 in [**BL99**], starting from $\mathbf{V} = \mathbf{L}$ and considering a $\mathbb{C}^{\omega}_{\omega_1}$-generic filter G over $\mathbf{L}$, one could get $\mathbf{L}[G] \models \neg\boldsymbol{\Delta}^1_2(\mathbb{L}) \wedge \neg\boldsymbol{\Sigma}^1_2(\mathbb{C})$. Hence, in particular,

$$\mathbf{L}[G] \models \Gamma(\mathbb{M}) \wedge \neg\Gamma(\mathbb{L}) \wedge \neg\Gamma(\mathbb{C}).$$

2.2.4 A word about the inaccessible

Before concluding this chapter, it is noteworthy to give a survey on the topic concerning the use of the inaccessible cardinal to get a particular regularity property for every set of reals. The general question turning out is the following:

Question. Is Solovay's inaccessible always necessary to get $\Gamma(\mathbb{P})$?

Obviously, the answer depends on which $\mathbb{P}$ one deals with. In some previous sections we have already seen cases in which such an answer is negative. For instance, as we have already said, since $\mathbb{C}_{\omega_1}$ is strongly Cohen homogeneous, it follows that one can obtain the Sacks measurability of all projective sets simply by a finite support ω_1-iteration of $\mathbb{C}$. Moreover, we have already remarked that

$$\Vdash_{\mathbb{C}} \text{"}\exists T \text{ Silver tree of Cohen reals"}$$

and therefore the same iteration gives a model where all projective sets are Silver measurable.

More complicate is the situation when one deals with Baire property and Lebesgue measurability. As we cited in the introduction, one of the most surprising results in this area, due to Shelah, underlines a huge difference between these two regularity properties. About Lebesgue measurability, one can show that

$$\mathbf{\Sigma}^1_3(\mathbb{R}) \Rightarrow \forall z \in \omega^\omega, \mathbf{L}[z] \models \text{“}\omega_1^{\mathbf{V}} \text{ is inaccessible”}.$$

(For a proof, one can also see [**Ra84**]). In this article Raisonnier also shows

$$\mathbf{\Sigma}^1_2(\mathbb{R}) \wedge \mathbf{\Sigma}^1_3(\mathbb{C}) \Rightarrow \forall z \in \omega^\omega, \mathbf{L}[z] \models \text{“}\omega_1^{\mathbf{V}} \text{ is inaccessible”}.$$

Nevertheless, the assumption $\mathbf{\Sigma}^1_2(\mathbb{R})$ cannot be dropped. In fact, this is the gap between Lebesgue measurability and Baire property, as Shelah proved in [**Sh84**]. The way to construct the model for **BP** without inaccessible was sketched out in the first chapter, when we introduced the amalgamation. The key point is represented by a nice property of the Amoeba forcing for category $\mathbb{UM}$. Such a property is called *sweetness* and it is a strenghtening of the σ-centeredness. The sweetness is precisely the property which allows the preservation of ccc under amalgamation, or better, it is itself preserved under amalgamation. Moreover, iteration with $\mathbb{UM}$ preserves sweetness, and so when one adds cofinally often iterations with $\mathbb{UM}$ in the construction of theorem 25, one obtains $\mathbf{B}_{\omega_1}$ to be ccc, without any need of the inaccessible κ. The crucial difference with Lebesgue measurability is that, the Amoeba forcing for measure $\mathbb{A}$ is not sweet, and so the same construction cannot be done.

The main question which is still open is whether Laver measurability needs an inaccessible or not. Such a question was explicitly asked by Brendle and Löwe in [**BL99**] and its importance is that one can see Mathias forcing as a uniform version of Laver forcing (see section 1.2 in [**Br95**]), and hence the use of the inaccessible to get $\Gamma(\mathbb{L})$ is strictly related to the famous open problem:

> **Question.** Does the statement “every subset of $[\omega]^\omega$ has the Ramsey property” have the consistency strenght of **ZFC**?

A failed attempt to get $\Gamma(\mathbb{L})$ without inaccessible. We conclude this section showing an example to understand which complications turn out when one tries to prove $\Gamma(\mathbb{L})$. The only method which is known so far is Shelah's machinery. As we have said above, the crucial property is the sweetness.

Definition 72. A forcing notion P is *sweet* if and only if there is $D \subseteq P$ dense and a sequence $\langle \sim_n : n \in \omega \rangle$ of equivalence relations on D such that:

1. for every $p, q \in D$, $n \in \omega$, if $p \sim_n q$ then $p \sim_{n+1} q$, and $\sim_n$ has countably many equivalence classes;
2. for every $p, q \in D$, $n \in \omega$, if $p \sim_n q$ then there exists $r \in [p]_n$ such that $r \leq p, q$;

3. for every $p \in D$ and for every sequence $\langle p_n : n \in \omega \rangle$ such that $p \sim_n p_n$, there exists $q_n \in [p]_n$ such that for every $j \geq n$, $q_n \leq p_j$;

4. for every $p, q \in D$, $q \leq p$, for every $n \in \omega$, there exists $k \in \omega$ such that

$$\forall p' \in [p]_k \exists q' \in [q]_n (q' \leq p').$$

Now consider the following forcing

$$\mathbb{KT} = \{(T, N) : T \subset \omega^{<\omega} \text{ is a finite tree} \wedge N \text{ is a nowhere dense tree}\},$$

ordered by

$$(T', N') \leq (T, N) \Leftrightarrow T' \sqsupseteq^{\text{right}} T \wedge N' \supseteq N \wedge \forall t' \in T'(t' \notin N),$$

where

$$T' \sqsupseteq^{\text{right}} T \Leftrightarrow \forall t' \in T' \setminus T(t' \rhd t \Rightarrow t'(|t|) > \max\{s(|t|) : s \in \textsc{Succ}(t, T))\}).$$

(In other words, every new node t' in T' extending $t \in T$ has to take value at $|t|$ to the right of all of the values taken by the immediate successors of t in T). Note that this is a sort of "dual" version of $\mathbb{UM}$.

If we set $T_G = \bigcup\{T : (T, N) \in G\}$, then, clearly,

$$\Vdash_{\mathbb{KT}} \dot{T}_G \text{ is a Laver tree.}$$

The two points that one should check are the following:

Hope 1: $\mathbb{KT}$ is sweet.

Hope 2: any branch through T_G is Cohen.

About Hope 1, it is clear that $\mathbb{KT}$ is σ-centered, i.e., $\mathbb{KT}$ can be written as a countable union of F_n's $\subseteq \mathbb{KT}$ such that every $p, q \in F_n$ are compatible in F_n. We want to show even more, indeed $\mathbb{KT}$ is sweet. For every $n \in \omega$, let

$$(T', N') \sim_n (T, N) \Leftrightarrow T = T' \wedge N' {\restriction} j_n = N {\restriction} j_n,$$

where $j_n = \max\{\min\{|s'| : s' \trianglerighteq s_j \wedge s' \notin N'\} : s_j \in T\}$, where $\langle s_i : i \in \omega \rangle$ is the enumeration of all the finite sequences $s \in \omega^{<\omega}$.

It is clear that conditions 1, 2 and 4 of definition 72 are satisfied. About condition 3, one can easily note that, given a sequence $\langle (T_n, N_n) \in \mathbb{KT} : n \in \omega \rangle$, such that for every n, $p_n \sim_n p_{n+1}$, we get

$$N^* = \bigcup_{n \in \omega} N_n {\restriction} j_n \text{ is nowhere dense.}$$

In fact, given $s \in \omega^{<\omega}$ arbitrarily, $s = s_n$ for some $n \in \omega$, there exists $s' \trianglerighteq s_n$ such that $s' \notin N_n$. Hence, $s' \notin N^*$ as well, by the choice of j_n. One has thus shown that $\mathbb{KT}$ is sweet. Actually, to use it into Shelah's machinery, one should check that $Q * \mathbb{KT}$ is sweet, whenever Q is sweet; however, the above technic to demonstrate that $\mathbb{KT}$ is sweet gives a good hint to show that the iteration is still sweet.

It is therefore clear, since we said at the beginning that this is a failed attempt, that what is wrong is Hope 2. In fact, one can note that there are branches $z \in [T_G]$ which are not shifted away from any ground model nowhere dense set. The best one can obtain is

$$\Vdash_{\mathbb{KT}} \exists T \subseteq \dot{T}_G, T \text{ is a Miller tree}(\forall z(z \in [T] \Rightarrow z \text{ is Cohen})).$$

Hence, we only obtain a different proof of the fact that $\Gamma(\mathbb{M})$ does not need an inaccessible, which was already indirectly known, since it is implied by **BP**. Nevertheless, even if this example fails to give a definitive answer concerning $\Gamma(\mathbb{L})$, it is anyway interesting and it seems to be propaedeutic for further studies around this problem.

2.2.5 Conclusions and open questions

The work done throughout this section allows to settle some combinations of the Γ-RP-diagram, like those in 2.2.1 and 2.2.2. However, many issues are still open; among those, we state the following unsolved questions:

Question 1. $\Gamma(\mathbb{C}) \Rightarrow \Gamma(\mathbb{L})$?

Question 2. $\Gamma(\mathbb{L}) \Rightarrow \Gamma(\mathbb{C})$?

Furthermore, from the results of sections 2.2.1, Sy Friedman suggested me an even more general and deeper question: let $\mathbb{P}, \mathbb{Q}$ be arboreal proper forcings, such that $\mathbb{Q}$ is ω^ω-bounding and $\mathbb{P}$ adds unbounded reals.

Question 3. Can one always get a model $\mathbf{M} \models \Gamma(\mathbb{Q}) \wedge \neg\Gamma(\mathbb{P})$?

Chapter 3

Generalized Cantor space 2^κ

In the previous chapters, the space where our investigations took place was the Cantor space 2^ω (or the Baire space ω^ω). In this last brief chapter of the thesis we will deal with the generalized version of the Cantor space 2^κ, for κ uncountable cardinal.

Throghout the chapter, 2^κ will be equipped with the topology generated by basic clopen sets of the form

$$[s] = \{x \in 2^\kappa : x \rhd s\}, \text{ for every } s \in 2^{<\kappa} .$$

Since we want such a family to be of size κ, we assume $2^{<\kappa} = \kappa$. As usual one defines the family of Borel sets on 2^κ as the smallest family containing all such $[s]$'s and closed under complements and unions of size κ. We will still indicate it with Bor. Analogously one can define the family of projective sets Proj. The first main gap between this generalized case and the standard one is that the family of $\mathbf{\Delta}^1_1$ strictly contains the family of Borel sets (we will see the proof in the next section).

Another difference with the standard case is represented by the notion of perfect tree. In fact, except for the case κ inaccessible, the perfect trees on κ are somehow "fat", in the sense that there are levels $\alpha < \kappa$ such that $2^\alpha = \kappa$, and this gives the unpleasant consequence that κ-Sacks forcing $\mathbb{S}_\kappa$ is not κ^κ-bounding. A detailed work on the uncountable Sacks forcing can be found in [**Ka80**].

Finally, another wide gap consists of the difficulty in defining a reasonable notion of measure on 2^κ, where reasonable roughly means able to measure at least all Borel sets. One can obviously consider 2^κ equipped with the product measure m, but such a choice would not be appropriate for our purpose; in fact, in this case, the family of measurable sets would be only a σ-algebra, and so even the open sets in 2^κ would not be measurable w.r.t. m. Our work in the second section will aim to solve this problem, and that will also give us a way for introducing a suitable notion of Lebesgue measurability and the corresponding κ-random forcing.

3.1 Category

Once that a notion of topology is settled on a certain space, one can always define a notion of Baire property. In our specific case the definition is exactly the natural generalization of definition 1, i.e, to be equal to a Borel set modulo a meager set. Some properties of the category, which hold in the standard case, extend to this generalized case as well. Among these, one can remark the Baire category theorem, Fubini's theorem, the fact that every Borel set has the Baire property (these results are stated and proved in [**FHK10**], [**H96**] and [**HS01**]). Note also that the associated notion of forcing is equivalent to the κ-Cohen forcing $\mathbb{C}^\kappa$ consisting of all sequences $t \in 2^{<\kappa}$, ordered by extension.

Nevertheless, the differences with the standard case are deeper and more interesting than the analogies. The first wide gap is represented by the following result.

Theorem 73. *Let* $\textsc{Cub} = \{x \in 2^\kappa : \exists C \subseteq \kappa$ *closed unbounded* $(\alpha \in C \Rightarrow x(\alpha) = 1)\}$. *Then* $\textsc{Cub}$ *is* $\boldsymbol{\Sigma}^1_1$ *and does not have the Baire property.*

The proof can be found in [**HS01**], theorem 4.2. Such a result underlines a huge difference with the standard case, since one can directly prove in **ZFC** that all analytic sets in 2^ω have the Baire property.

Another interesting remark is that in **L**, one can construct a Δ^1_1 set without Baire property. Such a construction in essentially the same of the standard case, but with the difference that one can define a Δ^1_1 well-ordering of **L**. This is possible because "to be well-founded" is not only Π^1_1, but is Borel (in particular closed), and therefore the usual well-ordering of **L** can be defined in order to be Δ^1_1. However, this consistency result for $\boldsymbol{\Delta}^1_1$ sets cannot be shifted to a theorem of **ZFC**, as the following result shows.

Remark 74. Let $\mathbb{C}^\kappa(\kappa^+)$ be a κ^+-iteration with $< \kappa$-support and let G be $\mathbb{C}^\kappa(\kappa^+)$-generic over **L**. Then $\mathbf{L}[G] \models \boldsymbol{\Delta}^1_1(\mathbb{C}^\kappa)$.

Note that, as an immediate consequence, one obtains $\textsc{Bor} \subsetneq \boldsymbol{\Delta}^1_1$.

This result was proved by Philip Lucke and Philip Schlicht. Furhtermore, Sy Friedman proved that $\boldsymbol{\Delta}^1_1(\mathbb{C}^\kappa)$ holds in Silver's model as well, i.e, the model obtained by collapsing an inaccessible to κ^+. Such a result was a joint work with Tapani Hyttinen and Vadim Kulikov, presented in [**FHK10**].

The fact that an iteration of $\mathbb{C}^\kappa$ was sufficient to get $\boldsymbol{\Delta}^1_1(\mathbb{C}^\kappa)$ has inspired the following result.

Theorem 75. *Let* $\mathbb{C}^\kappa(\mathbf{L}[z])$ *be the set of* κ*-Cohen generic sequences over* $\mathbf{L}[z]$. *Then*

$$\boldsymbol{\Delta}^1_1(\mathbb{C}^\kappa) \Rightarrow \forall z \in 2^\kappa(\mathbb{C}^\kappa(\mathbf{L}[z]) \neq \emptyset)$$

Proof. Remind that, in 2^ω, the two statements "to be a Borel code" and "$x \in \mathbf{L}$" are Π^1_1 and Σ^1_2, respectively. On the contrary, in 2^κ the situation is different, as we already remarked in the previous page, since the two analogous statements are closed and Π^1_1, respectively, and therefore, the good well-ordering $\leq^{\mathbf{L}}$ is Δ^1_1 in this generalized case (instead of Δ^1_2). Finally, we may also note that, for any Borel code c, the formula "x is a code for a meager set" is Σ^1_1; in fact, it is equivalent to

$$\exists \langle c_\alpha : \alpha \in \kappa \rangle, c_\alpha\text{'s are codes for closed nowhere dense and } B_x \subseteq \bigcup_{\alpha \in \kappa} B_{c_\alpha},$$

where B_x represents the Borel set associated with the Borel code x. Hence, the result easily follows, since "c is a code for a closed nowhere dense set" is equivalent to

$$B_c \text{ is closed} \wedge \forall \eta \in 2^{<\kappa} \exists \eta' \in 2^{<\kappa} (B_c \cap [\eta'] = \emptyset),$$

which is obviously Borel.

All of these observations allow us to define two $\Sigma^1_1(z)$ sets, for any $z \in 2^\kappa$, in the following way. (Note the analogy, from now on, with the standard case).

First of all, we have to consider a formula $\varphi^z(v, a)$ consisting of the conjuction of the following:

(i) v is a code for a meager set;

(ii) a is an enumeration of all κ-reals $x <^{\mathbf{L}[z]} v$.

It is clear that, from the above observations, such a formula $\varphi^z(v, a)$ is $\Sigma^1_1(z)$. One can now define a new notion of ordering on 2^κ in the following way:

$$v \trianglelefteq w \Leftrightarrow \exists c \in 2^\kappa \exists a \in 2^\kappa (\varphi^z(v, a) \wedge v \in B_c \wedge w \notin \bigcup_{\alpha \in \kappa} B_{(a)_\alpha});$$

$$v \triangleleft w \Leftrightarrow \exists c \in 2^\kappa \exists a \in 2^\kappa (\varphi^z(v, a) \wedge v \in B_c \wedge w \notin B_c \cup \bigcup_{\alpha \in \kappa} B_{(a)_\alpha}),$$

where $(a)_\alpha$ represents the α-th element in the enumeration. Roughly speaking, a κ-real v is less than w if the first Borel meager set (w.r.t. the ordering $\leq^{\mathbf{L}[z]}$) containing v is $\leq^{\mathbf{L}[z]}$ the first one containing w. Also note that such an ordering is $\Sigma^1_1(z)$. We can now define the two $\Sigma^1_1(z)$ sets mentioned at the beginning:

$$X_z = \{(v, w) : v \notin \mathbb{C}^\kappa(\mathbf{L}[z]) \wedge v \triangleleft w\}$$

$$Y_z = \{(v, w) : v \notin \mathbb{C}^\kappa(\mathbf{L}[z]) \wedge w \trianglelefteq v\}.$$

The following remark is crucial.

Fact 76. *X_z is either meager or without Baire property (the same for Y_z).*

The proof is the same of the standard case, by using Fubini's theorem and the fact that only κ-many elements precede any element in the ordering $\leq^{\mathbf{L}[z]}$.

To conclude the proof is sufficient to note that, if $\mathbb{C}^\kappa(\mathbf{L}[z])$ is empty, then X_z and Y_z are one the complement of the other, and so, in particular, they are both $\Delta^1_1(z)$. However, this is a contradiction, because we are assuming $\boldsymbol{\Delta}^1_1(\mathbb{C}^\kappa)$ and so the two sets should be on the one hand one the complement of the other, and on the other hand they should be both meager. $\square$

3.2 Measure

Troughout this section κ will be an uncountable regular cardinal. The aim is to define a notion of measure on 2^κ, i.e., a function $\mu : \mathcal{F} \to \Omega$, where $\mathcal{F} \supseteq$ Bor, Ω is a linearly ordered set, such that:

(a) if $A, B \in \mathcal{F}, A \subseteq B$, then $\mu(A) \leq \mu(B)$;

(b) if $\{A_\alpha : \alpha < \kappa\}$ is a sequence of elements of $\mathcal{F}$, then

$$\mu(\bigcup_{\alpha<\kappa} A_\alpha) \leq \sum_{\alpha<\kappa} \mu(A_\alpha);$$

(c) if $\{A_\alpha : \alpha < \kappa\}$ is a sequence of disjoint elements of $\mathcal{F}$, then

$$\mu(\bigcup_{\alpha<\kappa} A_\alpha) = \sum_{\alpha<\kappa} \mu(A_\alpha).$$

(Note that we will also define a notion of $\sum_{\alpha<\kappa}$ in order to make such a measure well-defined).

Note that the product measure does not satisfy our requirements, since it only gives rise to a σ-algebra, which is not closed under unions of size $\kappa > \omega$.

Let $LO = \{\lambda_\alpha : \alpha < \kappa\}$ be the list of limit ordinals below κ such that $2^{\lambda_\alpha} = \kappa$ and let

$$W_\alpha = \langle s^\alpha_\xi : \xi < \kappa \rangle$$

be an enumeration of the elements $s \in 2^{\lambda_\alpha}$. Further, we will denote with $\mathcal{S}$ the family of all basic clopen sets.

The following elementary, but noteworthy, observation should be kept in mind to understand the definition of measure μ and for next developments.

Remark 77. Every $[s] \in \mathcal{S}$ can be easily seen as a union of κ-many disjoint $[t_\beta]'s$ in $\mathcal{S}$. In fact, given $s \in 2^\gamma$, there are κ-many incompatible t_β's in 2^λ, where λ is any limit ordinal $> \gamma$, such that $[s] = \bigcup_{\beta<\kappa}[t_\beta]$.

Such a remark inspires the following definition. (The reason for which π_α^β below is defined as a subset of $\kappa\cdot\kappa$ will be clear later on, and it is essentially because of the fact that μ will take values in $2^{\kappa\cdot\kappa}$).

Definition 78. Let $s \in 2^\gamma$ and λ_α be the least limit ordinal $\geq \gamma$. For every $\beta \geq \alpha$, one defines

$$\sigma_s^\beta =^{\mathbf{def}} \{t \in 2^{\lambda_\beta} : t \trianglerighteq s\}$$

and

$$\pi_s^\beta =^{\mathbf{def}} \{\kappa\cdot\beta + \xi : s_\xi^\beta \in \sigma_s^\beta\}.$$

(Note that if $\gamma = \lambda_\alpha$, then $\sigma_s^\alpha = \{s\}$).

We are now able to define our measure μ for basic clopen sets $[s] \in \mathcal{S}$, which will take values in $2^{\kappa\cdot\kappa}$. From now on, we use the notation μ_s in place of the more cumbersome $\mu([s])$.

Let $s \in 2^\gamma$, $\lambda_\alpha = \min\{\lambda : \lambda \text{ is limit } \wedge \lambda \geq \gamma\}$ and $W_\alpha = \langle s_\xi^\alpha : \xi < \kappa\rangle$ be the according well-order of 2^{λ_α}, introduced above. Then set

$$\mu_s(\delta) =^{\mathbf{def}} \begin{cases} 1 & \exists\beta \geq \alpha \text{ such that } \delta \in \pi_s^\beta \\ 0 & \text{otherwise .} \end{cases}$$

Notation: For every $s \in 2^{\lambda_\alpha}$, for some $\lambda_\alpha \in LO$, let δ_s be the unique element in π_s^α. Sometimes, we will indicate μ_s with i_{δ_s}.

Now, we define the suitable notion of sum on $2^{\kappa\cdot\kappa}$.

Definition 79. Given a set $\{x_\gamma : \gamma < \rho\} \subseteq 2^{\kappa\cdot\kappa}$, with $\rho \leq \kappa$, one defines $\sum_{\gamma<\rho} x_\gamma$ as follows: for every $\delta \in \kappa\cdot\kappa$, let $\Big(\sum_{\gamma<\rho} x_\gamma\Big)(\delta) = 1$ iff

either $\exists\gamma < \rho, x_\gamma(\delta) = 1$,

or $\exists\alpha < \kappa\exists s \in 2^{\lambda_\alpha}$ s.t. $\delta \in \pi_s^\alpha \wedge \forall\delta' \in \pi_s^{\alpha+1}, \Big(\sum_{\gamma<\rho} x_\gamma\Big)(\delta') = 1.$

For a reason which will be clear later on (see remark 83), it is also necessary to identify different elements of $2^{\kappa\cdot\kappa}$. To do that we need to consider, for every $s \in 2^{\lambda_0}$, the set $d_s = \bigcup_{\beta<\kappa}\pi_s^\beta$; note that, for every $s \in 2^{\lambda_0}$, $d_s \subseteq \kappa\cdot\kappa$, and $\{d_s : s \in 2^{\lambda_0}\}$ forms a partition of $\kappa\cdot\kappa$. One defines an equivalence relation on each 2^{d_s} as follows:

$\sim_s$: given $x, y \in 2^{d_s}$, one defines $x \sim_s y$ iff for every $\beta < \kappa$, there exists η_β such that $\kappa\cdot\beta \leq \eta_\beta < \kappa\cdot(\beta+1)$ and

$$\forall\delta\Big((\eta_\beta \leq \delta < \kappa\cdot(\beta+1) \wedge \delta \in \pi_s^\beta) \Rightarrow x(\delta) = y(\delta) = 1\Big).$$

Note that such equivalence relations $\{\sim_s : s \in 2^{\lambda_\alpha}\}$ induces an identification on elements in $2^{\kappa\cdot\kappa}$, that is not properly an equivalence relations (note that such an identification makes somehow pleonastic the second condition in the above definition of sum).

From now on, we will indicate with Ω the set $2^{\kappa\cdot\kappa}$ endowed with these equivalence relations. Furthermore, we will consider the lexicographical order on Ω, in order to make Ω linearly ordered, i.e, one sets, for every $x, y \in \Omega$,

$$x \leq y \Leftrightarrow x \leq_{\text{lex}} y.$$

Lastly, we will use the convention $\mathbf{0} = [\langle 0, 0, \dots \rangle]$ and $\mathbf{1} = [\langle 1, 1, \dots \rangle]$.

It is clear that, under these definitions, if $[s] \subseteq [t]$, then $\mu_s \leq \mu_t$. Moreover, one has the following result.

Fact 80. *Let $F \subseteq \mathcal{S}$ be a family of disjoint basic clopen sets. Then*

$$\mu\Big(\bigcup_{[s]\in F} [s]\Big) = \sum_{[s]\in F} \mu_s.$$

(Note that the assumption for the basic clopen sets to be disjoint could be dropped.)

One can now introduce a notion of outer measure μ^*, exactly in the same fashion of Lebesgue measure.

Definition 81. For every $X \subseteq 2^\kappa$, set

$$\mu^*(X) = \inf\{\mu(O) : O \in \mathcal{C}(X)\},$$

where $\mathcal{C}(X)$ is the set of open coverings of X, i.e., more precisely,

$$\mathcal{C}(X) = \{O : \exists\langle s_\gamma : \gamma < \kappa\rangle, s_\gamma \in 2^{<\kappa} \text{ s.t } O = \bigcup_{\gamma<\kappa} [s_\gamma] \wedge O \supseteq X\}.$$

(Remind that inf is meant with respect to the $\leq_{\text{lex}}$-order).

Trivial Remarks:

- for every $s \in 2^{<\kappa}$, $\mu_s = \mu^*([s])$.
- for every $X, Y \subseteq 2^\kappa$, $X \subseteq Y$, we have $\mu^*(X) \leq \mu^*(Y)$.

We introduce the family $\mathcal{F}$, which will be our family of measurable sets (in the standard case it is called Caratheodory's family).

Definition 82. One says that a subset $X \subseteq 2^\kappa$ is in $\mathcal{F}$ iff $\forall A \subseteq 2^\kappa$, one has

$$\mu^*(A) = \mu^*(A \cap X) + \mu^*(A \cap X^c).$$

It is not hard to see that $\mathcal{F}$ is closed under complementation and finite unions.

The remaining task is to show that such a family extends $\mathcal{S}$ and it is closed under unions of size $\leq \kappa$, in order to show that $\mathcal{F} \supseteq$ Bor. Moreover one has to show that μ^* satisfies the $\leq\kappa$-additivity for every $\leq\kappa$-union of disjoint sets in $\mathcal{F}$.

Before doing that, we need to step back for proving the $\leq\kappa$-subadditivity of the measure μ^*. The following preliminary observations are needful for the proof.

Remark 83. Let $\langle \delta_\xi : \xi < \kappa \rangle$ be a sequence of subsets of $\kappa \cdot \kappa$ such that:

i if $\xi < \xi'$ then $\delta_\xi \subseteq \delta_{\xi'}$;

ii if $\xi < \xi'$ then $m_\xi < m_{\xi'}$, where $m_\xi = \min\{\alpha : \alpha \in \delta_\xi\}$.

We will call it *nice sequence* . Let i_{δ_ξ} be the corresponding element in $2^{\kappa\cdot\kappa}$, i.e., $i_{\delta_\xi}(\delta) = 1$ iff $\gamma \in \delta_\xi$.

For every $x, y \in \Omega$ one has

$$\text{If } \forall \xi < \kappa (x + i_{\delta_\xi} \geq y) \text{ then } x \geq y.$$

(Note the importance of the equivalence relations introduced before to make that true).

Remark 84. (*Nasty behaviour of* $\sum_{\alpha<\kappa}$). Given $\{x_\alpha : \alpha < \kappa\}$ and $\{y_\alpha : \alpha < \kappa\}$ two sequences of element of Ω, even if for every $\alpha < \kappa$, $x_\alpha < y_\alpha$, it may happen

$$\sum_{\alpha<\kappa} x_\alpha > \sum_{\alpha<\kappa} y_\alpha.$$

Nevertheless, that does not happen if one further requires that, for each $\alpha < \kappa$,

$$\forall \delta \in \kappa \cdot \kappa (x_\alpha(\delta) = 1 \Rightarrow y_\alpha(\delta) = 1).$$

Lemma 85. *Let* $\{X_\alpha : \alpha < \rho\}$ *be a family of subsets of* 2^κ *(not necessarily in* $\mathcal{F}$*),* $\rho \leq \kappa$*. Then*

$$\mu^*\Big(\bigcup_{\alpha<\rho} X_\alpha\Big) \leq \sum_{\alpha<\rho} \mu^*(X_\alpha).$$

(w.l.o.g. one can assume $\mu^*(X_\alpha) \neq \mathbf{1}$*, for every* $\alpha < \rho$*, otherwise the (in)equality would be obvious).*

Proof. By remark 83, we have to define a nice sequence $\langle \delta_\xi : \xi < \kappa \rangle$ such that, for every $\xi < \kappa$,

$$\mu^*\Big(\bigcup_{\alpha<\rho} X_\alpha\Big) \leq \sum_{\alpha<\rho} \mu^*(X_\alpha) + i_{\delta_\xi}.$$

Fix $\xi < \kappa$. Consider a cofinal sequence $\langle \delta_\xi^\alpha : \alpha < \rho \rangle$ such that, for every $\alpha < \rho$, one can find $O_\alpha \in \mathcal{C}(X_\alpha)$ such that $\mu(O_\alpha) \leq \mu^*(X_\alpha) + i_{\delta_\xi^\alpha}$, with the further condition that, whenever $\mu(O_\alpha)(\eta) = 1$, then

$$\mu^*(X_\alpha)(\eta) = 1 \vee i_{\delta_\xi^\alpha}(\eta) = 1.$$

Note that this construction can be done, since $\mu^*(X_\alpha) \neq \mathbf{1}$, for $\alpha < \rho$. The reason to choose those O_α's so carefully is because of remark 84.

Note also that each $O_\alpha = \bigcup_{\gamma<\kappa}[t_\gamma^\alpha]$, for some $t_\gamma^\alpha \in 2^{<\kappa}$. Hence,

$$\bigcup_{\alpha<\rho}\bigcup_{\gamma<\kappa}[t_\gamma^\alpha] \in \mathcal{C}\Big(\bigcup_{\alpha<\rho} X_\alpha\Big)$$

We therefore get the following inequalities:

$$\mu^*\Big(\bigcup_{\alpha<\rho} X_\alpha\Big) \leq \mu^*\Big(\bigcup_{\alpha<\rho}\bigcup_{\gamma<\kappa}[t_\gamma^\alpha]\Big) = \sum_{\alpha<\rho}\sum_{\gamma<\kappa} \mu_{t_\gamma^\alpha} = \sum_{\alpha<\rho} \mu(O_\alpha) \leq$$

$$\leq \sum_{\alpha<\rho} \big(\mu^*(X_\alpha) + i_{\delta_\xi^\alpha}\big) = \sum_{\alpha<\rho} \mu^*(X_\alpha) + \sum_{\alpha<\rho} i_{\delta_\xi^\alpha} \leq \sum_{\alpha<\rho} \mu^*(X_\alpha) + i_{\delta_\xi},$$

where one can arbitrarily choose i_{δ_ξ} in order to be $\geq \sum_{\alpha<\rho} i_{\delta_\xi^\alpha}$.

□

We now prove that $\mathcal{F} \supseteq \mathcal{S}$ and it is closed under $\leq\kappa$-unions.

Fact 86. *For every $s \in 2^{<\kappa}$, for every $A \subseteq 2^\kappa$, one has*

$$\mu^*(A) = \mu^*(A \cap [s]) + \mu^*(A \setminus [s]).$$

Proof. By subadditivity, we only have to check $\geq$. W.l.o.g., one can assume $\mu^*(A) \neq \mathbf{1}$. Let $\langle \delta_\xi : \xi < \kappa \rangle$ be a sequence as in remark 83, with the further condition that for every $\xi < \kappa$, $\mu^*(A)(\delta_\xi) = 0$. We aim to show, for every $\xi < \kappa$,

$$\mu^*(A) + i_{\delta_\xi} \geq \mu^*(A \cap [s]) + \mu^*(A \setminus [s]).$$

Fix $\xi < \kappa$. Let $O = \bigcup_{\gamma<\kappa}[t_\gamma]$ such that $O \in \mathcal{C}(A)$ and $\mu(O) \leq \mu^*(A) + i_{\delta_\xi}$. Then, let $Z_\gamma^0 = [t_\gamma] \cap [s]$ and $Z_\gamma^1 = [t_\gamma] \setminus [s]$, for every $\gamma < \kappa$. Then set

$$O_0 = \bigcup_{\gamma<\kappa} Z_\gamma^0 \text{ and } O_1 = \bigcup_{\gamma<\kappa} Z_\gamma^1.$$

It is clear that $O_0 \in \mathcal{C}(A \cap [s])$ and $O_1 \in \mathcal{C}(A \setminus [s])$. Hence, the following inequalities conclude the proof:

$$\mu^*(A \cap [s]) + \mu^*(A \setminus [s]) \leq \mu(O_0) + \mu(O_1) = \sum_{\gamma<\kappa} \mu(Z_\gamma^0) + \sum_{\gamma<\kappa} \mu(Z_\gamma^1) =$$

$$= \sum_{\gamma<\kappa} \mu_{t_\gamma} = \mu(O) \leq \mu^*(A) + i_{\delta_\xi}.$$

□

Fact 87. *Let* $\{X_\alpha : \alpha < \kappa\}$ *be a family of sets in* $\mathcal{F}$*. Then* $\bigcup_{\alpha<\kappa} X_\alpha \in \mathcal{F}$*.*

Proof. Let $X = \bigcup_{\alpha<\kappa} X_\alpha$. Pick $A \subseteq 2^\kappa$ arbitrarily, such that $\mu(A) \neq \mathbf{1}$. Note that, by hypothesis

$$\forall\alpha < \kappa \forall B \subseteq 2^\kappa, \mu^*(B) = \mu^*(B \cap X_\alpha) + \mu^*(B \cap X_\alpha^c). \tag{3.1}$$

Let $\langle \delta_\xi : \xi < \kappa \rangle$ be the usual nice sequence and fixed $\xi < \kappa$. Pick $O = \bigcup_{\gamma<\kappa}[t_\gamma]$ such that $O \in \mathcal{C}(A)$ and $\mu(O) \leq \mu^*(A) + i_{\delta_\xi}$. Let

$$Z_\gamma^\alpha = [t_\gamma] \cap X_\alpha \text{ and } W_\gamma^\alpha = [t_\gamma] \cap X_\alpha^c$$

and note that

$$A \cap X \subseteq \bigcup_{\alpha<\kappa} \bigcup_{\gamma<\kappa} Z_\gamma^\alpha \text{ and } A \cap X^c \subseteq \bigcup_{\alpha<\kappa} \bigcup_{\gamma<\kappa} W_\gamma^\alpha.$$

Furthermore, by (3.1), $\mu^*(Z_\gamma^\alpha) + \mu^*(W_\gamma^\alpha) = \mu_{t_\gamma}$. Then

$$\mu^*(A \cap X) + \mu^*(A \cap X^c) \leq \sum_{\alpha<\kappa} \sum_{\gamma<\kappa} \mu^*(Z_\gamma^\alpha) + \sum_{\alpha<\kappa} \sum_{\gamma<\kappa} \mu^*(W_\gamma^\alpha) =$$

$$\sum_{\alpha<\kappa} \sum_{\gamma<\kappa} \mu_{t_\gamma} = \sum_{\alpha<\kappa} \mu(O) = \mu(O) \leq \mu^*(A) + i_{\delta_\xi}.$$

□

It remains to show the $\leq \kappa$-additivity for sets in $\mathcal{F}$. We first prove the ρ-additivity, for any $\rho < \kappa$.

Fact 88. *Let* $\rho < \kappa$*. For every* $\{X_\alpha : \alpha < \rho\}$ *family of sets in* $\mathcal{F}$*, one has*

$$\mu^*\Big(\bigcup_{\alpha<\rho} X_\alpha\Big) = \sum_{\alpha<\rho} \mu^*(X_\alpha).$$

Proof. The proof is by induction on $\rho < \kappa$. We already remarked that for finite unions the result holds, and moreover, such a proof also works for successor case. Let ρ be any limit cardinal such that $\omega \leq \rho < \kappa$ and assume for every $\beta < \rho$ one has the β-additivity. By subadditivity, one only has to show $\geq$.

Let $\delta \in \kappa \cdot \kappa$ such that $\sum_{\alpha<\rho} \mu^*(X_\alpha)(\delta) = 1$. Then, since the sum has size $< \kappa$, one finds $\eta < \rho$ such that $\sum_{\alpha<\eta} \mu^*(X_\alpha)(\delta) = 1$. By η-additivity, we know

$$\sum_{\alpha<\eta} \mu^*(X_\alpha) = \mu^*\Big(\bigcup_{\alpha<\eta} X_\alpha\Big),$$

and hence

$$\mu^*\Big(\bigcup_{\alpha<\rho} X_\alpha\Big)(\delta) = \mu^*\Big(\bigcup_{\alpha<\eta} X_\alpha\Big)(\delta) = 1.$$

□

Note that such a proof does not work for $\rho = \kappa$, since there could be some δ for which the value is 0 for every partial cut $< \kappa$ of the sum, but which becomes 1 at limit κ. Nevertheless, there is a very simple argument showing that one can get the κ-additivity, once we have the ρ-additivity, for every $\rho < \kappa$.

In fact one can easily note that

$$\mu^*\Big(\bigcup_{\alpha<\kappa} X_\alpha\Big) \geq \lim_{\rho<\kappa} \mu^*\Big(\bigcup_{\alpha<\rho} X_\alpha\Big),$$

since $\mu^*\Big(\bigcup_{\alpha<\kappa} X_\alpha\Big)(\delta) = 0$ trivially implies, for every $\rho < \kappa$,

$$\mu^*\Big(\bigcup_{\alpha<\rho} X_\alpha\Big)(\delta) = 0.$$

Thus, by ρ-additivity, we get

$$\mu^*\Big(\bigcup_{\alpha<\kappa} X_\alpha\Big) \geq \lim_{\rho<\kappa} \sum_{\alpha<\rho} \mu^*(X_\alpha) = \sum_{\alpha<\kappa} \mu^*(X_\alpha).$$

This concludes the demonstration that measure μ is well-defined and respects all requirements listed at the beginning. Note that, from now on, we will use μ also for indicating the outer measure μ^* for those elements in $\mathcal{F}$. Hence $\mu : \mathcal{F} \to \Omega$ is monotone and $\leq \kappa$-additive. It should be clear that μ is not translation-invariant (nevertheless, this should not represent a great obstacle in future developments, since the standard Lebesgue measure for ω^ω is not translation-invariant as well).

Once defined a measure on 2^κ, one can introduce many concepts related to it, in the same fashion of the standard case.

Definition 89. Let $X \subseteq 2^\kappa$. One says that X is *null* (or *measure zero*) iff there exists $\langle \delta_\xi : \xi < \kappa \rangle$ as in remark 83 and $\langle O_\xi : \xi < \kappa \rangle$ sequence of open sets such that, for every $\xi < \kappa$, $\mu(O_\xi) \leq i_{\delta_\xi}$ and $X \subseteq O_\xi$.

Remark 90. If $\{X_\alpha : \alpha < \kappa\}$ is a family of null sets, then also $\bigcup_{\alpha<\kappa} X_\alpha$ is null. In fact, given a nice sequence $\langle \delta_\xi : \xi < \kappa \rangle$, one can consider, for every ξ, another nice sequence $\langle \delta_\xi^\alpha : \alpha < \kappa \rangle$ such that $\sum_{\alpha<\kappa} i_{\delta_\xi^\alpha} \leq i_{\delta_\xi}$ and open sets O_ξ^α's such that, for every $\alpha < \kappa$, $\mu(O_\xi^\alpha) \leq i_{\delta_\xi^\alpha}$ and $X_\alpha \subseteq O_\xi^\alpha$. Thus, one obtains

$$\forall \xi < \kappa, \mu\Big(\bigcup_{\alpha<\kappa} O_\xi^\alpha\Big) \leq i_{\delta_\xi} \text{ and } X_\alpha \subseteq \bigcup_{\alpha<\kappa} O_\xi^\alpha.$$

In particular, it follows that the ideal

$$\mathcal{N}_\kappa =^{\mathbf{def}} \{X \subseteq 2^\kappa : X \text{ is null}\}$$

is κ-complete. By definition, it is straightforward that every set of size $\leq \kappa$ is null.

Now, we want to prove that the notion of null is very different from that one of meager.

Remark 91. There is $X \subset 2^\kappa$ null set, such that X^c is meager. To see that, fix an enumeration $\langle s_\gamma : \gamma < \kappa \rangle$ of $2^{<\kappa}$. Then, pick a nice sequence $\langle \delta_\xi : \xi < \kappa \rangle$ and for every $\xi < \kappa$, let

$$D_\xi = \bigcup_{\gamma < \kappa} [t^\xi_\gamma],$$

where each $t^\xi_\gamma \in 2^{<\kappa}$ such that $\forall \eta (\mu_{t^\xi_\gamma}(\eta) = 1 \Rightarrow \eta \in \delta_\xi)$ and $[t^\xi_\gamma] \subseteq [s_\gamma]$. It is clear that $\mu(D_\xi) \leq i_{\delta_\xi}$ and it is open dense. Thus $X = \bigcap_{\xi < \kappa} D_\xi$ has the required properties.

Once that one has a notion of null set, one can introduce, in standard way, a notion of Lebesgue measurability.

Definition 92. For every $X \subseteq 2^\kappa$, one says that X is *measurable* iff there exists $B \in \mathrm{BOR}$ such that $X \triangle B \in \mathcal{N}_\kappa$.

It is not hard to show that the family of all measurable subsets of 2^κ coincides with $\mathcal{F}$.

The following well-known facts for Lebesgue measure on 2^ω shift to 2^κ as well:

1. If $X \subseteq 2^\kappa$ is measurable then for every nice sequence $\langle \delta_\xi : \xi < \kappa \rangle$ there are O_ξ open set and C_ξ closed set such that $C_\xi \subseteq X \subseteq O_\xi$ and $\mu(O_\xi \setminus C_\xi) \leq i_{\delta_\xi}$.
2. If $X \subseteq 2^\kappa$ is measurable then there are $F \in \mathbf{\Pi}^0_2$ and $G \in \mathbf{\Sigma}^0_2$ such that $G \subseteq X \subseteq F$ and $\mu(F) = \mu(G) = \mu(X)$.
3. **AC** implies the existence of a non-measurable set.

Point 1 and 2 are immediate from the definition of μ^* given before. About point 3, one can easily build a non-measurable set X by using an enumeration of positive measure closed sets $\langle C_\alpha : \alpha < 2^\kappa \rangle$, as the following standard recursive construction shows: for every $\alpha < 2^\kappa$, pick

$$x_\alpha \in C_\alpha \wedge \forall \gamma < \alpha, x_\alpha \neq x_\gamma.$$

$$y_\alpha \in C_\alpha \wedge \forall \gamma \leq \alpha, y_\alpha \neq x_\gamma,$$

and then let $X_\alpha = \bigcup_{\gamma < \alpha} X_\gamma \cup \{x_\alpha\}$ and $Y_\alpha = \bigcup_{\gamma < \alpha} Y_\gamma \cup \{y_\alpha\}$. Finally, set

$$X = \bigcup_{\alpha < 2^\kappa} X_\alpha \text{ and } Y = \bigcup_{\alpha < 2^\kappa} Y_\alpha.$$

Clearly $X^c \supseteq Y$ and therefore neither X nor X^c can contain any closed set, which implies X is not measurable, by point 1.

These last elementary observations show that some natural properties of Lebesgue measure are preserved for our notion of measure as well. Another natural definition is that one of random forcing $\mathbb{R}^\kappa$.

Definition 93. One defines the *random forcing* $\mathbb{R}^\kappa$ as the poset consisting of Borel sets of positive measure, ordered by inclusion. By point 1 above, it is clear that an equivalent formulation is

$$\mathbb{R}^\kappa = \{C \subseteq 2^\kappa : C \text{ is closed} \wedge \mu(C) > 0\}.$$

A standard proof also shows that $\mathbb{R}^\kappa$ adds a generic $z_G \in 2^\kappa$, which we call random κ-real, and we denote with $\mathbb{R}^\kappa(\mathbf{V})$ the set of random κ-reals over $\mathbf{V}$.

The following result points out a crucial difference with the standard random forcing. (Note the similarity with the uncountable Sacks forcing).

Fact 94. *$\mathbb{R}^\kappa$ is not κ^κ-bounding.*

Proof. First of all, note that, for every $\gamma < \kappa$, for every $x \in \kappa^\kappa \cap \mathbf{V}$, the set

$$D_x^\gamma = \{C \in \mathbb{R}^\kappa : \exists \gamma' \geq \gamma \forall \xi \leq x(\gamma') + 1(\mu(C)(\kappa \cdot \gamma' + \xi) = 0)\}$$

is dense in $\mathbb{R}^\kappa$. This follows from the fact that for any C of positive measure there have to be cofinally many β's such that the set

$$\{\xi : \kappa \cdot \beta \leq \xi < \kappa \cdot (\beta + 1) \wedge \mu(C)(\xi) = 1\}$$

has size κ. Hence, if one defines

$$z_G(\gamma) = \min\{\xi : \exists C \in G(\mu(C)(\kappa \cdot \gamma + \xi) = 1)\},$$

where G is $\mathbb{R}^\kappa$-generic over $\mathbf{V}$, one obtains

$$\Vdash_{\mathbb{R}^\kappa} \forall \gamma \exists \gamma' \geq \gamma (z_G(\gamma') > x(\gamma')).$$

□

Nevertheless, even if $\mathbb{R}^\kappa$ adds unbounded κ-reals, by remark 91, we know that the random κ-real is not Cohen (nevertheless, $\mathbb{R}^\kappa$ may add Cohen reals anyway).

Similarly to Baire property, a possible connection between $\mathbb{R}^\kappa$ and Lebesgue measurability is represented by the following.

Conjecture. $\mathbf{\Delta}^1_1(\mathbb{R}^\kappa) \Rightarrow \forall x \in 2^\kappa(\mathbb{R}^\kappa(\mathbf{L}[x]) \neq \emptyset$.

The idea to prove that is essentially the same of fact 75. However, we do not know if Fubini's theorem holds for our generalized measure. We conclude with another interesting open question. It is known that the club filter Cub does not have the Baire property. Hence, the following question rises rather spontaneously.

Question 4. Is Cub Lebesgue measurable?

Bibliography

[**BJ95**] Tomek Bartoszyński, Haim Judah, *Set Theory-On the structure of the real line*, ***AK Peters Wellesley*** (1999).

[**BL99**] Jörg Brendle, Benedikt Löwe, *Solovay-type characterizations for forcing-algebra*, ***Association for Simbolic Logic***, Vol. 64 (1999), pp 1307-1323.

[**Br95**] Jörg Brendle, *Strolling through paradise*, ***Fundamenta Matematicae***, 148 (1995), pp 1-25.

[**BLH05**] Jörg Brendle, Benedikt Löwe, Lorenz Halbeisein, *Silver measurability and its relation to other regularity properties*, ***Mathematical Proceedings of the Cambridge Philosophical Society***, Vol. 138 (2005), pp 138-149.

[**DT98**] Carlos Augusto Di Prisco, Stevo Todorcevic, *Perfect-Set properties in* $\mathbf{L}(\mathbb{R})[U]$, ***Advanced in Mathematics***, Vol. 139 (1998), pp 240-259.

[**FHK10**] Sy D. Friedman, Tapani Hyttinen, Vadim Kulikov, *Generalized Baire space and classification theory.*

[**FS10**] Sy D. Friedman, David Schrittesser, *Projective measure without projective Baire*, Schrittesser's PhD thesis.

[**Ha03**] Lorenz Halbeisen, *Making doughnuts of Cohen reals*, ***Mathematical Logic Quarterly***, Vol. 49 (2003), pp 173-178.

[**HS01**] Aapo Halko, Saharon Shelah, *On strong measure zero subsets of* 2^κ, ***Fundamenta Matematicae***, 170(3):219-229, (2001).

[**H96**] Aapo Halko, *Negligible subsets of the generalized Baire space* $\omega_1^{\omega_1}$, ***Ann.Acc.Sci.Fenn.Math.Diss.***, (107):38, (1996).

[**Ik10**] Daisuke Ikegami, *Forcing absoluteness and regularity properties*, ***Annals of Pure and Applied Logic***, Vol. 161 (1999), pp 879-894.

[**Jech**] Thomas Jech, *Set theory*, ***Academic Press***, (1978).

[**JR93**] Haim Judah, Andrej Roslanowsky, *On Shelah's amalgamation*, ***Israel Mathematical Conference Proceedings***, Vol. 6 (1993), pp 385-414.

[**Ka80**] Akiro Kanamori, *Perfect-set forcing for uncountable cardinals*, ***Annals of Mathematical Logic***, 19 (1980), pp 97-114.

[**Kan**] Akiro Kanamori, *The higher infinite*, ***Springer***.

[**K12**] Yurii Khomskii, *Regularity properties and definability in the real number continuum*, ***ILLC Dissertation Series DS-2012-04***.

[**LR95**] Grzegorz Łabedzki, Miroslav Repický, *Hechler reals*, ***The Journal of Simbolic Logic***, Vol. 60 (1995), pp 444-458.

[**L96**] Grzegorz Łabedzki, *A topology generated by eventually different functions*, ***Proceedings of the 24th Winter School on Abstract Analysis. Charles University, Praha, 1996. Acta Universitatis Carolinae-Mathematica et Physica***, Vol. 37 n.2, pp 37-53.

[**Ra84**] Jean Raisonnier, *A mathematical proof of S.Shelah's theorem on the measure problem and related results*, ***Israel Journal of Mathematics***, Vol. 48 (1984), pp 48-56.

[**Sh84**] Saharon Shelah, *Can you take Solovay's inaccessible away?*, ***Israel Journal of Mathematics***, Vol. 48 (1985), pp 1-47.

[**Sh85**] Saharon Shelah, *On measure and category*, ***Israel Journal of Mathematics***, Vol. 52 (1985), pp 110-114.

[**So70**] Robert M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, ***Annals of Mathematics***, Vol. 92 (1970), pp 1-56.

[**Za00**] Jindřich Zapletal, *Descriptive set theory and definable forcing*, ***Memoirs of the American Mathematical society***, Num. 793 (200).

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Abstract

The paper is centered around the study of regularity properties of the real line. The notion of regularity is presented in a rather general way, by using arboreal forcings. In particular, we focus on questions concerning the separation of different regularity properties. More precisely, in some cases, given $\mathbb{P}, \mathbb{Q}$ arboreal forcings, we construct a model where all sets of reals are $\mathbb{P}$-measurable and a non-$\mathbb{Q}$-measurable set exists. A similar work is done for statements concerning the 2nd level of projective hierarchy. Finally, we also deal with questions about measure and category for the generalized Cantor space 2^κ, for κ uncountable cardinal. In particular, we introduce a new notion of measure on such a space, which allows us to define the corresponding notion of measurability and the related uncountable random forcing.

Die Arbeit befasst sich mit dem Studium von Regularitaetseigenschaften der reellen Zahlen. Der Begriff Regularitaet wird, dank Verwendung von sogenannten *arboreal forcings* allgemein eingefuehrt. Insberondere fokussieren wir uns auf die Frage der Trennung verschiedener Regularitaetseigenschaften. Genauergesagt, falls $\mathbb{P}$, $\mathbb{Q}$ arboreal forcings sind, konstruieren wir ein Modell indem saemtliche Teilmengen der reellen Zahlen $\mathbb{P}$-messbar sind, aber zugleich eine Menge existiert, die nicht $\mathbb{Q}$-messbar ist. Mit aehnlichen Mitteln werden auch Aussagen in der zweiten Ebene der projektiven Hierarchie untersucht. Schliesslich betrachten wir noch einige Fragen bezueglich Mass und Kategorie im verallgemeinerten Cantor Raum 2^κ, fuer ein uberabzaehlbares κ. Wir fuehren einen neuen Begriff fuer Mass in diesem Raum ein, der es uns erlaubt analoge Begriffe fuer Messbarkeit und ueberabzaehlbarem random forcing zu enfwickeln.

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